SIAM J. FINANCIAL MATH.
Vol. 12, No. 2, pp. 672–689

Pricing Variance Swaps on Time-Changed Markov Processes*

Peter Carr†, Roger Lee‡, and Matthew Lorig§

Abstract. We prove that the variance swap rate (fair strike) equals the price of a co-terminal European-style contract when the underlying is an exponential Markov process, time-changed by an arbitrary continuous stochastic clock, which has arbitrary correlation with the driving Markov process, provided that the payoff function G of the European contract satisfies an ordinary integro-differential equation, which depends only on the dynamics of the Markov process, not on the clock. We present examples of Markov processes where the function G that prices the variance swap can be computed explicitly. In general, the solutions G are not contained in the logarithmic family previously obtained in the special case where the Markov process is a Lévy process.

Key words. variance swap, time change, Markov process, integro-differential

AMS subject classifications. 91G20, 60J25

DOI. 10.1137/20M1344597

1. Introduction.

Consider a forward price F that evolves in continuous time. Let time zero be the valuation time for a derivative security written on the path of F, with a fixed maturity date $T > 0$. Assume that $F_0 > 0$ is a known constant, and that the F process is strictly positive over a time interval $[0, T]$. As a result, the log price process $X := \log F$ is well defined, and derivative securities expiring at T can also be written on the path of X. In particular, we focus on a continuously monitored variance swap, which pays the difference between the terminal quadratic variation of the log price process $[\log F]_T$ and a constant determined at inception. For brevity, we will refer to a continuously monitored variance swap as a VS in what follows. As with any swap, the constant that is determined at inception is chosen so that there is no initial cost for entering into the VS. The objective of this paper is to give additional conditions on the dynamics of F under which this constant can be determined from an initial observation of the T-maturity implied volatility smile.

Earlier papers by Neuberger (1990) and Dupire (1993) show that continuity of F suffices for pricing a VS relative to the co-terminal smile. Carr, Lee, and Wu (2012) weaken the continuity hypothesis by showing that the log price X can be specified as a Lévy process running on an unspecified continuous clock. When the Lévy process is specified as Brownian motion with drift $(-1/2)$, the earlier results of Neuberger (1990) and Dupire (1993) arise as a special case. The more general formulation of Carr, Lee, and Wu (2012) allows for the variance and jump-intensity to depend on the level of X through a local time-change (see Remark 4.3). However, the local variance and Lévy kernel must have the same functional dependence on

*Received by the editors June 11, 2020; accepted for publication (in revised form) February 21, 2021; published electronically May 27, 2021.
https://doi.org/10.1137/20M1344597

†Department of Finance and Risk Engineering, NYU Tandon, New York, NY 11201 USA (petercarr@nyu.edu).
‡Department of Mathematics, University of Chicago, Chicago, IL 60637 USA (rogerlee@math.uchicago.edu).
§Department of Applied Mathematics, University of Washington, Seattle, WA 98195-3925 USA (mlorig@uw.edu).

X (up to a scaling constant). Additionally, while the arrival rate of each jump size in X is allowed to depend on the level of X, the ratio of the arrival rates at any two jump sizes is constant in that previous paper.

This paper weakens the stationary independent increments property of the Lévy process used by Carr, Lee, and Wu (2012). We allow that X could be specified as a time-homogeneous Markov process running on an unspecified continuous clock. As a result (i) the variance and jump-intensity may have distinct X-dependence and (ii) the ratio of the arrival rates at any two jump sizes of X can depend on the current level of X.

In effect, we allow the background process to have nearly the full generality of *general* Markov processes, whose jump times are not predictable, as discussed in Remark 2.1. We allow that general background Markov process to undergo a time-change by an unspecified continuous stochastic clock which may have arbitrary correlation or dependence on the background process. In this setting, we prove that European-style payoff functions G price the VS, in the sense that the VS rate (fair strike) equals the price of a contract paying $G(\log F_T) - G(\log F_0)$, provided that G satisfies an ordinary integro-differential equation that depends only on the dynamics of the Markov driver, not on the clock.

Our results are related to the semiparametric approach taken by Lorig, Lozano-Carbassé, and Mendoza-Arriaga (2016), who consider the pricing of a VS when the underlying forward price F is modeled as Feller diffusion time-changed by an unspecified Lévy subordinator. For fully parametric approaches to VS pricing in models with jumps and stochastic volatility, we refer the reader to Itkin and Carr (2010); Wendong and Kuen (2014); Filipović, Gourier, and Mancini (2016); and Cui, Kirkby, and Nguyen (2017). For model-independent bounds on (discrete and continuous) VS prices, see Hobson and Klimmek (2012), Nabil (2014), and Henry-Labordère and Touzi (2016, Example 5.7).

The rest of this paper proceeds as follows. Section 2 specifies dynamics for the forward price process and verifies that these dynamics can arise from time-changing the solution of a stochastic differential equation. Section 3 states and proves our main result (Theorem 3.5), which establishes that the VS has the same value as a European-style claim whose payoff function solves an ordinary integro-differential equation (OIDE). Section 4 provides examples of price dynamics for which we can solve the OIDE explicitly. Section 5 concludes.

2. Time-changed Markov dynamics.

2.1. Assumptions. With respect to a "calendar-time" filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, assume that X is a semimartingale with predictable characteristics (B, A, ν), relative to a truncation function h (to be definite, let $h(z) := z\mathbf{1}_{\{|z|\leq 1\}}$), which satisfy

$$B_t = \int_0^t b_h(X_{s-})\mathrm{d}\tau_s, \qquad A_t = \int_0^t a^2(X_{s-})\mathrm{d}\tau_s, \qquad \nu(\mathrm{d}t, \mathrm{d}z) = \mathrm{d}\tau_t \times \mu(X_{t-}, \mathrm{d}z), \tag{2.1}$$

where τ is a real-valued, continuous, and increasing process that is null at zero, a is a Borel function, and for each fixed $x \in \mathbb{R}$ the $\mu(x, \cdot)$ is a Lévy measure, and

$$\sup_{x\in\mathbb{R}} |a(x)| < \infty, \qquad \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} z^2\mu(x, \mathrm{d}z) < \infty, \qquad \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} (\mathrm{e}^z - 1 - z)\mu(x, \mathrm{d}z) < \infty, \tag{2.2}$$

with

$$b_h(x) := -\frac{1}{2}a^2(x) - \int_{\mathbb{R}} (\mathrm{e}^z - 1 - h(z))\,\mu(x, \mathrm{d}z). \tag{2.3}$$

The intuition of the *Lévy kernel* or *transition kernel* μ is that it assigns to each point x in the state space a "local" Lévy measure $\mu(x, \cdot)$. Jumps of size in any interval J arrive with intensity $\mu(x, J)$ when X is at x.

Define the underlying forward price process $F = \{F_t\}_{t\in[0,T]}$ by

$$F_t = \exp(X_t).$$

Regarding $\mathbb{P}$ as a risk-neutral measure, we have chosen b_h in (2.3) to ensure F is a local martingale. If τ_T is integrable, then Lemma 3.4 will imply that F is a true martingale.

2.2. Time-change of an SDE solution. This section verifies that the assumptions of section 2.1 hold in the case when X comes from time-changing the solution of a stochastic differential equation (SDE) driven by a Brownian motion and a Poisson random measure. With respect to a filtration $\{\mathcal{G}_u\}_{u\geq 0}$ (the "business time" filtration), consider a Brownian motion W, and a Poisson random measure N with intensity measure $\mu_N(\mathrm{d}z)\mathrm{d}u$ for some Lévy measure μ_N. Assume that Y is a semimartingale that satisfies

$$\mathrm{d}Y_u = b(Y_u)\,\mathrm{d}t + a(Y_u)\,\mathrm{d}W_u + \int_{z\in\mathbb{R}} c(Y_{u-}, z)\,(N(\mathrm{d}u, \mathrm{d}z) - \mu_N(\mathrm{d}z)\mathrm{d}u),$$

where a is a bounded Borel function, b is given by

$$b(x) = -\frac{1}{2}a^2(x) - \int_{\mathbb{R}} (\mathrm{e}^z - 1 - z)\,\mu(x, \mathrm{d}z),$$

and c is a Borel function such that μ, defined for each Borel set J by

$$\mu(x, J) := \mu_N(\{z : c(x, z) \in J\backslash\{0\}\}),$$

satisfies

$$\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} z^2\mu(x, \mathrm{d}z) + \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} (\mathrm{e}^z - 1 - z)\mu(x, \mathrm{d}z) < \infty.$$

Then by Jacod and Shiryaev (1987, Prop. III.2.29), the semimartingale characteristics of Y are $(\widetilde{B}, \widetilde{A}, \widetilde{\nu})$, where

$$\widetilde{B}_u = \int_0^u b_h(Y_{v-})\mathrm{d}v, \qquad \widetilde{A}_u = \int_0^u a^2(Y_{v-})\mathrm{d}v, \qquad \widetilde{\nu}(\mathrm{d}u, \mathrm{d}z) = \mathrm{d}u \times \mu(Y_{u-}, \mathrm{d}z), \tag{2.4}$$

with b_h defined in (2.3).

