Semi-Analytical Solutions for Barrier and American Options Written on a Time-Dependent Ornstein–Uhlenbeck Process

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KEY FINDINGS

- For the first time the method of generalized integral transform, invented in physics for solving an initial-boundary value parabolic problem at $[0, y(t)]$ with a moving boundary $[y(t)]$, is applied to finance.
- Using this method, pricing of barrier and American options, where the underlying follows a time-dependent OU process (the Bachelier model with drift) are solved in a semi-analytical form.
- It is demonstrated that computationally this method is more efficient than the backward and even forward finite difference method traditionally used for solving these problems whereas providing better accuracy and stability.

ABSTRACT

In this article, we develop semi-analytical solutions for the barrier (perhaps, time-dependent) and American options written on the underlying stock that follows a time-dependent Ornstein–Uhlenbeck process with a lognormal drift. *Semi-analytical* means that given the time-dependent interest rate, continuous dividend and volatility functions, one need to solve a linear (for the barrier option) or nonlinear (for the American option) Volterra equation of the second kind (or a Fredholm equation of the first kind). After that, the option prices in all cases are presented as one-dimensional integrals of combination of the preceding solutions and Jacobi theta functions. We also demonstrate that computationally our method is more efficient than the backward finite difference method traditionally used for solving these problems, and can be as efficient as the forward finite difference solver while providing better accuracy and stability.

TOPICS

Derivatives, options, statistical methods*

The Ornstein–Uhlenbeck process with time-dependent coefficients is very popular among practitioners for modeling interest rates and credit because it is relatively simple, allows negative interest rates (which recently has become a relevant feature), and can be calibrated to the given term-structure of interest rates and to the prices or implied volatilities of caps, floors, or European swaptions, since the mean-reversion level and volatility are functions of time. Among this class, the

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most known are the Hull-White and Vasicek models (see Brigo and Mercurio 2006 and references therein).

The Hull-White model is a one-factor model for the stochastic short interest rate r_t of the form

$$dr_t = k[\theta(t) - r_t]dt + \sigma(t)dW_t, \tag{1}$$

where t is the time, $k > 0$ is the constant speed of mean-reversion, $\theta(t)$ is the mean-reversion level, $\sigma(t)$ is the volatility of the process, W_t is the standard Brownian motion under the risk-neutral measure. This model can also be used for pricing Equity or FX derivatives if one assumes that the mean-reversion level vanishes, while the mean-reversion rate is replaced either by $q(t) - r(t)$ for Equities, or by $r_f(t) - r_d(t)$ for FX, where $r(t)$, $q(t)$ are the deterministic interest rate and continuous dividends, and $r_d(t)$, $r_f(t)$ are the deterministic domestic and foreign interest rates.

Without loss of generality, in this article we mostly concentrate on the Equity world, whereas application of this technique to the Hull-White model is considered in Itkin and Muravey (2020). Since the process in Equation (1) is Gaussian, the model is tractable for pricing European plain vanilla options. However, for exotic options (e.g., liquid barrier options) or for American options, these prices are not known yet in closed form. Therefore, various numerical methods are used to obtain them, which can sometimes be computationally expensive. Note that simple one-factor models of the type considered in this article are not well suited to replicate the implied volatility surface of the exotic options, and instead more sophisticated models that treat volatility as a stochastic variable should be used in this case. Still, construction of a semi-analytical solution even for our simple model is useful and is discussed in the Discussion section. Once this is done, the same method could be used for solving other problems implicitly related to pricing of barrier options—for example, analyzing the stability of a single bank and a group of banks in the structural default framework, (Kaushansky, Lipton, and Reisinger 2018), calculating the hitting time density (Alil, Patie, and Pedersen 2005; Lipton and Kaushansky 2020a), and finding an optimal strategy for pairs trading (Lipton and de Prado 2020). Also, the method could be used for solving various problems in physics, where it was originally developed for the heat equation (see Kartashov 2001, Friedman 1964, and references therein).

In this article, we construct a semi-analytical solution for the prices of barrier and American options written on the process in Equation (1). The results obtained in this article are new. Our approach to a certain degree is similar to that in Mijatovic (2010), although Mijatovic used a different underlying process (the lognormal model with local spot-dependent volatility, and constant interest rates and dividends, but time-dependent barriers). Therefore, our model is more general in the sense that all parameters of the model are time-dependent, including time-dependent barriers. Also, as compared with Mijatovic (2010), we do not use a probabilistic argument but rather a theory of partial differential equations (PDEs). At the end we demonstrate that computationally our method is more efficient than the backward finite difference (FD) method used to solve these problems, and it can be as efficient as the forward finite difference solver while providing better accuracy and stability.

The rest of the article is organized as follows. In the next section, we describe the pricing problem for the barrier options where the underlying follows the Bachelier model and show how to transform the pricing PDE to the heat equation. In the Solution of the Barrier Pricing Problem section, we describe the method of the Generalized Integral Transform and construct semi-analytical solutions for the direct and inverse problems using complex analysis. In the Pricing American Options section, we apply the same technique for pricing American options in semi-analytical form. Also, by using this approach the exercise boundary is found simultaneously with the option price. The Numerical Example section demonstrates the results of numerical experiments and tests. In the final section, various additional aspects and extensions of the proposed method are discussed.

PROBLEM FOR PRICING BARRIER OPTIONS

We start by specifying the dynamics of the underlying spot price S_t to be

$$dS_t = [r(t) - q(t)]S_t dt + \sigma(t) dW_t, \tag{2}$$

where now $r(t)$ is the deterministic short interest rate. This model is also known in the financial literature as the Bachelier model. See, for example, Thomson (2016) for a thorough discussion of pro and contra of this model. One can think about S_t, for example, as the stock price or the price of some commodity asset. Although in the Bachelier model the underlying value could become negative, which is not desirable for the stock price, this is fine for commodities under the modern market conditions when the oil prices have been several times observed to be negative (see, e.g., CME Clearing 2020). For the sake of certainty, next we reference S_t as the stock price.

In Equation (2) we don't specify the explicit form of $r(t)$, $q(t)$, $\sigma(t)$ but assume that they are known as a differentiable functions of time $t \in [0, \infty)$. The case of discrete dividends is discussed in the final section.

Further in this section we consider a contingent claim written on the underlying process S_t in Equation (2), which is the Up-and-Out barrier Call option. It is known that by the Feynman-Kac formula (Klebaner 2005) one can obtain a parabolic (*linear*) PDE whose solution gives the Up-and-Out barrier Call option price $C(S, t)$ conditional on $S_0 = S$, which reads

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2(t)\, \frac{\partial^2 C}{\partial S^2} + [r(t) - q(t)]S \frac{\partial C}{\partial S} = r(t)C. \tag{3}$$

This equation should be solved subject to the terminal condition at the option maturity $t = T$

$$C(S, T) = (S - K)^+, \tag{4}$$

and the boundary conditions

$$C(0, t) = 0, \qquad C(H, t) = 0, \tag{5}$$

where $H = H(t)$ is the upper barrier. Note that for the arithmetic Brownian motion process, the domain of definition is $S \in (-\infty, H]$. However, here we move the boundary condition from minus infinity to zero; see the discussion in Itkin and Muravey (2020) about rigorous boundary conditions for this problem. This happens because in practice we can control the left boundary to make the probability of S dropping below 0 rare.

