Carbon Pricing under Competition and Regulation in the Extraction Industry*

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Abstract

Metals and minerals are critical in the construction of solar panels, batteries and electric vehicles. However, the emissions produced through extraction must be controlled and monitored. The goal of the paper is twofold: a) First establish a unique Nash equilibrium in the context of N producers of a commodity that creates emissions in the extraction process and a regulator who has established a cap-and-trade mechanism for these emissions. b) Then extend the problem to a multi-economy setting in which miners and oil companies operate in several regions where they face several types of regulations, such as the Cross Border Mechanism (CBAM) which became effective in Europe in May 2023. Accordingly, we study the EU-traded carbon instruments which are fairly liquid at this moment and propose a two-factor model that is calibrated to traded carbon Futures and options. Our claim is that, as explained by Damon et al. (2019) grandfathering gives today too much advantage to the first emitters at the expenses of new entrants, while the cap-and-trade mechanism, using Carbon Permits for instance, cannot be easily extended to a global economy. Carbon Derivatives, instead, provide a classical and efficient way to complete the emissions market.

Keywords: CO_2 Emissions; Cap-and-trade; Nash Equilibrium; Carbon Derivatives.

JEL Classification: C72, G12, H21, Q54.

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1 Introduction

The goal of the paper is twofold: a) First establish a unique Nash equilibrium in the context of N producers of a commodity who create emissions in the extraction process and a regulator who has established a cap-and-trade mechanism for these emissions. b) Then extend the problem to a multi-economy setting in which big miners and oil companies operate in several regions where they face several types of regulations, such as the Cross Border Mechanism (CBAM) that has become effective in Europe in May 2023. Our claim is that carbon offsets preserving forests for instance are a beautiful concept but not easy to construct, presently at least. Instead, carbon derivatives purchased at the international level provide a safer avenue. Accordingly, we study the EU-traded carbon instruments which are reasonably liquid at this moment and propose a two-factor model that we calibrate on Carbon Futures and options

Within a cap-and-trade framework, the government institutes an emissions cap and promulgates an amount of emission permits proportional to that cap. Firms need to own permits for every ton of CO_2 they emit into the atmosphere (COP21, 2015). Firms can buy or sell permits in a carbon market that ultimately sets the emissions price. As in the second part of our paper, those with higher emission costs have the option to buy permits from firms that can lessen their emissions at lower costs. Hence, at core, the cap-and-trade program is a market-based incentive that establishes a regulatory stipulation whereby companies involved in the scheme meet an aggregate emissions cap and the regulator allows the compliance requirement to be traded across companies. Initially, permits are granted at a low price for existing users – a practice called grandfathering (Damon et al., 2019). Within a specific industry, companies that are covered can use their rights to emit CO_2 to compensate for their emissions. These rights must reach an equilibrium between the companies and the regulator. On the one hand, extracting companies must maximize profits by optimizing their extractions rates while operating under the cap. On the other hand, the regulator must maximize its tax rate while complying with the Paris Accord to limit the level of pollution in the atmosphere.

In the first part of our paper, we exhibit a unique Nash Equilibrium under which the regulator optimizes the tax rate it levies on the extracting company to minimize CO_2 emissions, while the extracting companies optimize the extraction rate of their mining operations to maximize their profits. Thus, the strategic interaction between the participants leads to different outcomes from those companies that are not covered under the cap-and-trade system. This interplay leads to a unique Nash Equilibrium in which the actions of each participant are the best response to the other participant's best response.

However, as a market-based approach, the cap-and-trade generates a few macro-economic concerns. First, the provision of allowances is such that emissions are reduced only by companies with a relatively lower cost of doing so. Second, in the cap-and-trade framework, the mandatory condition for alleviating the emission of CO_2 in the most cost-effective manner is that all firms must reduce CO_2 at the same marginal cost, which is not easy to reach in a real market. Lastly, since market-based perspectives create incentives for innovative technologies that can lessen the cost of decreasing emissions, companies are not equally equipped to compete in such a market. For example, BP recently entered the offshore wind market through a joint venture with Norway's Equinor[1]. This partnership will allow BP to significantly reduce its carbon emission in the US through the development of offshore wind projects. As a result, BP will have access to extra carbon permits they could sell in the carbon market, thus purposely creating for itself an advantage in terms of

[1] Reuters, 2020.

a global ESG profile compared to a company like Exxon for example, which has fewer renewable projects. We argue in the second part of the paper that the excess of allowances created by BP for itself can be sold as Carbon derivatives in California (the most liquid one in the US) or in the EU. Along similar lines, Microsoft corporation and the Australian farm Wilmot Cattle entered two years ago a partnership allowing Microsoft to use grazing cattle to offset its carbon footprint (via carbon sequestration)[2]. According to the deal, Microsoft bought half a million dollars' worth of carbon credits from Wilmot Cattle. Evidently, this deal gave a large advantage to Wilmot Cattle in the Australian farming industry as it used renewable technology to reduce its carbon emissions and sell its excess of allowances in the carbon market of its choice. Returning to the US, after the August 2022 passage of the Inflation Reduction Act (IRA), incentives for renewable energy technologies such as carbon sequestration, charging stations, and electric vehicles (EVs) have grown substantially (Matlock and Chesnick, 2022), thus creating further areas of imperfect competition and opportunities in the carbon market. These examples emphasize the existence of a state of intrinsic market imperfection within the cap-and-trade framework.

We extend the analysis of Aïd and Biagini (2023) to a multi-economy setting that represents the reality of the three largest iron ore miners[3] for instance, and to the existence of liquid Carbon Derivatives accessible to all economic players. We argue like Spilker and Nugent (2022) that a voluntary Carbon Derivatives market has many virtues going forward as metal miners and oil producers are more sensitive to their ESG ratings than to a geographically undefined regulator. Our Carbon Derivatives model is presented in section 3. In the next section (section 2), we introduce our carbon regulation and competition model. We conclude in section 4.

2 A Carbon Regulation and Competition Model

This section presents a stochastic differential game model blended with the theory of optimal control. The control variables are the tax rate $u_0(t)$ chosen by player 0 (the regulator) and the extraction rates $u_1(t), \cdots, u_n(t)$ of player $i_{i=1,\cdots n}$ (the commodity company). While accounting for the carbon cap ζ_i, we examine the model within a long-term lease setting and derive the optimal tax and extraction rates under a unique Nash Equilibrium assumption.

2.1 Problem Formulation

One of the intricacies of the regulation of carbon markets lies in the ability for firms to trade allowances over several periods. For that matter, we first consider a time horizon T, $0 < T \leq \infty$. We assume that the carbon market is regulated through an emissions tax rate $(u_0(t),\ 0 < t \leq T)$ and that all emitters (companies), subjected to the tax set by the regulator, extract the commodity (oil, gas, copper metal, etc.) at a rate $u_i(t)$, $i = 1, \cdots n$, $0 < t \leq T$. As said above, the regulator (or government) is denoted as player 0; player i is one of n companies.

Besides, the regulator chooses to set a cap level (ζ_i) on the amount of CO_2 player i can emit when extracting the commodity. As in Mandell (2008), we take as given an emission cap-and-trade regulatory protocol under which carbon allowances are issued by player 0. Each permit typically allows any player i to emit CO_2 emissions at a level not to exceed ζ_i $(u_i < \zeta_i)$. The regulator sets up the cap ζ_i at a level corresponding to the extraction rate $(u_i(t),\ i = 1, \cdots n)$ of each of the

[2] Agriland, 2021.

[3] Vale, Rio Tinto, BHP

player i, $i = 1, \cdots n$; it also seeks to maximize the income tax ($u_0(t)$, $0 < t \leq T$) it levies on each one of them. In parallel, each player i aims at maximizing its shares of profits from its extracting operations. Concomitantly, both player 0 and each player i share profits from the sales of the commodity following a rule subject to the share portion θ_i. The regulator expects $1 - \theta_i$ percent of the profits while each of the emitters expects θ_i percent of them. At $0 < t \leq T$, we denote the price X_t of the extracted commodity and choose it to be a mean reverting process with stochastic volatility along the lines of (Schwartz, 1997) and Heston (1993). The evolution of the profit sharing agreement between the regulator and each of the emitters follows the rules of a differential game problem with the following state variables:

$$X(t) \in \mathbb{R}^+ \quad \text{and} \quad Y_1(t), \cdots, Y_n(t) \in [0, K].$$

For each player $i = 1 \cdots n$, $Y_i(t) \in [0, K]$ is the amount of commodity produced or extracted. $K < \infty$ is the total amount of commodity available at the beginning of the lease. The state space $\mathbb{R}^+ \times [0, K]$ encapsulates the domain of the commodity price X_t as well as the amount of commodity extracted $Y(t)$. $\xi(t)$ represents the main driving process of the volatility σ_t. The process $\sigma_t = f(\xi_t)$ is the stochastic volatility process that captures the main noise generated by the dynamics of our commodity price. The pricing model we apply is an extension of the mean reverting one-factor model proposed by (Schwartz, 1997) and of the stochastic volatility model of Heston (1993). Each player $i = 1 \cdots n$, acting as a controller, aims at maximizing its own profit throughout the duration of the lease. The resulting differential game problem is characterized by the processes $X(t), Y(t) = \big(Y_1(t), \cdots, Y_n(t)\big)$ and follows the dynamics below.

$$\begin{cases} dX(t) &= X(t)\Big(\kappa\Big(\mu - \ln\big(X(t)\big)\Big)dt + \sigma_t dW(t)\Big) \\ d\xi_t &= k(\alpha - \xi_t)dt + \beta\sqrt{\xi_t}dB(t), \qquad \sigma_t = f(\xi_t) \\ dY_1(t) &= u_1(t)dt \\ & \vdots \\ dY_n(t) &= u_n(t)dt \\ X(s) &= x, \quad \xi(s) = \xi, \quad Y(s) = y \quad \text{where} \quad y = (y_1, \cdots, y_n), \quad 0 \leq s \leq t < \infty \end{cases} \tag{2.1}$$

where κ the mean-reverting rate of the commodity, k the mean-reverting rate of the stochastic volatility, μ the long-run mean, and σ the volatility of the commodity price are constants. The parameters α and β are also constants and are defined in Heston (1993) as the long run variance and the volatility of volatility, respectively. We also define the function $f(\xi_t) = \sqrt{\xi_t}$. The correlation between $W(t)$ and $B(t)$ is such that $dW(t)dB(t) = \rho dt$ for some $\rho \in (-1, 1)$. $W(t)$ and $B(t)$ are Wiener processes defined on a probability space $(\Omega, \mathcal{F}, P)$. The set U_0 represents the appropriate tax rates or fees levied by the player 0 (regulator). The set U_i embodies the extraction rates from the mining operations of each player $i = 1, \cdots n$. U_0 and U_i are such that

$$\begin{cases} u_0(t) \in U_0 = [\underline{u}_0, \overline{u}_0] \\ u_i(t) \in U_i = [\underline{u}_i, \overline{u}_i] \end{cases}$$

The processes $u_0(t), u_1(t), \cdots, u_n(t)$ are control variables since in cap-and-trade systems, the tax rate on CO_2 emissions depends on the production of each of the n players, $i = 1, ..., n$. We have $u_0(t) \in U_0 = [\underline{u}_0, \overline{u}_0]$[4] and $u_i(t) \in U_i = [\underline{u}_i, \overline{u}_i]$[5].