Now let $\{\tau_t\}_{t\geq 0}$ be a continuous increasing family of finite $\mathcal{G}$-stopping times (which are *not* assumed to be independent of Y). Let the "calendar-time" filtration be defined by $\mathcal{F}_t := \mathcal{G}_{\tau_t}$, and let

$$X_t := Y_{\tau_t}.$$

By Kallsen and Shiryaev (2002b, Lemma 5), the $\mathcal{F}$-characteristics of X are (B, A, ν), where $A_t = \widetilde{A}_{\tau_t}$, $B_t = \widetilde{B}_{\tau_t}$, and ν is determined by

$$\int_{[0,t]\times\mathbb{R}} \mathbf{1}_J(z)\nu(\mathrm{d}s, \mathrm{d}z) = \int_{[0,\tau_t]\times\mathbb{R}} \mathbf{1}_J(z)\widetilde{\nu}(\mathrm{d}u, \mathrm{d}z) \tag{2.5}$$

for general Borel sets J and $t \geq 0$. By the first two equalities in (2.4) we have

$$\widetilde{A}_{\tau_t} = \int_0^{\tau_t} a^2(Y_{v-})\mathrm{d}v = \int_0^t a^2(X_{s-})\mathrm{d}\tau_s, \qquad \widetilde{B}_{\tau_t} = \int_0^{\tau_t} b_h(Y_{v-})\mathrm{d}v = \int_0^t b_h(X_{s-})\mathrm{d}\tau_s,$$

and by substituting the last equality in (2.4) into (2.5) and changing variables u to τ_s, we have

$$\int_{[0,t]\times\mathbb{R}} \mathbf{1}_J(z)\nu(\mathrm{d}s, \mathrm{d}z) = \int_{[0,t]}\int_{\mathbb{R}} \mathbf{1}_J(z)\mu(X_{s-}, \mathrm{d}z)\mathrm{d}\tau_s.$$

Therefore, (B, A, ν) satisfies (2.1). This verifies the hypotheses of section 2.1, as claimed.

Remark 2.1. Time-changes of SDE solutions are nearly as general as time-changes of *general* Markov processes whose jump times are not predictable.

To be precise, Çinlar and Jacod (1981) show that every strong Markov quasi-left-continuous semimartingale (which includes every Feller semimartingale) is a continuous time change of an SDE solution driven by Brownian motion and a Poisson random measure (on an enlarged probability space if needed). Thus, if X is a continuous time-change τ' of a *general* Feller semimartingale Y', then by Çinlar–Jacod, Y' is a continuous time change τ'' of an SDE solution Y, and therefore, X is a continuous time change $\tau' \circ \tau''$ of an SDE solution Y.

2.3. Notation. Let $C^n(\mathbb{R})$ denote the class of n-times continuously differentiable functions, and define the integro-differential operator $\mathcal{A}$ by

$$\begin{aligned}\mathcal{A}g(x) &:= b_h(x)g'(x) + \frac{a^2(x)}{2}g''(x) + \int_{\mathbb{R}} (g(x+z) - g(x) - g'(x)h(z))\ \mu(x, \mathrm{d}z)\\ &= \frac{a^2(x)}{2}(g''(x) - g'(x)) + \int_{\mathbb{R}} (g(x+z) - g(x) + (1-\mathrm{e}^z)g'(x))\ \mu(x, \mathrm{d}z)\end{aligned} \tag{2.6}$$

for all $g \in C^2(\mathbb{R})$ such that $g(x+z) - g(x) + (1-\mathrm{e}^z)g'(x) \in L^1(\mu(x, \mathrm{d}z))$ for all x.

In more concise notation,

$$\mathcal{A} = \frac{1}{2}a^2(x)\left(\partial^2 - \partial\right) + \int_{\mathbb{R}} \left(\mathrm{e}^{z\partial} - 1 + (1-\mathrm{e}^z)\partial\right)\ \mu(x, \mathrm{d}z), \tag{2.7}$$

where $\mathrm{e}^{z\partial}$ is the *shift operator* defined by $\mathrm{e}^{z\partial}g(x) := g(x+z)$. This use of ∂ to express translations in the jump part of the generator $\mathcal{A}$ follows Itkin and Carr (2012).

Let $C^{1+}(\mathbb{R})$ denote the union of $C^2(\mathbb{R})$ and the following set: all $C^1(\mathbb{R})$ functions g whose derivative is everywhere absolutely continuous, and whose second derivative (which therefore exists a.e.) is equal (a.e.) to a bounded function, which we will still denote by g'' or $\partial^2 g$.

Thus the definition of $\mathcal{A}$ extends, by relaxing the $g \in C^2(\mathbb{R})$ condition to $g \in C^{1+}(\mathbb{R})$, which still defines $\mathcal{A}g$ uniquely, up to sets of measure zero, via (2.6).

3. VS pricing. In what follows, each C will denote a constant (nonrandom and non-time-varying). Different instances of C, even within the same expression, may have different values.

Lemma 3.1. *Suppose that* $g \in C^{1+}(\mathbb{R})$ *and there exists* $p \in \mathbb{R}$ *such that*

$$\sup_{x\in\mathbb{R}} |g'(x)\mathrm{e}^{-px}| < \infty \qquad \text{and} \qquad \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} (\mathrm{e}^{pz} - 1 - pz)\ \mu(x, \mathrm{d}z) < \infty.$$

Then $g(X)$ *is a special semimartingale.*

Proof. By the form of Itô's rule in, for instance, Protter (2004, Theorem IV.70), $g(X)$ is a semimartingale.

By Kallsen and Shiryaev (2002a, Lemma 2.8), it suffices to check that the predictable process

$$\int_0^t \int_{\{z:|g(X_{s-}+z)-g(X_{s-})|>1\}} |g(X_{s-} + z) - g(X_{s-})|\mu(X_{s-}, \mathrm{d}z)\mathrm{d}\tau_s \tag{3.1}$$

is finite (hence of finite variation, as it is increasing in t).

In the case $p = 0$, we have $|g(x + z) - g(x)| \le C|z|$. In the case $p \ne 0$, we have

$$|g(x + z) - g(x)| \le \int_{x\wedge(x+z)}^{x\vee(x+z)} C\mathrm{e}^{p\zeta}\mathrm{d}\zeta = C\mathrm{e}^{px}|\mathrm{e}^{pz} - 1|.$$

In this case, for each $m > 0$, let $k(m)$ be such that

$$|\mathrm{e}^{pz} - 1|\mathbf{1}_{|\mathrm{e}^{pz}-1|>1/m} < (\mathrm{e}^{pz} - 1 - pz) + k(m)z^2$$

for all z, and let $M := \sup_{s\in[0,T]} \mathrm{e}^{pX_s} < \infty$ because X is càdlàg. Then

$$\int_{\{z:|g(X_{s-}+z)-g(X_{s-})|>1\}} |g(X_{s-} + z) - g(X_{s-})|\mu(X_{s-}, \mathrm{d}z)$$

is bounded in case $p = 0$ by $\sup_{x\in\mathbb{R}} \int_{\{z:|z|>1/C\}} C|z|\mu(x, \mathrm{d}z) < \infty$, and in case $p \ne 0$ by C times

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} \int_{\{z:|\mathrm{e}^{pz}-1|>1/(CM)\}} M|\mathrm{e}^{pz} - 1|\mu(x, \mathrm{d}z) \\
&\quad \le M \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} (\mathrm{e}^{pz} - 1 - pz)\mu(x, \mathrm{d}z) + Mk(CM) \sup_{x\in\mathbb{R}} \int_{\mathbb{R}} z^2\mu(x, \mathrm{d}z) < \infty.
\end{aligned}$$

These upper bounds do not depend on $s \in [0, t]$, which verifies that (3.1) is finite. ■

Lemma 3.2. *If* $\mathbb{E}\tau_T < \infty$, *then* $\mathbb{E}\sup_{t\in[0,T]} |X_t| < \infty$.

Proof. Let $B'_t := B_t + \int_{[0,t]\times\mathbb{R}}(z - h(z))\nu(\mathrm{d}u, \mathrm{d}z)$. We have $\mathbb{E}\sup_{t\in[0,T]} |B'_t| < \infty$ due to (2.2) and $\mathbb{E}\tau_T < \infty$.

