

MAXIMUM DRAWDOWN INSURANCE

PETER CARR

Courant Institute, NYU
251 Mercer Street, New York, New York 10012, USA
pcarr@nyc.rr.com

HONGZHONG ZHANG

Department of Statistics, Columbia University
1255 Amsterdam Avenue
New York, New York 10027, USA
hzhang@stat.columbia.edu

OLYMPIA HADJILIADIS*

Department of Mathematics
Brooklyn College and the Graduate Center, CUNY
New York, New York 10016, USA
ohadjiliadis@brooklyn.cuny.edu

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The *drawdown* of an asset is a risk measure defined in terms of the running maximum of the asset's spot price over some period $[0, T]$. The asset price is said to have drawn down by at least $\$K$ over this period if there exists a time at which the underlying is at least $\$K$ below its maximum-to-date. We introduce insurance against a large realization of maximum drawdown and a novel way to hedge the liability incurred by underwriting this insurance. Our proposed insurance pays a fixed amount should the maximum drawdown exceed some fixed threshold over a specified period. The need for this drawdown insurance would diminish should markets rise before they fall. Consequently, we propose a second kind of cheaper maximum drawdown insurance that pays a fixed amount contingent on the drawdown preceding a drawup. We propose double barrier options as hedges for both kinds of insurance against large maximum drawdowns. In fact for the second kind of insurance we show that the hedge is model-free. Since double barrier options do not trade liquidly in all markets, we examine the assumptions under which alternative hedges using either single barrier options or standard vanilla options can be used.

Keywords: Drawdown; drawup; static replication.

*Corresponding author.

1. Introduction

We introduce two financial products which protect an investor against an asset's price drawing down by a fixed amount by expiry or before it draws up by the same amount. Both claims are issued with a fixed positive strike K and a fixed finite maturity date T . In order to specify their payoff, we let S_t denote the spot price of some asset or portfolio which can be monitored continuously over the fixed time interval $[0, T]$. Let $M_t := \sup_{s \in [0, t]} S_s$ and $m_t := \inf_{s \in [0, t]} S_s$ be the continuously-monitored maximum and minimum of this asset price over $[0, t]$, respectively. Let $D_t := M_t - S_t$ be the level of the drawdown process at time $t \in [0, T]$. Similarly, let $U_t := S_t - m_t$ be the level of the drawup process at time $t \in [0, T]$. For a fixed $K > 0$, let τ_K^D and τ_K^U be the time at which the drawdown process D and the drawup process U first reaches K , respectively. The maximum drawdown of an asset or portfolio over a period $[0, T]$ is defined as $MD_T := \sup_{t \in [0, T]} D_t$. The payoff at T of the first claim is $1(MD_T \geq K)$ for some strike $K > 0$ and that of the second claim is $1(\tau_D(K) \leq \tau_U(K) \wedge T)$. The premium for both digital calls is analogous to insurance premium. Although insurance on maximum drawdown is not presently underwritten, recent events suggest an interest in synthesizing this insurance. In this work we present model-free static hedges of the second claim using one-touch knockouts, which are a type of double barrier options. Then under symmetry and continuity assumptions, we also derive semi-static hedges of both claims using one-touch knockouts, single barrier one-touches and vanilla options. The symmetry and continuity assumptions used are separately developed first under arithmetic models and subsequently under geometric and pure jump models. In geometric models the payoff of the two options introduced is associated with the relative drawdown, maximum drawdown and drawup, which are respectively defined as $D_t^r := \frac{M_t}{S_t}$, $MD_T^r := \sup_{t \in [0, T]} D_t^r$ and $U_t^r := \frac{S_t}{m_t}$. Throughout our work, we assume no frictions and no arbitrage.

Maximum drawdown was introduced as a risk measure in finance in the last decade (Sornette [25], Vecer [27, 28]). Portfolio sensitivities with respect to the maximum drawdown have been studied in Pospisil and Vecer [20]. The distributional properties of both the drawdown and the maximum drawdown have been studied in Douady *et al.* [12], Magdon-Ismail *et al.* [18], Pospisil and Vecer [19]. Probabilistic considerations involving both the drawdown and the drawup stopping times have been treated in Hadjilidiadis and Vecer [15], Pospisil *et al.* [21], Zhang and Hadjilidiadis [30, 31]. Optimal investment strategies under drawdown constraints have also been studied in Chekhlov *et al.* [9], Cvitanic and Karatzas [10], Grossman and Zhou [14].

Since its introduction the maximum drawdown has been commonly used as a measure of risk of holding an asset over a pre-specified period $[0, T]$. Consequently, a risk averse investor who is concerned that this risk measure realizes to a value larger than expected would presumably be interested in being compensated for large realizations of maximum drawdown. Moreover, the maximum drawdown provides a means to evaluate the risk of holding a hedge fund. An asset manager who knows

in advance that his portfolio risk is being evaluated wholly or in part by the portfolio's maximum drawdown is exposed to large positive realizations of maximum drawdown. In particular, it is not uncommon for managers who experience large maximum drawdowns to see their funds under management rapidly diminish. Since performance fees are typically proportional to funds under management, these fees would diminish accordingly. By purchasing a digital call before any such maximum drawdown is realized, a portfolio manager can insure against the loss of income.

The premium for this digital call can be cheapened if the payoff is lessened. One way to do this is to further introduce dependence of the terminal payoff on the time it takes maximum drawup to reach a level. If the investor holding the digital call is also long the underlying asset, then it seems reasonable that the investor would be willing to give up some of the payoff if a drawup occurs first, in return for reduced premium. Since $1(\tau_K^D \leq T) = 1(MD_T \geq K)$, we have:

$$1(\tau_K^D \leq \tau_K^U \wedge T) = 1(MD_T \geq K) - 1(\tau_K^U \leq \tau_K^D \leq T).$$

Consider a claim that pays $1(\tau_K^D \leq \tau_K^U \wedge T)$ dollars at T . In words, the claim pays one dollar at its expiry date T if and only if a drawdown of size K precedes the earlier of a drawup of the same size and expiry. For brevity, we refer to this claim as a digital call on a K -drawdown preceding a K -drawup. Such a payoff would be of interest to anyone who is more concerned about the downside than the upside, or at least more so than the market is. The payoff from the digital call on the K -drawdown preceding a K -drawup will be smaller than the payoff from a co-terminal digital call on maximum drawdown with strike K because of the possibility that a K -drawup precedes a K -drawdown.

A financial intermediary who provides a digital call on maximum drawdown or K -drawdown preceding a K -drawup to clients, is typically faced with the problem of hedging the exposure and marking the position after the sale. If there exists a hedging strategy which perfectly replicates the payoff of such a digital call under a set of reasonable assumptions, then the mark-to-market value of this replicating portfolio can be used to mark the position of this digital call. A hedging strategy which achieves a perfect replication with the least possible time instances in which trading is involved is undoubtedly more robust than a dynamic hedging strategy which involves continuous trading. Such a replication is also known as static and was introduced in Breeden and Litzenberger [3]. It was further studied in Bowie and Carr [2], Carr and Chou [5], Carr and Madan [6], Carr *et al.* [7], Derman *et al.* [11], and Sbuelz [23]. A static replication hedging strategy which involves trading in fairly liquid instruments is a very powerful tool for hedging. An example of such an instrument is an one-touch knockout which is a type of a double barrier option liquidly traded in FX markets.

Although static hedging is not a new concept, the use of double barrier options as hedge instruments in this paper is one of its main innovations. In particular, in this paper, we show that there exist a robust static hedge of a digital call on

the K -drawdown preceding a K -drawup. The hedge uses positions in one-touch knockouts. We then develop simple sufficient conditions on the underlying asset price dynamics which allow semi-robust replicating strategies to hedge a digital call on maximum drawdown with one-touch knockouts. One-touch knockouts do trade liquidly in the over-the-counter (OTC) currency options market. Our strategy replicates perfectly under a symmetry condition, provided that the running maximum increases only continuously.¹ One-touch knockouts are not necessarily available for all currency pairs, thus hedging and marking requires the development of additional simple sufficient conditions on the underlying asset price dynamics which allow alternative replicating strategies. In particular, if we enforce symmetry condition and additionally assume the running minimum decreases continuously,¹ then we can develop replicating strategies that use only single barrier one-touches, or even path-independent options. Note that for all above strategies, hedging requires only occasional trading, typically only when maxima or minima change. As vanilla options are not necessarily available for all currency pairs, one can always impose further dynamical restrictions and resort to classical dynamic hedging. Whenever a model allows the payoff of vanilla options to be dynamically replicated with the underlying asset, it can be used in conjunction with our results to replicate the payoff of calls on maximum drawdown with the same instruments.

The remainder of this paper is structured in the following way. In Sec. 2, after introducing all the instruments we need, we develop a model-free static replication of a digital call on the K -drawdown preceding a K -drawup using one-touch knockouts. In Sec. 3, we impose an assumption of continuity¹ and symmetry to develop a semi-static replication of a digital call on maximum drawdown with one-touch knockouts. In Sec. 4, we reinforce the symmetry assumption in order to develop a semi-static portfolio of one-touches to replicate the payoffs of both target digital calls. While in Sec. 5, we present a semi-static portfolio of binary options on the spot to replicate the target payoffs under another symmetry assumption. In Sec. 6, we proceed to geometric models and present a static replication strategy for the digital call on the K -drawdown preceding K -drawup using one-touch knockouts. In Sec. 7 through 8, under appropriate geometric symmetry assumptions, we develop semi-static replication of both target digital calls with consecutively more liquid instruments. In Sec. 9, we discuss on how to extend previous results to certain stochastic processes with discrete state space. Finally, we summarize the paper with some closing remarks in Sec. 10.

2. Model-Free Static Replication of Digital Call on K -Drawdown Preceding a K -Drawup with One-Touch Knockouts

Suppose that we have some fixed well-defined target payoff of a contingent claim in mind. Although super-replicating strategies are worthy of attention, in this paper

¹In Sec. 9, we have also been able to prove equivalent results when the underlying is a purely jump process with fixed jump size.

we focus on exact replicating strategies. Hence, we consider a trading strategy in other assets which is replicating, non-anticipating, and self-financing. Such a trading strategy is said to be *robust* if these three properties all hold irrespective of the dynamics of all assets in the economy. The only assumption made is that the market price of all our holdings at expiry is equal to their intrinsic value.

Let $B_t(T)$ be the price of a default-free zero coupon bond paying one dollar with certainty at T . We assume that $B_t(T) > 0$ for all $t \in [0, T]$ and hence no arbitrage implies the existence of a probability measure \mathbb{Q}^T associated with this numeraire. The measure \mathbb{Q}^T is equivalent to the statistical probability measure and hence is usually referred to as an equivalent martingale measure. Under \mathbb{Q}^T , the ratios of non-dividend paying asset prices to B are martingales. We will use \mathbb{Q}^T to describe the arbitrage-free values of options in this paper. The conditional form of this measure on the σ -algebra $\mathcal{F}_t = \sigma\{S_s; s \leq t\}$ generated by the underlying S_t , will be denoted by \mathbb{Q}_t^T .

Let us denote by $DC_t^{MD}(K, T)$ the value at time $t \in [0, T]$ of a digital call on maximum drawdown, and by $DC_t^{D < U}(K, T)$ the value at time $t \in [0, T]$ of a digital call on the K -drawdown preceding a K -drawup. Their arbitrage-free prices are:

$$DC_t^{MD}(K, T) := B_t(T)\mathbb{Q}_t^T(MD_T \geq K), \tag{2.1}$$

$$DC_t^{D < U}(K, T) := B_t(T)\mathbb{Q}_t^T(\tau_K^D \leq \tau_K^U \wedge T). \tag{2.2}$$

In this section, our hedging instruments will be bonds, one-touch knockouts and their spreads.

