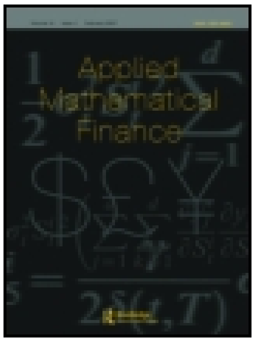

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Spiking the Volatility Punch

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ABSTRACT
An alternative volatility index called SPIKES has been recently introduced. Like VIX, SPIKES aims to forecast S&P 500 volatility over a 30-day horizon and both indexes are based on the same theoretical formula; yet, they differ in several ways. While some differences are introduced in response to the controversy surrounding possible VIX manipulation, others are due to the choice of the S&P500 exchange-traded fund (ETF), named SPY, as a substitute for the S&P500 (SPX) Index itself. Indeed, options on the SPX, used for VIX computation, are European-style, whereas options on the SPY ETF, used for SPIKES computation, are American-style.

Overall, the difference is mainly due to the early exercise premium of the component options and the dividend timing of the underlying SPY versus SPX and we assess the magnitude of these separate contributions under the benchmark Black, Merton and Scholes setting. By applying both the finite difference method and newly-derived approximation formulas we show that the new SPIKES index will track the VIX index as long as 30-day US interest rates and annualized dividend yields continue to be range-bound between 0 and 10% per year. Hence, after more that 20 years of supremacy, VIX may have found its first competitor.

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Volatility modelling; VIX; SPIKES; American Options; Exercise boundary

1. An introduction to VIX and SPIKES

The primary volatility index quoted in the financial press has been VIX, since it was first created in 1993. The construction of VIX was changed in 2003 to take advantage of conceptual breakthroughs in theoretical replicating strategies for over-the-counter variance swaps, see Britten-Jones and Neuberger (2000); Demeterfi et al. (1999), and discussion in Carr and Wu (2009); Jiang and Tian (2005), among others. Since its re-design in 2003, VIX[2] has been the cost of a portfolio of out-of-the money (OTM) options written on the SPX. The weighting scheme combines the mid-point of bid and ask SPX option quotes, updated every minute, at almost all of the OTM strikes and for two maturities that straddle 30 days. The across-strike weights are designed to equate the Gamma-cash contribution from each option, while the across-maturity weights are designed to target a 30-day forecasting horizon. A detailed description on how the VIX index is constructed in practice is given in CBOE VIX white paper (2009).

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The emphasis on bid and ask quotes rather than on trades has led prominent academics to question whether VIX can be manipulated, see Griffin and Shams (2018). The excessive volumes of out-of-the-money SPX options on VIX settlements were found to have no other explanation. In response to the controversy surrounding possible VIX manipulation, an alternative volatility index called SPIKES has been recently introduced and quoted on the Miami International Securities Exchange (MIAX). Currently, futures and options both trade on SPIKES, the former on the Minneapolis Grain Exchange (MGEX) and the latter on MIAX Option Exchange. The futures have monthly expiration for the six nearest months. The options are European-style and have monthly expiration also for the six nearest months. They can be bought or sold individually or as a combination.

In this paper, we analyse the SPIKES volatility index, primarily by comparing it to VIX, which is much more well known. Like VIX, SPIKES aims at forecasting S&P500 volatility over a 30-day horizon. SPIKES uses the same weighting scheme across strikes and maturities as VIX, but differs from VIX in several ways:

(1) Dividend Timing Difference: SPIKES is calculated from prices of OTM options written on the SPY ETF, rather than from OTM SPX option prices. The underlying SPY ETF pays dividends quarterly,[1] whereas the 500 stocks comprising S&P500 pay dividends much more frequently.
(2) Early Exercise Premium: SPY options have an American-style exercise feature, whereas SPX options have a European-style exercise feature.
(3) Trade Prioritization: SPIKES uses a patented price-dragging technique to select option prices in the portfolio which gives priority to trades.
(4) Weight Updating: the weights on OTM option prices in SPIKES are updated on the order of seconds, rather than minutes.
(5) Modified truncation and rolling methodology: the SPY option market has 1-cent increments versus the 5-cents increments allowed for SPX options and the rolling into a new expiration is of 3 days for SPIKES versus 9 days for VIX.

Besides, the SPY options, whose prices are combined into SPIKES, trade on several US exchanges. This makes SPIKES harder to manipulate than VIX, which combines the prices of SPX options trading only on the CBOE.

Points (3) to (5) of the above list are related to so-called market conventions and do not affect the difference between the theoretical counterparts of the two indexes, both based on the theory of variance swap replication. Hence, we focus only on the differences between SPIKES and VIX arising from different dividend timing and exercise styles. We decompose the difference between SPIKES and VIX (squared) into the sum of the dividend timing difference (DTD) and the early exercise premium (EEP), the precise definitions of which are given below. Note that, under zero interest rates the total contribution of EEPs in SPIKES vanishes when computed in days where no quarterly dividends are due during the lifetime of the constituent options (this happens overall for 8 out of 12 months), and becomes positive again in the days preceding an expected dividend. On the contrary, in the case of positive interest rates, the total contribution is always positive since the EEP is positive for put options; by symmetry, the same would happen also under the (hypothetical) case of negative interest rates since the EEP would be positive for call options.

On average, VIX is an upward biased forecast of subsequent realized volatility, Bekaert and Hoerova (2014); Carr and Wu (2009); Jiang and Tian (2005), among others. Since the DTD can either add to or subtract from the non-negative EEP, at least in principle, it is not obvious ex-ante whether or not SPIKES has a higher upward bias than VIX. Historically, the difference between SPIKES and VIX has been positive, but small in magnitude. One can use an option valuation model to gauge how large this difference can become as US dividend yields and interest rates rise from their current positive but historically low levels.

In a hypothetical case of all 500 stocks in the S&P500 paying zero dividends, the DTD is clearly zero, while the EEP is positive. An increase in dividend yields creates a non-zero DTD which again can have either sign. In standard valuation models such as Black, Merton and Scholes (Black and Scholes, 1973), the American Call EEP increases with dividends and decreases with interest rates, while the American Put EEP has the opposite behaviour in dividends and interest rates. As a result, an increase or decrease in dividends has a mixed effect on the overall EEP across Calls and Puts, as does an increase or decrease in interest rates. The primary purpose of this paper is to use the benchmark Black, Merton, Scholes (BMS) model to calculate the potential impact of a rise in interest rates and/or dividend yields on the magnitudes of the DTD and the EEP, and thereby to assess whether SPIKES is likely to continue tracking VIX in the future. Our numerical results indicate that the difference is likely to remain small so long as 30-day interest rates and annualized dividend yields both remain below 10% per year.

One can also consider the use of an alternative option pricing model that is consistent with the volatility skew. We thought it prudent to begin with the perhaps overly simple but familiar BMS setting. We could then use the results of this preliminary investigation to assess whether a more complicated but more realistic option valuation model would change our qualitative conclusions, we leave this issue to future research.

The rest of the paper is outlined as follows: Section 2 includes the preliminary data analysis; Section 3 details the computation of the two indexes and highlights their main differences from a theoretical point of view. In Section 4 we derive our main results which are a closed-form approximation for the early exercise premium of a single American Put and for the aggregate contribution of all Put components to the theoretical SPIKES index (squared). Section 5 is devoted to a numerical illustration and Section 6 to concluding remarks and possible future directions of our research. Technical details and proofs are finally summed up in the Appendix section.

2. Data analysis from 2005 to 2018

An analysis of data from 2005 to 2018 shows that the inclusion of the early exercise premiums as well as the difference dividend timing in the SPIKES calculation has had a negligible impact on the average historical level of SPIKES, as evidenced in Figure 1, where both the historical values for VIX and the back-calculated values for SPIKES are plotted. It is clear from the picture that SPIKES tracks VIX during the whole period. The simplest explanation for this negligible difference is that all of the options used in the volatility index calculations are OTM, hence with a small EEP. It also helps that US interest rates have averaged lower in this period than they did in the late 1970s, when inflation was larger. It is possible that a return of higher interest rates or a sharp increase in dividend payouts would increase the gap between SPIKES and VIX.

Table 1 reports some basic statistics for SPIKES and VIX levels and percentage log-differences (returns). The mean level of SPIKES at 18.90 has been slightly higher than that of VIX at 18.70; however, a two-sample t-test cannot reject the null hypothesis of equal mean between the two-time series, with a p-value of 0.34. The two volatility indices also have very similar standard deviations and skewness. The percentage changes in the two volatility indices are virtually indistinguishable over daily, weekly, and monthly horizons. This is confirmed, once again, by the outcome of a t-test for which the null of equal means cannot be rejected at all frequencies. In addition, the two volatility indices have very similar, strongly negative correlations with the S&P 500 over the daily, weakly and monthly horizons. Moreover, the correlation between both level and percentage changes in the two volatility indices is nearly 1.