Defining M_t by

$$X_t = X_0 + M_t + B'_t,$$

we have, by Jacod and Shiryaev (1987, Proposition II.2.29), that M is a local martingale satisfying

$$\mathbb{E}[M,M]_T = \mathbb{E}\int_0^T a^2(X_s)\mathrm{d}\tau_s + \mathbb{E}\int_0^T\int_{\mathbb{R}} z^2\mu(X_{s-},\mathrm{d}z)\mathrm{d}\tau_s < \infty$$

because $\mathbb{E}\tau_T < \infty$. By Burkholder–Davis–Gundy, $\mathbb{E}\sup_{t\in[0,T]}|M_t| < \infty$, which implies the result. ■

Lemma 3.3. *Suppose τ_T is bounded and $p \in \mathbb{R}$ satisfies*

$$\sup_{x\in\mathbb{R}}\int_{\mathbb{R}}(\mathrm{e}^{pz}-1-pz)\mu(x,\mathrm{d}z) < \infty. \tag{3.2}$$

Let

$$Z_t := \exp(pX_t - K_t),$$
$$K_t := \int_0^t \frac{1}{2}(p^2-p)a^2(X_s)\mathrm{d}\tau_s + \int_0^t\int_{\mathbb{R}}[(\mathrm{e}^{pz}-1-pz) - p(\mathrm{e}^z-1-z)]\mu(X_{s-},\mathrm{d}z)\mathrm{d}\tau_s.$$

Then Z is a martingale, and

$$\mathbb{E}\sup_{t\in[0,T]}\exp(pX_t) < \infty. \tag{3.3}$$

Proof. Let N be the integer-valued random measure associated with the jumps of X. Let $\widetilde{N} := N - \nu$.

By Kallsen and Shiryaev (2002a, Theorem 2.19), the process Z is the stochastic exponential of the local martingale

$$pX_t^c + \int_{[0,t]\times\mathbb{R}}(\mathrm{e}^{pz}-1)\widetilde{N}(\mathrm{d}s,\mathrm{d}z),$$

where X^c is the continuous martingale part of X. By the boundedness of τ_T and assumptions (2.2) and (3.2), it follows that

$$p^2\int_0^T a^2(X_s)\mathrm{d}\tau_s + \int_0^T\int_{\mathbb{R}}(\mathrm{e}^{pz}-1)^2\wedge(\mathrm{e}^{pz}-1)\ \mu(X_{s-},\mathrm{d}z)\mathrm{d}\tau_s$$

is bounded. So by Lepingle and Mémin (1978), the process Z is a martingale and $\mathbb{E}\sup_{t\in[0,T]}Z_t < \infty$, which implies (3.3) because $\sup_{t\in[0,T]}K_t$ is bounded. ■

Let us define two conditions that may be satisfied by (τ_T, g), where $g \in C^{1+}(\mathbb{R})$. The first is

$$\mathbb{E}\tau_T < \infty \quad \text{and} \quad \sup_{x\in\mathbb{R}}|g'(x)| + \operatorname*{ess\,sup}_{x\in\mathbb{R}}|g''(x)| < \infty, \tag{3.4}$$

and the second is τ_T, which is bounded, and

$$\exists p\in\mathbb{R} \text{ with } \sup_{x\in\mathbb{R}}\int_{\mathbb{R}}(\mathrm{e}^{pz}-1-pz)\mu(x,\mathrm{d}z) + \operatorname*{ess\,sup}_{x\in\mathbb{R}}\mathrm{e}^{-px}(|g(x)|+|g'(x)|+|g''(x)|) < \infty. \tag{3.5}$$

Lemma 3.4. *Assume that g is a sum of finitely many $C^{1+}(\mathbb{R})$ functions, each of which satisfies (3.4) or (3.5). Let*

$$\Gamma_t := g(X_t) - g(X_0) - \int_0^t \mathcal{A}g(X_{s-})\mathrm{d}\tau_s, \qquad t \in [0,T].$$

Then Γ is a martingale.

Proof. We prove for the case when the g satisfies (3.4) or (3.5). The case when g is the sum of such functions follows immediately by linearity.

Either one of the conditions (3.4) or (3.5) implies that $\mathcal{A}g$ is well defined.

To show that Γ is a local martingale, note that Jacod and Shiryaev (1987, Theorem II.2.42c) extends as follows. They assume g bounded, only to show that $g(X)$ is a special semimartingale, but the conditions in Lemma 3.1 suffice for that conclusion. Moreover, they assume $g \in C^2$, only to use Itô's lemma, but C^{1+} suffices here by Protter (2004, Theorem IV.70) and its first corollary.

To show that Γ is a true martingale, it suffices, by Protter (2004, Theorem I.51), to show that $\mathbb{E}\sup_{t\in[0,T]}|\Gamma_t| < \infty$. In case (3.4), let $p := 0$. In both cases, by (2.2), we have

$$|g'(x)| \int_{\mathbb{R}} (\mathrm{e}^z - 1 - z)\mu(x,\mathrm{d}z) < C\mathrm{e}^{px}, \tag{3.6}$$

and by Taylor's theorem and $|g''(x+z)| \le C\mathrm{e}^{px+|p|}$ for $|z|<1$, we have

$$\int_{|z|<1} |g(x+z) - g(x) - g'(x)z|\mu(x,\mathrm{d}z) \le C\mathrm{e}^{px+|p|}\int_{|z|<1} z^2\mu(x,\mathrm{d}z) \le C\mathrm{e}^{px}, \tag{3.7}$$

and by (2.2),

$$\int_{|z|>1} |g(x+z) - g(x) - g'(x)z|\ \mu(x,\mathrm{d}z) \le C\mathrm{e}^{px}\int_{|z|>1} (\mathrm{e}^{pz} + 1 + |z|)\ \mu(x,\mathrm{d}z) \le C\mathrm{e}^{px}, \tag{3.8}$$

where each C does not depend on x. Combining (3.6), (3.7), (3.8), and the bounds on g' and g'', we have

$$\sup_{t\in[0,T]}\left|\int_0^t \mathcal{A}g(X_{s-})\mathrm{d}\tau_s\right| \le \int_0^T |\mathcal{A}g(X_{s-})|\mathrm{d}\tau_s \le C\tau_T \sup_{t\in[0,T]} \mathrm{e}^{pX_t},$$

which is integrable in case (3.4) because $\mathbb{E}\tau_T < \infty$, and in case (3.5) by Lemma 3.3. The remaining component of Γ has magnitude

$$|g(X_t) - g(X_0)| \le \begin{cases} C(1+|X_t|) & \text{in case (3.4)}, \\ C(1+\mathrm{e}^{pX_t}) & \text{in case (3.5)}, \end{cases}$$

which has integrable supremum by Lemmas 3.2 and 3.3. ∎

In conclusion, we relate $\mathbb{E}\,[\log F]_T$ to the value of a European-style contract.

Theorem 3.5. *Assume that the forward price F, the log-price X, and the clock τ satisfy the assumptions of section 2.1. Assume that G is a sum of finitely many $C^{1+}(\mathbb{R})$ functions, each of which satisfies (3.4) or (3.5), and that $\mathcal{A}\,G$ satisfies (for a.e. x)*

$$\mathcal{A}\,G(x) = a^2(x) + \int_{\mathbb{R}} z^2 \mu(x, \mathrm{d}z). \tag{3.9}$$

Then G prices the VS, meaning that

$$\mathbb{E}[\log F\text{\_}T = \mathbb{E}\,G(\log F_T) - G(\log F_0). \tag{3.10}$$

Thus, if $\mathbb{P}$ is a martingale measure for VS and G contracts, then the fair strike of the VS (equivalently, the forward price of the floating leg of the VS) is (3.10).

Remark 3.6. The sum of finitely many functions is more general than a single function; for instance, G may be the sum of two functions, one satisfying (3.5) for some $p > 0$, and the other for some $p < 0$.

Remark 3.7. Functions G that satisfy the conditions of Theorem 3.5, and therefore price the VS, are not unique. Indeed, if G does, then so does $G(\cdot) + C_0 + C_1 \exp(\cdot)$, where C_0, C_1 are any constants. Adding the latter two terms does not affect the valuation $\mathbb{E}G(\log F_T) - G(\log F_0)$, because $\mathbb{E}F_T = F_0$.