Our goal now is to build a series of transformations to transform Equation (3) to the heat equation.

Transformation to the Heat Equation

To transform the PDE Equation (3) to the heat equation, we first make a change of the dependent and independent variables as follows:

$$S \to x/g(t), \qquad C(S, t) \to e^{f(x,t)} u(x, t), \tag{6}$$

where new functions $f(x,t)$, $g(t)$ has to be determined in such a way that the equation for u is the heat equation. This can be done by substituting Equation (6) into Equation (3) and providing some tedious algebra. The result reads

$$\begin{aligned} f(x,t) &= k(t) - \frac{g'(t) + g(t)(r(t)-q(t))}{2g(t)^3\sigma(t)^2} x^2, \\ k(t) &= \frac{1}{2}\log\left(\frac{g(t)}{g(0)}\right) + \frac{1}{2}\int_0^t [3r(s) - q(s)]\, ds. \end{aligned} \tag{7}$$

The function $g(t)$ solves the following ordinary differential equation

$$\begin{aligned} 0 &= b(t)g(t) - g''(t) + 2g'(t)\frac{\sigma'(t)}{\sigma(t)} + 2\frac{g'(t)^2}{g(t)}, \\ b(t) &= 2(r(t)-q(t))\frac{\sigma'(t)}{\sigma(t)} - [(r(t)-q(t))^2 + r'(t) - q'(t)]. \end{aligned} \tag{8}$$

The Equation (8) by substitution

$$g(t) \to e^{\int_0^t w(s)ds} \tag{9}$$

can be further transformed to the Riccati equation

$$w'(t) = b(t) + w(t)^2 + 2w(t)\frac{\sigma'(t)}{\sigma(t)}. \tag{10}$$

This equation cannot be solved analytically for arbitrary functions $r(t)$, $q(t)$, $\sigma(t)$ but can be efficiently solved numerically. Also, in some cases it can be solved in closed form. For instance, if $|r(t) - q(t)|t = \varepsilon \ll 1$ (which at the current market is a typical case), then $b(t)$ can be reduced to $b(t) = 2(r(t) - q(t))\sigma'(t)/\sigma(t)$. Then assuming in the first approximation on ε

$$|\, w(t)\,| \gg |\, r(t) - q(t)\,|, \tag{11}$$

we obtain the solution

$$w(t) = \frac{\sigma^2(t)}{D - \int_0^t \sigma^2(s)ds}, \tag{12}$$

where D is an integration constant. Thus, Equation (11) can be rewritten as

$$\mathrm{V}(t)\left[1 + \frac{V}{\bar{V}}\varepsilon\right] \gg D(r(t) - q(t)),$$

where $\mathrm{V}(t) = \sigma^2(t)$ is the normal variance, and $\bar{V}(t) = \frac{1}{t}\int_0^t \sigma^2(s)ds$ is the average normal variance. Thus, our solution in Equation (12) is correct if $\varepsilon \ll 1$ and $V\varepsilon/\bar{V} \ll 1$, because then D can always be chosen to obey the inequality $\mathrm{V}(t) \gg D(r(t) - q(t))$, $\forall t \in [0,T]$.

With these explicit definitions, Equation (3) transforms to the form

$$\frac{1}{2}\sigma(t)^2 e^{2\int_0^t w(s)ds} \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0. \tag{13}$$

The next step is to make a change of the time variable

$$\tau(t) \to \frac{1}{2}\int_t^T \sigma^2(s) e^{2\int_0^s w(m)dm} ds, \tag{14}$$

so Equation (13) finally takes the form of the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}. \tag{15}$$

Equation (15) should be solved subject to the terminal condition

$$u(x,0) = (xe^{-\int_0^T w(s)ds} - K)^+ e^{-f(x,T)}, \tag{16}$$

and the boundary conditions

$$u(0,\tau) = 0, \qquad u(y(\tau),\tau) = 0, \qquad y(\tau) = H(t(\tau))g(t(\tau)). \tag{17}$$

These conditions directly follow from Equations (4) and (5), whereas $y(\tau)$ is now a time-dependent upper barrier.[1] The function $t(\tau)$ is the inverse map of Equation (14). It can be computed for any $t \in [0,T]$ by substituting it into Equation (14), then finding the corresponding value of $\tau(t)$, and finally inverting.

Solution of the Barrier Pricing Problem

The PDE in Equations (15), (16), and (17) is a parabolic equation whose solution should be found at the domain with moving boundaries. These types of problems have been known in physics for a long time. Similar problems arise, for example, in the field of nuclear power engineering and safety of nuclear reactors, in studying combustion in solid-propellant rocket engines, in laser action on solids, in the theory of phase transitions (the Stefan problem and the Verigin problem [in hydromechanics]), in the processes of sublimation in freezing and melting, and in the kinetic theory of crystal growth (see Kartashov 1999 and references therein). Analytical solutions of these problems require nontraditional, and sometimes sophisticated, methods. Those methods were actively elaborated on by the Russian mathematical school in the 20th century starting from A. V. Luikov, and then by B. Ya. Lyubov, E. M. Kartashov, and many others.

As applied to mathematical finance, one of these methods—the method of heat potentials—was actively used by A. Lipton and his coauthors, who solved various problems of mathematical finance by using this approach (see Lipton 2001, Lipton and de Prado 2020, and references therein). Another method that we use in this article is the method of a generalized integral transform. Next we closely follow Kartashov (2001) when give an exposition of the method.

We start by introducing an integral transform of the form

$$\bar{u}(p,\tau) = \int_0^{y(\tau)} u(x,\tau)\sinh(x\sqrt{p})dx, \tag{18}$$

[1] Therefore, we can also naturally solve the same problem with the time-dependent upper barrier $H = H(t)$, as this just changes the definition of $y(t)$.

where $p = a + i\omega$ is a complex number with $\mathbb{R}(p) \ge \beta > 0$, and $-\frac{\pi}{4} < \arg(\sqrt{p}) < \frac{\pi}{4}$. Let's multiply both parts of Equation (15) by $\sinh(x\sqrt{p})$ and then integrate on x from zero to $y(\tau)$:

$$\int_0^{y(\tau)} \frac{\partial u}{\partial \tau} \sinh(x\sqrt{p})dx = \int_0^{y(\tau)} \frac{\partial^2 u}{\partial x^2} \sinh(x\sqrt{p})dx.$$

Integrating by parts, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial \tau}\int_0^{y(\tau)} u(x,\tau)\sinh(x\sqrt{p})dx - u(y(\tau),\tau)\sinh(y(\tau)\sqrt{p})y'(\tau) \\
&= \frac{\partial u(x,\tau)}{\partial x}\sinh(x\sqrt{p})\Big|_0^{y(\tau)} + \sqrt{p}u(x,\tau)\cosh(x\sqrt{p})\Big|_0^{y(\tau)} + p\int_0^{y(\tau)} u(x,\tau)\sinh(x\sqrt{p})dx.
\end{aligned}$$