[4] $-1 \leq \underline{u}_0 \leq 1$ and $0 \leq \overline{u}_0 \leq 1$. $-1 \leq \underline{u}_0 < 0$ or negative tax would be considered a tax subsidy.
[5] $\underline{u}_i, \overline{u}_i > 0$.

Definition 2.1. *The tax rate $u_0(\cdot)$ taking values on $[\underline{u}_0, \overline{u}_0]$ and the extraction rates $u_1(\cdot), \cdots, u_n(\cdot)$ taking values on $[\underline{u}_i, \overline{u}_i]_{i=1\cdots n}$ are admissible controls with respect to the initial data $(s, x, y_i) \in [0, T] \times \mathbb{R} \times [0, K]$ if:*

- *For each player $i = 1 \cdots n,$, $Y(t) = \big(Y_1(t), \cdots, Y_n(t)\big) \in [0, K]$ and for $t \in [0, T]$, equation (2.1) has a unique solution with $X(s) = x$, $Y(s) = y_{y=(y_1,\cdots,y_n)}$, and $X(t) \in \mathbb{R}^+$.*
- *The processes $u_0(\cdot)$, $u_1(\cdot), \cdots, u_n(\cdot)$ are $\{\mathcal{F}_t\}_{t\geq 0}$-adapted with $\mathcal{F}_t = \sigma\big\{B(s), W(s); s \leq t\big\}$.*

Assumption 1. Each of the n players has an emission cap $\zeta_{i=1,\ldots,n} \in (0, 1]$ set by player 0. This cap can also be considered as an incentive parameter.
Assumption 2. Each of the n players $(i = 1, ..., n)$ knows its own emission cap ζ_i but does not know the emission cap of the other $n - 1$ players.
Assumption 3. The marketplace in which carbon permits are being traded is frictionless and liquid.

2.1.1 The Carbon Cap

In our carbon model, the admissibility condition requires that we take into consideration the extraction and taxation rates that depend on the information available to the players up to time t (see Definition 2.1). For $u_{i=1,\ldots,n}$, the extraction or production rate of the commodity extracted by player i at any given time t during the lifetime of the contract, we hold that the extraction cost function $C_i(x, u_i)$ of each player i is measurable and defined by (2.2).

$$C_i(x, u_i) = \eta_i x u_i + c_{i0} - \lambda_i \log(u_i^2), \tag{2.2}$$

where $c_{i0} \geq 0$ is the initial cost incurred by player i for setting up the extraction operations of the commodity. $0 < \eta_i < 1$ is the proportion of revenue allocated towards the cost. Convexity is a fundamental property of cost functions, which property can be verified for equation (2.2) with respect to u_i. While c_{i0} and λ_i are constant, for any player $i = 1 \cdots n$, the emission or penalty function is given by $p_i(u_i)$ and is evaluated via (2.3) or more specifically (2.4). In order to foster a nonpolluting environment, the government entity pegs the total amount of CO_2 emissions to the extraction volume u_i for each player i such that if u_i exceeds the cap ζ_i, the emitter is taxed by the amount $\lambda_i \ln\big(\frac{u_i}{\zeta_i}\big)$. $\lambda_i > 0$ plays two roles; in (2.2), it is a cost control parameter specific to each extracting firm i (PWC, 2012) while in (2.3) and (2.4) it represents an incentive parameter, proportional to the carbon allowances auction quote (Cramton and Kerr, 2002). Typically, within a cap-and-trade mechanism, the total amount of the cap is divided into permits, each of which allows the company to emit one ton of CO_2. The regulator allocates these permits to companies for free or through an auction. Over a period, the cap normally gets smaller, bestowing a growing incentive for companies to shrink their emissions more efficiently, while retaining their production costs down[6]. If u_i is less than ζ_i, then player i is allowed to trade its excess permits with other companies in the carbon market. As mentioned above, we amply elaborate on this in the second part of the paper.

$$p_i(u_i) = \begin{cases} 0 & \text{if } u_i \leq \zeta_i \\ \lambda_i \ln\big(\frac{u_i}{\zeta_i}\big) & \text{if } u_i > \zeta_i. \end{cases} \tag{2.3}$$

[6] Environmental Defense Fund, 2020.

Although the penalty function (2.3) expresses the capping mechanism, it is not continuous and thereby not differentiable. To palliate this hurdle, we propose another function (2.4)[7] which has similar features but is differentiable.

$$\hat{p}_i(u_i) = \lambda_i \ln\Big(\frac{u_i}{\zeta_i}\Big). \tag{2.4}$$

The penalty function (2.4) has two roles. First, if the extraction rate u_i is well below the regulated cap ($u_i \ll \zeta_i$), then $\hat{p}(u_i) < 0$ and (2.4) acts as an incentive function. In times of commodity scarcity, commodity companies sometimes received incentives and tax exoneration (Staats, 1974). Equation (2.4) encapsulates this important empirical fact. Secondly, if $u_i > \zeta_i$ then $\hat{p}_i(u_i) > 0$. In this case, the commodity company is penalized for its excess commodity production and (2.4) acts as a penalty function. Diverging from traditional policy approaches such as command and control, by setting $u_i < \zeta_i$, we capture the fact that the cap set by the regulator characterizes the maximum amount of allowable CO_2 player i can emit(Cramton and Kerr, 2002). In doing so, we ensure that effective enforcement will deter all the n players from emitting above the cap level (see Figure 4).

2.1.2 The Setup

At any given time $t \in [0, T]$, each of the n players produces the commodity at the following total profit rate

$$P_i\big(X(t), u_i(t)\big) = X(t)u_i(t) - \Big(C_i\big(X(t), u_i(t)\big) + \hat{p}_i(u_i)\Big),\ i = 1, \cdots, n, \tag{2.5}$$

where C_i is defined in equation (2.2). Equation (2.5) implies that the pre-tax profit rate function of each extracting company is

$$\theta_i P_i\big(X(t), u_i(t)\big),\ i = 1, \cdots, n.$$

Accordingly, the profit rate function of the government entity without the tax revenue is

$$\sum_{i=1}^{n}(1-\theta_i)P_i\big(X(t), u_i(t)\big),\ i = 1, \cdots, n.$$

So, the total income tax the government entity levies on each of the n commodity companies is given by

$$\sum_{i=1}^{n} u_0(t)\theta_i P_i\big(X(t), u_i(t)\big),\ i = 1, \cdots, n,$$

where u_0 is the tax rate. As a result, the government (player 0)'s profit rate function is given by (2.6) while the post-tax profit rate function of each of the companies (player i) is expressed by (2.7).

$$L_0\big(X(t), u_0(t)\big) = \sum_{i=1}^{n} P_i\big(X(t), u_i(t)\big)\big((1-\theta_i) + u_0(t)\theta_i\big),\ i = 1, \cdots, n;\ t < T. \tag{2.6}$$

$$L_i\big(X(t), u_0(t), u_i(t)\big) = \theta_i P_i\big(X(t), u_i(t)\big)\big(1 - u_0(t)\big),\ i = 1, \cdots, n;\ 0 \le t < T. \tag{2.7}$$

At the end of the lease, we assume that there are no extraction revenues. Hence, the profit rate of each commodity company is either zero or equal to the cost of closing the mine. Thus, the terminal

[7] Equation (2.4) is obviously a concave function with respect to u_i.

profit rate of the government is the market value of the remaining commodity reserve and is given by

$$\Phi_0\big(X(T)\big)= X(T)\sum_{i=1}^{n}\big(K-Y_i(T)\big),\ i=1,\cdots,n.$$

Meanwhile, we define the terminal profit rate of each commodity company by

$$\Phi_i\big(X(T),Y_i(T)\big)= X(T)Y_i(T),\ i=1,\cdots,n.$$

Without loss of generality, for $i=0,\cdots,n$, we hold that the running profit rate function L_i and the terminal profit rate function Φ_i are Lipschitz continuous on their corresponding bounded sets. As a result, given a discount rate $r>0$, the payoff functionals of player 0 and of each of the n players are given by (2.8) and (2.9), respectively[8].

$$\begin{aligned} J_0(s,x,y,\xi;u_0,u_{-0}) =&E\bigg[\int_s^T e^{-r(t-s)}L_0\big(X(t),u_0(t),u_{-0}(t)\big)dt \\ &+e^{-r(T-s)}\Phi_0\big(X(T)\big)\bigg|X(s)=x,Y_i(s)=y_i,\xi(s)=\xi\bigg],\ i=1,\cdots,n. \end{aligned} \tag{2.8}$$

$$\begin{aligned} J_i(s,x,y,\xi;u_i,u_{-i}) =&E\bigg[\int_s^T e^{-r(t-s)}L_i\big(X(t),u_i(t),u_{-i}(t)\big)dt \\ &+e^{-r(T-s)}\Phi_i\big(X(T),Y_i(T)\big)\bigg|X(s)=x,Y_i(s)=y_i,\xi(s)=\xi\bigg],\ i=1,\cdots,n. \end{aligned} \tag{2.9}$$

Player 0, knowing the strategy space and the profit rate function of each of the n players, attunes its tax rate $u_0(\cdot)$ in order to maximize its payoff functional (2.8). Equivalently, by choosing a suitable strategy with knowledge of the space of admissible controls of player 0 and of its profit rate function, each of the n players maximizes its payoff functional (2.9) through the adjustment of its extraction rate $u_i(\cdot)$. Player 0 and each of the n players do not have any information related to the strategy either of them is currently using. This interaction corresponds to a non-cooperative differential game in which co-operation only arises at an equilibrium. Our aim is to find such an equilibrium, namely a non-cooperative Nash equilibrium $u^*=(u_0^*,u_1^*,...,u_n^*)$ such that, for $y=(y_1,\cdots,y_n)$,

$$J_i(s,x,y,\xi;u_i^*,u_{-i}^*)\geq J_i(s,x,y,\xi;u_i,u_{-i}^*),\forall\, u_i(\cdot)\in\mathcal{U}_i(s,x,y_i,\xi),\ i=0,\cdots,n. \tag{2.10}$$

If the payoff functional of each player is at least equal to the payoff functional that would be earned by wielding another strategy while the behavior of the other player stays constant, then the optimal strategy profile $u^*=(u_0^*,u_1^*,...,u_n^*)$ or unique Nash Equilibrium would be obtained. We prove its existence in the next section.