Proof of Theorem 3.5. We have

$$\begin{aligned}
\mathbb{E}\,[X]_T &= \mathbb{E}\biggl(\int_0^T a^2(X_t)\mathrm{d}\tau_t + \int_0^T \int_{\mathbb{R}} z^2 N(\mathrm{d}t, \mathrm{d}z) \biggr) \\
&= \mathbb{E} \int_0^T \biggl(a^2(X_{t-}) + \int_{\mathbb{R}} z^2 \mu(X_{t-}, \mathrm{d}z) \biggr) \mathrm{d}\tau_t \\
&= \mathbb{E} \int_0^T \mathcal{A}G(X_{t-})\mathrm{d}\tau_t \\
&= \mathbb{E}\,G(X_T) - G(X_0)
\end{aligned}$$

by Jacod and Shiryaev (1987, Theorems I.4.52 and II.1.8), eq. (3.9) and Lemma 3.4. ■

Theorem 3.5 allows us to value a VS relative to the T-maturity implied volatility smile as follows:

$$\underbrace{\mathbb{E}[\log F]_T}_{\text{A}} = \underbrace{\mathbb{E}\,G(\log F_T)}_{\text{B}} - \underbrace{G(\log F_0)}_{\text{C}}. \tag{3.11}$$

A = the amount agreed upon at time 0 to pay at time T when taking the long side of a VS.
B = the value of a European contract with payoff $G(\log F_T)$.
C = the value of $G(\log F_0)$ zero-coupon bonds.

As shown in Carr and Madan (1998), if h is a difference of convex functions, then for any $\kappa \in \mathbb{R}^+$ we have

$$h(F_T) = h(\kappa) + h'(\kappa)\Bigl((F_T - \kappa)^+ - (\kappa - F_T)^+\Bigr) + \int_0^\kappa h''(K)(K - F_T)^+ \mathrm{d}K$$

$$+\int_{\kappa}^{\infty} h''(K)(F_T - K)^+ \mathrm{d}K.$$

Here, h' is the left-derivative of h, and h'' is the second derivative, which exists as a generalized function. Taking expectations,

(3.12)
$$\mathbb{E}\, h(F_T) = h(\kappa) + h'(\kappa)\Big(C(T,\kappa) - P(T,\kappa)\Big) + \int_0^{\kappa} h''(K)P(T,K)\mathrm{d}K + \int_{\kappa}^{\infty} h''(K)C(T,K)\mathrm{d}K,$$

where $P(T,K)$ and $C(T,K)$ are, respectively, the prices of put and call options on F with strike K and expiry T. Knowledge of F_0 and the T-expiry smile implies knowledge of the initial prices of T-expiry European options at all strikes $K > 0$. Thus, the quantity B in (3.11) is uniquely determined from the T-expiry volatility smile by applying (3.12) to $h = G \circ \log$, assuming one can determine the function G. Therefore, to price a VS relative to co-terminal calls and puts, what remains is to find a solution G of the OIDE (3.9).

4. Examples.

In this section we provide examples, in the setting of section 2.2, of local variance and Lévy kernel pairs (a^2, μ) such that solutions G of OIDE (3.9) can be obtained explicitly. In one of the examples, moreover, we investigate the ratio between the values of the VS and the log contract.

4.1. Constant relative jump intensity.

Theorem 4.1. *Assume the local variance $a^2(x)$ and Lévy kernel $\mu(x, \mathrm{d}z)$ are of the forms*

$$a^2(x) = \gamma^2(x)\,\sigma^2, \qquad \mu(x,\mathrm{d}z) = \gamma^2(x)\,\nu(\mathrm{d}z),$$

where $\sigma \geq 0$ is a constant, ν is a Lévy measure, and γ is a positive bounded Borel function. Assume $\mathbb{E}\tau_T < \infty$. Then

(4.1)
$$G(x) := -Q\,x$$

prices the VS, where

(4.2)
$$Q := \frac{\sigma^2 + \mu_2}{\sigma^2/2 + \varphi_0}, \qquad \varphi_0 := \int_{\mathbb{R}} (\mathrm{e}^z - 1 - z)\nu(\mathrm{d}z), \qquad \mu_2 := \int_{\mathbb{R}} z^2 \nu(\mathrm{d}z).$$

Proof. One can verify directly that G in (4.1) satisfies (3.4) and (3.9). ■

Remark 4.2. In particular, the constant Q in two extreme cases is as follows:

(4.3) No jumps $(\nu \equiv 0)$: $Q = 2,$

(4.4) Pure jumps $(\sigma = 0)$: $Q = \mu_2/\varphi_0.$

Remark 4.3. Dynamics of this form arise by time-changing a Lévy process Y_u using the clock

$$\tau_t := \inf\left\{u \geq 0 : \int_0^u \frac{1}{\gamma^2(Y_v)}\mathrm{d}v \geq t\right\}.$$

See, for instance, Küchler and Sørensen (1997, Proposition 11.6.1). Thus the payoff function (4.1) in this case should, and indeed does, match the payoff function obtained by Carr, Lee, and Wu (2012) for time-changed Lévy processes.

4.2. Fractional linear relative jump intensity. Let ν be a Lévy measure such that $2\varphi_0 < \mu_2 < \infty$, with notation as in (4.2). (The case of $\mu_2 < 2\varphi_0$ admits a similar solution, but with $\beta < 0$.)

Let $\alpha, \beta \in \mathbb{R}$ such that

$$0 < \beta < 1 - \frac{2\varphi_0}{\mu_2}.$$

Let

$$\gamma_3 := -\frac{\alpha}{2\beta} - \frac{1}{\beta}, \qquad \gamma_0 := -\frac{\alpha}{2\beta} + \frac{(\beta-1)\mu_2}{2\beta\varphi_0} < \gamma_3.$$

Let γ_1 and γ_2 satisfy $\gamma_0 < \gamma_1 < \gamma_2 < \gamma_3$. Define the C^1 function

$$G(x) := \begin{cases} \alpha\gamma_1 + \beta\gamma_1^2 + (x-\gamma_1)(\alpha + 2\beta\gamma_1), & x < \gamma_1, \\ \alpha x + \beta x^2, & \gamma_1 \le x \le \gamma_2, \\ \alpha\gamma_2 + \beta\gamma_2^2 + (x-\gamma_2)(\alpha + 2\beta\gamma_2), & x > \gamma_2. \end{cases} \tag{4.5}$$

We can and do take $\partial^2 G(x) = 2\beta \mathbf{1}_{x\in[\gamma_1,\gamma_2]}$ in the sense of Theorem 3.5.

Let a be a positive, bounded Borel function, and let

$$c(x) := \frac{a^2(x)}{2} \times \frac{\partial^2 G(x) - \partial G(x) - 2}{\int_{\mathbb{R}}(G(x) - G(x+z) + (\mathrm{e}^z - 1)\partial G(x) + z^2)\nu(\mathrm{d}z)}. \tag{4.6}$$

Lemma 4.4. *The function c is positive and bounded.*

Proof. The denominator in (4.6) has a positive lower bound because

$$\begin{aligned} \int_{\mathbb{R}} (G(x) + z\partial G(x) - G(x+z))\nu(\mathrm{d}z) + \varphi_0 \partial G(x) + \mu_2 &\ge \varphi_0 \partial G(x) + (1-\beta)\mu_2 \\ &\ge \varphi_0(\alpha + 2\beta\gamma_0) + (1-\beta)\mu_2 > 0. \end{aligned}$$

To show that the numerator $\partial^2 G - \partial G - 2$ in (4.6) is positive and bounded, we verify in three intervals. For $x \in (\gamma_1, \gamma_2)$, the numerator is $2\beta - \alpha - 2 - 2\beta x > 2\beta - \alpha - 2 - 2\beta\gamma_3 = 2\beta > 0$, and is moreover bounded above. In the other two intervals, the result follows from

$$-\alpha - 2\beta\gamma_1 - 2 > -\alpha - 2\beta\gamma_2 - 2 > -\alpha - 2\beta\gamma_3 - 2 = 0,$$

where the first two expressions are the numerator for $x \le \gamma_1$ and $x \ge \gamma_2$, respectively. ■

Theorem 4.5. *Assume the local Lévy kernel μ is given by*

$$\mu(x, \mathrm{d}z) = c(x)\nu(\mathrm{d}z),$$

where c, ν, G, and the local variance a^2 are related by (4.5) *and* (4.6)*. Assume $\mathbb{E}\tau_T < \infty$. Then G prices the VS.*

Proof. We have that G satisfies (3.4) and, by (4.6), the OIDE (3.9). ■

Remark 4.6. We describe these dynamics as "fractional linear relative jump intensity" because, for x such that $\{x+z : z \in \operatorname{supp}(\nu)\} \cup \{x\} \subset [\gamma_1, \gamma_2]$, the relative jump intensity

$$\frac{c(x)}{a^2(x)} = \frac{\beta - \alpha/2 - 1 - \beta x}{(1-\beta)\mu_2 + (\alpha + 2\beta x)\varphi_0}$$

is a ratio of polynomials linear in the underlying log-price.