With allowance for the boundary conditions in Equation (17) and the definition in Equation (18), we obtain the following Cauchy problem:

$$\begin{aligned}
&\frac{\partial \bar{u}}{\partial \tau} - p\bar{u} = \Psi(\tau)\sinh(y(\tau)\sqrt{p}), \\
&\bar{u}(p,0) = \int_0^{y(0)} u(x,0)\sinh(x\sqrt{p})dx, \qquad \Psi(\tau) = \frac{\partial u(x,\tau)}{\partial x}\Big|_{x=y(\tau)}.
\end{aligned} \tag{19}$$

Equation (19) can be solved explicitly, assuming that $\Psi(\tau)$ is known. The solution reads

$$\bar{u}e^{-p\tau} = \int_0^{\tau} \Psi(k)e^{-pk}\sinh(y(k)\sqrt{p})dk + \int_0^{y(0)} u(x,0)\sinh(x\sqrt{p})dx. \tag{20}$$

As $\mathbb{R}(p) \ge \beta > 0$, and $\bar{u}(x,\tau) < \infty$, the function $\bar{u}e^{-p\tau} \to 0$ at $\tau \to \infty$. Therefore, letting τ tend to ∞, we obtain an equation that makes a connection between the moving boundary $y(\tau)$ and $\Psi(\tau)$:

$$\int_0^{\infty} \Psi(\tau)e^{-p\tau}\sinh(y(\tau)\sqrt{p})d\tau = -\int_0^{y(0)} u(x,0)\sinh(x\sqrt{p})dx. \tag{21}$$

Using the definitions in Equation (16) and Equation (7), the integral in the RHS of Equation (21) can be represented as

$$\begin{aligned}
F(p) &\equiv -e^{-\int_0^T w(s)ds}\int_{K_1}^{y(0)} (x-K_1)\sinh(x\sqrt{p})e^{k(T)-a(T)x^2}dx \\
&= e^{-\int_0^T w(s)ds}\frac{e^{k(T)-x\left[a(T)x+\sqrt{p}\right]}}{8a^{3/2}(T)}\Bigg\{2\sqrt{a(T)}(e^{2\sqrt{p}x}-1) - \sqrt{\pi}e^{\frac{\left[2a(T)x+\sqrt{p}\right]^2}{4a(T)}} \\
&\quad\cdot\left[\left[\sqrt{p}-2a(T)K_1\right]\operatorname{erf}\left(\frac{2a(T)x-\sqrt{p}}{2\sqrt{a(T)}}\right) + \left[\sqrt{p}+2a(T)K_1\right]\operatorname{erf}\left(\frac{2a(T)x+\sqrt{p}}{2\sqrt{a(T)}}\right)\right]\Bigg\}\Bigg|_{K_1}^{y(0)}, \\
K_1 &= Ke^{\int_0^T w(s)ds}, \quad a(t) = \frac{g'(t)+g(t)(r(t)-q(t))}{2g(t)^3\sigma(t)^2}.
\end{aligned} \tag{22}$$

Thus, Equation (21) takes the form

$$\int_0^\infty \Psi(\tau)e^{-p\tau}\sinh(y(\tau)\sqrt{p})d\tau = F(p), \tag{23}$$

where F(p) is known from Equation (22).

Equation (23) is a linear Fredholm integral equation of the first kind (Polyanin and Manzhirov 2008). The solution $\Psi(\tau)$ can be found numerically on a grid by solving a system of linear equations. In other words, given functions $r(t)$, $q(t)$, $\sigma(t)$, we can compute first $w(t)$, then $g(t)$, and finally $\tau(t)$ (or $t(\tau)$), thus determining the moving boundary $y(\tau)$. Next we can solve Equation (23) for $\Psi(\tau)$ and substitute it into Equation (20) to obtain the generalized transform of $u(x,\tau)$ in the explicit form. Therefore, if this transform can be inverted back, we solved the problem of pricing Up-and-Out barrier Call options.

The Inverse Transform

In this section the description of inversion is borrowed from Kartashov (2001). Because that book has not been translated into English, we provide a wider exposition of the method. Also, the book contains various typos that are fixed here.

As known from a general theory of the heat equation, the solution of the heat equation $\mathcal{L}u(x,\tau)=0$, $\mathcal{L}=\partial/\partial\tau-\nu\,\partial^2/\partial x^2$ at the space domain $0<x<l$, where l is a constant, can be expressed via Fourier series of the form, (Polyanin 2002)

$$u(x,\tau)=\sum_{n=1}^{\infty}\alpha_n e^{-\nu\gamma_n^2 t}\sin\left(\frac{n\pi x}{l}\right),$$

where $\psi(x)=\sin(n\pi x/l)$ are the eigenfunctions of the heat operator $\mathcal{L}$, and $\gamma_n = n\pi/l$ are its eigenvalues.

Therefore, by analogy we look for the inverse transform of $\bar{u}$, or for the solution of Equation (20) in terms of $u(x,\tau)$, to be a generalized Fourier series of the form (Kartashov 2001)

$$u(x,\tau)=\sum_{n=1}^{\infty}\alpha_n(\tau)e^{-\left(\frac{n\pi}{y(\tau)}\right)^2\tau} sin\left(\frac{n\pi x}{y(\tau)}\right), \tag{24}$$

where $\alpha(\tau)$ are some functions to be determined. Note that this definition automatically respects the vanishing boundary conditions for $u(x,\tau)$. We assume that this series converges absolutely and uniformly $\forall x \in [0, y(\tau)]$ for any $\tau>0$.

Applying this generalized integral transform to both parts of Equation (18) and integrating, we obtain

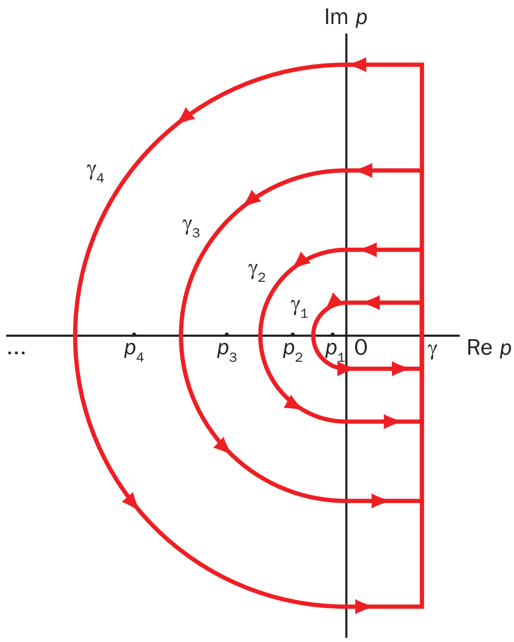
$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n\alpha_n(\tau)e^{-(n\pi/y(\tau))^2\tau}}{p+(n\pi/y(\tau))^2}=\frac{y(\tau)}{\pi\sinh(\sqrt{p}\,y(\tau))}\bar{u}(p,\tau). \tag{25}$$