As reported in Table 2, when SPIKES log-differences are regressed on VIX log-differences with no intercept, the estimated slope as well as the $\mathbb{R}^2$ of the regression are both very close to 1. This result holds for daily, weekly, and monthly horizons. The table also shows the result of regressing SPY on SPX after adjusting for the different dividend payout times.

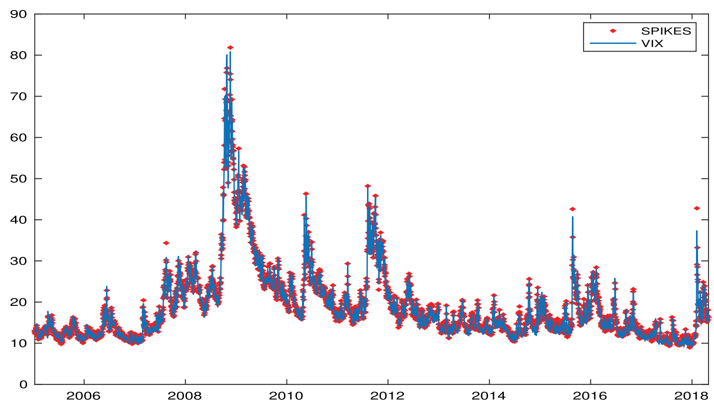

Figure 1. Historical data for VIX (blu solid line) and back-calculated values for SPIKES (red stars), from 2005 to 2018.

In Figure 2, the regression fit of SPIKES returns on VIX returns is plotted for weekly and monthly log-differences, to confirm that the two indices have moved together.

Finally, Figure 3 displays the mean difference between SPIKES and VIX by calendar month; the difference is highest in the months preceding those where the dividend dates fall (February, May, August, and November). This is expected, as an effect of both the different dividend timing DTD and the early exercise premium embedded in American component options.

Table 1. Descriptive statistics for VIX, SPIKES and their returns, from 2005 to 2018.

Statistics	Index	Daily Ret.	Weekly Ret.	Monthly Ret.
Vol. Index	VIX, SPIKES	VIX, SPIKES	VIX, SPIKES	VIX, SPIKES
Mean	18.70, 18.90	0.28%, 0.28%	1.1%, 1.1%	3.10%, 3.00%
Std. Dev.	9.20, 9.20	123%, 123%	115%, 112%	96%, 94%
Skewness	2.50, 2.50	2.20, 2.90	2.80, 2.80	2.70, 2.90
SPX Corr.	−51%, −50%	−72%, −71%	−71%, −71%	−69%, −69%
VIX Corr.	1, 99.9%,	1, 97.4%,	1, 98.5%	1, 99.1%

3. Computing VIX, SPIKES and their difference

The computation of both SPIKES and VIX is based on a theoretical result on the replication of a variance swap via a static position in OTM European-style index options combined with dynamic trading in futures written on their underlying, Britten-Jones and Neuberger (2000); Demeterfi et al. (1999). To ease one's understanding of the construction of the two volatility indices, we introduce two concepts called *TVIX* and *TSPIKES* which stand for theoretical VIX and theoretical SPIKES, respectively.

For $TVIX^2$, the cost of the theoretical replicating portfolio is:

$$TVIX^2 = \frac{365}{30}\frac{1}{B}\left[\int_0^f \frac{2}{K_p^2} p(K_p)dK_p + \int_f^\infty \frac{2}{K_c^2} c(K_c)dK_c\right] \tag{1}$$

where 365/30 is an annualization factor based on calendar days, B is the price of a zero-coupon bond paying 1 USD in 30 days, and f is the 30-day forward price of the S&P500 index, which, in practice, is approximated by the futures price. In (1), $p(K_p)$ and $c(K_c)$ respectively denote market prices of 30 day European-style OTM Puts and Calls written on the S&P500 Index struck at $K_p \in [0, f]$ and $K_c > f$ respectively.

Assuming no frictions, deterministic interest rates, and a strictly positive and continuous futures price process, $TVIX^2$ is the cost of replicating a fictitious variance swap paying the quadratic variation of the futures log-price at maturity. Academics sometimes wrongly describe this replication strategy as model-free, but what they should be writing is that this replication is not as model-dependent as the standard approach for replicating path-dependent derivatives. Under either stochastic interest rates and/or jumps in price and/or non-negative futures prices, the terminal quadratic variation of the futures log-price cannot be theoretically replicated without further assumptions.

The magnitude of the variance swap replication failure increases as we move from theory towards practice, see Jiang and Tian (2007) for a comprehensive discussion. In practice, the actual variance swap has discrete monitoring, most often daily, and often

Table 2. The R^2 value when SPIKES returns are regressed on VIX returns (top row) and SPY returns are regressed on SPX returns (bottom row).

Asset Pair	Daily Return R^2	Weekly Return R^2	Monthly Return R^2
SPIKES vs. VIX	0.95	0.97	0.98
SPY vs. SPX	0.98	0.99	1.00

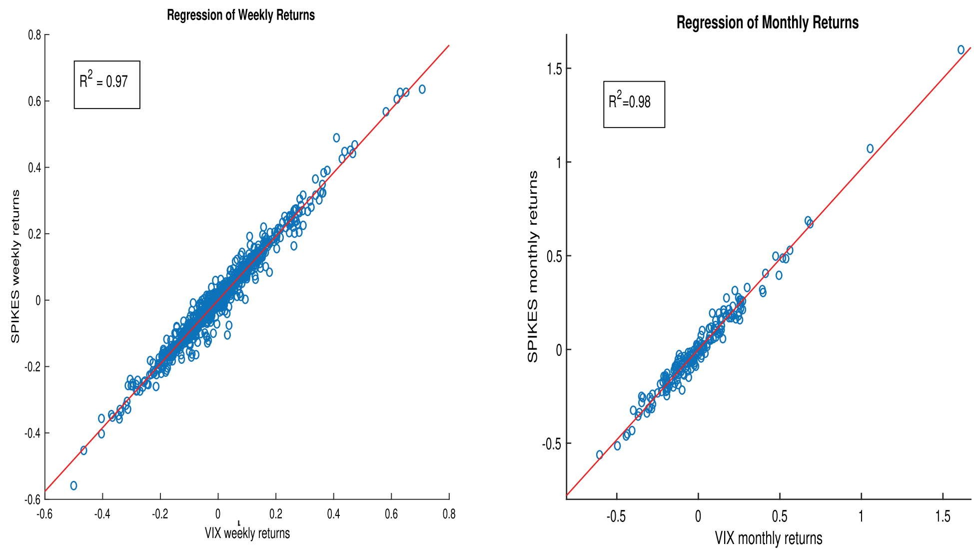

Figure 2. Linear regressions of SPIKES returns on VIX returns.

squares discrete-time log-price relatives of SPX, not its futures price. Since option prices are only observed at discrete strikes, the above $TVIX^2$ integral formula has to be approximated by either fitting an implied volatility smile across strikes or by replacing the integral with a sum arising from the truncation and discrete spacing of strikes. When a sum is used, a correction term needs to be added to capture the fact that f does not fall on a strike. Since observed option maturities are only rarely exactly 30 days, a further approximation error is introduced by the necessity of interpolating across two maturities straddling 30 days. When:

- the annualization factor is based on minutes rather than days
- the integral is replaced by a sum with a correction term

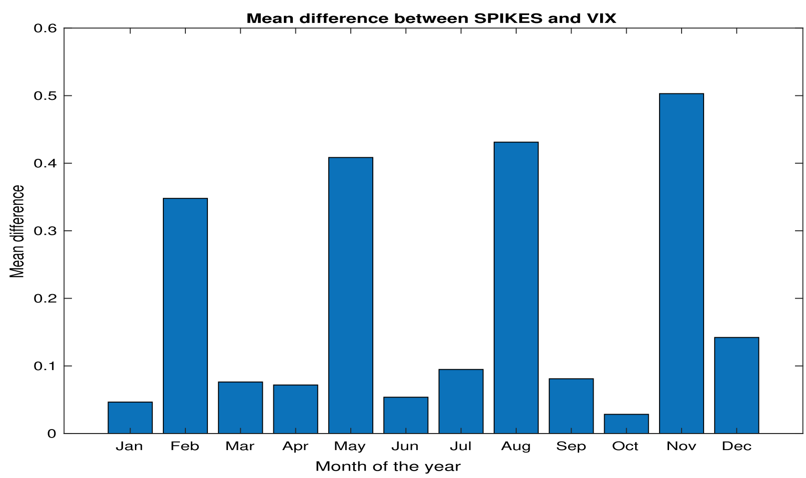

Figure 3. Mean difference between SPIKES and VIX, by calendar months.