4.3. Lévy mixture with state-dependent weights. Assume the local variance $a^2(x)$ and Lévy kernel $\mu(x, \mathrm{d}z)$ are of the forms

$$a^2(x) = \alpha\sigma_0^2(x) + \delta\beta\sigma_1^2(x), \quad \mu(x,\mathrm{d}z) = \frac{\sigma_0^2(x)}{2}\nu_0(\mathrm{d}z) + \delta\frac{\sigma_1^2(x)}{2}\nu_1(\mathrm{d}z), \quad \frac{\sigma_1^2(x)}{\sigma_0^2(x)} = \mathrm{e}^{cx} =: \mathrm{e}_c(x), \tag{4.7}$$

where $\alpha, \beta, \delta \geq 0$ and ν_0, ν_1 are Lévy measures with

$$\int_{\mathbb{R}} \left|\mathrm{e}^{\lambda z} - 1 + (1-\mathrm{e}^z)\lambda\right| \nu_i(\mathrm{d}z) < \infty \qquad \forall \lambda \in \mathbb{C}, \qquad i \in \{0,1\}. \tag{4.8}$$

Let us first derive a candidate solution G to (3.9) from an ansatz and then verify the validity of the solution.

Inserting the expressions for a^2 and μ from (4.7) into (3.9) and dividing by $\frac{1}{2}\sigma_0^2(x)$, we have

$$(\mathcal{A}_0 + \delta\mathrm{e}_c\mathcal{A}_1)G = I_0 + \delta\mathrm{e}_c I_1, \tag{4.9}$$

where I_0 and I_1 are constants defined by

$$I_0 = 2\alpha + \int_{\mathbb{R}} z^2\, \nu_0(\mathrm{d}z), \qquad I_1 = 2\beta + \int_{\mathbb{R}} z^2\, \nu_1(\mathrm{d}z),$$

and using the notation of (2.7), the operators $\mathcal{A}_0$ and $\mathcal{A}_1$ are given by

$$\mathcal{A}_0 = \alpha\left(\partial^2 - \partial\right) + \int_{\mathbb{R}} \left(\mathrm{e}^{z\partial} - 1 + (1-\mathrm{e}^z)\partial\right)\nu_0(\mathrm{d}z),$$
$$\mathcal{A}_1 = \beta\left(\partial^2 - \partial\right) + \int_{\mathbb{R}} \left(\mathrm{e}^{z\partial} - 1 + (1-\mathrm{e}^z)\partial\right)\nu_1(\mathrm{d}z).$$

Assume a solution G of (4.9) has a power series expansion in δ:

$$G = \sum_{n=0}^{\infty} \delta^n G_n, \tag{4.10}$$

where the functions $\{G_n\}_{n\geq 0}$ are, at this point, unknown. Inserting the expansion (4.10) into OIDE (4.9) and collecting terms of like order in δ results in the following sequence of nested OIDEs:

$$\mathcal{O}(1): \qquad \mathcal{A}_0 G_0 = I_0,$$

$$(4.11) \qquad \mathcal{O}(\delta): \qquad \mathcal{A}_0 G_1 + \mathrm{e}_c \mathcal{A}_1 G_0 = \mathrm{e}_c I_1,$$
$$\mathcal{O}(\delta^n): \qquad \mathcal{A}_0 G_n + \mathrm{e}_c \mathcal{A}_1 G_{n-1} = 0, \qquad n \geq 2.$$

Noting that

$$\mathcal{A}_0 \mathrm{e}_\lambda = \phi_\lambda \mathrm{e}_\lambda, \qquad \phi_\lambda = \alpha\left(\lambda^2 - \lambda\right) + \int_{\mathbb{R}} \Big(\mathrm{e}^{\lambda z} - 1 + (1-\mathrm{e}^z)\lambda\Big)\nu_0(\mathrm{d}z) \qquad \forall \lambda \in \mathbb{C},$$
$$\mathcal{A}_1 \mathrm{e}_\lambda = \chi_\lambda \mathrm{e}_\lambda, \qquad \chi_\lambda = \beta\left(\lambda^2 - \lambda\right) + \int_{\mathbb{R}} \Big(\mathrm{e}^{\lambda z} - 1 + (1-\mathrm{e}^z)\lambda\Big)\nu_1(\mathrm{d}z) \qquad \forall \lambda \in \mathbb{C},$$

one can easily verify, by direct substitution into (4.11), a solution G_0 given by

$$(4.12) \qquad G_0(x) := -Q_0 x, \qquad Q_0 := \frac{2\alpha + \int_{\mathbb{R}} z^2 \nu_0(\mathrm{d}z)}{\alpha + \int_{\mathbb{R}} (\mathrm{e}^z - 1 - z)\nu_0(\mathrm{d}z)},$$

and solutions $\{G_n\}_{n \geq 1}$ given, for $c \neq 0$, by

(4.13)
$$G_n(x) := Q_1 \frac{\mathrm{e}_{nc}(x)}{\phi_{nc}} \prod_{k=1}^{n-1} \frac{-\chi_{kc}}{\phi_{kc}}, \quad Q_1 := 2\beta + \int_{\mathbb{R}} z^2 \nu_1(\mathrm{d}z) - Q_0\left(\beta + \int_{\mathbb{R}} (\mathrm{e}^z - 1 - z)\nu_1(\mathrm{d}z)\right).$$

Thus we have a formal series expansion, defined by (4.10), (4.12), and (4.13), for a function G that solves OIDE (3.9). The following conditions suffice for validity of this expansion.

Theorem 4.7. *Assume that the local variance $a^2(x)$ and Lévy kernel $\mu(x, \mathrm{d}z)$ are given by* (4.7). *Assume further that ν_0 and ν_1 satisfy* (4.8), *and $c \neq 0$, and*

$$(4.14) \qquad \lim_{n\to\infty} \frac{n^2 c^2 + J(nc;\nu_1)}{J((n+1)c;\nu_0)} = 0, \qquad \textit{where} \qquad J(x;\nu) := \int_{\mathbb{R}} \nu(\mathrm{d}z)(\mathrm{e}^{xz} - 1 - xz).$$

Then the function G is well defined on $\mathbb{R}$ by (4.10), *with* (4.12) *and* (4.13), *and solves OIDE* (3.9).

Proof. The summation in (4.10) can be written as

$$(4.15) \qquad -Q_0 x + Q_1 \sum_{n=1}^{\infty} a_n u^n(x), \qquad \text{where } a_n = \frac{1}{\phi_{nc}} \prod_{k=1}^{n-1} \frac{-\chi_{kc}}{\phi_{kc}} \quad \text{and} \quad u(x) = \delta \mathrm{e}_c(x).$$

The infinite sum is a power series in u, with coefficients $\{a_n\}_{n\geq 1}$ satisfying, by (4.14),

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{-\chi_{nc}}{\phi_{(n+1)c}} = 0,$$

which implies that the sum in (4.15) has infinite radius of convergence, and G is well-defined on $\mathbb{R}$ by (4.10), with (4.12) and (4.13). As every power series can be differentiated and integrated term-by-term within its radius of convergence, G solves OIDE (3.9). ■

Remark 4.8. If $\alpha = 0$, $\beta > 0$, $\nu_1 \equiv 0$, and $c > 0$ (resp., $c < 0$), then any Lévy measure ν_0 with support on the positive (resp., negative) axis will satisfy (4.14).