Before describing the payoff of one-touch knockouts, it will be helpful to introduce terminology that indicates exactly where the spot price is when a barrier option knocks in or knocks out. For concreteness, we will focus on a lower barrier L . Let τ_L^S be the first hitting time of the spot price process S to the barrier $L < S_0$. If S never hits L , we set $\tau_L^S = \infty$. Recall that m_t is the continuously-monitored running minimum of this asset price over $[0, t]$. The payoff $1(\tau_L^S \leq T)$ is the same as the payoff $1(m_T \leq L)$.

A barrier L is said to be skipfree, when

$$S_{\tau_L^S} = L. \tag{2.3}$$

When we instead have $S_{\tau_L^S} < L$ we say that a barrier has been crossed. While when we have $S_{\tau_L^S} \leq L$, we say that the barrier L has been hit. When we have both $S_{\tau_L^S} = L$ and $m_T = L$, we say that the barrier L has been grazed. An one-touch knockout is issued with an in-barrier V , an out-barrier W , and a fixed maturity date T . We assume that the spot stays in between V and W when the one-touch knockout is issued. For concreteness, we will focus on the case in which the out-barrier W is the higher barrier. To describe the payoff of an one-touch knockout formally, let τ_V^S and τ_W^S be the first hitting times of the spot process S to V and W respectively. As usual, if S never reaches a barrier, then

we set the first hitting time to infinity. The arbitrage-free value of an one-touch knockout is:

$$\begin{aligned}
 OTKO_t(V, W, T) &:= B_t(T) \mathbb{Q}_t^T(\tau_V^S \leq \tau_W^S \wedge T) \\
 &= B_t(T) \mathbb{Q}_t^T(\tau_V^S \leq T, M_{\tau_V^S} < W). \tag{2.4}
 \end{aligned}$$

In words, the one-touch knockout pays one dollar at its maturity date T if and only if the spot price S hits the in-barrier V before hitting the out-barrier W and this first hitting time to V occurs before the expiry T . Notice that the one-touch knockout also pays one dollar at T if $\tau_V^S \leq \tau_W^S \leq T$. In words, the out-barrier W is extinguished when the in-barrier V is first hit.

Sometimes it is convenient to modify the knockout condition of an one-touch knockout. For example, we consider the following payoff ²

$$OTKO_t(V, W^+, T) := B_t(T) \mathbb{Q}_t^T(\tau_V^S \leq T, M_{\tau_V^S} \leq W). \tag{2.5}$$

This claim pays out one dollar at expiry if and only if the spot price S hits the in-barrier V before crossing the out-barrier W and this first hitting time to V occurs before the expiry T .

The last claim which we want to make use of is a sequential double-touch whose payoff is the result of differentiating the payoff of an one-touch knockout in (2.4) with respect to its higher out-barrier W . This claim has a positive payoff if and only if the underlying spot price first touches W and then hits V from above before maturity. We accordingly refer to this claim as a ricochet-upper-first down-and-in:

$$\begin{aligned}
 RUFDI_t(V, W, T) &= \lim_{\epsilon \rightarrow 0^+} \frac{OTKO_t(V, W + \epsilon, T) - OTKO_t(V, W, T)}{\epsilon} \\
 &= B_t(T) E_t^{\mathbb{Q}^T} \{1(\tau_V^S \leq T) \delta(M_{\tau_V^S} - W)\}. \tag{2.6}
 \end{aligned}$$

Notice that a ricochet-upper-first down-and-in is itself a spread of two one-touch knockouts with slightly different upper out-barriers and identical lower in-barriers set at V .

In what follows, we focus on the payoffs from a digital call written on the K -drawdown preceding a K -drawup. We present a trading strategy in other assets which is replicating, non-anticipating, and self-financing.

Theorem 2.1 (Robust replication: I). *Under frictionless markets, no arbitrage implies that the digital call on the K -drawdown preceding a K -drawup can be valued*

²Throughout the paper, the notation x^+ (x^- , resp.) means the right (left, resp.) limit of x .

relative to the prices of bonds, one-touch knockouts and their spreads:

$$\begin{aligned}
 DC_t^{D<U}(K, T) &= 1(\tau_K^D \leq \tau_K^U \wedge t)B_t(T) + 1(t < \tau_K^D \wedge \tau_K^U \wedge T) \\
 &\quad \times \left\{ OTKO_t(M_t - K, M_t^+, T) + \int_{M_t^+}^{(m_t+K)^-} RUFDI_t(H - K, H, T)dH \right\}. \tag{2.7}
 \end{aligned}$$

for any $t \in [0, T]$ and $K > 0$.

Proof. Suppose that a digital call on the K -drawdown preceding a K -drawup has been sold at time 0. In order to develop a static hedge, we condition on being at some time t before expiry and before a drawdown or drawup of size K has been realized:

$$t \in [0, \tau_K^D \wedge \tau_K^U \wedge T).$$

Then the maximum-to-date M_t and the minimum-to-date m_t are both known constants that bracket the current spot S_t . The fact that neither a drawdown nor a drawup of size K has yet occurred implies that $M_t - m_t < K$. As a result, we have:

$$M_t - K < m_t \leq S_t \leq M_t < m_t + K.$$

Let us focus on the running maximum at time τ_K^D .³ Since the running maximum is an increasing process, we must have:

$$\begin{aligned}
 \{\tau_K^D \leq \tau_K^U \wedge T\} &= \{\tau_K^D \leq \tau_K^U \wedge T, M_{\tau_K^D} \geq M_t\} \\
 &= \{\tau_K^D \leq \tau_K^U \wedge T, M_{\tau_K^D} \in [M_t, m_t + K)\}.
 \end{aligned}$$

This is because, if $M_{\tau_K^D} = M$ for some $M \geq m_t + K$, then either $\tau_M^S < t$ and hence $\tau_K^D \wedge \tau_K^U \leq t$, or else $\tau_M^S \in [t, \tau_K^D)$ in which case $\tau_K^U \leq \tau_K^D$. Moreover, by restricting $M_{\tau_K^D}$ to the interval $[M_t, m_t + K)$, we can't have a K -drawup precede a K -drawdown, since if $\tau_K^U \leq \tau_K^D \leq T$, then $M_{\tau_K^D} > m_t + K$. So we can further obtain that

$$\begin{aligned}
 \{\tau_K^D \leq \tau_K^U \wedge T\} &= \{\tau_K^D \leq \tau_K^U \wedge T, M_{\tau_K^D} \in [M_t, m_t + K)\} \\
 &= \{\tau_K^D \leq T, M_{\tau_K^D} \in [M_t, m_t + K)\},
 \end{aligned}$$

We now present a key result that allows the digital call to be replicated with one-touch knockouts. Observe that if and when the unit payoff of the digital call is realized, the stock price has to be visiting a new low level:

$$\begin{aligned}
 \{\tau_K^D \leq \tau_K^U \wedge T\} &= \{\tau_K^D \leq T, M_{\tau_K^D} \in [M_t, m_t + K)\} \\
 &= \{\tau_K^D \leq T, \tau_K^D = \tau_{M_{\tau_K^D} - K}^S, M_{\tau_K^D} \in [M_t, m_t + K)\}. \tag{2.8}
 \end{aligned}$$

³As a convention, we set $M_{\tau_K^D} = \infty$ if $\tau_K^D = \infty$.

As a consequence of (2.8), the payoff of a digital call has the following representation:

$$\begin{aligned}
 1(\tau_K^D \leq \tau_K^U \wedge T) &= 1(\tau_K^D \leq T, \tau_K^D = \tau_{M_t^S - K}^S, M_{\tau_K^D} = M_t) \\
 &\quad + \int_{M_t^+}^{(m_t+K)^-} 1(\tau_K^D \leq T, \tau_K^D = \tau_{H-K}^S) \delta(M_{\tau_K^D} - H) dH \\
 &= 1(\tau_{M_t-K}^S \leq T, M_{\tau_{M_t-K}^S} = M_t) + I,
 \end{aligned}$$

where:

$$I := \int_{M_t^+}^{(m_t+K)^-} 1(\tau_{H-K}^S \leq T) \delta(M_{\tau_{H-K}^S} - H) dH.$$

Under no arbitrage assumption, taking expectations of (2.9) under \mathbb{Q}_t^T implies that:

$$\begin{aligned}
 DC_t^{D < U}(K, T) &= OTKO_t(M_t - K, M_t^+, T) \\
 &\quad + \int_{M_t^+}^{(m_t+K)^-} RUFDI_t(H - K, H, T) dH,
 \end{aligned}$$

for all $t \in [0, \tau_K^D \wedge \tau_K^U \wedge T)$.

If and when $\tau_K^D \wedge \tau_K^U < T$, then at that time, we do not hold any spreads of one-touch knockouts, the one-touch knockout in the portfolio either knocks into a bond if $\tau_K^D \leq \tau_K^U$, or knocks out if $\tau_K^D \geq \tau_K^U$. As a consequence, the digital call can be valued at any $t \in [0, T]$. Hence, we have (2.7). □

We have shown a robust hedge of the digital call on K -drawdown preceding a K -drawup. This hedge portfolio (2.7) can be set up with one-touch knockouts and their spreads, which do trade liquidly in the OTC currency option market. However, to obtain a replicating portfolio of the digital call on maximum drawdown with tradable assets, we need to place structure on the spot price process. We proceed to develop this in the next section.

3. Semi-Static Replication of Digital Call on Maximum Drawdown with One-Touch Knockouts

In this section we place structure on S , the stochastic process governing the spot price of the underlying asset. In particular, we assume that the running maximum can only increase continuously¹ whenever $MD_t < K$. Of course, this condition is already met if the process is continuous or spectrally negative. We also impose a symmetry condition on the process between the first time that a new maximum M_t is established and the first exit time of the corridor $(M_t - K, M_t + K)$. To be more specific, recall that τ_B^S denotes the first hitting time of the spot price process S to a barrier B . Let $\tau(M, K)$ be the first exit time of a corridor centered at M with lower

barrier $M - K$ and higher barrier $M + K$. Then whenever the underlying spot price process is at its maximum to date M_t with $MD_t < K$, we have

$$\mathbb{Q}_t^T(\tau(M_t, K) = \tau_{M_t - K}^S) = \mathbb{Q}_t^T(\tau(M_t, K) = \tau_{M_t + K}^S). \tag{3.1}$$

In words, first exiting on the left before T has the same risk-neutral probability as first exiting on the right before T . This condition is met by symmetric Lévy processes such as symmetric stable processes which includes standard Brownian motion. It is also met by the Ocone martingales (see for example Ocone[17]) as well as any process constructed as the difference of two independent identically distributed processes.

We will need to impose both of our assumptions in order to replicate a digital call on maximum drawdown using just bonds and one-touch knockouts. The set of stochastic processes that satisfy both assumptions are said to satisfy **A1**:

A1: Continuity of the maximum and exit symmetry. *While $MD_t < K$, the running maximum is continuous. Moreover, at times $\tau(u) := \tau_u^S \wedge \tau_K^D \wedge T$ for all $u > S_0$, the risk-neutral probability of first exiting at $M_{\tau(u)} - K$ before T is the same as the risk-neutral probability of first exiting at $M_{\tau(u)} + K$ before T .*

Suppose that we attempt to replicate the payoff of a digital call on maximum drawdown. At time t when $MD_t > K$, we simply hold a bond, but while $MD_t < K$, we attempt a semi-dynamic strategy by rolling up the lower barrier of the one-touch being held each time the running maximum increases. No other instruments are held. While this strategy is replicating, it is not yet self-financing as it costs money to move up the lower barrier of an one-touch closer to the spot price. To finance the rollup of the barriers of this one-touch until $\tau_K^D \wedge T$, we assume that **A1** holds, i.e. we rely on the continuity of the running maximum and the exit symmetry assumed present when the maximum ticks up. For $t \in [0, \tau_K^D \wedge T)$, suppose that we also hold an upper barrier one-touch struck K dollars above the maximum-to-date. While this augmentation finances the rollup of the lower barrier one-touch being held, it no longer replicates the desired payoff, since a path that first hits $M_{\tau_K^D} - K$ and then hits $M_{\tau_K^D} + K$ will trigger payoffs from both one-touches. For $t \in [0, \tau_K^D \wedge T)$, suppose we further alter the strategy by imposing a knockout barrier at the lower level $M_t - K$ on the one-touch struck at $M_t + K$, and a knockout barrier at the higher level $M_t + K$ on the one-touch struck at $M_t - K$. Then we are using two one-touch knockouts. It is easily seen that, when the underlying satisfies **A1**, the latest strategy self-finances and replicates the payoff of a digital call on maximum drawdown. In particular, we have:

Theorem 3.1 (Semi-static pricing using one-touch knockouts). *Under frictionless markets and assumption **A1**, no arbitrage implies that the digital call on maximum drawdown can be valued relative to the prices of bonds and one-touch*

knockouts as:

$$DC_t^{MD}(K, T) = 1(MD_t \geq K)B_t(T) + 1(MD_t < K) \times \{OTKO_t(M_t - K, M_t + K, T) + OTKO_t(M_t + K, M_t - K, T)\}, \tag{3.2}$$

for $t \in [0, T]$ and $K > 0$.