- the maturity interpolation is linear

then the approximation of $TVIX^2$ is called VIX^2. As an aside, one can develop both a theory and a target ”variance swap like” payoff such that VIX^2, ignoring the correction term mentioned above, is the exact replication cost, as opposed to an approximation of the replication cost of a theoretical or exact variance swap.

TSPIKES is based on the same theoretical formula and weighting scheme as *TVIX*, but where European-style options on the S& *P*500 Index (SPX) are necessarily replaced by American-style options on the S&*P*500 ETF (SPY). Hence:

$$TSPIKES^2 = \frac{365}{30}\frac{1}{B}\left[\int_0^F \frac{2}{K_p^2}P(K_p)dK_p + \int_F^\infty \frac{2}{K_c^2}C(K_c)dK_c\right] \tag{2}$$

where $P(K_p)$ and $C(K_c)$ respectively denote market prices of the American-style OTM Puts and Calls written on the SPY ETF, struck at $K_p \in [0, F]$ and $K_c > F$ respectively, and F is the 30-day forward price of the SPY ETF. However, unlike with $TVIX^2$, there is no known theory under which $TSPIKES^2$ is the initial cost of replicating a theoretical variance swap.

As for *TVIX*, the *TSPIKES* integral formula can also be approximated in practice by an annualization factor, by truncation and discrete spacing of strikes, and by linear interpolation across two maturities straddling 30 days. This approximation of $TSPIKES^2$ is called SPIKES^2.

In variance swap replication theory, the market prices of the European options are observed directly. As a result, the only role of the underlying forward price is to separate OTM Put strikes from OTM Call strikes. The underlying forward price is not needed to calculate option premia from a model as the option premia are supposed to be directly observed. Likewise, the American option premia in *TSPIKES* are supposed to be directly observed. In fact, when future values of SPIKES are to be compared to future values of VIX, we do not observe future prices of the component options. However, we can use an option pricing model to project these future option prices onto future relevant stochastic state variables such as spot SPX or SPY, and/or an ATM implied volatility of SPX or SPY. We can then use the model to compare future levels of SPIKES to future levels of VIX, conditional on given numerical values of the relevant state variables.

When an American option on SPY is exercised early or at maturity, its payoff relates to the spot value of SPY, not its futures price. When option pricing models are used to determine the early exercise premium of a SPY option, it is easier to evolve the single spot price of the underlying rather than to evolve the entire term structure of forward or futures prices. It is also easier to assume that the implied volatilities are constant across moneyness and calendar time rather than to assume they vary with moneyness and stochastically over time.

If we assume that the only relevant stochastic state variable is the spot and that it has constant proportional carrying costs and constant instantaneous volatility over time, we are actually using the BMS model to price options. Consider first the pricing of European-style SPX options; we assume for the rest of the paper that the dividends from the 500 stocks in SPX are continuously paid over time and that the annualized dividend yield of SPX is constant at some known level $\gamma \geq 0$. In BMS setting, the risk-free interest rate and the volatility are also constant at some known levels r and $\sigma > 0$ and so it is the proportional carrying cost $r - \gamma \in \mathbb{R}$. Then the value of $TVIX^2$, when the current value of the underlying SPX is at some known level $X > 0$, is given by:

$$TVIX^2 = \frac{365}{30}\frac{1}{B}\left[\int_0^f \frac{2}{K_p^2} p^{bs}(X,\gamma,K_p)dK_p + \int_f^\infty \frac{2}{K_c^2} c^{bs}(X,\gamma,K_c)dK_c\right], \tag{3}$$

where $p^{bs}(X,\gamma,K_p)$ and $c^{bs}(X,\gamma,K_c)$ denote, respectively, the BMS model value of a European Put and Call for strikes $K_p \in [0,f]$ and f being the forward price in T of the underlying SPX.

Note that $TVIX^2$ is independent of the inputs X, γ, r, and T that enter into the relative pricing of each constituent SPX option and, in BMS model, is simply the constant instantaneous variance rate σ^2. However, when we move from $TVIX^2$ to VIX^2, VIX^2 gains dependence on X, γ, r, and Tdue to the discreteness of strikes.

The theoretical counterpart of squared SPIKES, i.e. $TSPIKES^2$, is defined by:

$$TSPIKES^2 = \frac{365}{30}\frac{1}{B}\left[\int_0^F \frac{2}{K_p^2} P^{bs}(Y,q,K_p)dK_p + \int_F^\infty \frac{2}{K_c^2} C^{bs}(Y,q,K_c)dK_c\right] \tag{4}$$

where $P^{bs}(Y,q,K_p)$ and $C^{bs}(Y,q,K_c)$ respectively denote the BMS model value of an American Put and Call when SPY is at Y > 0, with constant quarterly paid dividend yield $q \geq 0$, and for strikes $K_p \in [0,F]$ and $K_c \geq F$, F being the forward price of SPY.

Let $(p/c)^{bs}(Y,q,K)$ be the theoretical value of a European Put or Call on SPY. Suppose we subtract and add $(p/c)^{bs}(Y,q,K)$ in (4). Then we can decompose the difference between $TSPIKES^2$ and $TVIX^2$ into two parts:

$$TSPIKES^2 - TVIX^2 = \epsilon^x(Y,q) + \delta^d(X,Y;\gamma,q) \tag{5}$$

where $\epsilon^x(Y,q)$ is the non-negative SPY options' aggregate early exercise premium

$$\epsilon^x(Y,q) = \frac{365}{30}\frac{1}{B}\Big[\int_0^F \frac{2}{K_p^2}[P^{bs}(Y,q,K_p) - p^{bs}(Y,q,K_p)]dK_p + \int_F^\infty \frac{2}{K_c^2}[C^{bs}(Y,q,K_c) - c^{bs}(Y,q,K_c)]dK_c\Big]$$

and $\delta^d(X,Y;\gamma,q)$ the dividend timing difference :

$$\delta^d(X,Y;\gamma,q) = \frac{365}{30}\frac{1}{B}\left[\int_0^F \frac{2}{K_p^2} p^{bs}(Y,q,K_p)dK_p + \int_F^\infty \frac{2}{K_c^2} c^{bs}(Y,q,K_c)dK_c\right] - TVIX^2.$$

Note that while the decomposition in (5) is model free, the computation of the two addends depends on the assumption on the underlying assets X and Y. It is important to display the theoretical relationship between the two underlying assets SPX and SPY, with SPX having a constant continuously paid, continuously compounded, annualized dividend yield $\gamma \geq 0$ and SPY having a constant quarterly-paid, quarterly-compounded annualized dividend rate $q \geq 0$. Consider the one-year price relatives $\frac{X_1}{X_0}$ and $\frac{Y_1}{Y_0}$ when time 0 is just after SPY has paid a quarterly dividend. Let D_0 be the price at this time of a pure discount bond paying 1 USD in 1 year. Under the forward measure $\mathbb{Q}_1$, we have

$D_0 E_0^{\mathbb{Q}_1} \frac{X_1}{X_0} = e^{-\gamma}$, while $D_0 E_0^{\mathbb{Q}_1} \frac{Y_1}{Y_0} = \left(\frac{1}{1+q/4}\right)^4$. Equating the two expressions and solving for q one gets $q = 4(e^{\gamma/4} - 1)$. Notice that $q \geq \gamma$ since the γ is compounded more often.

Under our model assumptions it is possible to show that $Y_t = X_t \exp\left(\gamma(t - T_D(t))^+\right)$ where $T_D(t)$ is the dividend date just before time $t \geq 0$ and X_t and Y_t coincide at any ex-dividend date, just after the discrete dividend is paid. In Figure 4, we plot an example of a price path for Y and X to illustrate the jump effect of the difference in the dividend schemes on the price dynamics. The dividend timing difference is due to the price jump occurring when a dividend date exists within the one-month time horizon; its magnitude can be expressed, within our framework, as a function of the proportional dividend rate q (Carr and Wu, 2009; Klassen 2009). Precisely, we have $\delta^d(X, Y; \gamma, q) = \delta^d(\gamma, q) = 2\left[\log\left(1 + \frac{q}{4}\right) - \frac{q/4}{1+q/4}\right]$.