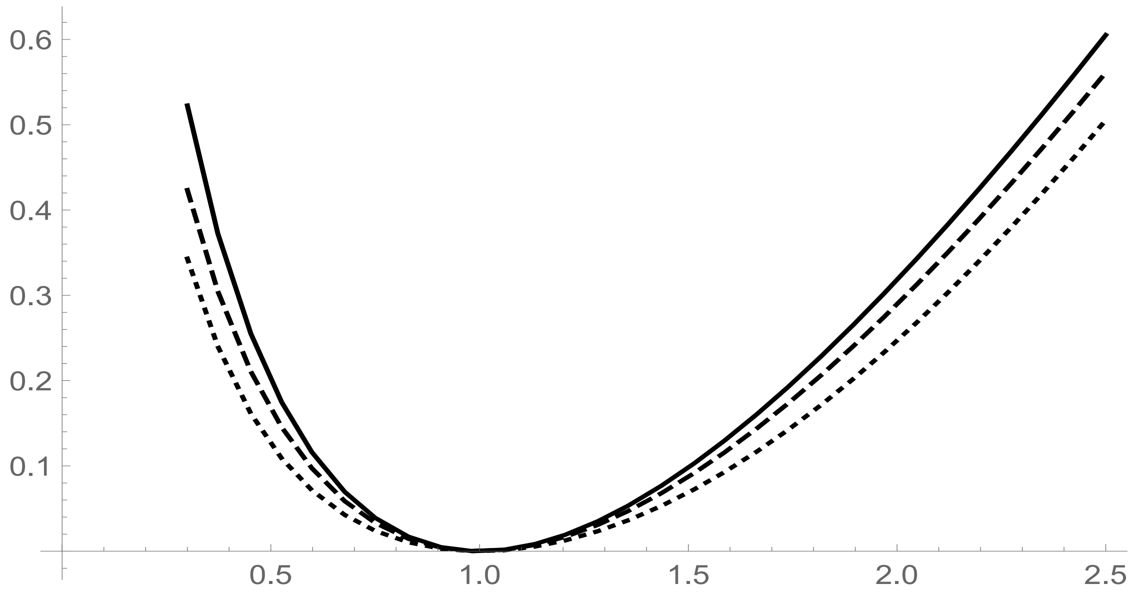

Figure 1. *In this figure, we set $F_0 = 1$, $\alpha = 1$, $\beta = 0$, $\delta = 0.35$, $\nu_0 \equiv 0$, and $\nu_1 = \delta_{z_0}$, where $z_0 = 2.5$, and we plot $h(F_T)$ as a function of F_T with $c = 0$ (solid), $c = -1$ (dashed), and $c = -2$ (dotted). Note that, when $c = 0$, we are in the setting of section 4.1 and thus h is a* log *contract plus an affine function.*

Remark 4.9. If $\alpha > 0$, $\beta = 0$, $\nu_0 \equiv 0$, and $c > 0$ (resp., $c < 0$), then a Lévy measure ν_1 will satisfy (4.14) only if the support of ν_1 lies strictly within the negative (resp., positive) axis.

Remark 4.10. In the particular case where the forward price F is a time-change of an exponential Lévy process with variance α and Lévy measure ν_0, the function G_0 prices the VS. In the more general class of models in (4.7), which can be seen as a regular δ-perturbation around the time-changed exponential Lévy case, the candidate function G for pricing the VS by Theorem 3.5 becomes, by (4.15), a δ-perturbation around G_0.

In Figures 1 and 2, using a variety of different model parameters, we plot

$$h(F_T) := G(\log F_T) - G(\log F_0) + A(F_T - F_0), \qquad A = \frac{-1}{F_0} G'(\log F_0) \tag{4.16}$$

as a function of F_T, where G is defined by (4.10), (4.12), and (4.13). Note that, if G prices the VS, then h prices the VS for any constant A. The particular value of A in (4.16) ensures that $h'(F_0) = 0$.

4.3.1. Ratio of the VS value to the log contract value. Although the purpose of this paper is to compute the value of a VS relative to the G contract (and to solve for G), it is also interesting to compute the ratio of the value of the VS to the value of a European log contract. To this end, for a function G that prices a VS, let

$$\mathcal{Q}(T, F_0) := \frac{\mathbb{E}\, G(\log F_T) - G(\log F_0)}{-\mathbb{E} \log(F_T/F_0)} = \frac{\mathbb{E}\,[\log F]_T}{-\mathbb{E} \log(F_T/F_0)}.$$

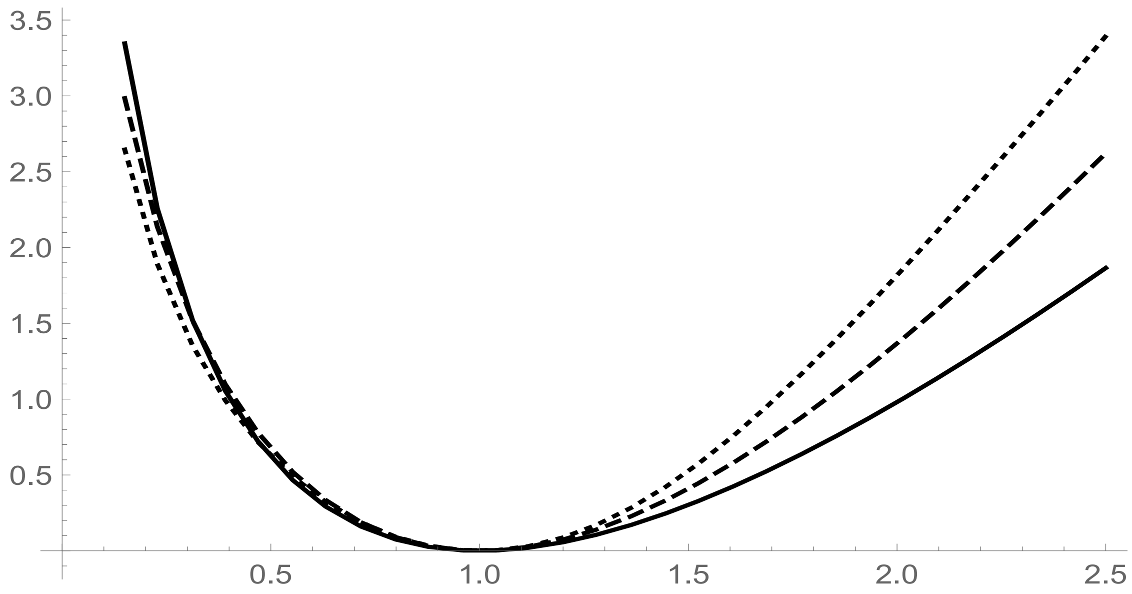

Figure 2. *In this figure, we set $F_0 = 1$, $\alpha = 1$, $\beta = 0$, $\delta = 1$, $\nu_0 \equiv 0$, and $\nu_1 = \delta_{z_0}$, where $z_0 = -2.5$, and we plot $h(F_T)$ as a function of F_T with $c = 0$ (solid), $c = 2$ (dashed), and $c = 4$ (dotted). Note that, when $c = 0$, we are in the setting of section 4.1 and thus h is a* log *contract plus an affine function.*

In Carr, Lee, and Wu (2012) the authors find that if $F_t = \exp(\widehat{Y}_{\tau_t})$, where $\widehat{Y}$ is a *Lévy process*, then the ratio $\mathcal{Q}(T, F_0)$ is a constant Q which is *independent* of the initial value F_0 of the underlying and the time to maturity T (see Theorem 4.1 and Remark 4.3 of section 4.1). This is in contrast to empirical results from the same paper, which show in a study of S&P500 data that the ratio $\mathcal{Q}(T, F_0)$ is not constant. In the more general time-changed Markov setting considered in the present paper, the ratio $\mathcal{Q}(T, F_0)$ can, in general, depend on the current value F_0 of the underlying and the time to maturity T. Below, we derive a formal approximation for the ratio $\mathcal{Q}(T, F_0)$ for one specific example of (a^2, μ) which is of the form (4.7).

Assumption 4.11. Throughout this section, we assume $F_t = \exp(Y_{\tau_t})$, where τ is a continuous time change *independent* of Y and the Laplace transform $L(t, \lambda) := \mathbb{E}\, e^{\tau_t \lambda}$ is known. Let the Markov process Y have local variance $a^2(x)$ and Lévy kernel $\mu(x, dz)$ of the form (4.7) with

$$\alpha = 1, \qquad \beta = 0, \qquad \sigma_0^2(x) = 2\omega^2, \qquad \sigma_1^2(x) = 2\omega^2 e_c(x), \qquad \nu_0 \equiv 0, \qquad \nu_1 \equiv \nu,$$

where $\omega, c > 0$. Assume, moreover, that the Lévy measure ν satisfies the conditions of Theorem 4.7. Thus, the function G, defined by (4.10), (4.12), and (4.13), solves (3.9) . In accordance with Remark 4.9, jumps must be downward, i.e., $\nu(\mathbb{R}^+) = 0$.

We compute an approximation for $\mathcal{Q}(T, F_0)$ in three *Steps*, described below.