The LHS of this equation is regular everywhere except simple poles on the negative semi-axis (see Exhibit 1),

$$p_n=-\left(\frac{n\pi}{y(\tau)}\right)^2, \quad n=1,2,\ldots$$

EXHIBIT 1

Contours of Integration in a Complex Plane: p_n, $n = 1, 2$ … Are Simple Poles of the LHS of Equation (25), γ_n–the Integration Contours



Let us sequentially integrate both sides of Equation (25) on p along contours $\gamma_1, \gamma_2, \ldots$. The contour γ_n consists of the vertical line $\gamma > 0$, the half-round of radius $R_n = [\pi 2/(2y(\tau)](2n^2 + 2n + 1)$ (the contour γ_n crosses the $Re(p)$ axis in the middle point between p_n and $p_n + 1$ with the center in the origin), and two horizontal lines $Y = \pm[\pi 2/(2y(\tau))]\ (2n^2 + 2n + 1)$. It means that the circle R_n doesn't hit any pole of the LHS of Equation (25). Then by the Cauchy's residual theorem (Mitrinovic and Keckic 1984), the integral taken along the contour γ_n is equal to 2pi times the sum of residuals of the LHS of Equation (25) that lie inside γ_n.

As poles are simple, and the function under the integral in the LHS of Equation (25) has the form $F_1(p)/F_2(p)$, the residual of such a function is (Mitrinovic and Keckic 1984)

$$\mathrm{Res}[F_1(p)/F_2(p); p_k] = F_1(p)/F_2'(p)\Big|_{p=p_k}$$

The preceding analysis is the basis for running a residual machinery to calculate all the coefficients $\alpha_n(\tau)$.

Residual Machinery

Let us denote via $\mathcal{I}_k$ the following contour integral

$$\mathcal{I}_k = \frac{1}{2\pi i}\oint_{\gamma_k} \frac{\bar{u}(p,\tau)}{\sinh(\sqrt{p}\,y(\tau))}dp.$$

Next we show that all coefficients α_n, $n = 1, \ldots, \infty$ can be expressed via these integrals.

1. **Coefficient** $\alpha_1(\tau)$. Integrating Equation (25) along the contour γ_1 gives

$$\alpha_1(\tau)e^{-(\pi/y(\tau))^2\tau}\oint_{\gamma_1}\frac{1}{p+(\pi/y(\tau))^2}dp + \sum_{n=2}^{\infty}(-1)^{n+1}n\alpha_n(\tau)e^{-(n\pi/y(\tau))^2\tau}\oint_{\gamma_1}\frac{1}{p+(n\pi/y(\tau))^2}dp = \frac{y(\tau)}{\pi}\oint_{\gamma_1}\frac{\bar{u}(p,\tau)dp}{\sinh(\sqrt{p}\,y(\tau))}.$$

Observe, that

$$\oint_{\gamma_1}\frac{1}{p+(\pi/y(\tau))^2}dp = 2\pi i, \qquad \oint_{\gamma_1}\frac{1}{p+(n\pi/y(\tau))^2}dp = 0, \quad n \geq 2,$$

where the second result is due to the Cauchy integral theorem (Mitrinovic and Keckic 1984). Then

$$\alpha_1(\tau, y) = \frac{y(\tau)}{\pi} e^{(\pi/y(\tau))^2\tau}\mathcal{I}_1. \tag{26}$$

2. **Coefficient** $\alpha_2(\tau)$. By analogy, integrating the second equation of Equation (25) along the contour γ_2 we obtain

$$\alpha_1(\tau)e^{-(\pi/y(\tau))^2\tau}\oint_{\gamma_2}\frac{dp}{p+(\pi/y(\tau))^2} - 2\alpha_2(\tau)e^{-(2\pi/y(\tau))^2\tau}\oint_{\gamma_2}\frac{dp}{p+(2\pi/y(\tau))^2}$$
$$+\sum_{n=3}^{\infty}(-1)^{n+1}\alpha_n(\tau)e^{-(n\pi/y(\tau))^2\tau}\oint_{\gamma_2}\frac{dp}{p+(n\pi/y(\tau))^2} = \frac{y(\tau)}{\pi}\oint_{\gamma_2}\frac{\bar{u}(p,\tau)}{\sinh\left(\sqrt{p}\,y(\tau)\right)}dp,$$

whence using again the residual theorem and Equation (26) we find

$$\alpha_2(\tau, y) = -\frac{y(\tau)}{2\pi}e^{(2\pi/y(\tau))^2\tau}\left[\mathcal{I}_2 - \mathcal{I}_1\right].$$

3. **Coefficient** $\alpha_n(\tau)$. Proceeding in a similar manner, we obtain a general formula for the coefficients α_n, $n \ge 1$

$$\alpha_n(\tau) = (-1)^{n+1}\frac{y(\tau)}{n\pi}e^{(n\pi/y(\tau))^2\tau}\left[\mathcal{I}_n - (1-\delta_{n,1})\mathcal{I}_{n-1}\right], \tag{27}$$

where $\delta_{n,1}$ is the Kronecker symbol.

The Final Solution

To calculate the integrals in the RHS of Equation (27), we rewrite them in the explicit form by using the solution for $\bar{u}(p,\tau)$ previously found in Equation (20),

$$\mathcal{I}_j = \frac{1}{2\pi i}\oint_{\gamma_k}\frac{e^{p\tau}}{\sinh(y(\tau)\sqrt{p})}\left[\int_0^{\tau}\Psi(s)e^{-ps}\sinh(y(s)\sqrt{p})ds + \int_0^{y(0)}u(x,0)\sinh(x\sqrt{p})dx\right]dp.$$

As $\sinh(x)$ is a periodic complex function with the period $\pi k/i$, the RHS of this equation is regular everywhere except simple poles, where $\sinh(\sqrt{p}\,y(\tau))$ vanishes. It is easy to checks that these poles are exactly p_i, $i = 1,\ldots,k$. Therefore, we again can directly apply the Cauchy residual theorem. Computing residuals, after some algebra we obtain

$$\alpha_n(\tau) = \frac{2}{y(\tau)}\left[\int_0^{y(0)}u(x,0)\sin\left(\frac{n\pi x}{y(\tau)}\right)dx + \int_0^{\tau}e^{(n\pi/y(\tau))^2 s}\Psi(s)\sin\left(\frac{n\pi y(s)}{y(\tau)}\right)ds.\right] \tag{28}$$

Thus, from Equation (24) and Equation (28) we find the final solution

$$u(x,\tau) = \frac{2}{y(\tau)}\left[\sum_{n=1}^{\infty}e^{-\left(\frac{n\pi}{y(\tau)}\right)^2\tau}\sin\left(\frac{n\pi x}{y(\tau)}\right)\int_0^{y(0)}u(x,0)\sin\left(\frac{n\pi x}{y(\tau)}\right)dx\right.$$
$$\left.+\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{y(\tau)}\right)\int_0^{\tau}e^{(n\pi/y(\tau))^2(s-\tau)}\Psi(s)\sin\left(\frac{n\pi y(s)}{y(\tau)}\right)ds\right].$$