The overall contribution $\epsilon^x(Y, q)$ to the difference between $TSPIKES^2$ and $TVIX^2$, as indicated in (5), is obtained by adding the total EEPs arising from the American Calls and American Puts. Specifically, we can write:

$$\epsilon^x(Y, q) = TotalEEP_{put} + TotalEEP_{call} \tag{6}$$

where $TotalEEP_{put}, TotalEEP_{call}$ denote the aggregate contribution of Puts and Calls, respectively. The total contribution by Calls $TotalEEP_{call}$ can be computed explicitly. Since the SPIKES index is based on a 30-day horizon, there can be at most one quarterly dividend payment from the underlying SPY ETF; when no dividend is expected during the lifetime of the options (this happens overall on $\frac{2}{3}$ of the days in a year) this contribution vanishes, otherwise the EEP of an American-style Call is given, in the BMS setting, by the EEP of a Bermudan Option with exercise times T_D, T and can be computed in closed form (Villiger, 2006). The cumulative contribution $Total\text{EEP}_{call}$ is then obtained by simple integration. Hence, in what follows we will focus on the computation of $Total\text{EEP}_{put}$. To this end, we develop an explicit approximation for the individual EEP of an American-style Put which, suitably integrated, leads to an explicit

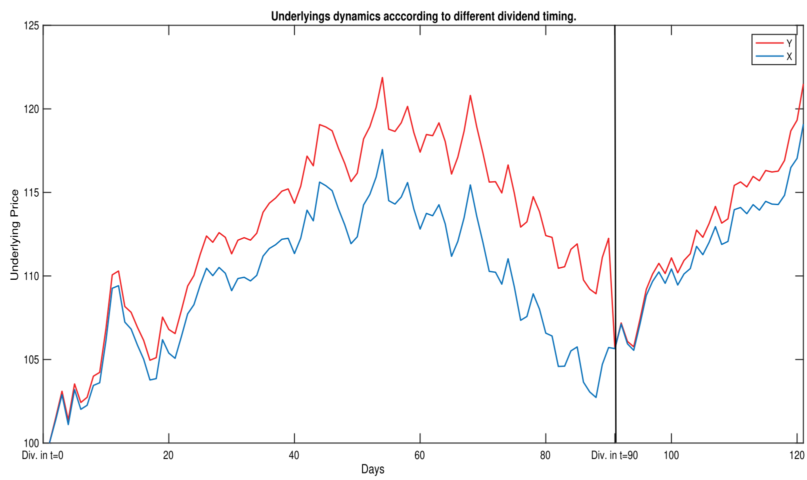

Figure 4. One possible path (120 days) of SPX and SPY under Black, Merton and Scholes model, when taking into account the different dividend schemes: $\gamma = 0.02, \sigma = 0.20, r = 0.01, SPX_0 = SPY_0 = 100$, $T_D = 0, 90$ days.

formula for its weighted integral $Total\text{EEP}_{put}$. As a byproduct we obtain the pricing formula of an American put option on an underlying paying discrete dividends in a BMS setting, thus complementing the outcomes in Villiger (2006) which hold for American calls. In what follows we will remove the call-put subscripts from the strike price which will be simply denoted by K.

4. Closed-form approximation for the early exercise premium of an American-style put option

The underlying dynamics in Black and Scholes (1973) model can be generalized to the case when the risky asset pays a single proportional dividend before maturity; if the continuously compounded interest rate is constant at $r > 0$ and Y_t is the spot price at time t, representing the price of the underlying SPY, the dynamics of the process Y under the risk-neutral probability measure Q, in $t \in [0, T]$, is given by:

$$dY_t = rY_t dt + \sigma Y_t dW_t - \delta(t - T_D) D_t dt \tag{7}$$

where $D_t = qY_t$, q is the (annualized) proportional dividend rate and δ is the Dirac delta function. As already remarked, if the underlying stock pays no dividends between the valuation time t and the option's maturity date T, then an American Call with expiration date T and strike price K has the same price as a European one i.e. $C(t, T, K) = c(t, T, K)$, where C and c denote the price of an American and a European Call, respectively. In contrast, American Puts have a positive early exercise premium.

4.1. The individual early exercise premium

It is well known, see Carr, Jarrow, and Myneni (1992), that the initial value of an American Put's EEP with expiry T, strike price K, and a non-dividend-paying underlying priced at Y_0, may be represented as:

$$EEP(0, T, K) = \mathbb{E}^Q\left[rK\int_0^T e^{-rt}\mathbf{1}_{\{Y_t < B(t,K)\}}dt\right], \tag{8}$$

where $B(t, K)$ is the early exercise boundary for strike K and maturity T, which has no known exact formula, but solves an integral equation. Specifically, under Black and Scholes (1973) assumptions, Carr, Jarrow, and Myneni (1992) proved that

$$EEP(0, T, K) = rK\int_0^T e^{-rt} N\left(\frac{\ln\left(\frac{B(t,K)}{Y_0}\right) - (r - \sigma^2/2)t}{\sigma\sqrt{t}}\right) dt. \tag{9}$$

Indeed, the discrete nature of the dividend makes it possible to split the integral in (8) and write the early exercise premium for an American put as:

$$
\begin{aligned}
EEP(0,T,K) &= \mathbb{E}^Q\left[rK\int_0^{T_D} e^{-rt}\mathbf{1}_{\{Y_t^c < B(t,K)\}}dt + rK\int_{T_D}^{T} e^{-rt}\mathbf{1}_{\{Y_t^x < B(t,T)\}}dt\right] \\
&= \mathbb{E}^Q\left[rK\int_0^{T_D} e^{-rt}\mathbf{1}_{\{Y_t^c < B(t,T)\}}dt\right] + e^{-rT_D}\mathbb{E}^Q\left[rK\int_{T_D}^{T} e^{-r(t-T_D)}\mathbf{1}_{\{Y_t^x < B(t,T)\}}dt\right],
\end{aligned}
\tag{10}
$$

where the cum-dividend stock price Y^c appears in the indicator function in the first integral, and the ex-dividend stock price Y^x appears in the indicator function in the second integral.

Theoretically, increasing the level of the short interest rate r raises each American Put's EEP, after accounting for the increase in the early exercise boundary $B(t,K), t \in [0,T]$. As a result, the overall effect of an increase in the interest rate r on the EEP should be positive. The magnitude of this effect is obtained by computing the above integrals, once we have an expression for the early exercise boundary $B(t,K)$.

We assume that the early exercise boundary $B(t,K), t \in [0,T]$ is approximated by an exponential function of time, as in Ju (1998), both before and after the ex-dividend date. Precisely, the true early exercise boundary is approximated by $Y_p = \{Y_p(t), 0 < t < T\}$, defined as:

$$
Y_p(t) \approx \begin{cases} Ce^{-ht}, & \text{if } 0 < t < T_D \\ Le^{gt}, & \text{if } T_D < t < T \end{cases}
\tag{11}
$$

where C, L, h, g are positive constants.

Theorem 4.1 *Assume we are under the BMS model setting and that:*

- **a1**. *the price changes of the risky asset Y in the time interval $[0,T]$ are described by the dynamics in* (7) *with constant parameters r, σ, q and $0 < T_D < T$;*
- **a2**. *the early exercise boundary for American put options is approximated by the piece-wise exponential function in* (11).

Then the early exercise premium for an American Put with strike K and expiration date T is approximated by

$$
\begin{aligned}
EEP(0,T,K) =& \tilde{f}(Y_0^c; C, -h) - e^{-rT_D}\mathbb{E}^Q\left[\tilde{f}(Y_{T_D}^c e^{hT_D}; C, -h)\right] \\
&+ e^{-rT_D}\mathbb{E}^Q\left[\tilde{f}(Y_{T_D}^x e^{-gT_d}; L, g)\right] - e^{-rT}\mathbb{E}^Q\left[\tilde{f}(Y_T^x e^{-gT}; L, g)\right],
\end{aligned}
\tag{12}
$$

with $\tilde{f} : \mathbb{R}_+^3 \to \mathbb{R}_+$ defined as:

$$
\tilde{f}(\tilde{y}; l, u) = \frac{2rK}{\sigma^2(p_+ - p_-)}\left[\mathbf{1}_{\{\tilde{y} < l\}}\left(\frac{1}{p_+} - \frac{1}{p_-} - \frac{1}{p_+}\left(\frac{\tilde{y}}{l}\right)^{p_+}\right) - \mathbf{1}_{\{\tilde{y} > l\}}\frac{1}{p_-}\left(\frac{\tilde{y}}{l}\right)^{p_-}\right],
\tag{13}
$$

where $\tilde{y} = y\exp(-ut)$ and p_+, p_- are the roots of $\mathcal{P}(p) = \frac{\sigma^2}{2}p^2 + (r - u - \frac{\sigma^2}{2})p - r$.

Proof. A detailed proof is given in the Appendix.