2.2 Nash Equilibrium

Definition 2.2. *If $u^*=(u_0^*,u_1^*,\cdots,u_n^*)$ is the Nash equilibrium of our differential game problem (2.1), then, for $i=1,\cdots,n$, the functions defined on $[0,T]\times\mathbb{R}^+\times[0,K]^n$*

$$V_0(s,x,y,\xi)=\sup_{u_0\in\mathcal{U}_0} J_0(s,x,y,\xi;u_0,u_{-0}^*) \tag{2.11}$$

[8] $u_{-i}=(u_j)_{j\in\mathbb{N},j\neq i}$ is the action profile of all players except i with $(u_j)_{j\in\mathbb{N}}=(u_0,\cdots,u_n)$.

$$V_i(s,x,y,\xi) = \sup_{u_i \in \mathcal{U}_i} J_i(s,x,y,\xi;u_i,u^*_{-i}) \tag{2.12}$$

are called value functions of player 0 and of each of the n players, respectively.

The value functions (2.11) and (2.12) must satisfy the Hamilton Jacobi Isaacs equations, the solutions of which will provide the sufficient and necessary condition for the existence and determination of a unique Nash Equilibrium $u^* = (u^*_0, u^*_1, \cdots, u^*_n)$. To resolve the Hamilton Jacobi Isaacs equations, we must first define the Hamiltonians corresponding to the value functions (2.11) and (2.12). These Hamiltonians are respectively given by (2.13) and (2.14).

$$\begin{aligned} H_0\Big(s,x,y,\xi;V_0,\nabla V_0,\frac{\partial^2 V_0}{\partial x^2},\frac{\partial^2 V_0}{\partial x\partial \xi},\frac{\partial^2 V_0}{\partial \xi^2}\Big) = rV_0 - \sup_{u_0\in U_0}\Bigg\{ & \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V_0}{\partial x^2} + \kappa\big(\mu - \ln(x)\big)x\frac{\partial V_0}{\partial x} \\ & + \sum_{i=1}^{n} u^*_i \frac{\partial V_0}{\partial y_i} + \kappa\big(\mu - \ln(x)\big)x\frac{\partial V_0}{\partial x} \\ & + k(\alpha-\xi)\frac{\partial V_0}{\partial \xi} + \rho\beta\xi x\frac{\partial^2 V_0}{\partial x\partial \xi} \\ & + \frac{1}{2}\beta^2\xi\frac{\partial^2 V_0}{\partial^2 \xi} + L_0(x,u_0,u^*_{-0})\Bigg\}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} H_i\Big(s,x,y,\xi;V_i,\nabla V_i,\frac{\partial^2 V_i}{\partial x^2},\frac{\partial^2 V_i}{\partial x\partial \xi},\frac{\partial^2 V_i}{\partial \xi^2}\Big) = rV_i - \sup_{u_i\in U_i}\Bigg\{ & \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V_i}{\partial x^2} + \kappa\big(\mu - \ln(x)\big)x\frac{\partial V_i}{\partial x} \\ & + \sum_{j=1,j\neq i}^{n} u^*_j \frac{\partial V_i}{\partial y_j} + u_i\frac{\partial V_i}{\partial y_i} \\ & + \kappa\big(\mu - \ln(x)\big)x\frac{\partial V_i}{\partial x} + k(\alpha-\xi)\frac{\partial V_i}{\partial \xi} \\ & + \rho\beta\xi x\frac{\partial^2 V_i}{\partial x\partial \xi} + \frac{1}{2}\beta^2\xi\frac{\partial^2 V_i}{\partial^2 \xi} + L_i(x,u_i,u^*_{-i})\Bigg\}. \end{aligned} \tag{2.14}$$

For $i = 1, \cdots, n$, the Hamilton Jacobi Isaacs equations corresponding to the Hamiltonians (2.13) and (2.14) are

$$\begin{cases} \dfrac{\partial V_0}{\partial s} = H_0\Big(s,x,y,\xi;V_0,\nabla V_0,\dfrac{\partial^2 V_0}{\partial x^2},\dfrac{\partial^2 V_0}{\partial x\partial \xi},\dfrac{\partial^2 V_0}{\partial \xi^2}\Big), & (s,x,y) \in [0,T) \times \mathbb{R}^+ \times [0,K]^n \\ \dfrac{\partial V_i}{\partial s} = H_i\Big(s,x,y,\xi;V_i,\nabla V_i,\dfrac{\partial^2 V_i}{\partial x^2},\dfrac{\partial^2 V_i}{\partial x\partial \xi},\dfrac{\partial^2 V_i}{\partial \xi^2}\Big), & (s,x,y) \in [0,T) \times \mathbb{R}^+ \times [0,K]^n \\ V_0(T,x,y_i) = \Phi_0(x), & x \in \mathbb{R}^+ \\ V_i(T,x,y_i) = \Phi_i(x,y), & (x,y) \in \mathbb{R}^+ \times [0,K]^n \end{cases} \tag{2.15}$$

We can show that (2.15) has a unique viscosity solution by applying the classical results on viscosity solutions (Crandall and Lions, 1983), (Pemy, 2022).

2.3 The Long Term Lease Framework

It is well established that regulators often lease lands and implement their carbon offset program. For example, the Biden administration recently approved a 30 year lease eight billion dollars Willow oil project that will allow ConocoPhillips' to produce more than 600 million barrels of crude oil in the state of Alaska[9]. Emitters also benefit from long-term leases through the securing of better financing terms resulting in preferable investment decisions. To better supervise lands use, lease contracts usually limit the total amount of lands for commodity extraction. They also specify the estimated associated extraction costs. For example, according to a New York Times article[10], in March 2023, in order to block oil and gas leases on more than 13 million of the 23 million acres that make up the National Petroleum Reserve in the state of Alaska, the Interior Department issued new restrictive rules.

In our analysis, at the beginning of the lease, each of the n player, $(i = 1, \cdots n)$, enters a long-term contract with player 0 or the regulator. This contract guarantees that the state variables remain within the state space throughout the duration of the lease[11]. Most extracting leases are divided into a primary and a secondary period. However, when extracting operations are not immediately effectuated, primary leases mandated renewable ten-year terms which often call for delay rental payments (Fambrough, 2015), (Smith, 2018). Hence, there are numerous instances where primary terms of commodity leases are extended over several years. To capture this contingency, in equation (2.1), we let $T \to \infty$[12] and obtain

$$\begin{cases} dX(t) &= X(t)\Big(\kappa\Big(\mu - \ln\big(X(t)\big)\Big)dt + \sigma_t dW(t)\Big) \\ d\xi_t &= k(\alpha - \xi_t)dt + \beta\sqrt{\xi_t}dB(t), \qquad \sigma_t = f(\xi_t), \\ X(0) &= x, \quad \xi(0) = \xi, \qquad 0 \le t < \infty. \end{cases} \tag{2.16}$$

For r the risk-free rate, the corresponding payoff functionals for the government entity and for each of the commodity firms i are given by (2.17) as follows

$$J_i(x, \xi; u_i, u^*_{-i}) = E\left[\int_0^\infty e^{-rt} L_i\Big(X(t), u_i(t), u_{-i}(t)\Big)dt \,\middle|\, X(0) = x, \xi(0) = \xi\right], i = 0, \cdots, n. \tag{2.17}$$

Equivalently, the value functions of the government entity and of each of the n commodity firms $i = 1, \cdots, n$ are denoted by

$$V_0(x, \xi) = \sup_{u_0 \in \mathcal{U}_0} J_0(x, \xi; u_0, u^*_{-0}) \tag{2.18}$$

$$V_i(x, \xi) = \sup_{u_i \in \mathcal{U}_i} J_i(x, \xi; u_i, u^*_{-i}) \tag{2.19}$$

The Hamiltonians associated with the value functions (2.18) and (2.19) are given by (2.20) and (2.21) for player 0 and player i, respectively.

$$\begin{aligned} H_0\Big(x, \xi; V_0, \frac{\partial V_0}{\partial x}, \frac{\partial V_0}{\partial \xi}, \frac{\partial^2 V_0}{\partial x^2}, \frac{\partial^2 V_0}{\partial \xi \partial x}, \frac{\partial^2 V_0}{\partial \xi^2}\Big) =& rV_0 - \sup_{u_0 \in U_0}\Big\{\frac{1}{2}\xi x^2 \frac{\partial^2 V_0}{\partial x^2} + \kappa\big(\mu - \ln(x)\big)x\frac{\partial V_0}{\partial x} \\ &+ k(\alpha - \xi)\frac{\partial V_0}{\partial \xi} + \rho\beta\xi x\frac{\partial^2 V_0}{\partial x \partial \xi} + \frac{1}{2}\beta^2\xi\frac{\partial^2 V_0}{\partial^2 \xi} \\ &+ L_0(x, u_0, u^*_{-0})\Big\}, \end{aligned} \tag{2.20}$$

[9] Financial Times, 2023.
[10] New York Times, 2023.
[11] $X(t) \in \mathbb{R}^+$ and $Y_1(t), \cdots, Y_n(t) \in [0, K]$.
[12] See for example Golosov et al. (2014)

$$H_i\Big(x,\xi;V_i,\frac{\partial V_i}{\partial x},\frac{\partial V_i}{\partial \xi},\frac{\partial^2 V_i}{\partial x^2},\frac{\partial^2 V_i}{\partial \xi\partial x},\frac{\partial^2 V_i}{\partial \xi^2}\Big) = rV_i - \sup_{u_i\in U_i}\Big\{\frac{1}{2}\xi x^2\frac{\partial^2 V_i}{\partial x^2} + \kappa\big(\mu-\ln(x)\big)x\frac{\partial V_i}{\partial x} + k(\alpha-\xi)\frac{\partial V_i}{\partial \xi} + \rho\beta\xi x\frac{\partial^2 V_i}{\partial x\partial \xi} + \frac{1}{2}\beta^2\xi\frac{\partial^2 V_i}{\partial^2 \xi} + L_i(x,u_i,u^*_{-i})\Big\}. \tag{2.21}$$