4. Semi-Static Replication with One-Touches

In the last two sections we derived static and semi-static hedges of the target digital calls with one-touch knockouts and their spreads. Since one-touch knockouts are relatively illiquid at present, this section presents an alternative semi-static hedge which just uses single-barrier one-touches under symmetry and continuity assumptions. In order to proceed we first define a single barrier one-touch. Let $L \in \mathbb{R}$ be the barrier of a single-barrier one-touch with expiry T whose price at time $t \in [0, T]$ is denoted by $OT_t(L, T)$. We define

$$OT_t(L, T) := B_t(T)Q_t^T(\tau_L^S \leq T). \tag{4.1}$$

Suppose that the spot starts inside the corridor between V and W , where V and W are the in-barrier and out-barrier of an one-touch knockout respectively. Let τ be the first exit time of the above corridor, then we impose the following assumption:

A2: Skip-freedom and hitting symmetry. *The spot S cannot exit the corridor between V and W by a jump. If the first exit time $\tau \leq T$, then we have*

$$Q_\tau^T(\tau_{S_\tau - \Delta}^S \leq T) = Q_\tau^T(\tau_{S_\tau + \Delta}^S \leq T), \tag{4.2}$$

for any $\Delta > 0$.

Under our assumptions, we claim that the payoff of an one-touch knockout with in-barrier V and out-barrier W is replicated by a portfolio of one-touches:

Proposition 4.1 (Semi-static pricing of one-touch knockouts: I). *Under frictionless markets and assumption A2, no arbitrage implies that $t \in [0, \tau_V^S \wedge \tau_W^S \wedge T]$*

$$OTKO_t(V, W, T) = OT_t(V, T) + \sum_{n=1}^{\infty} [OT_t(V - 2n\Delta, T) - OT_t(V + 2n\Delta, T)], \tag{4.3}$$

where $\Delta = W - V$.

Proof. Suppose an one-touch knockout with in-barrier V and out-barrier W has been sold at time 0. In order to hedge this position, an investor takes a long position on a series of one-touches with barriers at $V, V - 2\Delta, V - 4\Delta, \dots$ and also takes a

short position on a series of one-touches with barriers at $V + 2\Delta, V + 4\Delta, \dots$. If neither barrier is hit by T , then all one touches expire worthless. If $\tau_V^S \leq \tau_W^S \wedge T$, then at τ_V^S , the one-touch with barrier V becomes a bond, while **A2** implies that all of the other one-touches can be costlessly liquidated. The reason is that for each $n = 1, 2, \dots$, the long position in the one-touch with barrier $V - 2n\Delta$, is canceled by the short position in the one-touch with barrier $V + 2n\Delta$. On the other hand, if $\tau_W^S \leq \tau_V^S \wedge T$, then at τ_W^S , **A2** implies that all of the one-touches can be costlessly liquidated. The reason is that since $V = W - \Delta$, the portfolio can also be considered as long a series of one-touches with barriers at $W - \Delta, W - 3\Delta, W - 5\Delta, \dots$, while also being short a series of one-touches with barriers at $W + \Delta, W + 3\Delta, W + 5\Delta \dots$. Hence, for each $n = 1, 2, \dots$, the long position in the one-touch with barrier $W - (2n - 1)\Delta$, is canceled by the short position in the one-touch with barrier $W + (2n - 1)\Delta$. Since the value of the one-touch portfolio matches the payoff of the one-touch knockout when (S, t) exits $(V \wedge W, V \vee W) \times [0, T]$, no arbitrage forces the values prior to exit to be the same. □

Recall that Theorem 2.1 stated that the payoff of a digital call on the K -drawdown preceding a K -drawup can be statically replicated by one-touch knockouts, and Theorem 3.1 stated that under **A1**, the payoff of a digital call on maximum drawdown can be dynamically replicated by rolling up the barriers of one-touch knockouts. If **A2** holds for all barriers of one-touch knockouts being held, then the target digital calls can be replicated just by rolling up the barriers of a portfolio of single barrier one-touches.

In Sec. 4.1 and 4.2, we will separately develop portfolios of one-touches which can be used to replicate the payoff of a digital call on maximum drawdown and the payoff of a digital call on the K -drawdown preceding a K -drawup, respectively.

4.1. Hedging digital call on maximum drawdown with one-touches

In this subsection we develop a semi-static replication of a digital call on maximum drawdown using one-touches. By Theorem 3.1 and Proposition 4.1, we just need to ensure **A2** holds for all barriers of one-touch knockouts being held. For this purpose we impose structure on the spot price process:

A3: Continuity of the maximum, drawdown, and hitting symmetry. *While $MD_t < K$, the running maximum is continuous, and the drawdown cannot jump up by more than $K - D_t$. Moreover, at times $\tau(u) := \tau_u^S \wedge \tau_K^D \wedge T$ for all $u > S_0$, the risk-neutral probability of hitting $S_{\tau(u)} - \Delta$ before T is the same as the risk-neutral probability of hitting $S_{\tau(u)} + \Delta$ before T , for any $\Delta > 0$.*

From Proposition 4.1, it is not difficult to see that **A3** also implies **A1**. In fact, under **A3**, at times $\tau(u) := \tau_u^S \wedge \tau_K^D \wedge T$ for $u > S_0$, evaluating (4.3) at $V = M_{\tau(u)} \mp K$

and $W = M_{\tau(u)} \pm K$, we obtain

$$\begin{aligned}
 & OTKO_{\tau(u)}(M_{\tau(u)} \mp K, M_{\tau(u)} \pm K, T) \\
 &= OT_{\tau(u)}(M_{\tau(u)} \mp K, T) \\
 &+ \sum_{n=1}^{\infty} \{OT_{\tau(u)}(M_{\tau(u)} \mp (4n + 1)K, T) - OT_{\tau(u)}(M_{\tau(u)} \pm (4n - 1)K, T)\},
 \end{aligned}
 \tag{4.4}$$

which implies that

$$OTKO_{\tau(u)}(M_{\tau(u)} - K, M_{\tau(u)} + K, T) = OTKO_{\tau(u)}(M_{\tau(u)} + K, M_{\tau(u)} - K, T).
 \tag{4.5}$$

As a result, we have:

Theorem 4.1 (Semi-static pricing using one-touches: I). *Under frictionless markets and assumption **A3**, no arbitrage implies that the digital call on maximum drawdown can be valued relative to the prices of bonds and one-touches as:*

$$\begin{aligned}
 DC_t^{MD}(K, T) &= 1(MD_t \geq K)B_t(T) + 1(MD_t < K) \left\{ OT_t(M_t - K, T) \right. \\
 &+ \sum_{n=1}^{\infty} OT_t(M_t - (4n + 1)K, T) - \sum_{n=1}^{\infty} OT_t(M_t + (4n - 1)K, T) \\
 &\left. - \sum_{n=1}^{\infty} OT_t(M_t - (4n - 1)K, T) + \sum_{n=0}^{\infty} OT_t(M_t + (4n + 1)K, T) \right\},
 \end{aligned}
 \tag{4.6}$$

for any $t \in [0, T]$ and $K > 0$.

Proof. Suppose a digital call on maximum drawdown has been sold at time 0. In order to hedge this position, consider a strategy of always holding the replicating portfolio of one-touches on the right hand side of (4.6). This semi-dynamic trading strategy is followed until the earlier of expiry and the first hitting time of running drawdown to the strike K . If the running drawdown increase to K before T , then a bond of maturity T is held afterwards.

Since we assume that the running maximum is continuous, the above replicating portfolio never yields a payout due a hit of barriers higher than M_t . When the running maximum increases continuously with $t < \tau_K^D \wedge T$, assumption **A3** guarantees that it costs nothing to move the barriers of one-touches being held. Hence, the first time to receive a cash flow from the above portfolio is at time $\tau := \tau_K^D$. If $\tau > T$, then all one-touches expire worthless, as does the target claim. If $\tau \leq T$, then at

$\tau, S_\tau = M_\tau - K$, Proposition 4.1 and assumption **A3** imply that, the portfolio of one-touches has the same value as

$$OTKO_\tau(M_\tau - K, M_\tau + K, T) + OTKO_\tau(M_\tau + K, M_\tau - K, T) = B_\tau(T).$$

We conclude that in all cases, the payoff of the digital call is matched by the liquidation value of a non-anticipating self-financing portfolio of bonds and one-touches. Furthermore, the right hand side of (4.6) is the cost of setting up the replicating portfolio at time t . Hence, no arbitrage implies that this cost is also the price of the target claim. \square

4.2. Hedging digital call on the K -drawdown preceding a K -drawup with one-touches

In this subsection we develop a semi-static replication of a digital call on the K -drawdown preceding a K -drawup using one-touches. By Theorem 2.1 and Proposition 4.1, we just need to ensure **A2** holds for all barriers of one-touch knockouts being held. For this purpose we impose structure on the spot price process:

A3': Continuity of the maximum, minimum, and hitting symmetry. *While $t \leq \tau_K^D \wedge \tau_K^U \wedge T$, the running maximum and the running minimum are continuous. Moreover, at times $\theta(u) := \tau_u^D \wedge \tau_u^U \wedge T$ for all $u \in (0, K]$, the risk-neutral probability of hitting $S_{\theta(u)} - \Delta$ before T is the same as the risk-neutral probability of hitting $S_{\theta(u)} + \Delta$ before T , for any $\Delta > 0$.*

Assumption **A3'** is sufficient for applying Proposition 4.1. Evaluating (4.3) at $V = M_t - K$ and $W = M_t$, we obtain

$$\begin{aligned} &OTKO_t(M_t - K, M_t, T) \\ &= OT_t(M_t - K, T) + \sum_{n=1}^{\infty} \{OT_t(M_t - (2n + 1)K, T) \\ &\quad - OT_t(M_t + (2n - 1)K, T)\}, \end{aligned} \tag{4.7}$$

for $K > 0$ and $t \in [0, \tau_{M_t - K}^S \wedge \tau_{M_t}^S \wedge T]$. Differentiating (4.3) with respect to W , and evaluating at $V = H - K$ and $W = H$ implies that for $K > 0$ and $t \in [0, \tau_{H - K}^S \wedge \tau_H^S \wedge T]$:

$$\begin{aligned} &RUFDI_t(H - K, H, T) \\ &= -2 \sum_{n=1}^{\infty} n \left(\frac{\partial}{\partial H} OT_t(H - (2n + 1)K, T) + \frac{\partial}{\partial H} OT_t(H + (2n - 1)K, T) \right), \end{aligned} \tag{4.8}$$

since K is a constant.