Step 1. *Derive an approximation for $u(t, x; \varphi) := \mathbb{E}_x\, \varphi(Y_t)$.*

Formally, the function u satisfies the Kolmogorov backward equation

$$(4.17) \qquad (-\partial_t + \mathcal{A})u = 0, \qquad u(0, \cdot; \varphi) = \varphi,$$

where $\mathcal{A}$, the generator of Y, is given by

$$\mathcal{A} = \omega^2 \mathcal{A}_0 + \delta \mathrm{e}_c \omega^2 \mathcal{A}_1. \tag{4.18}$$

Now, suppose that the function u has a power series expansion in δ,

$$u = \sum_{n=0}^{\infty} \delta^n u_n, \tag{4.19}$$

where the functions $\{u_n\}_{n\geq 0}$ are unknown. Inserting expressions (4.18) and (4.19) into (4.17) and collecting terms of like powers of δ, we obtain a sequence of nested partial integro-differential equations (PIDEs) for the unknown functions $\{u_n\}_{n\geq 0}$,

$$\begin{aligned} &\mathcal{O}(1): && (-\partial_t + \omega^2 \mathcal{A}_0) u_0 = 0, && u_0(0,\cdot;\varphi) = \varphi, \\ &\mathcal{O}(\delta^n): && (-\partial_t + \omega^2 \mathcal{A}_0) u_n = -\mathrm{e}_c \omega^2 \mathcal{A}_1 u_{n-1}, && u_n(0,\cdot;\varphi) = 0, \qquad n \geq 1. \end{aligned}$$

The solution to this nested sequence of PIDEs is given in Jacquier and Lorig (2013, eq. (5.2)). We have

$$u_n(t,x;\varphi) = \int_{\mathbb{R}} \left(\sum_{k=0}^{n} \frac{\mathrm{e}^{t\omega^2 \phi_{\mathtt{i}\lambda+kc}} \mathrm{e}_{\mathtt{i}\lambda+nc}(x)}{\prod_{j\neq k}^{n} (\omega^2 \phi_{\mathtt{i}\lambda+kc} - \omega^2 \phi_{\mathtt{i}\lambda+jc})} \right) \left(\prod_{k=0}^{n-1} \omega^2 \chi_{\mathtt{i}\lambda+kc} \right) \widehat{\varphi}(\lambda) \mathrm{d}\lambda, \tag{4.20}$$

where an empty product is defined to equal one $\prod_{k=0}^{-1}(\cdots) := 1$, and $\widehat{\varphi}$ denotes the distributional generalization of the Fourier transform defined for integrable functions φ by

$$\widehat{\varphi}(\lambda) := \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(x) \mathrm{e}^{-\mathtt{i}\lambda x} \mathrm{d}x.$$

Inserting expression (4.20) into the sum (4.19) and truncating at order N yields $\bar{u}_N$, our Nth *order approximation of* u. Explicitly,

$$\begin{aligned} \bar{u}_N(t,x;\varphi) &:= \sum_{n=0}^{N} \delta^n u_n(t,x;\varphi) \\ &= \int_{\mathbb{R}} \sum_{n=0}^{N} \delta^n \left(\sum_{k=0}^{n} \frac{\mathrm{e}^{t\omega^2 \phi_{\mathtt{i}\lambda+kc}} \mathrm{e}_{\mathtt{i}\lambda+nc}(x)}{\prod_{j\neq k}^{n} (\omega^2 \phi_{\mathtt{i}\lambda+kc} - \omega^2 \phi_{\mathtt{i}\lambda+jc})} \right) \left(\prod_{k=0}^{n-1} \omega^2 \chi_{\mathtt{i}\lambda+kc} \right) \widehat{\varphi}(\lambda) \mathrm{d}\lambda. \end{aligned} \tag{4.21}$$

Step 2. Derive an approximation for $v(t,x;\varphi) := \mathbb{E}_x \varphi(Y_{\tau_t})$.

Using the independence of τ and Y, we have

$$v(t,x;\varphi) := \mathbb{E}_x \varphi(Y_{\tau_t}) = \mathbb{E}_x \mathbb{E}_x[\varphi(Y_{\tau_t})|\tau_t] = \mathbb{E}\, u(\tau_t,x;\varphi). \tag{4.22}$$

Replacing the function u in (4.22) with $\bar{u}_N$ yields $\bar{v}_N$, our Nth *order approximation of* v. Explicitly,

$$\begin{aligned} \bar{v}_N(t,x;\varphi) &:= \mathbb{E}\, \bar{u}_N(\tau_t,x;\varphi) \\ &= \int_{\mathbb{R}} \sum_{n=0}^{N} \delta^n \left(\sum_{k=0}^{n} \frac{L(t,\omega^2 \phi_{\mathtt{i}\lambda+kc}) \mathrm{e}_{\mathtt{i}\lambda+nc}(x)}{\prod_{j\neq k}^{n} (\omega^2 \phi_{\mathtt{i}\lambda+kc} - \omega^2 \phi_{\mathtt{i}\lambda+jc})} \right) \left(\prod_{k=0}^{n-1} \omega^2 \chi_{\mathtt{i}\lambda+kc} \right) \widehat{\varphi}(\lambda) \mathrm{d}\lambda, \end{aligned} \tag{4.23}$$

using (4.21) and $\mathbb{E}\mathrm{e}^{\lambda\tau_t} = L(t,\lambda)$.

Step 3. Derive an approximation for $\mathcal{Q}(T, F_0)$.

With G as given in Theorem 4.7, we have

$$\begin{aligned} \mathcal{Q}(T,F_0) &= \frac{\mathbb{E}G(\log F_T) - G(\log F_0)}{-\mathbb{E}\log(F_T/F_0)} \\ &= Q_0 + \frac{\sum_{n=1}^{\infty} b_n \Big(\mathbb{E}\mathrm{e}_{nc}(\log F_T) - \mathrm{e}_{nc}(\log F_0)\Big)}{-\mathbb{E}\log F_T + \log F_0} \\ &= Q_0 + \frac{\sum_{n=1}^{\infty} b_n \Big(\mathbb{E}\mathrm{e}_{nc}(Y_{\tau_T}) - \mathrm{e}_{nc}(\log F_0)\Big)}{-\mathbb{E}Y_{\tau_T} + \log F_0} \\ &= Q_0 + \frac{\sum_{n=1}^{\infty} b_n \Big(v(T,\log F_0;\mathrm{e}_{nc}) - \mathrm{e}_{nc}(\log F_0)\Big)}{-v(T,\log F_0;\mathrm{Id}) + \log F_0}, \end{aligned} \tag{4.24}$$

$$b_n := Q_1 \delta^n a_n = Q_1 \frac{\delta^n}{\phi_{nc}} \prod_{k=1}^{n-1} \frac{-\chi_{kc}}{\phi_{kc}},$$

where Id is the identity function $\mathrm{Id}(x) = x$. Replacing the function v wherever it appears in (4.24) by $\bar{v}_N$ and truncating the infinite sum at N terms produces $\bar{\mathcal{Q}}_N(T, F_0)$, our Nth *order approximation of* $\mathcal{Q}(T, F_0)$. Explicitly,

$$\bar{\mathcal{Q}}_N(T,F_0) := Q_0 + \frac{\sum_{n=1}^{N} b_n \Big(\bar{v}_N(T,\log F_0;\mathrm{e}_{nc}) - \mathrm{e}_{nc}(\log F_0)\Big)}{-\bar{v}_N(T,\log F_0;\mathrm{Id}) + \log F_0}. \tag{4.25}$$

The Fourier transforms of the complex exponential e_γ ($\gamma \in \mathbb{C}$) and the identity function Id, as needed to compute $\bar{v}_N(T, \log F_0; \mathrm{e}_{nc})$ and $\bar{v}_N(T, \log F_0; \mathrm{Id})$ in (4.25), are given by

$$\widehat{\mathrm{e}}_\gamma(\lambda) = \delta(\lambda + \mathtt{i}\gamma), \qquad \gamma \in \mathbb{C}, \qquad \widehat{\mathrm{Id}}(\lambda) = \mathtt{i}\delta'(\lambda), \tag{4.26}$$

where δ and δ' denote the Dirac delta function and its derivative, understood in the sense of distributions. Inserting (4.26) into (4.23) and integrating produces closed-form expressions for both $\bar{v}_N(T, \log F_0; \mathrm{e}_{nc})$ and $\bar{v}_N(T, \log F_0; \mathrm{Id})$.

Figure 3 plots $\bar{\mathcal{Q}}_N(T, F_0)$ as a function of F_0.

5. Conclusion.

In Carr, Lee, and Wu (2012), the authors model the forward price as the exponential of a Lévy process time-changed by a continuous increasing stochastic clock. In this setting, they show that a VS has the same value as a fixed number of European log contracts. The exact number of log contracts that price the VS depends only on the dynamics of the driving Lévy process, irrespective of the time-change.