This can also be rewritten as

$$u(x,\tau) = \frac{2}{y(\tau)}\left[\int_0^{y(0)} dz\, u(z,0)\sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{y(\tau)}\right)^2\tau} \sin\left(\frac{n\pi x}{y(\tau)}\right)\sin\left(\frac{n\pi z}{y(\tau)}\right)\right.$$
$$\left. + \int_0^{\tau} ds\, \Psi(s)\sum_{n=1}^{\infty} e^{(n\pi/y(\tau))^2(s-\tau)} \sin\left(\frac{n\pi y(s)}{y(\tau)}\right)\sin\left(\frac{n\pi x}{y(\tau)}\right)\right]. \tag{29}$$

We proceed with the observation that the sums in Equation (29) could be expressed via Jacobi theta functions of the third kind (Mumford et al. 1983). By definition,

$$\theta_3(z,\omega) = 1 + 2\sum_{n=1}^{\infty} \omega^{n^2}\cos(2nz). \tag{30}$$

Therefore,

$$\sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{y(\tau)}\right)^2\tau} \sin\left(\frac{n\pi x}{y(\tau)}\right)\sin\left(\frac{n\pi z}{y(\tau)}\right) = \frac{1}{4}[\theta_3(\phi_-(z),\omega_1) - \theta_3(\phi_+(z),\omega_1)],$$
$$\sum_{n=1}^{\infty} e^{(n\pi/y(\tau))^2(s-\tau)} \sin\left(\frac{n\pi y(s)}{y(\tau)}\right)\sin\left(\frac{n\pi x}{y(\tau)}\right) = \frac{1}{4}[\theta_3(\phi_-(y(s)),\omega_2) - \theta_3(\phi_+(y(s)),\omega_2)],$$
$$\omega_1 = e^{-\left(\frac{\pi\sqrt{\tau}}{y(\tau)}\right)^2}, \quad \omega_2 = e^{\left(\frac{\pi\sqrt{s-\tau}}{y(\tau)}\right)^2}, \quad \phi_-(z) = \frac{\pi(x-z)}{2y(\tau)}, \quad \phi_+(z) = \frac{\pi(x+z)}{2y(\tau)}. \tag{31}$$

A well-behaved theta function must have parameter $|\omega| < 1$ (Mumford et al. 1983). This condition holds at any $\tau > 0$.

Thus, Equation (29) transforms to a simpler form:

$$u(x,\tau) = \frac{1}{2y(\tau)}\left[\int_0^{y(0)} dz\, u(z,0)[\theta_3(\phi_-(z),\omega_1) - \theta_3(\phi_+(z),\omega_1)]\right.$$
$$\left. + \int_0^{\tau} ds\, \Psi(s)[\theta_3(\phi_-(y(s)),\omega_2) - \theta_3(\phi_+(y(s)),\omega_2)]\right]. \tag{32}$$

The RHS of Equation (29) depends on x via functions $\phi-$, $\phi+$. Since the theta function $\theta_3(z,\omega)$ is even in z, the boundary condition at $x = 0$ is satisfied. At $x = y(\tau)$ it is also satisfied as follows from Equation (31) if one reads it from right to left.

The result in Equation (32) to some extent is not a surprise, as it is known that the Jacobi theta function is the fundamental solution of the one-dimensional heat equation with spatially periodic boundary conditions (Ohyama 1995).

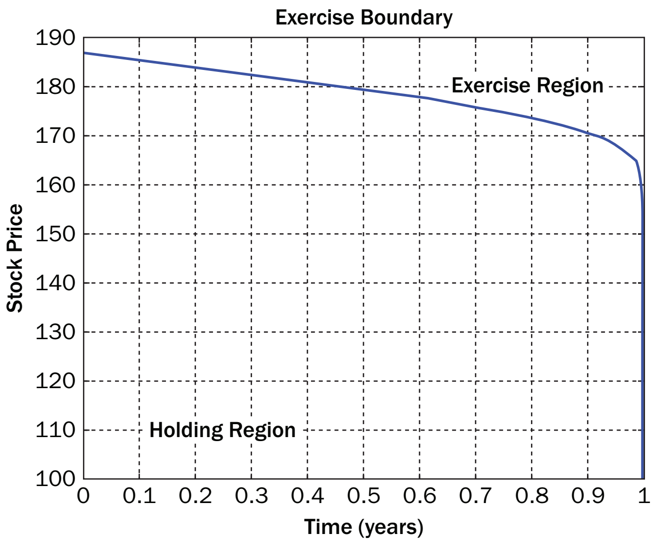
It is also worth mentioning that Equation (32) can be transformed to the Volterra equation of the second kind for $\Psi(\tau)$ by differentiating both parts on x and then letting $x = y(\tau)$. This equation can be solved instead of Equation (23), which is computationally easier than solving Equation (23) (Carr, Itkin, and Muravey 2020).

PRICING AMERICAN OPTIONS

We recall that an American option is an option that can be exercised at any time during its life. American options allow option holders to exercise the option anytime prior to and including its maturity date, thus increasing the value of the option to the holder relative to European options, which can be exercised only at maturity.

EXHIBIT 2

Typical Exercise Boundary for the American Call Option



The majority of exchange-traded options are American style. For a more detailed introduction, see Detemple (2006) and Hull (1997).

It is known that pricing American (or Bermudan) options requires solution of a linear complementary problem. Various efficient numerical methods have been proposed for doing that—for instance, when the underlying stock price S_t follows the time-dependent Black-Scholes model; these (finite difference) methods are discussed in Itkin (2017; see also references therein).

Another approach—elaborated in Andersen, Lake, and Offengenden (2016), for example, for the Black-Scholes model with constant coefficients—uses a notion of the exercise boundary $S_B(t)$. The boundary is defined in such a way that, for example, for the American Put option $P_A(S, t)$ at $S \le S_B(t)$ it is always optimal to exercise the option, therefore $P_A(S, t) = K - S$. For the complementary domain $S > S_B(t)$ the earlier exercise is not optimal, and in this domain $P_A(S, t)$ obeys the Black-Scholes equation. This domain is called the continuation (holding) region. The problem of pricing American options lies in the fact that $S_B(t)$ is not known in advance. Instead, we know only the price of the American option at the boundary. For instance, for the American Put we have $P_A(S_B(t), t) = K - S_B(t)$, and for the American Call, $C_A(S_B(t), t) = S_B(t) - K$. A typical shape of the exercise boundary for the Call option obtained with the parameters $K = 100$, $r = 0.05$, $q = 0.03$, $\sigma = 0.2$ is presented in Exhibit 2. The method proposed in Andersen, Lake, and Offengenden (2016) finds $S_B(t)$ by numerically solving an integral (Volterra) equation for $S_B(t)$. The resulting scheme is straightforward to implement and converges at a speed several orders of magnitude faster than existing approaches.