Note that expected values are calculated using the cum-dividend initial price for the first two terms and the ex-dividend price for the following others. The relevance of the above theorem is in the fact that the function $\tilde{f}$ is not path dependent; while the LHS of the expression in (A12) depends on the distribution of the whole path from 0 to T of the stochastic process Y, the RHS only depends, through $\tilde{f}$, on the random variables Y_{T_D}, Y_T. Further, it is straightforward to show that

Lemma 4.2. *Under the assumptions of Theorem 4.1, if no dividends are due in* $[t, T]$ *then we have*:

$$\begin{aligned}\mathbb{E}_t^Q\left[e^{-r(T-t)}\tilde{f}(\tilde{Y}_T; l, u)\right] &= Ke^{-r(T-t)}N\big(-d_2(t, \tilde{Y}_t, l, T)\big)\\ &\quad - \frac{2rK}{\sigma^2(p_+ - p_-)}\frac{1}{p_+}\left(\frac{\tilde{Y}_t}{l}\right)^{p_+} N\big(-d_2^+(t, \tilde{Y}_t, l, T)\big)\\ &\quad - \frac{2rK}{\sigma^2(p_+ - p_-)}\frac{1}{p_-}\left(\frac{\tilde{Y}_t}{l}\right)^{p_-} N\big(d_2^-(t, \tilde{Y}_t, l, T)\big) \qquad (14)\end{aligned}$$

where $\tilde{Y}_t = Y\exp(-ut)$, $d_2^-(t, \tilde{Y}_t, l, T) = d_2(t, \tilde{Y}_t, l, T) + p^-\sigma\sqrt{T-t}$, $d_2^+(t, \tilde{Y}_t, l, T) = d_2(t, \tilde{Y}_t, l, T) + p^+\sigma\sqrt{T-t}$ *and* $\mathbb{E}_t$ *denotes conditional expectation at time* t.

Proof. The proof is straightforward, given the known dynamics of the process Y and observing that $\frac{2r}{\sigma^2(p_+ - p_-)}\left(\frac{1}{p_+} - \frac{1}{p_-}\right) = 1$.

It is worth to remark that the EEP in 4.1 can be computed explicitly by applying Lemma 4.2 where Y is replaced by either the cum-dividend process Y^c or the ex-dividend process Y^x. In this latter case the conditional expectation in $t = T_D$ is computed first, and the unconditional expectation is obtained by using the tower rule.

In order to gauge the magnitude of the approximation error, we solve the partial differential equation (PDE) obtained from the dynamics in (7) for a discrete proportional dividend, qY at T_D, see also Göttsche and Vellekoop (2011). Notably, by applying the finite difference method to numerically solve the PDEs it is possible to get both the price of American options and the (true) early exercise boundary. In Figures 5 and 6, we plot the (true) early exercise boundary obtained by numerical integration, for several values of the risk-free rate r and the annualized dividend rate q, respectively. Indeed, the presence of a discrete dividend makes the boundary jump upwards at time T_D , being the boundarydecreasing in calendar time t before time T_D, and increasing afterwards. In Figure 7 we plot both the true and the approximated boundary for a specific example with $r = 2\%, q = 3\%, T_D = 1/2$ months and $K = 100, T = 1$ month. The exponential approximation is above the true boundary , hence the closed form for the put option EEP is an upward biased approximation of the actual one.

Figures 8 and 9, represent a zoom of the early exercise boundary for the period before and after the dividend date, respectively. In these pictures we also plot the linear approximations obtained as steps towards the final exponential approximation of the boundary; their precise computation is detailed in the Appendix.

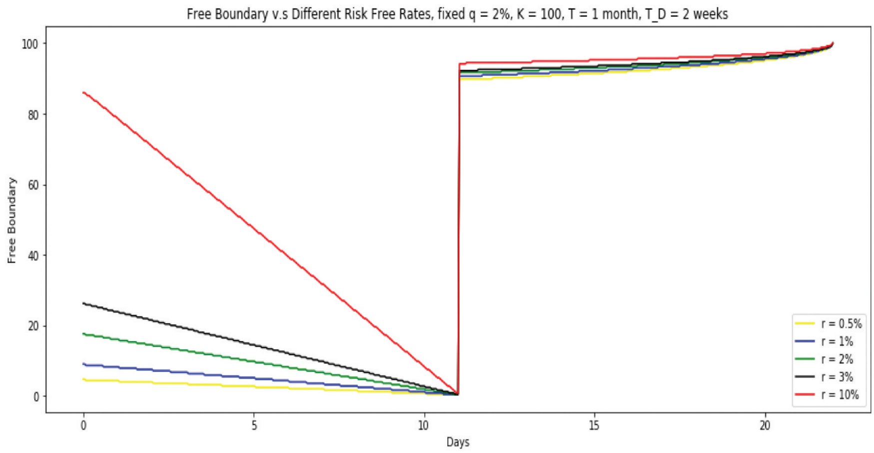

Figure 5. Early exercise boundary curves for different levels of the risk-free interest rate r, when the dividend rate q is fixed at q=3%. Option strike price and maturity are set at K = 100 and $T = 1$ month and the next dividend is expected in $T_D = 2$ weeks.

4.2. The cumulative early exercise premium

The representation of the EEP of a single American Put given in (13) allows us to compute directly the excess of $TSPIKES^2$ over $TVIX^2$ arising from the early exercise premium of component put options by integrating over all of the OTM strikes $K \in [0, F]$, i.e. $Total\ EEP_{Put}(0, T) = \int_0^F \frac{2}{K^2} EEP(0, T, K)dK$. It is possible to prove the following result:

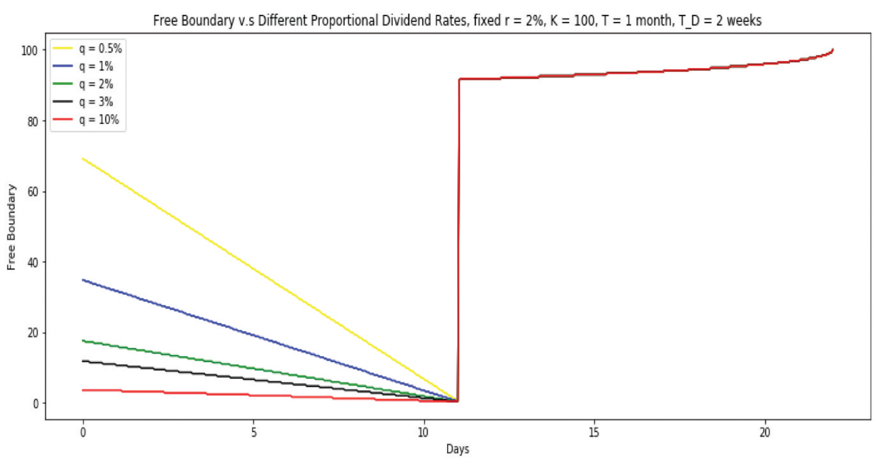

Figure 6. Early exercise boundary curves for different levels of the dividend rate q, when the risk free interest rate is fixed at r=2%. Option strike price and maturity are set at $K = 100$ and $T = 1$ month and the next dividend is expected in $T_D = 2$ weeks.

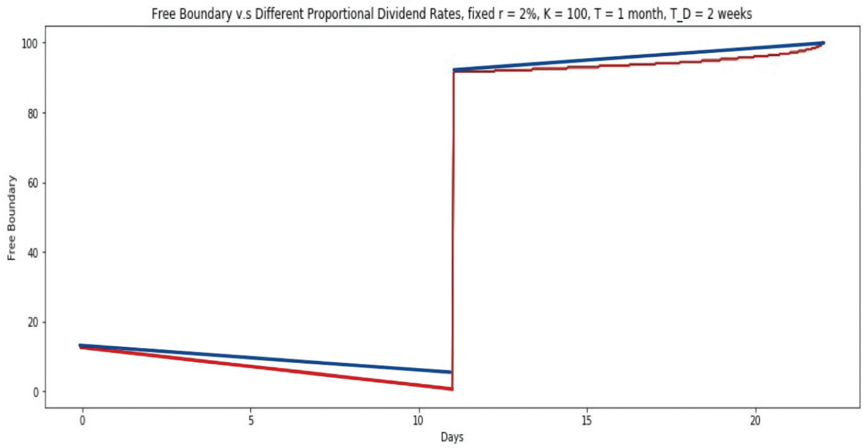

Figure 7. Early exercise boundary curve when $r = 2\%$. Option strike price and maturity are set at $K = 100$ and $T = 1$ month and the next dividend is expected in $T_D = 2$ weeks.