Substituting (4.7) and (4.8) in (2.7), and ignoring the left and right limits, it gives rise to:

Theorem 4.2 (Semi-static pricing using one-touches: II). *Under frictionless markets and assumption **A3'**, no arbitrage implies that the digital call on the K -drawdown preceding a K -drawup can be valued relative to the price of bonds and one-touches as:*

$$\begin{aligned}
 DC_t^{D<U}(K, T) &= 1(\tau_K^D \leq t \wedge \tau_K^U \wedge T)B_t(T) + 1(t < \tau_K^D \wedge \tau_K^U \wedge T) \\
 &\quad \times \left\{ \sum_{n=0}^{\infty} (2n + 1)[OT_t(M_t - (2n + 1)K, T) + OT_t(M_t + (2n + 1)K, T)] \right. \\
 &\quad \left. - \sum_{n=1}^{\infty} 2n [OT_t(m_t - 2nK, T) + OT_t(m_t + 2nK, T)] \right\}, \tag{4.9}
 \end{aligned}$$

for any $t \in [0, T]$ and $K > 0$.

Proof. Suppose that the digital call on the K -drawdown preceding a K -drawup has been sold at time 0. In order to hedge this position, consider a strategy of always holding the replicating portfolio of one-touches on the right hand side of (4.9). This semi-dynamic trading strategy is followed until the earlier of expiry and the first hitting times of the running drawdown/drawup to the strike K . If the running drawdown increases to K before τ_K^U and T , then a bond of maturity T is held afterwards.

Since we assume that the running maximum and the running minimum are continuous, the above replicating portfolio never yields a payout due to a hit of barriers outside the corridor $[m_t, M_t]$. When the running maximum increase or the running minimum decreases continuously with $t < \tau_K^D \wedge \tau_K^U \wedge T$, assumption **A3'** guarantees that it cost nothing to move the barriers of one-touches in the above portfolio. Hence, the first time to get a cash flow from the above portfolio is when $M_t - m_t = K$. Let us denote by τ the first time that $M_t - m_t \geq K$, then clearly $\tau = \tau_K^D \wedge \tau_K^U$. If $\tau > T$, then the one-touches expire worthless, as does the target claim. If $\tau \leq T$, then at τ , $M_\tau = m_\tau + K$, Proposition 3.1 and assumption **A3'** imply that, the portfolio of one-touches has the same value as the one-touch knockout $OTKO_\tau(M_\tau - K, M_\tau, T)$, whose payoff matches the target option, with value zero or the price of a bond. In the former case, $\tau = \tau_K^U$, the one-touches are liquidated for zero; while in the latter case, $\tau = \tau_K^D$, the liquidation proceeds are used to buy the bond. We conclude that in all cases, the payoff of the target digital call is matched by the liquidation value of a non-anticipating self-financing portfolio of bonds and one-touches. Furthermore, the right hand side of (4.9) is the cost of setting up the replicating strategy at time t . Hence, no arbitrage implies that this cost is also the price of the target claim. □

The hedging strategies in Theorem 4.1 and Theorem 4.2 would be easier to implement in practice than the hedges using one-touch knockouts because they do not involve integrating over barriers. If we enforce the symmetry assumption of the underlying spot price process, then it is possible to develop hedging strategies with only digital options on the underlying. We present these results in the next section.

5. Semi-Static Replication with Vanilla Options

In the previous section, we developed two semi-static hedges with a series of co-terminal single-barrier one-touches of the target digital calls. Since barrier options are not so liquid for most underlyings, this section presents another semi-static hedge which uses digital options on the underlying. The replication only succeeds under some symmetry and continuity assumptions, which we will make precise.

Let $B \in \mathbb{R}$ be the strike of a digital option on the underlying with expiry T . For $t \in [0, T]$, let $DP_t(B, T)$ and $DC_t(B, T)$ denote the prices at time t of a digital put and a digital call on spot respectively. We define

$$DP_t(B, T) := B_t(T)Q_t^T(S_T < B) + \frac{1}{2}B_t(T)E_t^{Q^T}\{\delta(S_T - B)\}, \tag{5.1}$$

$$DC_t(B, T) := B_t(T)Q_t^T(S_T > B) + \frac{1}{2}B_t(T)E_t^{Q^T}\{\delta(S_T - B)\}. \tag{5.2}$$

Notice that if S_T turn out to be at B , then both digital options pay 50 cents at expiry. We will make use of these digital options to replicate the payoff of an one-touch knockout. To this end, we develop semi-static replication of the target digital calls with vanilla options.

Consider a spot price process starting inside the corridor between V and W , where V and W are the in-barrier and out-barrier of an OKTO respectively. Let $\tau := \tau_V^S \wedge \tau_W^S$ be the first exit time of the above corridor, then we impose the following assumption:

A4: Skip-freedom and symmetry. *The spot S cannot exit the corridor between V and W by a jump. If the first exit time $\tau \leq T$, then at time τ , the conditional risk-neutral probability distribution of S_T is symmetric about S_τ .*

The above assumption is met by all continuous symmetric Lévy processes. The characterization of continuous martingales that satisfy the symmetry in **A4** can be found in Tehranchi [26].

Under assumption **A4**, we claim that the payoff of an one-touch knockout with skip-free in-barrier V and out-barrier W is replicated by a portfolio of digital options:

Proposition 5.1 (Semi-static pricing of one-touch knockouts: II). *Under frictionless markets and assumption **A4**, no arbitrage implies that for*

$$t \in [0, \tau_V^S \wedge \tau_W^S \wedge T],$$

(1) If $V < W$:

$$OTKO_t(V, W, T) = 2 \sum_{n=0}^{\infty} DP_t(V - 2n\Delta, T) - 2 \sum_{n=1}^{\infty} DC_t(V + 2n\Delta, T); \quad (5.3)$$

(2) If $V > W$:

$$OTKO_t(V, W, T) = 2 \sum_{n=0}^{\infty} DC_t(V - 2n\Delta, T) - 2 \sum_{n=1}^{\infty} DP_t(V + 2n\Delta, T); \quad (5.4)$$

where $\Delta = W - V$.

Proof. The idea of the proof is similar as that in Proposition 4.1. We skip the details here. □

In Sec. 5.1 and 5.2, we will separately develop portfolios of digital options which can be used to replicate the payoff of a digital call on maximum drawdown and the payoff of a digital call on the K -drawdown preceding a K -drawup, respectively.

5.1. Hedging digital call on maximum drawdown with vanilla options

In this subsection we develop a semi-static replication of a digital call on maximum drawdown using digital options on the underlying. By Theorem 3.1 and Proposition 5.1, we need to ensure **A4** holds for all barriers of one-touch knockouts being held. For this purpose we impose structure on the spot price process:

A5: Continuity of the maximum, drawdown, and symmetry. While $MD_t < K$, the running maximum is continuous, and the drawdown cannot jump up by more than $K - D_t$. Moreover, at times $\tau(u) := \tau_u^S \wedge \tau_K^D \wedge T$ for all $u > S_0$, the conditional risk-neutral probability distribution of S_T , is symmetric about $S_{\tau(u)}$.

From Proposition 5.1, it is not difficult to see that **A5** also implies **A1**. In fact, under **A5**, whenever the maximum increases continuously with $MD_t < K$, evaluating (5.3) and (5.4) at $V = M_t \mp K$ and $W = M_t \pm K$:

$$\begin{aligned} &OTKO_t(M_t - K, M_t + K, T) \\ &= 2 \sum_{n=0}^{\infty} \{DP_t(M_t - (4n + 1)K, T) - DC_t(M_t + (4n + 3)K, T)\}, \\ &OTKO_t(M_t + K, M_t - K, T) \\ &= 2 \sum_{n=0}^{\infty} \{DC_t(M_t + (4n + 1)K, T) - DP_t(M_t - (4n + 3)K, T)\}. \end{aligned}$$

Assumption **A4** implies that

$$OTKO_t(M_t - K, M_t + K, T) = OTKO_t(M_t + K, M_t - K, T).$$

As a result, we have:

Theorem 5.1 (Semi-static pricing using vanilla options: I). *Under frictionless markets and assumption **A5**, no arbitrage implies that the digital call on maximum drawdown can be valued relative to the prices of bonds and digital options as:*

$$\begin{aligned}
 DC_t^{MD}(K, T) &= 1(MD_t \geq K)B_t(T) + 1(MD_t < K) \\
 &\times \left\{ 2 \sum_{n=0}^{\infty} [DP_t(M_t - (4n + 1)K, T) + DC_t(M_t + (4n + 1)K, T)] \right. \\
 &\quad \left. - 2 \sum_{n=1}^{\infty} [DC_t(M_t + (4n - 1)K, T) + DP_t(M_t - (4n - 1)K, T)] \right\}, \tag{5.5}
 \end{aligned}$$

for any $t \in [0, T]$ and $K > 0$.

5.2. Hedging digital call on the K -drawdown preceding a K -drawup with vanilla options

In this subsection we develop a semi-static replication of a digital call on the K -drawdown preceding a K -drawup using one-touches. By Theorem 2.1 and Proposition 5.1, we need to ensure **A4** holds for all barriers of one-touch knockouts being held. For this purpose we impose structure on the spot price process:

A5': Continuity of the maximum, minimum, and symmetry. *While $t \leq \tau_K^D \wedge \tau_K^U \wedge T$, the running maximum and the running minimum are continuous. Moreover, at times $\theta(u) := \tau_u^D \wedge \tau_u^U \wedge T$ for all $u \in (0, K]$, the conditional risk-neutral probability distribution of hitting S_T is symmetric about $S_{\theta(u)}$.*

Assumption **A5'** is sufficient for applying Proposition 5.1. Evaluating (5.3) at $V = M_t - K$ and $W = M_t$, we obtain:

$$\begin{aligned}
 OTKO_t(M_t - K, M_t, T) \\
 = 2 \sum_{n=0}^{\infty} \{ DP_t(M_t - (2n + 1)K, T) - DC_t(M_t + (2n + 1)K, T) \}, \tag{5.6}
 \end{aligned}$$

for $K > 0$ and $t \in [0, \tau_{M_t - K}^S \wedge \tau_{M_t}^S \wedge T]$. Differentiating (5.3) with respect to W , and evaluating at $V = H - K$ and $W = H$ implies that for $K > 0$ and

$$t \in [0, \tau_{H-K}^S \wedge \tau_H^S \wedge T]:$$

$$\begin{aligned} & RUFDI_t(H - K, H, T) \\ &= -4 \sum_{n=1}^{\infty} n \left(\frac{\partial}{\partial H} DP_t(H - (2n + 1)K, T) + \frac{\partial}{\partial H} DC_t(H + (2n - 1)K, T) \right), \end{aligned} \tag{5.7}$$

since K is a constant.

Substituting (5.6) and (5.7) in (2.7), and ignoring the left and right limits, it gives rise to:

Theorem 5.2 (Semi-static pricing using vanilla options: II). *Under frictionless markets and assumption $A5'$, no arbitrage implies that the digital call on the K -drawdown preceding a K -drawup can be valued relative to the price of bonds and digital options as:*

$$\begin{aligned} & DC_t^{D < U}(K, T) \\ &= 1(\tau_K^D \leq t \wedge \tau_K^U \wedge T)B_t(T) + 1(t < \tau_K^D \wedge \tau_K^U \wedge T) \\ &\quad \times \left\{ \sum_{n=0}^{\infty} (4n + 2)[DP_t(M_t - (2n + 1)K, T) + DC_t(M_t + (2n + 1)K, T)] \right. \\ &\quad \left. - 4 \sum_{n=1}^{\infty} n [DP_t(m_t - 2nK, T) + DC_t(m_t + 2nK, T)] \right\}, \end{aligned} \tag{5.8}$$

for $t \in [0, T]$ and $K > 0$.

6. Model-Free Static Replication of Digital Call on K -Relative Drawdown Preceding a K -Relative Drawup with One-Touch Knockouts in Geometric Models

In the previous sections we developed static and semi-static replications under certain arithmetic symmetry assumptions. However, there are obvious financial limitations of this setup. For example, it requires no carrying cost for the underlying asset; the price of the underlying can be negative with positive probability. In what follows we will consider a more complicated setup, which allows carrying cost and keeps price positive.