In terms of this article, the continuation region is a domain with the moving boundary, where the option price solves the corresponding PDE. In case of our model in Equation (2), this is the PDE in Equation (3). Therefore, this problem is, by nature, similar to that for the barrier options considered in the first section of the article, but the difference is as follows:

- For the barrier option pricing problem, the moving boundary (the time-dependent barrier) is known, as this is stated in Equation (17). But the Option Delta $\frac{\partial u}{\partial x}$ at the boundary $x = y(\tau)$ is not, and should be found by solving the linear Fredholm equation in Equation (23). Also, the problem is solved subject to the vanishing condition at the barrier (the moving boundary) for the option value.
- For the American option pricing problem the moving boundary is not known. However, the option Delta $\frac{\partial u}{\partial x}$ at the boundary $x = y(\tau)$ is known (it follows from the conditions $\frac{\partial C_A}{\partial S}\big|_{S=S_B(t)} = 1$ and $\frac{\partial P_A}{\partial S}\big|_{S=S_B(t)} = -1$ expressed in variables x and $y(\tau)$ according to their definitions in Section 1). Also the boundary condition for the American Call and Put at the exercise boundary (the moving boundary) differs from that for the Up-and-Out barrier option, namely, it is $C_A(S_B(t), t)) = S_B(t) - K$ for the Call, and $P_A(S_B(t), t)) = K - S_B(t)$ for the Put.

Because of the similarity of these two problems, it turns out that the American option problem can be solved for the continuation region together with the simultaneous finding of the exercise boundary by using the same approach that we proposed for solving the barrier option pricing problem. However, due to the highlighted differences, some equations slightly change. Also, it is worth mentioning that a similar approach, which uses a method of heat potentials, has been developed in Lipton and Kaushansky (2020b).

Solution of the American Call Option Pricing Problem

Because the PDE we need to solve is the same as in Equation (3), we do same transformations as in the first section of this article and come up to the same heat equation as in Equation (15). It should be solved subject to the terminal condition

$$u(x,0) = \left(xe^{-\int_0^T w(s)ds} - K\right)^+ e^{-f(x,T)},$$

and the boundary conditions

$$u(0,\tau) = 0, \qquad u(y(\tau),\tau) \equiv \psi_1(\tau) = y(\tau) - K,$$
$$\Psi(\tau) \equiv \frac{\partial u}{\partial x}\Big|_{x=y(\tau)} = \frac{e^{-f(y(\tau),t)}}{g^2(t)}\left[1 + a_1(t)y(\tau)(y(\tau)-K)\right], \quad a_1(t) = \frac{g'(t) + g(t)(r(t)-q(t))}{g(t)\sigma(t)^2},$$

where $t = t(\tau)$. We underline once again that the function $y(\tau)$ here is not known yet, while $\Psi(\tau)$ is known. These problems with the free boundaries are also well known in physics.

We proceed by using the same transformation in Equation (18) and by analogy with Equation (19) obtain the following Cauchy problem:

$$\frac{\partial \bar{u}}{\partial \tau} - p\bar{u} = \Psi(\tau)\sinh(x\sqrt{p}) + \psi_1(\tau)\sqrt{p},$$
$$\bar{u}(p,0) = \int_0^{y(0)} u(x,0)\sinh(x\sqrt{p})dx.$$

This problem can be solved explicitly to yield (Kartashov 2001),

$$\bar{u}e^{-p\tau} = \int_0^\tau \Psi(\tau)e^{-p\tau}\sinh(y(\tau)\sqrt{p})d\tau + \int_0^{y(0)} u(x,0)\sinh(x\sqrt{p})dx + \sqrt{p}\int_0^\tau e^{-p\tau}\psi_1(\tau)d\tau.$$

Accordingly, instead of Equation (21) we obtain

$$\int_0^\infty e^{-p\tau}\left[\Psi(\tau)\frac{\sinh(y(\tau)\sqrt{p})}{\sqrt{p}} + y(\tau)\right]d\tau = \frac{K}{p} - \frac{1}{\sqrt{p}}\int_0^{y(0)} u(x,0)\sinh(x\sqrt{p})dx = \frac{K}{p} + \frac{F(p)}{\sqrt{p}}.$$

This is a nonlinear Fredholm equation of the first kind but now with respect to the function $y(\tau)$. It can also be solved numerically (iteratively).

The next step is to reduce our problem to that with homogeneous boundary conditions. This can be done by change of the dependent variable

$$u(x,\tau) = W(x,\tau) + \Theta(x,\tau), \quad \Theta(x,\tau) = (1 - x/y(\tau)]\psi_1(\tau).$$

The function $W(x,\tau)$ solves he same heat equation with the same terminal condition and with the homogeneous boundary conditions. Therefore, it can be solved by using the method of generalized integral transform described in the Solution of the Barrier Pricing Problem section. The solution reads

$$
u(x,\tau) = \Theta(x,\tau) + \sum_{n=1}^{\infty} \alpha_n(\tau) e^{-\left(\frac{n\pi}{y(\tau)}\right)^2 \tau} \sin\left(\frac{n\pi x}{y(\tau)}\right),
$$

$$
\alpha_n(\tau) = \frac{2}{y(\tau)}\left[\int_0^{y(0)} [u(z,0) - \Theta(z,0)] \sin\left(\frac{n\pi z}{y(\tau)}\right) dz + \int_0^{\tau} e^{(n\pi/y(\tau))^2 s} \left[\Psi(s) + \frac{\psi_1(s)}{y(s)}\right] \sin\left(\frac{n\pi y(s)}{y(\tau)}\right) ds\right].
$$

Again, using the definition of the Jacobi theta function in Equation (30), this can be finally rewritten as

$$
\begin{aligned}
u(x,\tau) = \Theta(x,\tau) + \frac{1}{2y(\tau)}\Big[\int_0^{y(0)} dz\, [u(z,0) - \Theta(z,0)][\theta_3(\phi_-(z),\omega_1) - \theta_3(\phi_+(z),\omega_1)] \\
+ \int_0^{\tau} ds \left[\Psi(s) + \frac{\psi_1(s)}{y(s)}\right][\theta_3(\phi_-(y(s)),\omega_2) - \theta_3(\phi_+(y(s)),\omega_2)]\Big].
\end{aligned}
$$

NUMERICAL EXAMPLE

To test performance and accuracy of our method, in this section we provide a numerical example where a particular time dependence of $r(t)$, $q(t)$, $\sigma(t)$ is chosen as

$$
r(t) = r_0 e^{-r_k t}, \qquad q(t) = q_0, \qquad \sigma(t) = \sigma_0 e^{-\sigma_k t}. \tag{33}
$$

Here r_0, q_0, σ_0, r_k, σ_k are constants. With this model, Equation (10) can be solved analytically to yield

$$
w(t) = q_0 - r_0 e^{-r_k t}. \tag{34}
$$

Accordingly, from Equation (9) we find

$$
g(t) = \exp\left[q_0 t + \frac{r_0}{r_k}(e^{-r_k t} - 1)\right], \tag{35}
$$

and from Equation (7)

$$
f(x,t) = k(t) = \frac{1}{2}\left[\log\left(\frac{g(t)}{g(0)}\right) - q_0 t + 3\frac{r_0}{r_k}(1 - e^{-r_k t})\right]. \tag{36}
$$

The algorithm described in the first section was implemented in python. We did it for two reasons. First, we found neither any standard implementation of the Jacobi theta functions in Matlab nor any custom good one. Surprisingly, this is also not a part of numpy or scipy packages in python. However, they are available as a part of the python package mpmath, which is a free (BSD-licensed) Python library for real and complex floating-point arithmetic with arbitrary precision; see Johansson (2007). It has been developed by Fredrik Johansson since 2007, with help from many contributors.