Proposition 4.3 *Under the assumptions of Theorem 4.1 the cumulative contribution of the early exercise premiums of OTM Puts is approximated by*:

$$\begin{aligned} Total\ EEP_{Put}(0,T) = \tilde{G}(\tilde{Y}^c_0; a, -h) - e^{-rT_D}\mathbb{E}^Q\left[\tilde{G}(\tilde{Y}^c_{T_D}; a, -h)\right] + e^{-rT_D}\mathbb{E}^Q\left[\tilde{G}(\tilde{Y}^x_t; b, g)\right] \\ - e^{-r(T)}\mathbb{E}^Q\left[\tilde{G}(\tilde{Y}^x_T; b, g)\right] \end{aligned} \tag{15}$$

where $\tilde{Y}^c_s = Y^c_s \exp(hs), \tilde{Y}^x_s = Y^x_t \exp(-gs), s \geq 0$ *and* $a := C/K, b := L/K$ *are positive constants. Finally, the function* $G : \mathbb{R}^3_+ \to \mathbb{R}_+$ *is defined as*:

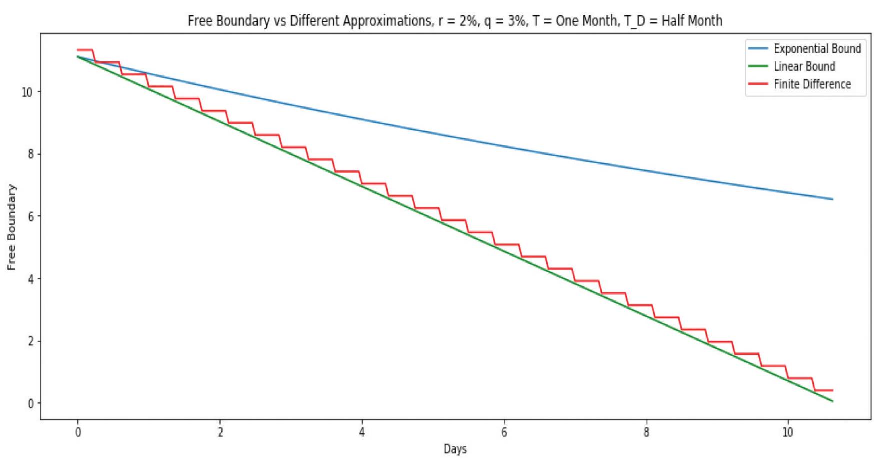

Figure 8. Early exercise boundary before the dividend date T_D: Finite difference method (Red), Linear approximation (Green), Exponential approximation (Blue).

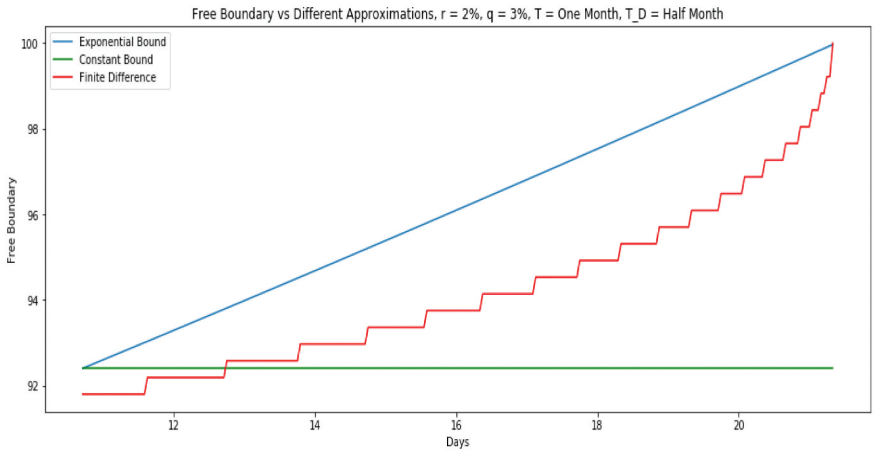

Figure 9. Early exercise boundary after the dividend date T_D: Finite difference method (Red), Constant approximation (Green), Exponential approximation (Blue).

$$\tilde{G}(\tilde{y}; v, u) = \frac{4r}{\sigma^2(p^+ - p^-)}$$

$$\times \left[\mathbf{1}_{\{\tilde{y} < vF\}} \left(\left(\frac{1}{p^+} - \frac{1}{p^-} \right) \log \frac{Fv}{\tilde{y}} + \frac{1}{(p^+)^2} \left(\frac{\tilde{y}^{p^+}}{(Fv)^{p^+}} - 1 \right) + \frac{1}{(p^-)^2} \right) + \mathbf{1}_{\{\tilde{y} > vF\}} \frac{\tilde{y}^{p^-}}{(vF)^{p^-}} \right], \tag{16}$$

where F is the forward price of the underlying Y for maturity T, $\tilde{y} = y \exp(-ut)$ and p_+, p_- are the roots of $\mathcal{P}(p) = \frac{\sigma^2}{2} p^2 + (r - u - \frac{\sigma^2}{2})p - r$.

Proof. It suffices to compute the following integral:

$$\tilde{G}(\tilde{y}, v, u) = \int_0^F \frac{2}{K^2} \tilde{f}(\tilde{y}, vK, u) dK$$

$$= \int_0^F \frac{2}{K^2} \frac{2rK}{\sigma^2(p^+ - p^-)} \left(\mathbf{1}_{\{\tilde{y} < vK\}} \left(\frac{1}{p^+} - \frac{1}{p^-} - \frac{1}{p^+} \frac{\tilde{y}^{p^+}}{(vK)^{p^+}} \right) - \mathbf{1}_{\{\tilde{y} > vK\}} \frac{1}{p^-} \frac{\tilde{y}^{p^-}}{vK^{p^-}} \right) dK$$

$$= \frac{4r}{\sigma^2(p^+ - p^-)} \times \int_0^F \frac{1}{K} \left(\mathbf{1}_{\{\tilde{y} < vK\}} \left(\frac{1}{p^+} - \frac{1}{p^-} - \frac{1}{p^+} \frac{\tilde{y}^{p^+}}{(vK)^{p^+}} \right) - \mathbf{1}_{\{\tilde{y} > vK\}} \frac{1}{p^-} \frac{\tilde{y}^{p^-}}{vK^{p^-}} \right) dK$$

Since all the addends in the integrating function are power functions in K, the final expression for $\tilde{G}$ is obtained by applying elementary integration rules.

The Total EEP can be approximated by a closed formula since the expectations appearing in (15) can be computed explicitly.

Lemma 4.4. *Under the assumptions of Theorem 4.1, if no dividends are due in* $[t, T]$ *then we have*:

$$\mathbb{E}_t^Q\left[e^{-r(T-t)}\tilde{G}(\tilde{Y}_T; v, u)\right] = 2e^{-rt}\left[\left(\left(\log\frac{Fe^{u(T-t)}v}{Y_t(1-q)e^{r-\sigma^2/2}} - \frac{2r}{\sigma^2(p^+-p^-)}\left(\frac{1}{(p^+)^2} - \frac{1}{(p^-)^2}\right)\right)\right.\right.$$

$$\times N(-d_2(t, T, Y_t(1-q), Fve^{ut})) + \sigma\sqrt{t}N'(-d_2(t, T, Y_t(1-q), Fve^{ut}))]$$

$$+\frac{4r}{\sigma^2(p^+-p^-)}\frac{1}{(p^+)^2}\left(\frac{Y_t(1-q)}{Fv}\right)^{p^+} N\left(-d_2(t, T, Y_t(1-q), Fve^{u(T-t)}) - \sigma p^+\sqrt{T-t}\right)$$

$$+\frac{4r}{\sigma^2(p^+-p^-)}\frac{1}{(p^-)^2}\left(\frac{Y_t(1-q)}{Fv}\right)^{p^-} N\left(d_2(t, T, Y_t(1-q), Fve^{u(T-t)}) + \sigma p^-\sqrt{T-t}\right). \tag{17}$$

where F *is the forward price of the underlying* Y *prevailing at time* t *for maturity* T, $\tilde{Y}_t = Y\exp(-ut)$, $d_2^-(t, \tilde{Y}_t, l, T) = d_2(t, \tilde{Y}_t, l, T) + p^-\sigma\sqrt{T-t}$, $d_2^+(t, \tilde{Y}_t, l, T) = d_2(t, \tilde{Y}_t, l, T) + p^+\sigma\sqrt{T-t}$, *and* $\mathbb{E}_t$ *denotes conditional expectation at time* t.

Proof. The proof is straightforward, given the known dynamics of the process Y in BMS setting and observing that $\frac{2r}{\sigma^2(p_+-p_-)}\left(\frac{1}{p_+} - \frac{1}{p_-}\right) = 1$

We remark that when there is no dividend paid during the lifetime of SPY options, then the overall contribution of Puts to the EEP reduces to:

$$Total\ EEP_{Put}(0, T) = \tilde{G}(\tilde{Y}_0, b, g) - e^{-r(T)}\mathbb{E}^Q\left[\tilde{G}(\tilde{Y}_T, b, g)\right]. \tag{18}$$

5. Numerical results

In this section, we derive the difference between $TSPIKES^2$ and $TVIX^2$ by computing the early exercise premiums for American Call and Put components via a Crank-Nicolson finite difference scheme. We consider different examples for which the dividend is paid 1 week, 2 weeks, and 3 weeks ahead, with options expiring in 1 month for both SPY and SPX. For each case, we fix $Y_0 = 100$, $\sigma = 0.2$, and $T = 1$ month. We let r and q vary, to test how interest rates and dividend yields affect the difference between $TSPIKES^2$ and $TVIX^2$. The numerical integration to obtain the two volatility indices is truncated at $K_{lowest} = 30$ for Puts and $K_{highest} = 200$ for Calls. The above difference is then compared with the closed-form approximation obtained by adding three terms: the exact dividend timing difference (DTD), the exact contribution of Calls to the total EEP, and the approximated contribution of Puts to the EEP, obtained by applying Proposition 4.3 and Lemma4.4.