The Hamilton Jacobi Isaacs equations corresponding to the Hamiltonians (2.20) and (2.21) are

$$\begin{cases} H_0\Big(x,\xi;V_0,\frac{\partial V_0}{\partial x},\frac{\partial V_0}{\partial \xi},\frac{\partial^2 V_0}{\partial x^2},\frac{\partial^2 V_0}{\partial \xi\partial x},\frac{\partial^2 V_0}{\partial \xi^2}\Big) = 0, & (x,\xi)\in\mathbb{R}\times\mathbb{R}^+, \\ H_i\Big(x,\xi;V_i,\frac{\partial V_i}{\partial x},\frac{\partial V_i}{\partial \xi},\frac{\partial^2 V_i}{\partial x^2},\frac{\partial^2 V_i}{\partial \xi\partial x},\frac{\partial^2 V_i}{\partial \xi^2}\Big) = 0, & (x,\xi)\in\mathbb{R}\times\mathbb{R}^+;\ i=1,\cdots,n. \end{cases} \tag{2.22}$$

Theorem 2.3. *There exists a unique Nash equilibrium $u^*=(u^*_0,u^*_1,....,u^*_n)$ and a threshold commodity price $\hat{x}$ defined as follows*

$$\hat{x} = \exp\left(\frac{\sum_{i=1}^n \theta_i\Big(\lambda_i - c_{i0} + \lambda_i\ln\big(\frac{\lambda_i\zeta_i}{1-\eta_i}\big)\Big)}{\sum_{i=1}^n\theta_i\lambda_i}\right), \tag{2.23}$$

such that

$$u^*_0(x) = \begin{cases} \overline{u}_0 & \text{if } x<\hat{x} \\ \underline{u}_0 & \text{if } x\geq\hat{x}, \end{cases} \tag{2.24}$$

and

$$u^*_i = \frac{\lambda_i}{(1-\eta_i)x}, \quad x\neq 0, \quad i=1,...,n. \tag{2.25}$$

Moreover, the solutions of the Hamilton Jacobi equations (2.22) are

$$V_0(x,\xi) = A_0\ln(x) + g_0(\xi), \quad V_i(x,\xi) = A_i\ln(x)+g_i(\xi),\ i=1,\cdots,n, \tag{2.26}$$

such that

$$A_0 = \frac{\sum_{i=1}^n \lambda_i\Big((1-\theta_i) + \overline{u}_0\mathbf{1}_{\{x,x<\hat{x}\}}\theta_i + \underline{u}_0\mathbf{1}_{\{x,x\geq\hat{x}\}}\theta_i\Big)}{r+\kappa}, \tag{2.27}$$

$$A_i = \begin{cases} \dfrac{\theta_i\lambda_i(1-\overline{u}_0)}{r+\kappa} & \text{if } x<\hat{x} \\ \dfrac{\theta_i\lambda_i(1-\underline{u}_0)}{r+\kappa} & \text{if } x\geq\hat{x}, \quad i=1,...,n, \end{cases} \tag{2.28}$$

and the functions g_i, $i=0,...,n$ satisfy the differential equations

$$\xi g''_i(\xi) + \frac{2k(\alpha-\xi)}{\beta^2}g'_i(\xi) - \frac{2r}{\beta^2}g_i(\xi) = \frac{\xi A_i}{\beta^2} + C_i, \tag{2.29}$$

where

$$C_0 = -\frac{2\kappa\mu A_0}{\beta^2} - \frac{2}{\beta^2}\sum_{i=1}^n\left(\lambda_i - c_{i0} + \lambda_i\ln\Big(\frac{\lambda_i\zeta_i}{1-\eta_i}\Big)\right)\big((1-\theta_i) + \overline{u}_0\mathbf{1}_{\{x,x<\hat{x}\}}\theta_i + \underline{u}_0\mathbf{1}_{\{x,x\geq\hat{x}\}}\theta_i\big). \tag{2.30}$$

For $i = 1, ..., n$ we have

$$C_i = \begin{cases} -\frac{2\kappa\mu A_i}{\beta^2} - \dfrac{2\theta_i}{\beta^2}\Bigg(\dfrac{\lambda_i}{1-\eta_i} - c_{i0} + \lambda_i \ln\Big(\dfrac{\lambda_i\zeta_i}{1-\eta_i}\Big)\Bigg)(1-\overline{u}_0) & \text{if } x < \hat{x} \\ -\frac{2\kappa\mu A_i}{\beta^2} - \dfrac{2\theta_i}{\beta^2}\Bigg(\dfrac{\lambda_i}{1-\eta_i} - c_{i0} + \lambda_i \ln\Big(\dfrac{\lambda_i\zeta_i}{1-\eta_i}\Big)\Bigg)(1-\underline{u}_0) & \text{if } x \geq \hat{x}. \end{cases} \tag{2.31}$$

We prove this crucial Theorem in Appendix A.1. Also, see Figure 2 and Figure 3. See Table 4 for a summary of all notations.

3 A Two-Factor Model for Pricing Carbon Derivative

As previously discussed, a fundamental caveat with the cap-and-trade protocol is that it can create an imperfect market through an unequal issuance of allowances. In this section, our attempt is to resolve this imperfection by constraining the allocation of carbon permits to the emission rate (u_t) and the underlying carbon permit (Z_t). Accordingly, we propose an original two-factor approach to price a carbon derivative by factoring in the emission rate, which we construe as a mean-reverting process, and a carbon permit, which is random and follows a geometric brownian process (Vasicek, 1977). Denoting σ_1 the volatility of the carbon permit and σ_2 the volatility of the emission rate, the dynamics of our novel carbon model are characterized by

$$\begin{cases} dZ_t &= Z_t\big((\mu_1 + \lambda u_t)dt + \sigma_1 dW_{1,t}\big) \\ du_t &= (m - bu_t)dt + \sigma_2 dW_{2,t} \\ Z_{t_0} &= z, \quad u_{t_0} = u, \quad 0 \leq t_0 \leq t < \infty, \quad b > 0, \end{cases} \tag{3.1}$$

where μ_1 measures the average rate of growth of the carbon permit, λ is a coefficient that measures how much the emission rate contributes to the drift of the carbon permit, m represents the mean of the mean reverting process, and b stands for the mean coefficient (measures how much the emission rate revolves around the mean). The processes $W_{1,t}$ and $W_{2,t}$ are correlated Wiener processes with correlation $\rho \in (-1, 1)$ such that $dW_{1,t}dW_{2,t} = \rho dt$. $W_{1,t}$ and $W_{2,t}$ are defined on the risk-neutral probability space $(\Omega, \mathcal{F}_t, \mathbb{Q})$ with filtration $\mathcal{F}_t = \sigma\Big(\{W_{t_0}, t_0 \leq t\}\Big)$. At strike price K and maturity t, the call written on a carbon permit is

$$C(t_0, t, z, u, K) = E^Q\Big[e^{-r(t-t_0)}\Big(\max(Z_t - K, 0)\Big)\Big]$$

and the price of a put option is

$$P(t_0, t, z, u, K) = E^Q\Big[e^{-r(t-t_0)}\Big(\max(K - Z_t, 0)\Big)\Big].$$

Lemma 3.1. *The processes Z_t and u_t that solve (3.1) are given as follows*

$$Z_t = Z_{t_0} \exp\left(\frac{\lambda}{b}u_{t_0} + (\mu_1 - \frac{1}{2}\sigma_1^2 + \frac{\lambda}{b}m)(t-t_0) - \frac{\lambda}{b}\Big(e^{-b(t-t_0)}u_{t_0} + \frac{m}{b}(1 - e^{-b(t-t_0)})\Big) + W_{3,\sigma_3(t_0,t)^2}\right) \tag{3.2}$$

and

$$u_t = e^{-b(t-t_0)}u_{t_0} + \frac{m}{b}(1 - e^{-b(t-t_0)}) + \sigma_2\int_{t_0}^t e^{-b(t-\xi)}dW_{2,\xi}, \tag{3.3}$$

with

$$\sigma_3(t_0,t)^2 = \sigma_1^2(t-t_0) + \frac{\lambda^2\sigma_2^2}{b^2}(\sqrt{t-t_0} - \sqrt{h(t_0,t)})^2 + 2\rho\frac{\sigma_1\sigma_2\lambda\sqrt{t-t_0}|\sqrt{t-t_0} - \sqrt{h(t_0,t)}|}{b} \quad (3.4)$$

and

$$h(t_0,t) = \tfrac{1}{2b}(1 - e^{2b(t-t_0)}).$$

3.1 Carbon Futures Contracts

Since carbon permits are essentially traded in futures and forwards, in this section, we derive the price of an European option on a Futures contract. To do so, we assume that a futures contract on a carbon permit has maturity T_1. We then consider a European option on this futures contract with maturity T. It is evident that $T_1 \geq T$. Let $(F(t;T_1))_t$ denote the futures price where t is a variable time in the future. By construction of Q, we have

$$F(t;T_1) = \mathbb{E}^Q\left[Z_{T_1}|\mathcal{F}_t\right]. \quad (3.5)$$

For K the strike price of a European call/put option, the European call price is

$$C(t,z,u,K,T) = \mathbb{E}^Q\Big[e^{-r(T-t)}\max\big(F(t;T_1) - K, 0\big)|Z_t = z, u_t = u\Big].$$

For a speculative initial time t_0, the European put price is

$$P(t,z,u,K,T) = \mathbb{E}^Q\Big[e^{-r(T-t_0)}\big(\max(K - F_{T_1}, 0\big)|F_{t_0} = z, u_{t_0} = u\Big].$$

Assuming that a carbon permit is a financial asset, the discounted process $e^{-rT}Z_T$ is a martingale under the equivanlent probability measure Q (Geman, 2005). Hence, we can simplify (3.5) as follows

$$F(t;T_1) = e^{r(T_1-t)}\mathbb{E}^Q\Big[e^{-r(T_1-t)}Z_{T_1}|\mathcal{F}_t\Big] = e^{r(T_1-t)}Z_t. \quad (3.6)$$

As a matter of fact, we have

$$C(t,z,u,K,T,T_1) \quad = \mathbb{E}^Q\Big[e^{-r(T-t)}\big(F(t;T_1) - K\big)^+|Z_t = z, u_t = u\Big] \quad (3.7)$$