This paper generalizes the underlying forward price dynamics to time-changed exponential Markov processes, where the background process may have a state-dependent (i.e., local) volatility and Lévy kernel, and where the stochastic time-change may have arbitrary dependence or correlation with the Markov process. In the time-changed Markov setting, we prove that the VS is priced by a European-style contract whose payoff depends only on the dynamics of the Markov process, not on the time-change. We explicitly compute the payoff function

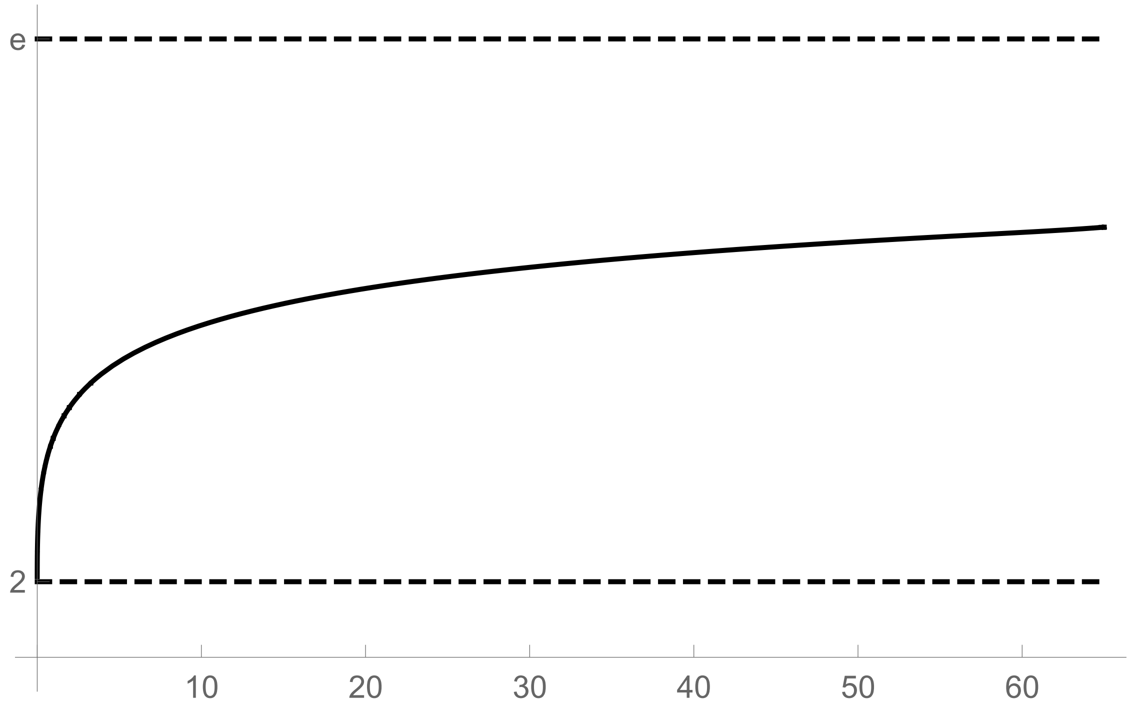

Figure 3. *A plot of $\bar{\mathcal{Q}}_N(T, F_0)$, our Nth order approximation of $\mathcal{Q}(T, F_0) := \frac{\mathbb{E}\,[\log F]_T}{-\mathbb{E}\,\log(F_T/F_0)}$ as a function of F_0 (solid line). In this plot, the forward price is given by $F_t = \exp(Y_t)$ (i.e., no time-change) and the Markov process Y has local variance $a^2(x) = 2\omega^2$ and Lévy kernel $\mu(x, \mathrm{d}z) = \delta\omega^2 \mathrm{e}^{cx}\nu(\mathrm{d}z)$, where $\nu = \delta_{z_0}$. We use the following parameters: $c = 0.395$, $\delta = 1.0$, $\omega = 0.3$, $z_0 = -1.0$, and $T = 1.0$. We fix $N = 35$. Note that as $F_0 \to 0$, the jump intensity goes to zero: $\delta\omega^2 F_0^c \to 0$. Accordingly, as $F_0 \to 0$, the ratio $\frac{\mathbb{E}\,[\log F]_T}{-\mathbb{E}\,\log(F_T/F_0)} \to 2$, which is what one would expect for a forward price process that experiences no jumps (see (4.3)). As $F_0 \to \infty$ and the jump-intensity increases, we expect the ratio $\frac{\mathbb{E}\,[\log F]_T}{-\mathbb{E}\,\log(F_T/F_0)} \to \mu_2/\varphi_0 = \mathrm{e}$, which is the corresponding ratio for a pure-jump Lévy-type process (see (4.4)). Note that if the Markov process Y were a Lévy process (i.e., with constant variance coefficient and Lévy measure), as in Carr, Lee, and Wu (2012), the ratio $\frac{\mathbb{E}\,[\log F]_T}{-\mathbb{E}\,\log(F_T/F_0)}$ would be a constant independent of F_0.*

that prices the VS for various driving Markov processes. When the Markov process is a Lévy process we recover the results of Carr, Lee, and Wu (2012).

For certain Markov processes, we also compute directly from model parameters an approximation for valuation of European-style contracts, showing the variation in the ratio of the VS value to the log contract value as a function of the current level of the underlying. This is in contrast to Carr, Lee, and Wu (2012), who show in the more restrictive time-changed Lévy process setting that this ratio is constant.

Acknowledgment. The authors are grateful to Feng Zhang and Stephan Sturm for their helpful comments.

REFERENCES

P. Carr, R. Lee, and L. Wu (2012), *Variance swaps on time-changed Lévy processes.* Finance Stoch., 16,

pp. 335–355.
P. Carr and D. Madan (1998), *Towards a theory of volatility trading*, in Option Pricing, Interest Rates and Risk Management, Handb. Math. Finance, Cambridge University Press, Cambridge, pp. 417–427.
E. Çinlar and J. Jacod (1981), *Representation of semimartingale Markov processes in terms of Wiener processes and Poisson random measures*, in Seminar on Stochastic Processes, Progr. Prob. Statist. 1, Birkhäuser Boston, Boston, pp. 159–242.
Z. Cui, J. L. Kirkby, and D. Nguyen (2017), *A general framework for discretely sampled realized variance derivatives in stochastic volatility models with jumps*, European J. Oper. Res., 262, pp. 381–400.
B. Dupire (1993), *Model art*, Risk, 6, pp. 118–124.
D. Filipović, E. Gourier, and L. Mancini (2016), *Quadratic variance swap models*, J. Financ. Econ., 119, pp. 44–68.
P. Henry-Labordère and N. Touzi (2016), *An explicit martingale version of the one-dimensional Brenier theorem*, Finance Stoch., 20, pp. 635–668.
D. Hobson and M. Klimmek (2012), *Model-independent hedging strategies for variance swaps*, Finance Stoch., 16, pp. 611–649.
A. Itkin and P. Carr (2010), *Pricing swaps and options on quadratic variation under stochastic time change models—Discrete observations case*, Rev. Deriv. Res., 13, pp. 141–176.
A. Itkin and P. Carr (2012), *Using pseudo-parabolic and fractional equations for option pricing in jump diffusion models*, Comput. Econ., 40, pp. 63–104.
J. Jacod and A. N. Shiryaev (1987), *Limit Theorems for Stochastic Processes*, Grundlehren Math. Wiss. 288, Springer-Verlag, Berlin.
A. Jacquier and M. Lorig (2013), *The smile of certain Lévy-type models*, SIAM J. Finan. Math., 4, pp. 804–830, https://doi.org/10.1137/12090246X.
J. Kallsen and A. Shiryaev (2002a), *The cumulant process and esscher's change of measure*, Finance Stoch., 6, pp. 397–428.
J. Kallsen and A. Shiryaev (2002b), *Time change representation of stochastic integrals*, Theory Probab. Appl., 46, pp. 522–528.
U. Küchler and M. Sørensen (1997), *Exponential Families of Stochastic Processes*, Springer Ser. Statist., Springer-Verlag, New York.
D. Lepingle and J. Mémin (1978), *Sur l'intégrabilité uniforme des martingales exponentielles*, Z. Wahrsch. Verw. Gebiete, 42, pp. 175–203.
M. Lorig, O. Lozano-Carbassé, and R. Mendoza-Arriaga (2016), *Variance swaps on defaultable assets and market implied time-changes*, SIAM J. Finan. Math., 7, pp. 273–307, https://doi.org/10.1137/140955380.
K. Nabil (2014), *Model-independent lower bound on variance swaps*, Math. Finance, 26, pp. 939–961.
A. Neuberger (1990), *Volatility Trading*, Institute of Finance and Accounting, London Business School, London.
P. Protter (2004), *Stochastic Integration and Differential Equations*, Stoch. Model. Appl. Probab. 21, Springer-Verlag, Berlin.
Z. Wendong and K. Y. Kuen (2014), *Closed form pricing formulas for discretely sampled generalized variance swaps*, Math. Finance, 24, pp. 855–881.