Also, we didn't find any standard implementation of solver for the Fredholm integral equation of the first kind in both python and Matlab. Therefore, we implemented a Tikhonov regularization method as this is described in Fuhry (2001). In particular, with the model used in this section, the function $F(p)$ reads

$$
F(p) = \frac{e^{-k(T)}}{p}\left[-\sqrt{p}(K_1 - y_0)\cosh(\sqrt{p}\,y_0) + \sinh(K_1\sqrt{p}) - \sinh(\sqrt{p}\,y_0)\right]. \tag{37}
$$

EXHIBIT 3
Parameters of the Test

r_0	q_0	σ_0	r_k	σ_k	H	S_0
0.02	0.01	$0.5 \cdot H$	0.1	0.2	90	60

The algorithm described in the first section was implemented in python. Finally to validate the results provided by our method, we implemented an FD solver for pricing Up-and-Out barrier options. This solver is based on the Crank–Nicolson scheme with a few Rannacher first steps and uses a nonuniform grid (for more detail, see, e.g., Itkin 2017). We implemented two solvers: one for the backward PDE, and the other for the forward PDE. But logically, because in this article we solved the backward PDE, it does make sense to compare our method with the backward solver. This implementation has been done in Matlab.

In our particular test, we choose parameters of the model as they are presented in Exhibit 3.

We recall that here $\sigma(t)$ is the normal volatility. Therefore, we choose its typical value by multiplying the log-normal volatility by the barrier level.

We run the test for a set of maturities $T \in [1/12, 0.3, 0.5, 1]$ and strikes $K \in [50, 55, 60, 65, 70, 75, 80]$. The Up-and-Out barrier Call option prices computed in such an experiment are presented in Exhibit 4.

In Exhibit 5, the relative difference between the Up-and-Out barrier Call option prices obtained by using our method and the FD solver are presented as a function of the option strike K and maturity T. Here, to provide a comparable accuracy, we run the FD solver with 101 nodes in space S and the time step $\Delta t = 0.01$.

It can be seen that the quality of the FD solution is not sufficient. Therefore, we reran it by using 201 nodes in space S and the time step $\Delta t = 0.001$. The relative difference between our semianalytic and the FD solutions in this case is presented in Exhibit 6.

We show that the agreement between prices obtained by using our method and the FD pricer is good, so the relative difference is about 1%. However, the cost for this improvement of the FD method is speed. In Exhibit 7, we compare the elapsed time of both methods. The no Ψ column has the following meaning. Since the volatility and the interest rate change with time relatively slow, contribution of the second integral in Equation (32) to the option price is negligible. Therefore, in this particular

EXHIBIT 4
Up-and-Out Barrier Call Option Price Computed by Our Method in the Test

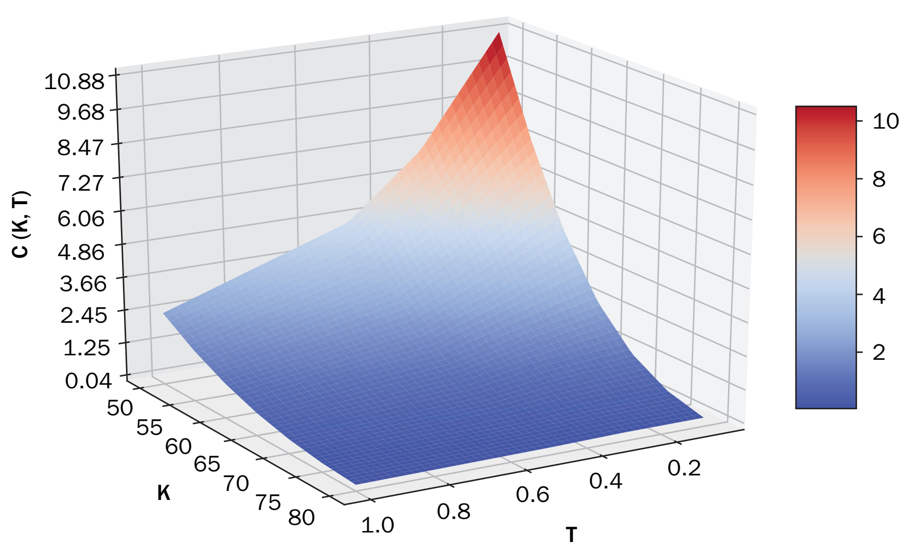


EXHIBIT 5

The Relative Difference in the Up-and-Out Barrier Call Option Prices Obtained by Using Our Method and the FD Solver with 101 Nodes in Space S and Time Step $\Delta t = 0.01$

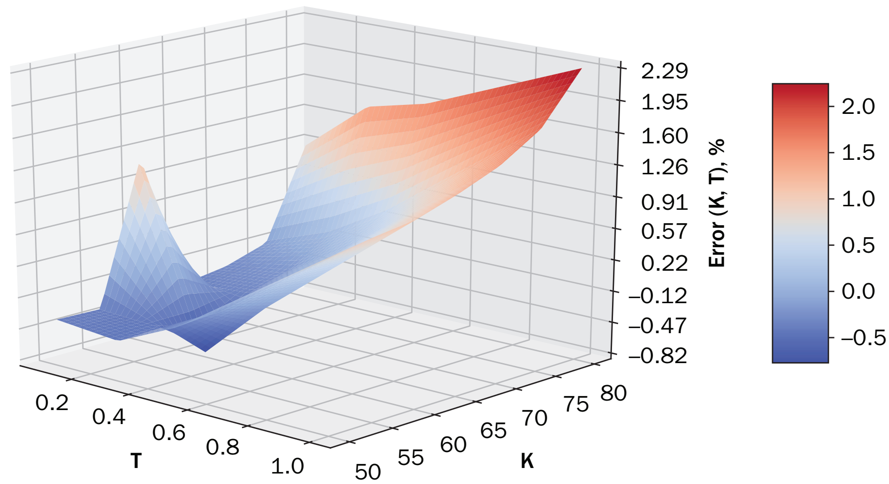


EXHIBIT 6

The Relative Difference in the Up-and-Out Barrier Call Option Prices Obtained by Using Our Method and the FD Solver with 201 Nodes in Space S and Time Step $\Delta t = 0.001$