Theorem 3.2. *The call option on a carbon Futures is given by*

$$\begin{aligned} C(t,u,K,T,T_1) \quad &= e^{-r(T-t)}\Bigg(F(t,T_1)e^{\kappa(t,T,u)+\frac{1}{2}\sigma_3^2(t,T)}N\Bigg(\frac{\ln\big(\frac{F(t,T_1)}{K}\big)+\kappa(t,T,u)}{\sigma_3(t,T)} + \sigma_3(t,T)\Bigg) \\ &-KN\Bigg(\frac{\ln\big(\frac{F(t,T_1)}{K}\big) + \kappa(t,T,u)}{\sigma_3(t,T)}\Bigg)\Bigg) \end{aligned} \quad (3.8)$$

and the put option on a carbon Futures is given by

$$\begin{aligned} P(t,u,K,T,T_1) \quad &= e^{-r(T-t)}\Bigg(KN\Bigg(-\frac{\ln\big(\frac{F(t,T_1)}{K}\big) + \kappa(t,T,u)}{\sigma_3(t,T)}\Bigg) \\ &-F(t,T_1)e^{\kappa(t,T,u)+\frac{1}{2}\sigma_3^2(t,T)}N\Bigg(-\frac{\ln\big(\frac{F(t,T_1)}{K}\big) + \kappa(t,T,u)}{\sigma_3(t,T)} - \sigma_3(t,T)\Bigg)\Bigg), \end{aligned} \quad (3.9)$$

where

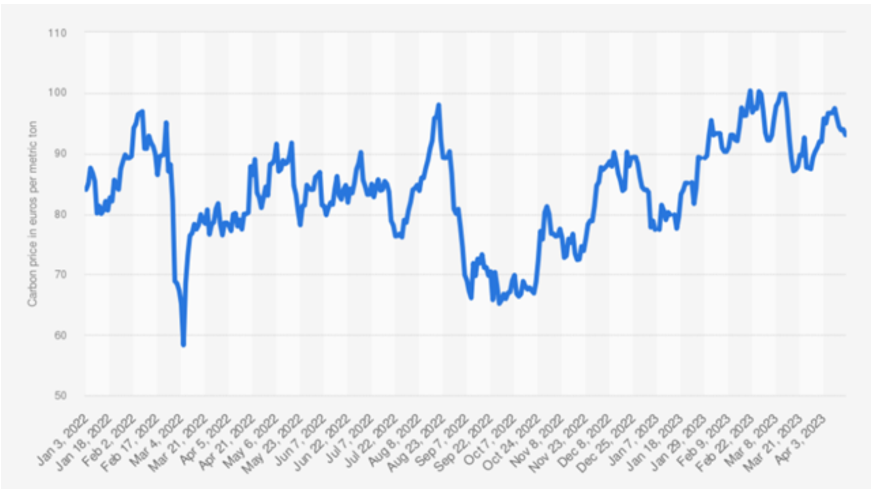
$$\kappa(t,T,u) := \frac{\lambda}{b}u + (\frac{\lambda}{b}m - \frac{1}{2}\sigma_1^2)(T-t) - \frac{\lambda}{b}\Big(e^{-b(T-t)}u + \frac{m}{b}(1-e^{-b(T-t)})\Big) \tag{3.10}$$

Figure 5 illustrates the application of this Theorem[13] *for which the proof is given in Appendix A.3.*

3.2 Data

We calibrate our model using ICE European Carbon Emission Allowance (EUA) Futures data from The ICE (2022). An European Carbon Emission Allowance (EUA) is a financial instrument that entitles the emission of one metric ton of CO_2 to participants under the European Union Emissions Trading System (EU-ETS). All EU-ETS participants have the obligation to surrender enough EUAs to cover all their emissions at the end of each compliance cycle. The carbon allowance or carbon permit price is the price of emitting one metric ton of CO_2. Figure 1 shows recent trends of EUA prices. From this Figure[14], we see that, since 2018, the average annual price of carbon permits in the EU has increased significantly.Data on emission rates are taken from Statista (2022). Given the unstructured nature of the initial dataset, we used Python 3.10 (64 bit) for massaging and extracting the information needed for the calibration. We formulate and write several m-files using Matlab R2022a (64 win).

Figure 1: EUA Futures Prices in the European Union 2022-2023



3.3 Calibration

For any given time $t \in [t_0, T_1]$, we let $F(t;T_1)$ denote the price at time t of the futures contract expiring at T_1. To calibrate our model, we assume that under the risk-neutral condition, F_T is the risk-neutral expectation of the carbon permit spot price Z_{T_1}. Hence, using (3.6) we have

$$F(t,T_1) \quad = E^Q\left[Z_{T_1}|\mathcal{F}_t\right] \tag{3.11}$$

[13] The computation of the expectation of F_t rests on the computation of the mean and variance of the process $W^*_{1,t}$.
[14] Ember, 2023.

So, by (3.2) and for $t_0 \leq t \leq T_1$ we have

$$Z_{T_1} = Z_{t_0} e^{k(t_0,t,u_{t_0})+W_{3,\sigma_3(t_0,t)^2}}$$

and we compute

$$F(t,T_1) = E^Q\left[Z_{T_1}|\mathcal{F}_t\right]$$

Therefore

$$\begin{aligned}
\mathbb{E}^Q\left[F(t,T_1)|\mathcal{F}_{t_0}=t_0,u_{t_0}=u\right] &= e^{r(T_1-t)}E^Q\left[Z_t|Z_{t_0}=z,u_{t_0}=u\right] \\
&= e^{r(T_1-t)}E^Q\left[Z_{t_0}e^{k(t_0,t,u_{t_0})+W_{3,\sigma_3(t_0,t)^2}}|Z_{t_0}=z,u_{t_0}=u\right] \\
&= e^{r(T_1-t)}E^Q\left[Z_{t_0}e^{r(t-t_0)+\kappa(t_0,t,u_{t_0})+W_{3,\sigma_3(t_0,t)^2}}|Z_{t_0}=z,u_{t_0}=u\right] \\
&= ze^{r(T_1-t)+r(t-t_0)}e^{\kappa(t_0,t,u)}E^Q\left[e^{W_{3,\sigma_3(t_0,t)^2)}}\right] \\
&= ze^{r(T_1-t_0)}e^{\kappa(t_0,t,u)+\frac{1}{2}\sigma_3(t_0,t)^2} \\
&= F(t_0,T_1)e^{\kappa(t_0,t,u)+\frac{1}{2}\sigma_3(t_0,t)^2}
\end{aligned} \tag{3.12}$$

because

$$\kappa(t_0,t,u_{t_0}) = k(t_0,t,u_{t_0}) - r(t-t_0). \tag{3.13}$$

Applying ICE futures quotes (The ICE, 2022) at different time horizons, we solve equation (3.14).

$$\begin{aligned}
\mathbb{E}^Q[F(t,T_1)] &= F(t_0,T_1)e^{\kappa(t_0,t,u_{t_0})+\frac{1}{2}\sigma_3(t_0,t)^2} \\
\log\left(\mathbb{E}^Q[F(t,T_1)]\right) &= \log\left(F(t_0,T_1)\right)+\kappa(t_0,t,u_{t_0})+\frac{1}{2}\sigma_3(t_0,t)^2
\end{aligned} \tag{3.14}$$

We subsequently use (3.14) to estimate our model parameters. To determine the remaining unknown parameters ($u, \lambda, b, m, \rho, \sigma_1$, and σ_2), we wield seven data points from (The ICE, 2022) to setup a system of equations. The latter produce the following estimates for our parameters: $u = 0.495, \lambda = 0.84, b = -1.194, m = 0.0023, \rho = 0.5116, \sigma_1 = 0.511$, and $\sigma_2 = 0.0285$.

Table 1: Settlement Price (ICE Futures vs our Model)
Carbon Call Option : Strike Price=50 - Strip: 12/27/2022

Settlement Date	Time to Maturity	Carbon Forward Price	Settlement Price (ICE)	Settlement Price (our Model)	ERROR
06/07/21	1.58	57.61	26.13	26.71	2.21%
06/08/21	1.576	58.6	26.91	27.46	2.06%
06/09/21	1.572	59.85	27.90	28.42	1.87%
06/10/21	1.568	59.86	27.90	28.47	2.05%
06/11/21	1.564	58.8	27.06	27.75	2.57%
06/14/21	1.56	58.98	27.19	27.93	2.74%
06/15/21	1.556	57.37	25.91	26.81	3.46%
06/16/21	1.552	57.36	25.60	26.84	4.88%
06/17/21	1.548	56.91	25.24	26.56	5.26%
06/18/21	1.544	57.89	26.00	27.32	5.07%

We implement the calibration using the EUA futures contract price (3.12)[15]. We calculate the maturity time by determining the number of trading days between the EUA futures contract

[15] See section **??** for a discussion on the characterizations of the EUA futures contracts.

Table 2: Settlement Price (ICE Futures vs our Model)
Carbon Put Option : Strike Price=65 - Strip: 12/25/2022

Settlement Date	Time to Maturity	Carbon Forward Price	SettIement Price (ICE)	Settlement Price (our Model)	ERROR
06/07/21	1.58	55.21	25.75	26.37	2.39%
06/08/21	1.576	56.2	25.35	25.97	2.43%
06/09/21	1.572	57.45	24.87	25.49	2.51%
06/10/21	1.568	57.46	24.86	25.41	2.25%
06/11/21	1.564	56.4	25.26	25.69	1.72%
06/14/21	1.56	56.58	25.17	25.56	1.55%
06/15/21	1.556	54.97	25.80	26.03	0.88%
06/16/21	1.552	54.96	25.51	25.96	1.79%
06/17/21	1.548	54.51	25.68	26.05	1.42%
06/18/21	1.544	55.49	25.29	25.64	1.41%

Table 3: Settlement Price (ICE Futures vs our Model)
Carbon Call Option : Strike Price=55 - Strip: 12/27/2022

Settlement Date	Time to Maturity	Carbon Forward Price	SettIement Price (ICE)	Settlement Price (our Model)	ERROR
06/07/21	1.58	57.61	24.29	25.33	4.28%
06/08/21	1.576	58.6	25.04	26.06	4.09%
06/09/21	1.572	59.85	25.99	26.98	3.80%
06/10/21	1.568	59.86	26.00	27.03	3.98%
06/11/21	1.564	58.8	25.18	26.33	4.57%
06/14/21	1.56	58.98	25.30	26.50	4.72%
06/15/21	1.556	57.37	24.08	25.41	5.53%
06/16/21	1.552	57.36	23.74	25.44	7.15%
06/17/21	1.548	56.91	23.40	25.16	7.52%
06/18/21	1.544	57.89	24.13	25.89	7.29%

settlement date and the Strip (maturity of the future contract) date[16] knowing that the number of months in a trading year is 250. Exerting the strike price of the carbon permit and applying our model parameters as well as the emission rate, we obtain the settlement prices (carbon call and put options). As an example, for a Strip corresponding to 12/27/2022 and a strike price equal to 50, our settlement prices are slightly higher (compared to The ICE (2022)) for both the call and put options with an error rate varying from 1.87% to 5.26% (for the call option) and from 4.59% to 6.45% (for the put option). We obtain similar results with different Strip dates and strike prices as shown in Tables 1, 2, and 3. Table 4 summarizes the calibration parameters.