As the spot price is always positive, it is much more convenient to consider the percentage drawdown as a measure of risk. Let $D_t^r := M_t/S_t$ be the level of the relative drawdown at time $t \in [0, T]$. For a fixed $K > 1$, let $\tau_K^{D^r}$ be the time at which the relative drawdown process D^r first reaches K . As usual, if D^r never reaches K , then we set $\tau_K^{D^r} = \infty$.

Similarly, let $U_t^r := S_t/m_t$ be the level of the relative drawup at time $t \in [0, T]$. For a fixed $K > 1$, let $\tau_K^{U^r}$ be the time at which the relative drawup process U^r first reaches K . If U^r never reaches K , then we set $\tau_K^{U^r} = \infty$.

We are interested in digital calls on maximum relative drawdown and digital calls on the K -relative drawdown preceding a K -relative drawup. To describe the payoff of these claims, let $MD_T^r = \sup_{t \in [0, T]} D_t^r$ be the maximum relative drawdown of the spot price process. A digital call on maximum relative drawdown pays one dollar at expiry if and only if the maximum relative drawdown exceeds K . We denote the price of the option at time t by

$$DC_t^{MD^r}(K, T) := B_t(T) \mathbb{Q}_t^T(MD_T^r \geq K). \tag{6.1}$$

A digital call on the K -relative drawdown preceding a K -drawup pays one dollar at expiry if and only if the relative drawdown reaches K before the earlier of expiry and the time at which the relative drawup first reaches K . We denote the price of this option at time t by

$$DC_t^{D^r}(K, T) := B_t(T) \mathbb{Q}_t^T(\tau_K^{D^r} \leq \tau_K^{U^r} \wedge T). \tag{6.2}$$

Analogous to the absolute drawdown setting in Sec. 2, we can replicate the payoff of the digital call on K -relative drawdown preceding a K -relative drawup with one-touch knockouts and ricochet-upper-first down-and-in claims. The argument is exactly the same as in Theorem 2.1. We present the following result without proof.

Theorem 6.1 (Robust replication: II). *Under frictionless markets, no arbitrage implies that the digital call on the K -relative drawdown preceding a K -relative drawup can be valued relative to the prices of roots, one-touch knockouts and their spreads:*

$$\begin{aligned} DC_t^{D^r}(K, T) &= 1(\tau_K^D \leq t \wedge \tau_K^U \wedge T) B_t(T) + 1(t < \tau_K^{D^r} \wedge \tau_K^{D^r} \wedge T) \\ &\quad \times \left\{ OTKO_t(M_t K^{-1}, M_t, T) + \int_{M_t^+}^{(m_t K)^-} RUFDI_t(HK^{-1}, H, T) dH \right\}, \end{aligned} \tag{6.3}$$

for any $t \in [0, T]$ and $K > 1$.

In the rest of the paper, we will develop semi-robust replications of the above two digital options under continuity and certain geometric symmetry assumptions on the dynamics of the spot price process.

7. Semi-static Replication with One-Touches in Geometric Models

In the last section we derived a static hedge of one target call with one-touch knockouts and their spreads. Because of illiquidity of one-touch knockouts, this section presents alternative semi-static hedges which just use single-barrier one-touches and lookbacks. The replications only succeed under certain symmetry and continuity assumptions. More specifically, suppose that the spot starts inside the corridor between V and W , where V and W are the in-barrier and out-barrier of

an one-touch knockout respectively. Let $\tau := \tau_V^S \wedge \tau_W^S$ be the first exit time of the above corridor, we assume that:

G1: Skip-freedom and geometric hitting symmetry. *The spot price process S cannot exit the corridor between V and W by a jump. Moreover, there exist a constant q , such that if the first exit time of the above corridor $\tau \leq T$, we have*

$$OT_\tau(S_\tau \Delta^{-1}, T) = \Delta^q \cdot OT_\tau(S_\tau \Delta, T), \tag{7.1}$$

for any $\Delta > 0$.

One of the most important model that satisfy **G1** is the geometric Brownian motion. For a proof of this fact, please refer to Remark 8.3 and the Appendix.

Under the above assumption, an one-touch knockout with in-barrier V and out-barrier W is replicated by a portfolio of one-touches. In particular, we have:

Proposition 7.1 (Semi-static pricing of one-touch knockouts: III). *Under frictionless markets and assumption **G1**, no arbitrage implies that, for any $t \in [0, \tau_V^S \wedge \tau_W^S \wedge T]$*

$$OTKO_t(V, W, T) = OT_t(V, T) + \sum_{n=1}^{\infty} [\Delta^{-nq} OT_t(V \Delta^{-2n}, T) - \Delta^{nq} OT_t(V \Delta^{2n}, T)], \tag{7.2}$$

where $\Delta = W/V \neq 1$.

Proof. Suppose an one-touch knockout with in-barrier V and out-barrier W has been sold at time 0. In order to hedge this position, consider a strategy of being long a series of one-touches with barriers at $V, V\Delta^{-2}, V\Delta^{-4}, \dots$, and also being short a series of one-touches with barriers at $V\Delta^2, V\Delta^4, \dots$. If neither barrier is hit by T , then all one touches expire worthless. If $\tau_V^S \leq \tau_W^S \wedge T$, then at τ_V^S , the one-touch with payoff with barrier at V knocks in, while assumption **G1** implies that all of the other one-touches can be costlessly liquidated. The reason is that for each $n = 1, 2, \dots$, the long position in the one-touches with barrier at $V\Delta^{-2n}$, is canceled by the short position in the one-touches with barrier at $V\Delta^{2n}$. Similarly, if $\tau_W^S \leq \tau_V^S \wedge T$, then at τ_W^S , assumption **G1** implies that all of the one-touches can be costlessly liquidated. The reason is that since $V = W\Delta^{-1}$, the portfolio can also be considered as long a series of one-touches with barriers at $W\Delta^{-1}, W\Delta^{-3}, W\Delta^{-5}, \dots$, while also being short a series of one-touches with barriers at $W\Delta, W\Delta^3, W\Delta^5 \dots$:

$$\sum_{n=0}^{\infty} [\Delta^{-nq} OT_t(W\Delta^{-2n-1}, T) - \Delta^{(n+1)q} OT_t(W\Delta^{2n+1}, T)].$$

Hence, for each $n = 0, 1, 2, \dots$, the long position in the one-touches with barrier at $W\Delta^{-2n-1}$, is canceled by the short position in the one-touch with barrier

at $W\Delta^{2n+1}$. Since the value of the one-touch portfolio matches the payoff of the one-touch knockout when (S, t) exits $(V \wedge W, V \vee W) \times [0, T]$, no arbitrage forces the values prior to exit to be the same. \square

In virtue of Theorem 6.1 and discussion in Sec. 4, Proposition 7.1 plays a crucial role to develop replicating strategies with one-touches for the digital call on K -relative drawdown preceding K -relative drawup. Moreover, we will see that, though not obvious, the digital call on maximum drawdown can also be replicated with one-touches and lookbacks.

7.1. Hedging digital call on maximum relative drawdown with one-touches and lookbacks in geometric models

In this subsection we develop a semi-static replication of a digital call on maximum drawdown using one-touches and lookbacks. For this purpose we impose the following assumption:

G2: Continuity of the maximum, drawdown, and hitting symmetry. *While $MD_t^r < K$, the running maximum is continuous, and the relative drawdown cannot jump up by more than $K - D_t^r$. Moreover, there exists a constant q , so that at times $\tau(u) := \tau_u^S \wedge \tau_K^{D^r} \wedge T$ for all $u > S_0$, the risk-neutral probability of hitting $S_{\tau(u)}\Delta^{-1}$ before T is the same as Δ^q times the risk-neutral probability of hitting $S_{\tau(u)}\Delta$, for any $\Delta > 0$.*

The above assumption is clearly satisfied by geometric Brownian motion and its independent continuous time-changes. The following result provides a semi-static replication for the digital call on maximum relative drawdown.

Theorem 7.1 (Semi-static pricing using one-touches: III). *Under frictionless markets and assumption G2, no arbitrage implies that the digital call on relative maximum drawdown can be valued relative to the prices of bonds, one-touches, and lookback options as:*

$$\begin{aligned}
 DC_t^{MD^r}(K, T) &= 1(MD_t^r \geq K)B_t(T) + 1(MD_t^r < K) \\
 &\times \left\{ \sum_{n=0}^{\infty} (K^{-2nq} OT_t(M_t K^{-4n-1}, T) + K^{(2n+1)q} OT_t(M_t K^{4n+1}, T) \right. \\
 &- K^{2(n+1)q} OT_t(M_t K^{4n+3}, T) - K^{-(2n+1)q} OT_t(M_t K^{-4n-3}, T)) \\
 &\left. + q [LBP_t(M_t, K, T) - LBC_t(M_t, K, T)] \right\}, \tag{7.3}
 \end{aligned}$$

for $t \in [0, T]$ and $K > 1$. Here the prices of the lookback put/call are given by,

$$\begin{aligned}
 &LBP_t(M, K, T) \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n}{K^{(n+1)q}} \int_0^{MK^{-(2n+3)}} \left(\frac{K^{2n+3}}{M/H}\right)^q P_n\left(\frac{q}{2} \log \frac{K^{2n+3}}{M/H}\right) OT_t(H, T) \frac{dH}{H},
 \end{aligned} \tag{7.4}$$

$$\begin{aligned}
 &LBC_t(M, K, T) \\
 &= \sum_{n=0}^{\infty} (-1)^n K^{(n+1)q} \int_{MK^{2n+1}}^{\infty} P_n\left(\frac{q}{2} \log \frac{MK^{2n+1}}{H}\right) OT_t(H, T) \frac{dH}{H},
 \end{aligned} \tag{7.5}$$

where $P_n(x)$ is a polynomial of degree n , satisfying

$$P_0(x) = 1, \quad P_n(0) = n + 1, \tag{7.6}$$

$$P'_{n+1}(x) = P'_n(x) + 2P_n(x). \tag{7.7}$$

Proof. Suppose a digital call on maximum relative drawdown has been sold at time 0. In order to hedge this position, consider a strategy of always holding the replicating portfolio on the right hand side of (7.3). This semi-dynamic trading strategy is followed until the earlier of expiry and the first hitting time of running relative drawdown to the strike K . If the running relative drawdown increases to K before T , then a bond of maturity T is held afterwards.

Since we assume that the running maximum can never increase by a jump, the above replicating portfolio never yields a payout due to a cross of the barriers higher than M_t . When the running maximum increases continuously with the maximum relative drawdown less than K , assumption **G2** can guarantee that it costs nothing to move the barriers of options in the above portfolio. Hence, the first time to receive a cash flow from the above portfolio is at time $\tau := \tau_K^{D^r}$. If $\tau > T$, then the spot price S_t is always within $(M_t/K, M_t]$, so all the one-touches expiry worthless, as does the target claim. On the other hand, if $\tau \leq T$, then at time τ , $S_\tau = M_\tau/K$, using assumption **G2** one can obtain that

$$LBP_\tau(M_\tau, K, T) = LBC_\tau(M_\tau, K, T).$$

Moreover, at time τ , Proposition 7.1 and assumption **G2** imply that the portfolio of one-touches has the same value as

$$OTKO_\tau(M_\tau/K, M_\tau K, T) + K^q \cdot OTKO_\tau(M_\tau K, M_\tau/K, T) = B_\tau(T).$$

We conclude that in all cases, the payoff of the target claim can be replicated by trading one-touches, lookbacks and bonds. The right hand side of (7.3) is the cost of setting up the replicating strategy at time t . Hence, no arbitrage implies that this cost is also the price of the call on maximum drawdown. \square

Remark 7.1. It is interesting to point out the difference between arithmetic and geometric settings. It is seen from Theorem 7.1 that it is not possible to replicate

a digital call on the relative maximum drawdown with just one-touch knockouts, because of the more complicated mechanism of barrier rolling.

7.2. Hedging digital call on the K -relative drawdown preceding a K -relative drawup with one-touches in geometric models

In this subsection we develop a semi-static replication of a digital call on the K -relative drawdown preceding a K -relative drawup using one-touches. The following assumption ensures the validity of the replication.