In Tables 3 and 4, we report the value in basis points of the total difference between (truncated) $TSPIKES^2$ and $TVIX^2$ for several pairs of r and q.

Table 3. *TSPIKES*^{2} -*TVIX*^{2} (in bp) computed with the finite difference method when dividends are paid during the lifetime of SPY options at time T_D=1,2,3 weeks.

TSPIKES^2 -*TVIX*^{2} Finite Difference	r	q	$T_D = 1$ week	$T_D = 2$ weeks	$T_D = 3$ weeks
	0.01	0.01	0.17	0.22	0.34
	0.01	0.03	0.51	0.97	1.56
	0.01	0.05	1.35	2.48	3.62
	0.01	0.1	6.31	9.51	11.98
	0.03	0.01	0.17	0.21	0.31
	0.03	0.03	0.48	0.87	1.46
	0.03	0.05	1.25	2.31	3.46
	0.03	0.1	5.93	9.13	11.69
	0.05	0.01	0.17	0.22	0.31
	0.05	0.03	0.47	0.81	1.38
	0.05	0.05	1.17	2.16	3.32
	0.05	0.1	5.59	8.78	11.41
	0.1	0.01	0.22	0.35	0.41
	0.1	0.03	0.50	0.76	1.26
	0.1	0.05	1.10	1.91	3.05
	0.1	0.1	4.91	8.03	10.81

It is clear from Table 3 that the true difference *TSPIKES*^{2}-*TVIX*^{2} is negligible. We draw the same qualitative conclusion from the outcomes displayed in Table 4, obtained by applying the closed formulas. As expected, the approximation is upward biased and the error is between 0.25 and 6 basis points for the considered examples. For any pair of interest and dividend rate, the difference increases when the dividend date gets closer to the expiration date, as anticipated from the theoretical discussion and consistently with Figure 3.

One naturally wonders whether OTM Calls or OTM Puts are more important in explaining the cumulative EEP across strikes, contributing to the difference between *TVIX*^{2} and *TSPIKES*^{2} . To answer to this question, we report in Table 5, the separate contributions of OTM Calls and OTM Puts to the total EEP, computed by applying the closed formulas, for the case when a single dividend is paid during SPY options' lifetime in $T_D = 3$ weeks. The total contribution of Calls is increasing in the dividend rate q and decreasing in the interest rate and, as expected, the total contribution of Puts does not vary with q and it is increasing with r; the dividend timing difference is increasing in q and does not change with r. Overall, the impact of a change in the dividend rate is higher than that in the interest rate.

6. Conclusions and future research

When SPIKES is back-calculated to 2005, it hardly differs from VIX. We used the benchmark BMS model to discuss the main causes of this difference and to asses whether it would remain negligible under other scenarios. To prove these results, we decomposed the difference between the theoretical counterparts of the two indexes in two main terms: the dividend timing difference and the total early exercise premium. The former is due to

Table 4. *TSPIKES*^{2} -*TVIX*^{2} (in bp) computed with the closed form approximation formula when dividends are paid during the lifetime of SPY options at time $T_D = 1, 2, 3$ *weeks*.

TSPIKES^2 -*TVIX*^{2} Closed Form	r	q	$T_D = 1$ week	$T_D = 2$ weeks	$T_D = 3$ weeks
	0.01	0.01	0.53	0.54	0.59
	0.01	0.03	1.09	1.51	2.03
	0.01	0.05	2.36	3.46	4.52
	0.01	0.1	9.36	12.53	14.92
	0.03	0.01	1.30	1.22	1.09
	0.03	0.03	1.83	2.11	2.46
	0.03	0.05	3.04	3.98	4.90
	0.03	0.1	9.75	12.83	15.16
	0.05	0.01	2.02	1.84	1.54
	0.05	0.03	2.53	2.64	2.83
	0.05	0.05	3.68	4.43	5.21
	0.05	0.1	10.12	13.08	15.33
	0.1	0.01	3.74	3.29	2.59
	0.1	0.03	4.24	3.91	3.66
	0.1	0.05	5.27	5.50	5.88
	0.1	0.1	11.10	13.63	15.66

the different dividend payment scheme for SPX and SPY which are the underlying assets of the constituent options, respectively, for VIX and SPIKES, and it is only dependent on the value of the annualized dividend rate q. The latter gathers the early exercise premiums arising from the American exercise style of SPY options against the European style of SPX options. The contribution of Calls can be computed in close form by applying the outcomes in Villiger (2006); we derive in this paper an analogous closed-form expression for the early exercise of a put option and for the aggregate contribution of put options on SPIKES. In the numerical illustration we show that the new SPIKES index

Table 5. Separate contributions of Calls EEP, Puts EEP, and the dividend timing to *TSPIKES*^{2} -*TVIX*^{2} (in bp) , when a single dividend is paid during SPY options' lifetime in $T_D = 3$ *weeks*.

r	q	Call Total Premium	Put Total Premium	Dividend Timing
0.01	0.01	0.17	0.36	0.06
0.01	0.03	1.11	0.36	0.56
0.01	0.05	2.63	0.36	1.54
0.01	0.1	8.52	0.36	6.05
0.03	0.01	0.11	0.92	0.06
0.03	0.03	0.98	0.92	0.56
0.03	0.05	2.44	0.92	1.54
0.03	0.1	8.19	0.92	6.05
0.05	0.01	0.06	1.42	0.06
0.05	0.03	0.86	1.42	0.56
0.05	0.05	2.25	1.42	1.54
0.05	0.1.	7.86	1.42	6.05
0.1	0.01.	0.00	2.53	0.06
0.1	0.03	0.58	2.53	0.56
0.1	0.05	1.82	2.53	1.54
0.1	0.1	7.08	2.53	6.05

will continue tracking the VIX index as long as 30-day US interest rates and annualized dividend yields continue to be range bound between 0 and 10% per year; we also compare the (true) difference computed by applying the finite difference method to solve the model PDE with the closed-form approximations proposed in this paper. We did not consider negative interest rates in the numerical application; indeed, both VIX and SPIKES are quoted in the United States where interest rates have remained positive to date and, most likely, should stay positive in the future. Even during the recent pandemic, the US Fed has refrained from pushing its benchmark rates below zero. Nevertheless, the theory tells us that in case of negative interest rates the contribution of American Calls to the total early exercise premium would increase as it may become convenient, under such circumstances, to exercise a call option before its expiration. However, Puts would never be exercised early and the contribution of American put options to the early exercise premium would vanish. Hence, we would not expect substantial changes to the overall difference between *TSPIKES* and *TVIX* if the interest rate r was assumed negative. We have also evidenced how the impact of the dividend rate is higher than that of the interest rate, since it affects both the dividend timing difference and the total EEP of Calls. Notably, during the historical period analysed in this paper the dividend rates for the SPY ETF have remained around 2% but for a single peak at 3.3%, as illustrated in Figure 10. Overall, we expect the gap between SPIKES and VIX to increase substantially only in case of an extremely positive performance of the stocks underlying the SPY ETF.