4 Conclusion

We first showed that a mechanism of cap-and-trade between N corporations producing emissions in the mining industry and a regulator can lead to a unique Nash equilibrium under proper assumptions. In the second part, we argue that the development of a liquid market of Carbon Derivatives – for which we have proposed a pricing methodology – can provide an answer while awaiting an international regulation that has to take place since mining and oil companies need to answer the concerns of their shareholders on their ESG ratings.

[16] In the EUA ETS market, the option matures two weeks before the futures.

Appendix

A Proofs

A.1 Proof of Theorem 2.3

We first apply the Hamilton Jacobi Isaacs equation (2.22) to the Hamiltonian (2.21), for $i = 1, \cdots, n$ as follows:

$$\begin{aligned} 0 &= H_i\Big(x, \xi; V_i, \frac{\partial V_i}{\partial x}, \frac{\partial V_i}{\partial \xi}, \frac{\partial^2 V_i}{\partial x^2}, \frac{\partial^2 V_i}{\partial x \partial \xi}, \frac{\partial^2 V_i}{\partial \xi^2}\Big) \\ &= r\big(A_i \ln(x) + g_i(\xi)\big) - \sup_{u_i \in U_i}\Big\{-\frac{1}{2}\xi A_i + \kappa\big(\mu - \ln(x)\big)A_i + k(\alpha - \xi)g_i'(\xi) + \frac{1}{2}\beta^2 \xi g_i''(\xi) \\ &\qquad + \theta_i\Big(xu_i - \eta_i x u_i - c_{i0} + \lambda_i \log(u_i^2) - \lambda_i \ln\Big(\frac{u_i}{\zeta_i}\Big)\Big)(1 - u_0^*)\Big\}. \end{aligned} \tag{A.1}$$

Consider the following expression

$$\theta_i\Big(xu_i - \eta_i x u_i - c_{i0} + \lambda_i \log(u_i^2) - \lambda_i \ln\Big(\frac{u_i}{\zeta_i}\Big)\Big)(1 - u_0^*),\ i = 1, \cdots, n. \tag{A.2}$$

The optimality condition requires that the derivative of expression (A.2) with respect to u_i be equal to zero. Thus, we have

$$\begin{aligned} &\theta_i\Big(x(1 - \eta_i) + 2\lambda_i \frac{1}{u_i} - \frac{\lambda_i}{u_i}\Big)(1 - u_0^*) = 0,\ i = 1, \cdots, n, \\ &u_i^* = \frac{\lambda_i}{(1 - \eta_i)x}, \quad x \neq 0, \quad i = 1, \cdots, n. \end{aligned} \tag{A.3}$$

Substituting (A.3) into (A.1) yields the desired results (2.28), (2.31) and (2.29) after simplifications. We now apply the Hamilton Jacobi Isaacs equation (2.22) to Hamiltonian (2.20) in the following manner

$$0 = H_0\Big(x, \xi; V_0, \frac{\partial V_0}{\partial x}, \frac{\partial V_0}{\partial \xi}, \frac{\partial^2 V_0}{\partial x^2}, \frac{\partial^2 V_0}{\partial x \partial \xi}, \frac{\partial^2 V_0}{\partial \xi^2}\Big) \tag{A.4}$$

Given that the operator H_0 is linear with respect to u_0, so the optimality depends on the sign of the quantity $\sum_{i=1}^{n} \theta_i\Big(xu_i^* - \eta_i x u_i^* - c_{i0} + \lambda_i \log(u_i^*)^2 - \lambda_i \ln\Big(\frac{u_i^*}{\zeta_i}\Big)\Big)$. Using the fact that for $i = 1, \cdots, n$, at $u_i^* = \frac{\lambda_i}{(1 - \eta_i)x}$, we can simplify this quantity as follows

$$\begin{aligned} &\sum_{i=1}^{n} \theta_i\Big(xu_i^* - \eta_i x u_i^* - c_{i0} + \lambda_i \ln(u_i^*)^2 - \lambda_i \ln\Big(\frac{u_i^*}{\zeta_i}\Big)\Big) \\ &= \sum_{i=1}^{n} \theta_i\Big(\lambda_i - c_{i0} + \lambda_i \ln\Big(\frac{\lambda_i \zeta_i}{1 - \eta_i}\Big) - \lambda_i \ln(x)\Big). \end{aligned}$$

We define the function

$$\varrho(x) = \sum_{i=1}^{n} \theta_i \Big(\lambda_i - c_{i0} + \lambda_i \ln\Big(\frac{\lambda_i \zeta_i}{1-\eta_i}\Big) - \lambda_i \ln(x) \Big). \tag{A.5}$$

Thus the optimal control u_0^* is obtained as follows

$$u_0^*(x) = \begin{cases} \overline{u}_0 & \text{if } \varrho(x) > 0 \\ \underline{u}_0 & \text{if } \varrho(x) \leq 0. \end{cases} \tag{A.6}$$

But, note that $\varrho(x) > 0$ if and only if

$$\sum_{i=1}^{n} \theta_i \Big(\lambda_i - c_{i0} + \lambda_i \ln\Big(\frac{\lambda_i \zeta_i}{1-\eta_i}\Big) - \lambda_i \ln(x) \Big) > 0$$

Set

$$\hat{x} = \exp\left(\frac{\sum_{i=1}^{n} \theta_i \Big(\lambda_i - c_{i0} + \lambda_i \ln\Big(\frac{\lambda_i \zeta_i}{1-\eta_i}\Big)\Big)}{\sum_{i=1}^{n} \theta_i \lambda_i} \right). \tag{A.7}$$

$$u_0^*(x) = \begin{cases} \overline{u}_0 & \text{if } x < \hat{x} \\ \underline{u}_0 & \text{if } x \geq \hat{x} \end{cases} \tag{A.8}$$

Finally, replacing (A.8) in (A.4) yields

$$\begin{aligned} 0 = {} & r\big(A_0 \ln(x) + g_0(\xi)\big) + \frac{1}{2}\xi A_0 - \kappa\big(\mu - \ln(x)\big)A_0 - k(\alpha - \xi) g_0'(\xi) - \frac{1}{2}\beta^2 \xi g_0''(\xi) \\ & - \sum_{i=1}^{n} \Big(x u_i^* - \eta_i x u_i^* - c_{i0} + \lambda_i \ln((u_i^*)^2) - \lambda_i \ln\Big(\frac{u_i^*}{\zeta_i}\Big) \Big) \big((1-\theta_i) + \overline{u}_0 \mathbf{1}_{\{x, x < \hat{x}\}} \theta_i \\ & + \underline{u}_0 \mathbf{1}_{\{x, x \geq \hat{x}\}} \theta_i \big). \end{aligned} \tag{A.9}$$

Consequently, in order to guarantee that (A.9) vanishes, A_0 should be

$$A_0 = \frac{\sum_{i=1}^{n} \lambda_i \Big((1-\theta_i) + \overline{u}_0 \mathbf{1}_{\{x, x < \hat{x}\}} \theta_i + \underline{u}_0 \mathbf{1}_{\{x, x \geq \hat{x}\}} \theta_i \Big)}{r + \kappa} \tag{A.10}$$

and $g_0(\xi)$, should solve the following second order linear differential equation

$$\begin{aligned} & \frac{1}{2}\beta^2 \xi g_0''(\xi) + k(\alpha - \xi) g_0'(\xi) - r g_0(\xi) = \frac{1}{2}\xi A_0 - \kappa \mu A_0 \\ & - \sum_{i=1}^{n} \Big(\lambda_i - c_{i0} + \lambda_i \ln\Big(\frac{\lambda_i \zeta_i}{1-\eta_i}\Big) \Big) \big((1-\theta_i) + \overline{u}_0 \mathbf{1}_{\{x, x < \hat{x}\}} \theta_i + \underline{u}_0 \mathbf{1}_{\{x, x \geq \hat{x}\}} \theta_i \big). \end{aligned} \tag{A.11}$$

Thus we have

$$\xi g_0''(\xi) + \frac{2k(\alpha - \xi)}{\beta^2} g_0'(\xi) - \frac{2r}{\beta^2} g'(\xi) = \frac{\xi}{\beta^2} A_0 + C_0 \tag{A.12}$$

where

$$C_0 = -\frac{2\kappa\mu A_0}{\beta^2} - \frac{2}{\beta^2} \sum_{i=1}^{n} \Big(\lambda_i - c_{i0} + \lambda_i \ln\Big(\frac{\lambda_i \zeta_i}{1-\eta_i}\Big) \Big) \big((1-\theta_i) + \overline{u}_0 \mathbf{1}_{\{x, x < \hat{x}\}} \theta_i + \underline{u}_0 \mathbf{1}_{\{x, x \geq \hat{x}\}} \theta_i \big). \tag{A.13}$$

The concavity of the Hamiltonian H_i with respect of $u_{i=1,\ldots,n}$ and the linearity of H_0 with respect to u_0 conveniently yield the uniqueness of the Nash equilibrium $(u_0^*, u_1^*, \ldots, u_n^*)$. □

A.2 Proof of Lemma 3.1

In order to price our carbon derivatives, we will introduce another process that will help us solve equation (3.1). Let $Y_t = \ln(Z_t) + \delta u_t$, using the Itô's Lemma we have

$$\begin{aligned} dY_t &= \tfrac{1}{Z_t} dZ_t - \tfrac{1}{2}\sigma_1^2 dt + \delta du_t \\ &= (\mu_1 - \tfrac{1}{2}\sigma_1^2 + \tfrac{\lambda}{b} m)dt + \sigma_3 dW_{3,t}, \end{aligned} \tag{A.14}$$

where

$$\sigma_3 = \sqrt{\sigma_1^2 + 2\rho\frac{\sigma_1\sigma_2\lambda}{b} + \frac{\sigma_2^2\lambda^2}{b^2}}$$

and

$$dW_{3,t} = \frac{\sigma_1 dW_{1,t} + \sigma_2 \frac{\lambda}{b} dW_{2,t}}{\sigma_3}.$$