G2': Continuity of the maximum, minimum, and hitting symmetry. *While $t \leq \tau_K^{Dr} \wedge \tau_K^{Ur} \wedge T$, the running maximum and the running minimum are continuous. Moreover, there exists a constant q , so that at times $\theta(u) := \tau_u^{Dr} \wedge \tau_u^{Ur} \wedge T$ for all $u \in (1, K]$, we have that*

$$OT_{\theta(u)}(S_{\theta(u)}\Delta^{-1}, T) = \Delta^q \cdot OT_{\theta(u)}(S_{\theta(u)}\Delta, T), \tag{7.8}$$

for any $\Delta > 0$.

Assumption **G2'** is sufficient for applying Proposition 7.1. Evaluating (7.2) at $V = M_t/K$ and $W = M_t^+$, we obtain,

$$\begin{aligned} &OTKO_t(M_t/K, M_t^+, T) \\ &= \sum_{n=0}^{\infty} \left\{ \frac{1}{K^{nq}} OT(M_t K^{-2n-1}, T) - K^{(n+1)q} OT_t(M_t K^{2n+1}, T) \right\}, \end{aligned} \tag{7.9}$$

for $K > 0$ and $t \in [0, \tau_{M_t/K}^S \wedge \tau_{M_t^+}^S \wedge T]$. Differentiating (7.2) with respect to W , and evaluating at $V = H/K$ and $W = H$ implies that for $K > 1$ and $t \in [0, \tau_{H/K}^S \wedge \tau_H^S \wedge T]$:

$$\begin{aligned} &RUFDI_t(H/K, H, T) \\ &= -2 \sum_{n=1}^{\infty} n \left(\frac{1}{K^{nq}} \frac{\partial}{\partial H} OT_t(HK^{-2n-1}, T) + K^{nq} \frac{\partial}{\partial H} OT_t(HK^{2n-1}, T) \right) \\ &\quad - \frac{q}{H} \sum_{n=1}^{\infty} n \left(\frac{1}{K^{nq}} OT_t(HK^{-2n-1}, T) + K^{nq} OT_t(HK^{2n-1}, T) \right), \end{aligned} \tag{7.10}$$

since K is a constant.

Substituting (7.9) and (7.10) in (6.3), and ignoring the left and right limits, it gives rise to:

Theorem 7.2 (Semi-static pricing using one-touches: IV). *Under frictionless markets and assumption **G2'**, no arbitrage implies that the digital call on the*

K-relative drawdown preceding a *K*-relative drawup can be valued relative to the prices of bonds and one-touches as:

$$\begin{aligned}
 DC_t^{D^r}(K, T) &= 1(\tau_K^D \leq t \wedge \tau_K^U \wedge T)B_t(T) + 1(t < \tau_K^D \wedge \tau_K^U \wedge T) \\
 &\quad \times \left\{ \sum_{n=0}^{\infty} (2n + 1) \left(\frac{1}{K^{nq}} OT(M_t K^{-2n-1}, T) + K^{(n+1)q} OT_t(M_t K^{2n+1}, T) \right) \right. \\
 &\quad - \sum_{n=1}^{\infty} 2n \left(\frac{1}{K^{nq}} OT_t(m_t K^{-2n}, T) + K^{nq} OT_t(m_t K^{2n}, T) \right) \\
 &\quad \left. - q \int_{M_t}^{m_t K} \sum_{n=1}^{\infty} n \left(\frac{OT_t(HK^{-2n-1}, T)}{K^{nq}} + \frac{OT_t(HK^{2n-1}, T)}{K^{-nq}} \right) \frac{dH}{H} \right\}, \quad (7.11)
 \end{aligned}$$

for any $t \in [0, T]$ and $K > 1$.

Proof. Suppose a digital call on the *K*-relative drawdown preceding a *K*-relative drawup has been sold at time 0. In order to hedge this position, consider a strategy of always holding the replicating portfolio of one-touches on the right hand side of (7.11). This semi-dynamic trading strategy is followed until the earlier of expiry and the first hitting times of running relative drawdown/drawup to the strike *K*. If the running relative drawdown increases to *K* before *T*, then a bond of maturity *T* is held afterwards.

Since we assume that the running maximum and the running minimum are continuous, the above replicating portfolio never yields a payout due to a hit of barriers outside the corridor $[m_t, M_t]$. When the running maximum increases or the running minimum decreases continuously with $t < \tau_K^{D^r} \wedge \tau_K^{U^r} \wedge T$, assumption **G2'** guarantees that it costs nothing to move the barriers of one-touches in the above portfolio. Hence, the first time to get a cash flow from the above portfolio is when $M_t/m_t = K$. Let us denote by τ the first time that $M_t/m_t \geq K$, then clearly $\tau = \tau_K^{D^r} \wedge \tau_K^{U^r}$. If $\tau > T$, then the one-touches expire worthless, as does the target claim. If $\tau \leq T$, then at τ , $M_\tau = m_\tau K$, by Proposition 7.1 and assumption **G2'**, the portfolio of one-touches has the same value as the one-touch knockout $OTKO_\tau(M_\tau/K, M_\tau, T)$, whose value matches the target option, with value either zero or the price of a bond. In the former case, $\tau = \tau_K^{U^r}$, the one touches are liquidated for zero; while in the latter case, $\tau = \tau_K^{D^r}$, the liquidation proceeds are used to buy the bond. We conclude that in all cases, the payoff of the target digital call is matched by the liquidation value of a non-anticipating self-financing portfolio of bonds and one-touches. Furthermore, the right hand side of (7.11) is the cost of setting up the replicating strategy at time t . Hence, no arbitrage implies that this cost is also the price of the target claim. □

8. Semi-Static Replication With Vanilla Options in Geometric Models

In the previous section we developed semi-static hedges with a series of co-terminal single-barrier options of the target calls. This section presents another semi-static hedge which uses more liquid vanilla options. The replications only succeed under some symmetry and continuity assumptions, which we will make precise.

Suppose that the spot starts inside the corridor between V and W , where V and W are the in-barrier and the out-barrier of an one-touch knockout respectively. Let τ be the first exit time of this corridor, we impose the following assumption:

G3: Skip-freedom and geometric symmetry. *The spot S cannot exit the corridor between V and W by a jump. Moreover, there exist a constant q , such that if the first exit time of the above corridor $\tau \leq T$, we have*

$$E_{\tau}^{\mathbb{Q}^T} \{ \delta(S_T - S_{\tau} \Delta^{-1}) \} = S_{\tau}^{-q} \cdot E_{\tau}^{\mathbb{Q}^T} \{ S_T^q \delta(S_T - S_{\tau} \Delta) \}, \tag{8.1}$$

for any $\Delta > 0$.

Remark 8.1. The symmetry in **G3** is often seen in finance literature Bowie and Carr [2], Carr and Chou [5], Carr *et al.* [7], Carr [8]. In particular, geometric Brownian motions and their independent time-changes all satisfy this assumption. The characterization of continuous martingales that satisfy this symmetry conditions is discussed in Tehranchi [26].

Remark 8.2. If we alternatively assume that a barrier B is skip-free and (8.1) holds at the first hitting time τ_B^S , then an one-touch with barrier at B can be replicated with vanilla options. This is the reflection principle, which we present below for completeness.

Lemma 8.1 (Reflection Principle). *Under frictionless market, an one-touch with skip-free barrier $B > 0$ can be replicated with vanilla options, provided that (8.1) holds at τ_B^S . In particular, for any $t \in [0, \tau_B^S \wedge T]$,*

(1) If $B < m_t$,

$$OT_t(B, T) = DP_t(B, T) + B^{-q} P_{q,t}(B, T); \tag{8.2}$$

(2) If $B > M_t$,

$$OT_t(B, T) = DC_t(B, T) + B^{-q} C_{q,t}(B, T), \tag{8.3}$$

where the generalized vanilla put/call prices $P_{q,t}/C_{q,t}$ are given by

$$P_{q,t}(B, T) := B_t(T) E_t^{\mathbb{Q}^T} \{ S_T^q [1(S_T < B) + 0.5\delta(S_T - B)] \}, \tag{8.4}$$

$$C_{q,t}(B, T) := B_t(T) E_t^{\mathbb{Q}^T} \{ S_T^q [1(S_T > B) + 0.5\delta(S_T - B)] \}. \tag{8.5}$$

Remark 8.3. We say a process is skip-free if every element in its state space is skip-free. If the spot price process is skip-free and satisfies **G3**, it is easily seen that condition (7.1) in assumption **G1** is also satisfied. In other words, for a skip-free process, condition (8.1) in **G3** is stronger than (7.1) in **G1**.

It is interesting to point out that, under **G3**, a one-touch knockout with in-barrier V and out-barrier W can be replicated by a portfolio of vanilla options:

Proposition 8.1 (Semi-static pricing of one-touch knockouts: IV). *Under frictionless markets and assumption **G3**, no arbitrage implies that, for $t \in [0, \tau_V^S \wedge \tau_W^S \wedge T]$,*

(1) *If $V < W$:*

$$\begin{aligned}
 OTKO_t(V, W, T) = & \sum_{n=0}^{\infty} \left(\frac{1}{\Delta^{nq}} DP_t(V\Delta^{-2n}, T) + \frac{\Delta^{nq}}{V^q} P_{q,t}(V\Delta^{-2n}, T) \right) \\
 & - \sum_{n=1}^{\infty} \left(\Delta^{nq} DC_t(V\Delta^{2n}, T) + \frac{1}{V^q \Delta^{nq}} C_{q,t}(V\Delta^{2n}, T) \right),
 \end{aligned} \tag{8.6}$$

(2) *If $V > W$:*

$$\begin{aligned}
 OTKO_t(V, W, T) = & \sum_{n=0}^{\infty} \left(\frac{1}{\Delta^{nq}} DC_t(V\Delta^{-2n}, T) + \frac{\Delta^{nq}}{V^q} C_{q,t}(V\Delta^{-2n}, T) \right) \\
 & - \sum_{n=1}^{\infty} \left(\Delta^{nq} DP_t(V\Delta^{2n}, T) + \frac{1}{V^q \Delta^{nq}} P_{q,t}(V\Delta^{2n}, T) \right),
 \end{aligned} \tag{8.7}$$

where $\Delta = W/V \neq 1$ and $P_{q,t}/C_{q,t}$ are defined in (8.4) and (8.5).

Proof. The idea of the proof is similar as that in Proposition 7.1. And we left it to the interested reader. □

Lemma 8.1 and Proposition 8.1 provide fundamentals of our replication results in this section. In Sec. 8.1 and 8.2 we will separately develop portfolios of vanilla options to replicate the payoff of a digital call on maximum relative drawdown and the payoff of a digital call on the K -relative drawdown preceding a K -relative drawup, respectively.

8.1. Hedging digital call on maximum relative drawdown with vanilla options in geometric models

In this subsection we develop a semi-static replication of a digital call on maximum relative drawdown using vanilla options. Let us first state the necessary assumptions regarding the dynamics of the spot price process.

G4: Continuity of the maximum, drawdown, and symmetry. While $MD_t^r < K$, the running maximum is continuous, and the relative drawdown cannot jump up by more than $K - D_t^r$. Moreover, there exists a constant q , so that at times $\tau(u) := \tau_u^S \wedge \tau_K^{D^r} \wedge T$ for all $u > S_0$, we have that

$$E_{\tau(u)}^{\mathbb{Q}^T} \{ \delta(S_T - S_{\tau(u)} \Delta^{-1}) \} = S_{\tau(u)}^{-q} \cdot E_{\tau(u)}^{\mathbb{Q}^T} \{ S_T^q \delta(S_T - S_{\tau(u)} \Delta) \}, \tag{8.8}$$

for any $\Delta > 0$.

If the spot price process is always continuous, then using Theorem 7.1 and Lemma 8.1, we can develop a replicating portfolio of vanilla options to hedge the digital call on maximum relative drawdown. However, we will show in the next theorem that, under the weaker assumption **G4**, such a portfolio is also possible.