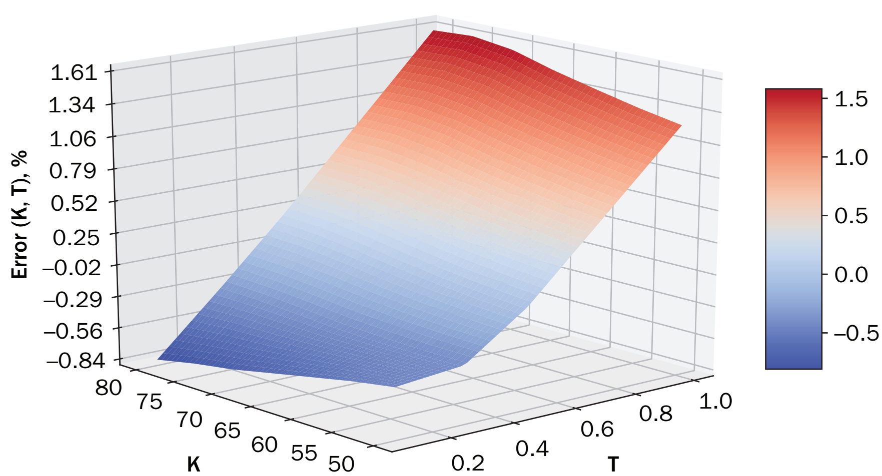


EXHIBIT 7

Elapsed Time in Seconds of Various Tests

Test	Semi Analytic	Semi Analytic, no Ψ	FD-101, $\Delta t = 0.01$	FD-201, $\Delta t = 0.01$	FD-201, $\Delta t = 0.001$
Elapsed time, seconds	0.23	0.16	0.36	0.65	3.6

case we can find the option price by computing only the first integral in Equation (32). Accordingly, we don't need to solve the Fredholm equation in Equation (23), which almost halves the elapsed time.

Finally, our tests show that linear algebra in python (numpy) at our machine is about three times slower than that in Matlab. Therefore, given the same accuracy, our method is about 30–40 times faster than the backward FD solver.

Of course, the forward FD solver is by an order of magnitude faster than the backward one if we need to simultaneously price multiple options of various strikes and maturities but written on the same underlying. However, for barrier options this approach requires a careful implementation, which often is not universal and has a lot of tricks.

Also, as can be seen from Exhibit 4, the relative difference—although small—increases when the strike and time to maturity increase. This happens because, in this case, the strike is closer to the barrier and the probability of reaching the barrier over the life of the option is higher. In this case, the FD solver—because the price has a kink at the barrier—provides a bigger difference.

DISCUSSION

In the first section of the article, our attention was drawn to the Up-and-Out barrier Call option C_{uao}. Obviously, using the barriers parity (Hull 1997), the price of the Down-and-Out barrier Call option C_{dao} can be found as $C_{dao} = C_{van} - C_{uao}$, where C_{van} is the price of the European Vanilla Call option. It is known that the latter is given by the corresponding formula for the process with constant coefficients, where those efficient constant coefficients $\bar{r}$, $\bar{q}$, $\bar{\sigma}$ are defined as

$$\bar{r} = \frac{1}{T}\int_0^T r(s)ds, \qquad \bar{q} = \frac{1}{T}\int_0^T q(s)ds, \qquad \bar{\sigma}^2 = \frac{1}{T}\int_0^T \sigma^2(s)ds.$$

Second, we underline that in addition to the model with time-dependent coefficients we also consider the barriers to be some arbitrary functions of time. Our method provides full coverage of this case, whereas constant barriers are just some particular case of the general solution.

The third and perhaps the most important point is about computational efficiency of our method. In addition to what was presented in the Numerical Example section, let's look at this problem from a theoretical pint of view. Suppose the barrier pricing problem is attacked by solving the forward PDE for a set of strikes K_i, $i = 1, \ldots, \bar{k}$ and a set of maturities T_j, $j = 1, \ldots, \bar{m}$ numerically by some FD method on a grid with N nodes in the space domain $S \in [0, H]$, and M nodes in the time domain $t \in [0, T_{\bar{m}}]$. Then the complexity of this method is known to be $O(MN + 4N)$. This should be compared with the complexity of our approach.

Let's assume that the Riccati equation in Equation (10) can be solved either analytically or, at least, approximately, as this is discussed in the Transformation to the Heat Equation section. Then the first computational step consists of solving the linear Fredholm equation in Equation (23) (or the corresponding Volterra equation). This can be done on a rarefied grid with $M_1 < M$ nodes and complexity $O(M_1^3)$. The intermediate values in t can be found (if necessary) by interpolation with the complexity $O(M_1^2)$. As the integral kernel doesn't depend on strikes K_i, this calculation can be done simultaneously for all strikes still preserving the complexity $O(M_1^3)$.

The final solution of the pricing problem is provided in the form of two integrals in Equation (32). Therefore, if we need the option price at a single value of S_0 (same as when solving the forward PDE), but for all strikes and maturities, the complexity is $O(2\bar{k}L(M_1 + N_1))$, where N_1 is the number of points in the x space, and $O(L)$ is the complexity of computing the Jacobi theta function $\theta_3(z, \omega)$. Normally, $M_1 \le N$, $L \ll N$, $N_1 \ll N$ for the typical values of N in the FD method (about 50–100 or even more). Thus, the total complexity of our method is fully determined by the solution of the Fredholm equation. Therefore, our method is slower than the corresponding FD method if $M_1 > (MN)^{1/3}$. For the American option, this situation is worse because instead of solving a linear Fredholm equation, we need to solve a nonlinear equation. This can be done iteratively, for example, using k iterations until the method converges to the given tolerance. Then the total complexity becomes $O(kM_1^3)$. However, our experiments show that using just $M_1 = 10$ points in p space could be sufficient, while further increase of M_1 doesn't change the results.

Also, the accuracy of the method in x can be increased if one uses high-order quadratures for computing the final integrals. For instance, one can use the Simpson instead of the trapezoid rule that doesn't affect the complexity of our method, while increasing the accuracy for the FD method is not easy (i.e., it significantly increases the complexity of the method; e.g., see Itkin 2017).

Another advantage of the approach advocated in this article, as was mentioned in Carr, Itkin, and Muravey (2020), is computation of option Greeks. Since the option prices are represented in closed form via integrals, the explicit dependence of prices on the model parameters is available and transparent. Therefore, explicit representations of the option Greeks can be obtained by a simple differentiation under the integrals. This means that the values of Greeks can be calculated simultaneously with the prices almost with no increase in time because differentiation under the integrals slightly changes the integrands, and these changes could be represented as changes in weights of the quadrature scheme used to numerically compute the integrals. Since the major computational time has to be spent for computation of densities, which contain special functions, they can be saved during the calculation of the prices and then reused for computation of Greeks.

It is worth mentioning that our method can also be extended to pricing American options where the underlying pays discrete dividends. Indeed, the constructed analytical solution covers the time interval starting from maturity T and up to the last ex-dividend date t_n using the final payoff as the terminal condition. Then, at t_n we have option prices $u(\tau, x)$ for all x. Due to continuity, shifting the underlying by the dividend amount Δx (so $x \mapsto \bar{x} = x + \Delta x$) and reinterpolating the prices to $u(\tau, \bar{x})$, we obtain the new terminal condition. Then the algorithms continue from t_n to t_{n-1} by replacing T with t_n and the terminal condition at t_n by $u(\tau, \bar{x})$. And so on. This same approach is used when solving this problem using the FD method (Itkin 2017; Tavella and Randall 2000).

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