Therefore,

$$Y_t = Y_{t_0} + (\mu_1 - \frac{1}{2}\sigma_1^2 + \frac{\lambda}{b} m)(t - t_0) + \sigma_1 W_{1,t-t_0} + \frac{\lambda\sigma_2}{b} W_{2,t-t_0},$$

and

$$\begin{aligned} Z_t &= e^{Y_t - \frac{\lambda}{b} u_t} \\ &= Z_{t_0} \exp\left(\frac{\lambda}{b} u_{t_0} + (\mu_1 - \frac{1}{2}\sigma_1^2 + \frac{\lambda}{b} m)(t - t_0) - \frac{\lambda}{b} u_t + \sigma_1 W_{1,t-t_0} + \frac{\lambda\sigma_2}{b} W_{2,t-t_0}\right). \end{aligned} \tag{A.15}$$

Note that

$$u_t = e^{-b(t-t_0)} u_{t_0} + \frac{m}{b}(1 - e^{-b(t-t_0)}) + \sigma_2 \int_{t_0}^{t} e^{-b(t-\xi)} dW_{2,\xi}. \tag{A.16}$$

Moreover, it is worth nothing from the definition of the Itô integral that the integral $\int_s^t e^{-b(t-\xi)} dW_{2,\xi}$ is Gaussian. It has expectation zero and its variance is given by the Itô isometry

$$E\left[\left(\int_{t_0}^{t} e^{-b(t-\xi)} dW_{2,\xi}\right)^2\right] = \int_{t_0}^{t} e^{-2b(t-\xi)} dt_0 = \frac{1}{2b}\left(1 - e^{-2b(t-t_0)}\right),$$

Consequently, the carbon permit process Z_t is

$$\begin{aligned} Z_t &= Z_{t_0} \exp\Bigg(\frac{\lambda}{b} u_{t_0} + (\mu_1 - \frac{1}{2}\sigma_1^2 + \frac{\lambda}{b} m)(t - t_0) - \frac{\lambda}{b}\left(e^{-b(t-t_0)} u_{t_0} + \frac{m}{b}(1 - e^{-b(t-t_0)})\right) \\ &\quad + \sigma_1 W_{1,t-t_0} + \frac{\lambda\sigma_2}{b} W_{2,t-t_0-h(t_0,t)}\Bigg). \end{aligned} \tag{A.17}$$

We have

$$\sigma_1 W_{1,t-t_0} + \frac{\lambda\sigma_2}{b} W_{2,t-t_0-h(t_0,t)} = W_{3,\sigma_3(t_0,t)^2} \tag{A.18}$$

with

$$\sigma_3(t_0,t)^2 = \sigma_1^2(t - t_0) + \frac{\lambda^2\sigma_2^2}{b^2}(t - t_0 - h(t_0,t)) + 2\rho\frac{\sigma_1\sigma_2\lambda\sqrt{t - t_0}\sqrt{t - t_0 - h(t_0,t)}}{b}. \tag{A.19}$$

In sum we have

$$Z_t = Z_{t_0} \exp\left(\frac{\lambda}{b} u_{t_0} + (\mu_1 - \frac{1}{2}\sigma_1^2 + \frac{\lambda}{b} m)(t - t_0) - \frac{\lambda}{b}\left(e^{-b(t-t_0)} u_{t_0} + \frac{m}{b}(1 - e^{-b(t-t_0)})\right) + W_{3,\sigma_3(t_0,t)^2}\right).$$

□

A.3 Proof of Theorem 3.2

We start by noticing that the futures call option premium is $C(t,z,u,K,T) = \mathbb{E}^Q[e^{-r(T-t)}(F(T,T_1)-K)^+ \big| Z_t = z, u_t = u]$ and that $F(t,T_1) = E^Q[Z_{T_1}|\mathcal{F}_t] = E^Q[ze^{k(t,T_1,u)+W_{3,\sigma_3^2(t,T_1)}}|Z_t = z, u_t = u]$. Now we can compute the value of a European call option as follows

$$C(t,z,u,K,T) \quad = e^{-r(T-t)}\mathbb{E}^Q\Big[\big(z\exp\big(r(T_1-T)+k(t,T,u)+W_{3,\sigma_3^2(t,T)}\big)-K\big)^+\Big]. \tag{A.20}$$

Let us find the density function of the process $X := z\exp\big((r-\delta)(T_1-T)+k(t,T,u)+W_{3,\sigma_3^2(t,T)}\big)$. The distribution function of X is

$$F(x) \quad = \mathbb{Q}\Big[W_{3,\sigma_3^2(t,T)} \le \ln\big(\frac{x}{z}\big)-(r-\delta)(T_1-T)-k(t,T,u)\Big],$$

so the density of X is

$$g(x) = \frac{1}{x\sigma_3(t,T)\sqrt{2\pi}}\exp\Bigg(-\frac{\big(\ln\big(\frac{x}{z}\big)-(r-\delta)(T_1-T)-k(t,T,u)\big)^2}{2\sigma_3^2(t,T)}\Bigg).$$

So we have

$$\begin{aligned} &C(t,z,u,K,T) \\ &= \frac{e^{-r(T-t)}}{\sigma_3(t,T)\sqrt{2\pi}}\int_K^\infty (x-K)\exp\Bigg(-\frac{\big(\ln\big(\frac{x}{z}\big)-(r-\delta)(T_1-T)-k(t,T,u)\big)^2}{2\sigma_3^2(t,T)}\Bigg)dx \end{aligned} \tag{A.21}$$

We set $y := \dfrac{\big(\ln\big(\frac{x}{z}\big)-r(T_1-T)-k(t,T,u)\big)}{\sigma_3(t,T)}$, thus $dy = \dfrac{dx}{x\sigma_3(t,T)}$ therefore we should have $x = ze^{r(T_1-T)+k(t,T,u)+y\sigma_3(t,T)}$.
Consequently (A.21) becomes

$$\begin{aligned} &C(t,z,u,K,T) \\ &= ze^{-r(T-t)+r(T_1-T)+k(t,T,u)+\frac{1}{2}\sigma_3^2(t,T)}N\Big(\frac{\ln\big(\frac{z}{K}\big)+r(T_1-T)+k(t,T,u)}{\sigma_3(t,T)}+\sigma_3(t,T)\Big) \\ &= -e^{-r(T-t)}KN\Big(\frac{\ln\big(\frac{z}{K}\big)+r(T_1-T)+k(t,T,u)}{\sigma_3(t,T)}\Big). \end{aligned} \tag{A.22}$$

Set $D_1(t,z,T,T_1,K,u) := \dfrac{\ln\big(\frac{z}{K}\big)+r(T_1-T)+k(t,T,u)}{\sigma_3(t,T)}+\sigma_3(t,T)$ and $D_2(t,z,T,T_1,K,u) := \dfrac{\ln\big(\frac{z}{K}\big)+r(T_1-T)+k(t,T,u)}{\sigma_3(t,T)}$ so we have Using the fact that under risk-neutral assumption $\mu_1 = r$, we have

$$k(t,T,u) \quad = r(T-t)+\kappa(t,T,u), \tag{A.23}$$

where

$$\kappa(t,T,u) := \frac{\lambda}{b}u+\big(\frac{\lambda}{b}m-\frac{1}{2}\sigma_1^2\big)(T-t)-\frac{\lambda}{b}\Big(e^{-b(T-t)}u+\frac{m}{b}\big(1-e^{-b(T-t)}\big)\Big). \tag{A.24}$$

$$\begin{aligned}&C(t,z,u,K,T)\\&=e^{-r(T-t)}\Bigg(F(t,T_1)e^{\kappa(t,T,u)+\frac{1}{2}\sigma_3^2(t,T)}N\Bigg(\frac{\ln\left(\frac{F(t,T_1)}{K}\right)+\kappa(t,T,u)}{\sigma_3(t,T)}+\sigma_3(t,T)\Bigg)\\&\qquad-KN\Bigg(\frac{\ln\left(\frac{F(t,T_1)}{K}\right)+\kappa(t,T,u)}{\sigma_3(t,T)}\Bigg)\Bigg)\end{aligned}\tag{A.25}$$

Now, we compute the price of the European put option

$$\begin{aligned}&P(t,z,u,K,T)\\&=\frac{e^{-r(T-t)}}{\sigma_3(t,T)\sqrt{2\pi}}\int_{-\infty}^{K}(K-x)\frac{1}{x}\exp\Bigg(-\frac{\left(\ln\left(\frac{x}{z}\right)-r(T_1-T)-k(t,T,u)\right)^2}{2\sigma_3^2(t,T)}\Bigg)dx\end{aligned}\tag{A.26}$$

We set $y:=\dfrac{\left(\ln\left(\frac{x}{z}\right)-r(T_1-T)-k(t,T,u)\right)}{\sigma_3(t,T)}$, thus $dy=\dfrac{dx}{x\sigma_3(t,T)}$. Hence, we should have $x=ze^{r(T_1-T)+k(t,T,u)+y\sigma_3(t,T)}$.
So, (A.26) becomes

$$\begin{aligned}&P(t,z,u,K,T)\\&=e^{-r(T-t)}KN\Bigg(\frac{\ln\left(\frac{K}{z}\right)-r(T_1-T)-k(t,T,u)}{\sigma_3(t,T)}\Bigg)\\&-ze^{-r(T-t)+r(T_1-T)+k(t,T,u)+\frac{1}{2}\sigma_3^2(t,T)}N\Bigg(\frac{\ln\left(\frac{K}{z}\right)-r(T_1-T)-k(t,T,u)}{\sigma_3(t,T)}-\sigma_3(t,T)\Bigg)\end{aligned}\tag{A.27}$$

Using the risk neutral assumption $\mu_1=r$, and using (A.23) and (A.24) we can simplify (A.27) as follows

$$\begin{aligned}&P(t,z,u,K,T)\\&=e^{-r(T-t)}\Bigg[KN\Bigg(\frac{\ln\left(\frac{K}{F(t,T_1)}\right)-\kappa(t,T,u)}{\sigma_3(t,T)}\Bigg)\\&\qquad-F(t,T_1)e^{\kappa(t,T,u)+\frac{1}{2}\sigma_3^2(t,T)}N\Bigg(\frac{\ln\left(\frac{K}{F(t,T_1)}\right)-\kappa(t,T,u)}{\sigma_3(t,T)}-\sigma_3(t,T)\Bigg)\Bigg].\end{aligned}\tag{A.28}$$