Theorem 8.1 (Semi-static pricing using vanilla options: III). *Under frictionless markets and assumption **G4**, no arbitrage implies that the digital call on maximum relative drawdown can be valued relative to the prices of bonds and vanilla options as:*

$$\begin{aligned} DC_t^{MD^r}(K, T) &= 1(MD_t^r \geq K)B_t(T) + 1(MD_t^r < K) \\ &\quad \times \left\{ \sum_{n=0}^{\infty} K^{-2nq} [DP_t(M_t K^{-4n-1}, T) + M_t^{-q} C_{q,t}(M_t K^{4n+1}, T)] \right. \\ &\quad + \sum_{n=0}^{\infty} K^{(2n+1)q} [DC_t(M_t K^{4n+1}, T) + M_t^{-q} P_{q,t}(M_t K^{-4n-1}, T)] \\ &\quad - \sum_{n=0}^{\infty} K^{2(n+1)q} [DC_t(M_t K^{4n+3}, T) + M_t^{-q} P_{q,t}(M_t K^{-4n-3}, T)] \\ &\quad - \sum_{n=0}^{\infty} K^{-(2n+1)q} [DP_t(M_t K^{-4n-3}, T) + M_t^{-q} C_{q,t}(M_t K^{4n+3}, T)] \\ &\quad \left. + q[VP_t(M_t, K, T) - VC_t(M_t, K, T)] \right\}, \tag{8.9} \end{aligned}$$

for $t \in [0, T]$ and $K > 1$. Here the prices of the vanilla put/call are given by,

$$\begin{aligned} VP_t(M, K, T) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{K^{(n+1)q}} \int_0^{MK^{-(2n+3)}} \left(\frac{K^{2n+3}}{M/H} \right)^q P_n \left(\frac{q}{2} \log \frac{K^{2n+3}}{M/H} \right) \\ &\quad \times [DP_t(H, T) + H^{-q} P_{q,t}(H, T)] \frac{dH}{H}, \tag{8.10} \end{aligned}$$

$$\begin{aligned} VC_t(M, K, T) &= \sum_{n=0}^{\infty} (-1)^n K^{(n+1)q} \int_{MK^{2n+1}}^{\infty} P_n \left(\frac{q}{2} \log \frac{MK^{2n+1}}{H} \right) \\ &\quad \times [DC_t(H, T) + H^{-q} C_{q,t}(H, T)] \frac{dH}{H}, \tag{8.11} \end{aligned}$$

where $P_{q,t}/C_{q,t}$ are given in (8.4) and (8.5), and polynomials $\{P_n(x)\}$ are defined in (7.6) and (7.7).

Proof. The proof is left to the reader. □

8.2. Hedging digital call on the K -relative drawdown preceding a K -relative drawup with vanilla options in Geometric Models

In this subsection we develop a semi-static replication of a digital call on the K -relative drawdown preceding a K -relative drawup using vanilla options. We strengthen assumption **G4** in last subsection in order to meet the self-financing requirement of our replication portfolio.

G4': Continuity of the maximum, minimum, and symmetry. While $t < \tau_K^{D^r} \wedge \tau_K^{U^r} \wedge T$, the running maximum and the running minimum are continuous. Moreover, there exists a constant q , so that at times $\theta(u) := \tau_u^{D^r} \wedge \tau_u^{U^r} \wedge T$ for all $u \in (1, K]$, we have that

$$E_{\theta(u)}^{\mathbb{Q}^T} \{ \delta(S_T - S_{\theta(u)} \Delta^{-1}) \} = S_{\theta(u)}^{-q} \cdot E_{\theta(u)}^{\mathbb{Q}^T} \{ S_T^q \delta(S_T - S_{\theta(u)} \Delta) \}, \tag{8.12}$$

for any $\Delta > 0$.

Assumption **G4'** is sufficient for applying Proposition 8.1. Evaluating (8.6) at $V = M_t/K$ and $W = M_t$, we obtain,

$$\begin{aligned} & OTKO_t(M_t/K, M_t, T) \\ &= \sum_{n=0}^{\infty} \left\{ \frac{1}{K^{nq}} \left[DP_t(M_t K^{-2n-1}, T) - \frac{C_{q,t}(M_t K^{2n-1}, T)}{M_t^q} \right] \right. \\ & \quad \left. + K^{(n+1)q} [M_t^{-q} P_{q,t}(M_t K^{-2n-1}, T) - DC_t(M_t K^{2n+1}, T)] \right\}, \tag{8.13} \end{aligned}$$

for $K > 1$.

Differentiating (8.6) with respect to W , and evaluating at $V = H/K$ and $W = H$ implies that for $K > 1$ and $t \in [0, \tau_{H/K}^S \wedge \tau_H^S \wedge T]$:

$$\begin{aligned} & RUFDI_t(H/K, H, T) \\ &= -2 \sum_{n=1}^{\infty} n \left(\frac{1}{K^{nq}} \frac{\partial}{\partial H} DP_t(HK^{-2n-1}, T) + \frac{K^{(n+1)q}}{H^q} \frac{\partial}{\partial H} P_{q,t}(HK^{-2n-1}, T) \right. \\ & \quad \left. + K^{nq} \frac{\partial}{\partial H} DC_t(HK^{2n-1}, T) + \frac{H^{-p}}{K^{(n-1)q}} \frac{\partial}{\partial H} C_{q,t}(HK^{2n-1}, T) \right) \\ & \quad - \frac{q}{H} \sum_{n=1}^{\infty} n \left(\frac{1}{K^{nq}} DP_t(HK^{-2n-1}, T) - \frac{K^{(n+1)q}}{H^q} P_{q,t}(HK^{-2n-1}, T) \right. \\ & \quad \left. + K^{nq} DC_t(HK^{2n-1}, T) - \frac{H^{-q}}{K^{(n-1)q}} C_{q,t}(HK^{2n-1}, T) \right), \tag{8.14} \end{aligned}$$

since K is a constant.

Substituting (8.13) and (8.14) in (6.3), it gives rise to:

Theorem 8.2 (Semi-static pricing using vanilla options: IV). *Under frictionless markets and assumption $\mathbf{G4}'$, no arbitrage implies that the digital call on the K -relative drawdown preceding a K -relative drawup can be valued relative to the prices of bonds and vanilla options as:*

$$\begin{aligned}
 DC_t^{D^r}(K, T) &= 1(\tau_K^D \leq t \wedge \tau_K^U \wedge T)B_t(T) + 1(t < \tau_K^D \wedge \tau_K^U \wedge T) \\
 &\times \left\{ \sum_{n=0}^{\infty} \frac{(2n+1)}{K^{nq}} (DP_t(M_t K^{-2n-1}, T) + M_t^{-q} C_{q,t}(M_t K^{2n+1}, T)) \right. \\
 &+ \sum_{n=0}^{\infty} \frac{(2n+1)}{K^{-(n+1)q}} (DC_t(M_t K^{2n+1}, T) + M_t^{-q} P_{q,t}(M_t K^{-2n-1}, T)) \\
 &- \sum_{n=1}^{\infty} \frac{2n}{K^{nq}} (DP_t(m_t K^{-2n}, T) + m_t^{-q} C_{q,t}(m_t K^{2n}, T)) \\
 &- \sum_{n=1}^{\infty} \frac{2n}{K^{-nq}} (DC_t(m_t K^{2n}, T) + m_t^{-q} P_{q,t}(m_t K^{-2n}, T)) \\
 &- q \int_{M_t}^{m_t K} \sum_{n=1}^{\infty} n \left(\frac{DP_t(HK^{-2n-1}, T)}{K^{nq}} + \frac{P_{q,t}(HK^{-2n-1}, T)}{K^{-(n+1)q} H^q} \right. \\
 &\left. + \frac{DC_t(HK^{2n-1}, T)}{K^{-nq}} + \frac{C_{q,t}(HK^{2n-1}, T)}{K^{(n-1)q} H^q} \right) \frac{dH}{H} \left. \right\}, \tag{8.15}
 \end{aligned}$$

for any $t \in [0, T]$ and $K > 1$.

Proof. The proof is left to the reader. □

9. Poisson Jump Processes

In Sec. 2–8 we developed static and semi-static replications of both digital options under certain continuity and symmetry assumptions. As it is pointed out earlier, the notion of continuity can be extended to skip-freedom so that purely jump models can be considered. In this section, we consider two different skip-free dynamical setups, increasing both complexity and financial realism. The first setup requires no carrying cost for the underlying asset and symmetry in the risk neutral price process. The second setup allows carrying costs and keeps prices positive. We refer to the two setups as the arithmetic case and the geometric case, respectively. In what follows we will develop replicating portfolio in both cases.

9.1. Arithmetic case

In this section, we require that the underlying has no carrying cost. This arises if the option we are concerned about is written on a forward price, or is written on a

spot price, but only under stringent conditions (see for example, Carr [8]). To cast the results of this section in their most favorable light, we will assume in this section that the barrier option is written on a forward price. The next section allows for nonzero carrying cost on the underlying asset.

Let F_t be the forward price at time $t \in [0, T]$. We assume that F is a continuous-time process. Under the risk-neutral measure \mathbb{Q}^T , F has representation

$$F_t = F_0 + a(N_{1,t} - N_{2,t}), \quad t \in [0, T], \tag{9.1}$$

where $a > 0$ is a constant, N_1 and N_2 are independent identically distributed doubly stochastic processes Brémaud [4], with jump intensity λ_t , which is independent of N_1 and N_2 . In words, the forward price F starts at $F_0 > 0$ and jumps up or down by the amount a according to an independent clock. Clearly, F will satisfy all arithmetic symmetry conditions **A1–A5'**, if we extend the notion of continuity to skip-freedom. It follows that⁴ we can construct replicating portfolios of one-touches or vanilla digital options once we have a replication with one-touch knockouts and their spreads in our hands.

Without loss of generality, let us assume that K is a positive integer multiple of a , so that overshoots are avoided. Since the replicating portfolio in Theorem 2.1 is purely static, one can easily extend (2.7) to the case in which the underlying is a skip-free process. More specifically, when the underlying process follows (9.1), a ricochet-upper-first down-and-in claim is a real spread of one-touch knockouts

$$\begin{aligned} RUFDI_t(H - K, H, T) &= B_t(T)E_t^{\mathbb{Q}^T} \{1(m_T \leq H - K)\delta(M_{\tau_{H-K}^F} - H)\} \\ &= OTKO_t(H - K, H + a, T) - OTKO_t(H - K, H, T), \end{aligned} \tag{9.2}$$

from which one immediately obtain the following counterpart of Theorem 2.1:

$$\begin{aligned} DC_t^{D < U}(K, T) &= 1(\tau_K^D \leq \tau_K^U \wedge t)B_t(T) + 1(t < \tau_K^D \wedge \tau_K^U \wedge T) \\ &\quad \times \left\{ OTKO_t(M_t - K, M_t + a, T) + \sum_{i=1}^{\frac{m_t + K - M_t}{a} - 1} \Lambda_t^{(i)} \right\}, \end{aligned} \tag{9.3}$$

where $\Lambda_t^{(i)}$ are the spreads of one-touch knockouts. That is,

$$\begin{aligned} \Lambda_t^{(i)} &= OTKO_t(M_t + ai - K, M_t + a(i + 1), T) \\ &\quad - OTKO_t(M_t + ai - K, M_t + ai, T), \end{aligned} \tag{9.4}$$

for any $t \in [0, T]$ and $K > 0$.

⁴This a consequence of Propositions 4.1 and 5.1.

Similarly, one can modify (3.2) slightly to obtain a replication of digital call on maximum drawdown:

$$\begin{aligned}
 DC_t^{MD}(K, T) &= 1(MD_t \geq K)B_t(T) + 1(MD_t < K) \\
 &\quad \times \{OTKO_t(M_t - K, M_t + K + a, T) \\
 &\quad + OTKO_t(M_t + K + a, M_t - K, T)\}, \tag{9.5}
 \end{aligned}$$

for $t \in [0, T]$ and $K > 0$. The portfolio on the right hand side of (9.5) obviously replicates the payoff of the digital call on maximum drawdown. Moreover, one can easily check that it is self-financing.

Let us now proceed to treat the complications that arise if we allow carrying costs on the underlying and if we further require that the underlying price process stays positive.

9.2. Geometric case

In this section, we will assume that all options are written on the spot price of some underlying asset. Let us consider a filtered risk-neutral probability space $(\Omega, \mathcal{F}, \mathbb{Q}^T)$, $\mathcal{F} = \cup_{t \in [0, T]} \mathcal{F}_t$. Let us denote by N_1 and N_2 two independent standard doubly stochastic processes, with positive jump arrival rates $\lambda_{1,t}$ and $\lambda_{2,t}$ under the risk-neutral measure \mathbb{Q}^T . We require that the trajectories of the intensities $\lambda_{1,t}$ and $\lambda_{2,t}$ are \mathcal{F}_0 -measurable, and the ratio $\lambda_{1,t}/\lambda_{2,t}$ is a constant. In particular, for a given positive constant $g > 0$, we assume that

$$\lambda_{1,t}(e^g - 1) + \lambda_{2,t}(e^{-g} - 1) = r_t - q_t, \tag{9.6}$$

where r_t is the riskfree rate and q_t is the dividend rate (see Carr [5]).

For a given positive constant S_0 , we also assume that the stochastic process governing the spot price of the underlying asset is given by

$$S_t = S_0 e^{g(N_{1,t} - N_{2,t})}, \quad t \in [0, T]. \tag{9.7}$$

In words, the spot price S starts at $S_0 > 0$ and jumps up by the amount $S_{t-}(e^g - 1) > 0$ or down by the amount $S_{t-}(e^{-g} - 1) < 0$ at independent exponential times. Note that equation (9.6) implies that the discounted stock price, with discount rate $r_t - q_t$, is a positive martingale.

Before developing any replication portfolio, let us first examine the symmetry properties of the spot price process. Under the risk neutral measure \mathbb{Q}^T , the log price is a difference of two independent Poisson processes.

$$d \log S_t = g(dN_{1,t} - dN_{2,t}), \quad t \in [0, T]. \tag{9.8}$$

One could employ Esscher transform (see for example, Brémaud [4], Shiryaev [24]) to construct a new probability measure equivalent to \mathbb{Q}^T , under which the log price

$\log S$ is a symmetric martingale. More specifically, let us define a constant

$$\pi = \frac{1}{2g} \log \frac{\lambda_{2,0}}{\lambda_{1,0}}. \tag{9.9}$$

Then we have a positive martingale

$$\begin{aligned} Y_t &= \exp\left(\pi g(N_{1,t} - N_{2,t}) - \int_0^t (\lambda_{1,s}(e^{\pi g} - 1) + \lambda_{2,s}(e^{-\pi g} - 1)) ds\right) \\ &= \left(\frac{S_t}{S_0}\right)^\pi \cdot \phi(t), \quad t \in [0, T], \end{aligned} \tag{9.10}$$

where $\phi(t) = \exp(-\int_0^t [\lambda_{1,s}(e^{\pi g} - 1) + \lambda_{2,s}(e^{-\pi g} - 1)] ds)$. Define a new measure \mathbb{P}^T by

$$E_t^{\mathbb{P}^T} \{Z\} = \frac{1}{Y_t} E_t^{\mathbb{Q}^T} \{ZY_T\}, \tag{9.11}$$

for any \mathcal{F}_T -measurable random variable Z . Under \mathbb{P}^T , the log price $\log S$ is a difference of two independent identically distributed doubly stochastic processes with jump intensity $e^{\pi g} \lambda_1$. Thus, at any time $t \in [0, T]$, for any $\Delta > 0$

$$E_t^{\mathbb{P}^T} \{\delta(S_T - S_t \Delta^{-1})\} = E_t^{\mathbb{P}^T} \{\delta(S_T - S_t \Delta)\}. \tag{9.12}$$

It follows that,

$$\begin{aligned} E_t^{\mathbb{Q}^T} \{\delta(S_T - S_t \Delta^{-1})\} &= E_t^{\mathbb{P}^T} \left\{ \left(\frac{S_T}{S_t}\right)^{-\pi} \cdot \frac{\phi(t)}{\phi(T)} \delta(S_T - S_t \Delta^{-1}) \right\} \\ &= E_t^{\mathbb{P}^T} \left\{ \left(\frac{S_T}{S_t}\right)^\pi \cdot \frac{\phi(t)}{\phi(T)} \delta(S_T - S_t \Delta) \right\} \\ &= E_t^{\mathbb{Q}^T} \left\{ \left(\frac{S_T}{S_t}\right)^{2\pi} \delta(S_T - S_t \Delta) \right\}, \end{aligned} \tag{9.13}$$

for all $\Delta > 0$. In other words, the spot price process will satisfy **G3**, if we extend the notion of continuity to skip-freedom.

By the discussion in Remark 8.3, and the fact that the spot price process is skip-free, it follows that S will satisfy all geometric symmetry conditions in **G1–G4'**. Therefore, it suffices to develop the counterparts of Theorem 6.1 and Theorem 7.1 for the model in (9.7).

Without loss of generality, let us assume that $\log K$ is a positive integer multiple of g , so that overshoots are avoided. Since the result in Theorem 6.1 is purely static, it can be easily extended to the model in (9.7). More specifically,

a ricochet-upper-first down-and-in claim is a real spread of one-touch knockouts

$$\begin{aligned} RUFDI_t(H/K, H, T) &= B_t(T)E_t^{\mathbb{Q}^T} \{1(m_T \leq H/K)\delta(M_{\tau_{H/F}^S} - H)\} \\ &= OTKO_t(H/K, He^g, T) - OTKO_t(H/K, H, T), \end{aligned} \tag{9.14}$$

from which one immediately obtains

$$\begin{aligned} DC_t^{D^r}(K, T) &= 1(\tau_K^{D^r} \leq \tau_K^{U^r} \wedge T)B_t(T) + 1(t < \tau_K^{D^r} \wedge \tau_K^{U^r} \wedge t) \\ &\quad \times \left\{ OTKO_t(M_t/K, M_t e^g, T) + \sum_{i=1}^{\frac{1}{g} \log \frac{M_t K}{M_t} - 1} \Gamma_t^{(i)} \right\}, \end{aligned} \tag{9.15}$$

where $\Gamma_t^{(i)}$ are the spreads of one-touch knockouts. That is,

$$\Gamma_t^{(i)} = OTKO_t(M_t e^{ig}/K, M_t e^{(i+1)g}, T) - OTKO_t(M_t e^{ig}/K, M_t e^{ig}, T) \tag{9.16}$$

for any $t \in [0, T]$ and $K > 0$.

Similarly, the result in Theorem 7.1 can be extended. In fact, one can show that, a digital call on maximum relative drawdown can be replicated with bonds, one-touch knockouts and lookbacks:

$$\begin{aligned} DC_t^{MD^r}(K, T) &= 1(MD_t \geq K)B_t(T) + 1(MD_t < K) \\ &\quad \times \{OTKO_t(M_t/K, M_t K e^g, T) + K^q \cdot OTKO_t(M_t K e^g, M_t/K, T) \\ &\quad + (1 - e^{-qg})[LBP_t(M_t, K, T) - LBC_t(M_t, K, T)]\}, \end{aligned} \tag{9.17}$$

for any $t \in [0, T]$ and $K > 0$. Here the prices of lookback put/call are given by

$$\begin{aligned} LBP_t(M, K, T) &= \sum_{n=0}^{\infty} \frac{(-1)^n e^{-[\frac{n}{2}]qg}}{K^{(n+1)q}} \times \sum_{i \leq \frac{1}{g} \log \frac{M}{K^{2n+3}}} \left(\frac{K^{2n+3}}{M e^{-ig}} \right)^q P_n \left(\log \frac{K^{2n+3}}{M e^{-ig}} \right) \\ &\quad \times OT_t(e^{(i-2[\frac{n+1}{2}]-1)g}, T), \end{aligned} \tag{9.18}$$

$$\begin{aligned} LBC_t(M, K, T) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{K^{-(n+1)q}} e^{[\frac{n+1}{2}]qg} \times \sum_{i \geq \frac{1}{g} \log(MK^{2n+1})+1} P_n \left(\log \frac{MK^{2n+1}}{e^{(i-1)g}} \right) \\ &\quad \times OT_t(e^{(i+2[\frac{n}{2}]+1)g}, T), \end{aligned} \tag{9.19}$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ are the floor and the ceiling functions (see for example Graham *et al.* [13]), and P_n is a function on the lattice $\mathbb{Z} \cdot g$, satisfying

$$P_0 = 1, \quad P_n(0) = n + 1, \tag{9.20}$$

$$P_{n+1}((i+1) \cdot g) - P_{n+1}(i \cdot g) = e^{qg} P_n((i+1) \cdot g) - P_n(i \cdot g). \tag{9.21}$$

10. Conclusion

In this article we developed static replications of a digital call on maximum drawdown and a digital call on the K -drawdown preceding a K -drawup. We then developed semi-static replications of these options using consecutively more liquid instruments under appropriate symmetry and continuity assumptions. We considered two different dynamical setups, increasing in complexity and financial realism. In both cases, our portfolio is self-financing, and only needs occasional trading, typically when the maximum or the minimum changes. Finally, we extend the replication results to the case in which the underlying process is driven by the difference of two independent Poisson processes. We showed that the previous semi-static trading strategies continue to replicate the payoffs of these claims with slight modifications.

Appendix

In the appendix we prove that the geometric Brownian motion model satisfies geometric symmetry in (8.1). And because of the continuity of sample paths and discussion in Remark 8.3, we further conclude that the geometric Brownian motion model satisfies all assumptions **G1–G4'**.

Let us begin with a filtered risk-neutral probability space $(\Omega, \mathcal{F}, \mathbb{Q}^T)$, $\mathcal{F} = \cup_{t \in [0, T]} \mathcal{F}_t$. We consider the logarithm of a spot price process given by a drifted Brownian motion

$$d \log S_t = \nu dt + \sigma dW_t, \quad S_0 = 1, \quad t \in [0, T], \tag{A.1}$$

where ν and $\sigma > 0$ are real constants, W is a standard Brownian motion adapted to the filtration $\{\mathcal{F}_t\}$. Let us denote by $q = -\frac{2\nu}{\sigma^2}$, then it is easily seen that

$$Y_t = e^{q\sigma W_t - \frac{1}{2}q^2\sigma^2 t} = \left(\frac{S_t}{S_0}\right)^q, \quad t \in [0, T], \tag{A.2}$$

is a positive martingale. Define a new measure \mathbb{P}^T by

$$E_t^{\mathbb{P}^T} = \frac{1}{Y_t} E_t^{\mathbb{Q}^T} \{ZY_T\}, \tag{A.3}$$

for any \mathcal{F}_T -measurable random variable Z . Using Girsanov Theorem, we know that, under \mathbb{P}_t^T , the log price follows

$$d \log S_u = -\nu du + \sigma dW_u, \quad u \in [t, T]. \tag{A.4}$$

Thus, at any fixed time $t \in [0, T]$, for any $\Delta > 0$

$$\begin{aligned} E_t^{\mathbb{Q}^T} \{\delta(S_T - S_t\Delta^{-1})\} &= E_t^{\mathbb{P}^T} \{\delta(S_T - S_t\Delta)\} = \frac{1}{Y_t} E_t^{\mathbb{Q}^T} \{Y_T \delta(S_T - S_t\delta)\} \\ &= E_t^{\mathbb{Q}^T} \left\{ \left(\frac{S_T}{S_t}\right)^q \delta(S_T - S_t\Delta) \right\}. \end{aligned} \tag{A.5}$$

This proves that the geometric Brownian motion model satisfies geometric symmetry in (8.1), and thus satisfies all assumptions **G1–G4'**.

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