□

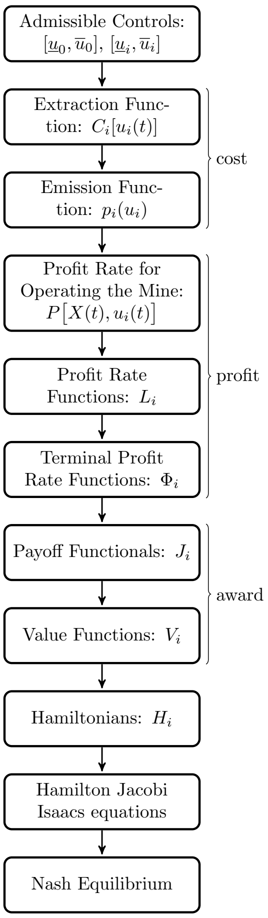
B Tables

Table 4: **Parametrization**

SECTION 2
$\mathbf{0 < t \leq T}$ – given time within the time horizon
$\mathbf{s}$ – hypothetical initial state ($s < t \leq T$)
$\zeta_\mathbf{i}$ – carbon cap on the maximum amount of CO_2 the extracting firm can emit
$\mathbf{X_t}$ – price or market value of the commodity at time t
$\mathbf{x}$ – value of $X(t)$ at $t = s$
$\theta_\mathbf{i}$ – share portion between commodity company and government
$\mathbf{K} < \infty$ – total amount of commodity available at the beginning of the lease
$\mathbf{Y_i(t)} \in [\mathbf{0}, \mathbf{K}]$ – amount of commodity produced or extracted
$\mathbf{y_i}$ – value of $Y_i(t)$ at $t = s$
$\xi(\mathbf{t})$ – main process driving volatility σ_t
$\sigma_\mathbf{t} = \mathbf{f}(\xi_\mathbf{t})$ – stochastic volatility process
ξ – value of ξ_t at $t = s$
κ – mean-reverting rate of the commodity
$\mathbf{k}$ – mean-reverting rate of the stochastic volatility
μ – long-run mean
σ – volatility of the commodity price
α – long run variance
β – volatility of volatility
$\mathbf{r}$ – risk-free rate
$\mathbf{W(t)}, \mathbf{B(t)}$ – Wiener processes
ρ – correlation between $W(t)$ and $B(t)$
$(\mathbf{\Omega}, \mathcal{F}, \mathbf{P})$ – probability space
$[\underline{\mathbf{u}}_\mathbf{0}, \overline{\mathbf{u}}_\mathbf{0}]$ – lower bound, upper bound (tax rate)
$[\underline{\mathbf{u}}_\mathbf{i}, \overline{\mathbf{u}}_\mathbf{i}]$ – lower bound, upper bound (extraction rate)
$\mathcal{F}_\mathbf{t} = \sigma\{\mathbf{B(s)}, \mathbf{W(s)}; \mathbf{s} \leq \mathbf{t}\}$ – Filtration (σ- algebra generated by $B(s)$ and $W(s)$
$\mathbf{C_i(x, u_i)}$ – extraction cost function of each player i
$\mathbf{c_{i0}} \geq \mathbf{0}$ – initial cost incurred by player i for setting up extraction operations
$\lambda_\mathbf{i} > \mathbf{0}$ – cost control parameter or incentive parameter
$\mathbf{0} < \eta_\mathbf{i} \leq \mathbf{1}$ – proportion of revenue allocated towards the extraction cost $C_i(u_i)$
$\mathbf{p_i(u_i)}$ – emission or penalty function
$\hat{\mathbf{p}}_\mathbf{i}(\mathbf{u_i})$ – continuous and differentiable $p_i(u_i)$

SECTION 3
$\mathbf{t}$ – variable time in the future ($0 < t < \infty$)
$\mathbf{t_0}$ – observed initial/current time ($0 \leq t_0 \leq t < \infty$)
$\mathbf{T}$ – maturity of the European option ($0 \leq t_0 \leq t \leq T < \infty$)
$\mathbf{T_1}$ – maturity of the futures contract ($0 \leq t_0 \leq t \leq T \leq T_1 < \infty$)
$\mathbf{Z_t}$ – carbon permit
$\mathbf{u_t}$ – emission rate
$\mathbf{z}$ – value of Z_t at $t = t_0$
$\mathbf{u}$ – value of u_t at $t = t_0$
$\mathbf{m}$ – mean of the mean reverting process
$\mathbf{b}$ – mean coefficient
$\mu_\mathbf{1}$ – measures of the average rate of growth of the carbon permit
$\sigma_\mathbf{1}$ – volatility of the carbon permit
$\sigma_\mathbf{2}$ – volatility of the emission rate
$\mathbf{K}$ – strike price
$\mathbb{Q}$ – probability measure
$\mathbf{F_{t,T}}$ – price of the futures contract expiring at T for a given time t
$\mathbf{r}$ – risk-free rate
$\mathbf{W_{1,t}}, \mathbf{W_{2,t}}$ – correlated Wiener processes
$(\mathbf{\Omega}, \mathcal{F}_\mathbf{t}, \mathbb{Q})$ – risk-neutral probability space
$\mathcal{F}_\mathbf{t} = \sigma(\{\mathbf{W_{t_0}}, \mathbf{t_0} \leq \mathbf{t}\})$ – filtration
$\rho \in (\mathbf{-1}, \mathbf{1})$ – correlation

C Figures



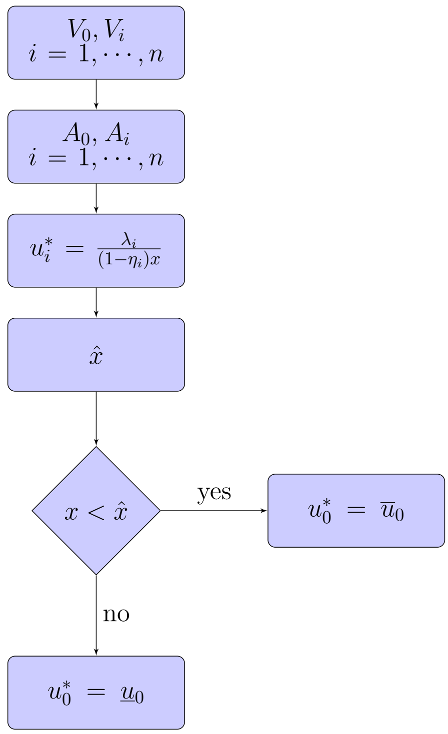


Figure 3: Carbon Regulation Model Algorithm

Figure 2: Carbon Regulation Model Overview

Figure 4: Carbon Competition: Tax, Cap, and Trade

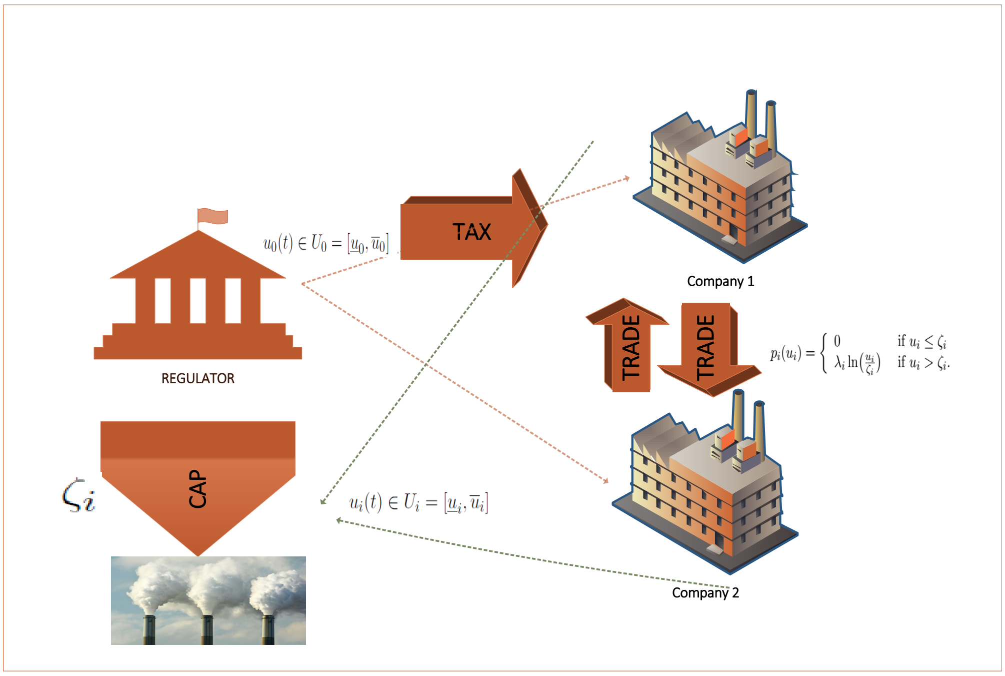
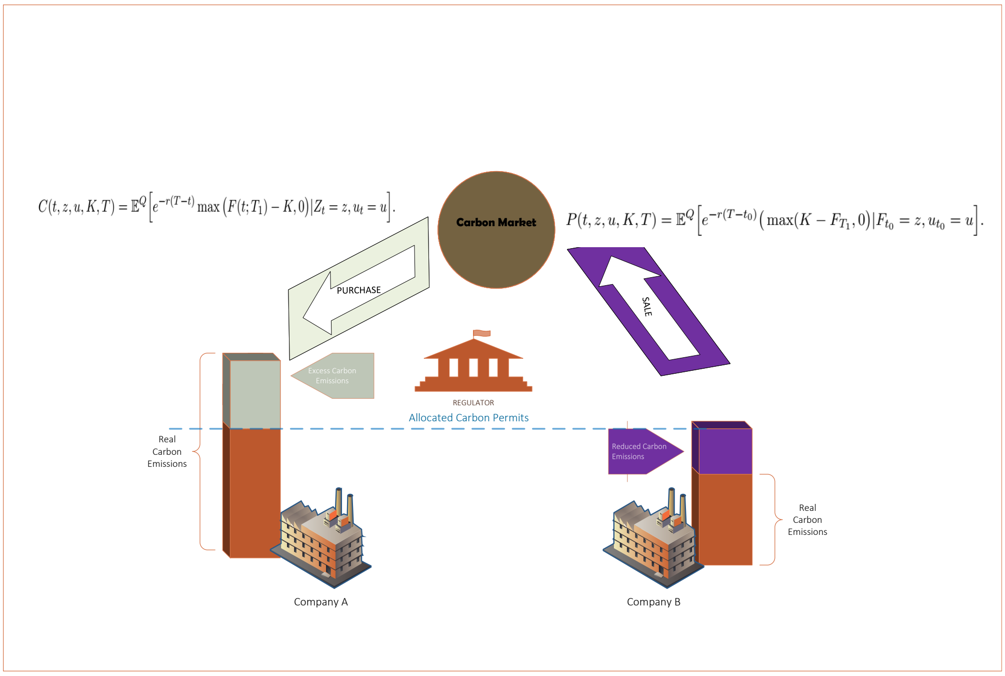


Figure 5: Carbon Competition: Carbon Derivative as Hedge



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