Infact, we believe that the prices of near and next maturity OTM options used to calculate both SPIKES and VIX respond primarily to the volatility of the underlying, along with interest rates and dividends. Future research can investigate whether the sensitivity of $TSPIKES^2$ and $TVIX^2$ with respect to the volatility parameter σ differs substantially, in line with the approach in Demeterfi et al. (1999). Given (5), it suffices to compute the derivative of the total EEP $\epsilon^x(Y, q)$ with respect to σ. This is easily approximated in closed form by computing the corresponding derivatives of the expressions for the Total EEP for Calls (Villiger, 2006) and that of the Total EEP for Puts given

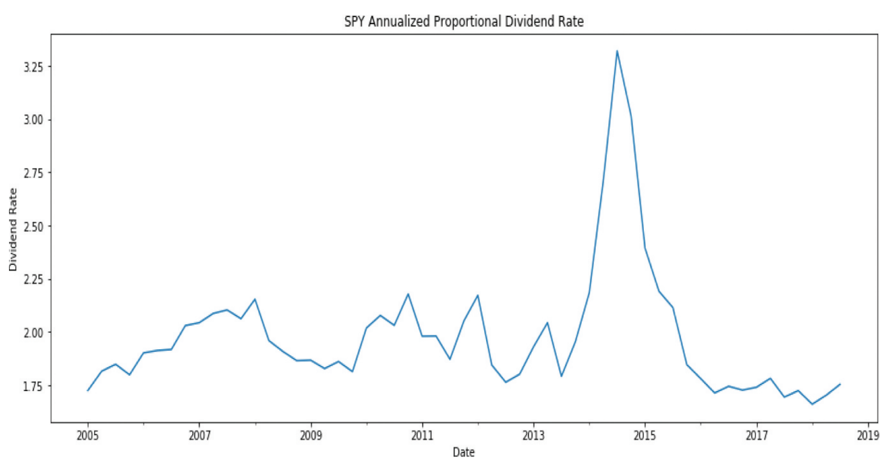

Figure 10. Historical values of the SPY dividend rate q, from 2005 to 2018.

in (16) or (18), whether ot not a discrete dividend is paid by the SPY ETF during the lifetime of the options. Preliminary results indicate that the numerical difference between the two Vegas is small, but not always negligible.

It is also worth to investigate whether the conclusions in this paper are robust to the assumptions that the interest rate, the dividend yield, and the variance rate are constant. When all three of these rates are allowed to be stochastic, the EEP should rise due to the extra volatility value arising from the optionality in American options and so should the gap between SPIKES and VIX. We leave this analysis to future research.

Note

1. According to the fund's prospectus, the SPDR S&P 500 ETF puts all dividends it receives from its underlying stocks into a non-interest-bearing account until the end of each quarter when they are distributed proportionally to the investors.

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Appendix

Appendix A. Proof of Theorem 4.1.

In this appendix we develop the necessary steps to approximate the exercise boundary and hence the EEP for the American-style Put options considered in the SPIKES Index computation. These are written on an underlying paying a single discrete proportional dividend qY at the dividend date $T_D < T$, being T their expiration date. It is worth noticing that the dividend date plays a crucial role in the approximation of the early exercise boundary for the options, which jumps at time T_D. Indeed, after time T_D, the option reduces to an American Put written on a non-dividend paying asset.

The proof is given by proceeding in steps:

(1) we restrict our attention to the time interval $(T_D, T]$, where no dividends are due, and consider a constant boundary L by setting $g = 0$ in (11).
(2) we apply a change of probability measure to derive a closed form formula when the exercise boundary is approximated by an exponentially growing boundary, $g \neq 0$, once again focusing on the time interval after the dividend date T_D;
(3) we generalize the outcomes of previous steps for approximating the exercise boundary and computing the EEP of an American-style Put for $t < T_D$.

Note that in the first two steps we assume that the early exercise boundary is flat at the level 0 for $t \leq T_D$. This corresponds to approximating the price of the American Put option with that of a hybrid American Put which can be exercised at any time in $(T_D, T]$.

Step 1 Since there are no dividends in the time interval $(T_D, T]$, the early exercise premium for the hybrid Put is just the present value of the interest earned on the strike price, while the stock price is below the early exercise boundary, as shown in Carr, Jarrow, and Myneni (1992), which we assume to be constant at L in the time interval$(T_D, T]$.

Hence, the price P^H at time $t < T_D$ of the hybrid Put is given by:

$$P^H(t, Y_t, T, K_p; L) = p(t, Y_t, T, K_p, L) \tag{A1}$$

$$+ e^{-r(T_D - t)}\mathbb{E}^Q\left[rK_p \int_{T_D}^{T} e^{-r(u-T_D)}\mathbf{1}_{\{Y_u < Y_p(u)\}}du\right]. \tag{A2}$$

The critical stock price at time T_D is defined as the level for Ysuch that the continuation value of the hybrid Put equals its exercise value; if we further assume that the exercise boundary is constant at this critical stock price after time T_D, then L implicitly solves:

$$P^H(T_D, L, T, K; L) = K - L. \tag{A3}$$

We know that when the underlying asset is non-dividend paying, the value at time $s \in (T_D, T]$, of interest on the strike while $Y < L$ is

$$\mathbb{E}_s^Q\left[rK\int_s^T e^{-r(u-s)}\mathbf{1}_{\{Y_u < L\}}du\right]. \tag{A4}$$

Consider a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with $f \in C^2$. By applying Ito's Lemma in integral form we get

$$\begin{aligned}
e^{-rT}f(Y_T) &= f(Y_s) + \int_s^T e^{-r(u-s)}f'(Y_u)dY_u \\
&\quad + \int_s^T e^{-r(u-s)}[\frac{f''(Y_u)}{2}\sigma^2 Y_u^2 - rf(Y_u)]du \\
&= f(Y_s) + \int_s^T e^{-r(u-s)}f'(Y_u)[dY_u - rY_u du] \\
&\quad + \int_s^T e^{-r(u-s)}[\frac{f''(Y_u)}{2}\sigma^2 Y_u^2 + rY_u f'(Y_u) - rf(Y_u)]du
\end{aligned}$$

and, taking conditional expectation at time s we get,

$$e^{-r(T-s)}\mathbb{E}_s^Q[f(Y_T)] = f(Y_s) + \mathbb{E}_s^Q\left[\int_s^T e^{-r(u-s)}[\frac{f''(Y_u)}{2}\sigma^2 Y_u^2 + rY_u f'(Y_u) - rf(Y_u)]du\right]. \tag{A5}$$

Hence, the value of the accrued interest on the strike for $Y < L$ may be written, in T_D as

$$\mathbb{E}_{T_D}^Q\left[rK\int_{T_D}^T e^{-r(u-T_D)}\mathbf{1}_{\{Y_u < L\}}du\right] = f(Y_{T_D}) - e^{-r(T-T_D)}\mathbb{E}_{T_D}^Q[f(Y_T)], \tag{A6}$$

if and only if there exists a function $f(Y)$ solving the following ordinary differential equation (ODE):

$$\frac{f''(Y)}{2}\sigma^2 Y^2 + rYf'(Y) - rf(Y) = -rK\mathbf{1}_{\{Y < L\}}. \tag{A7}$$

It is straightforward to show that a solution exists for any constant L, and it is given by

$$f(Y; L) = \frac{K_p}{r + \frac{\sigma^2}{2}}\left[\mathbf{1}_{\{Y < L\}}\left(r + \frac{\sigma^2}{2} - r\frac{Y}{L}\right) + \mathbf{1}_{\{Y > L\}}\frac{\sigma^2}{2}\left(\frac{Y}{L}\right)^{-2r/\sigma^2}\right]. \tag{A8}$$

The function $f(Y; L)$ can be seen as a final payoff at T whose time value at s matches the value of the interest earned on the strike price K when $Y < L$ between s and T. Finally, by discounting and applying the tower property for conditional expectation, the price of the hybrid Put with constant boundary at time t, for $0 \leq t \leq T_D$, is given

$$P^H(t, Y_t, T, K_p; L) = p(t, Y_t, T, K_p) + e^{-r(T_D-t)}\mathbb{E}_t^Q\left[f(Y_{T_D}; L) - e^{-r(T-T_D)}f(Y_T; L)\right]. \quad \text{(A9)}$$

To solve for L, we must compute the above price at time T_D i.e. conditioning on the information available at time T_D and then solve (A3). Note that we can write $L = bK_p$ for some positive constant b.

The early exercise premium in T_D for the hybrid Put can be computed explicitly:

$$EEP^H(T_D, T, K_p) = \mathbb{E}_{T_D}^Q\left[f(Y_{T_D}; L) - e^{-r(T-T_D)}f(Y_T; L)\right]$$

$$= \frac{K_p}{r + \frac{\sigma^2}{2}}\left[(r + \frac{\sigma^2}{2})\left[\mathbf{1}_{\{Y_{T_D} < L\}} - e^{-r(T-T_D)}N(-d_2(T_D, Y_{T_D}, T, L))\right]\right.$$

$$\left. - \frac{rY_{T_D}}{L}\left[\mathbf{1}_{\{Y_{T_D} < L\}} - N(-d_1(T_D, Y_{T_D}, T, L))\right]\right]$$

$$+ \frac{K}{r + \frac{\sigma^2}{2}}\frac{\sigma^2}{2}\left(\frac{Y_{T_D}}{L}\right)^{\frac{-2r}{\sigma^2}}\left[\mathbf{1}_{\{Y_{T_D} > L\}} - N\left(d_2(T_D, Y_{T_D}, T, L) - \frac{2r}{\sigma}\sqrt{T - T_D}\right)\right]. \quad \text{(A10)}$$

Finally, the early exercise premium at time $t = 0$ for the hybrid Put is simply the discounted value of the premium in (A10):

$$EEP^H(0, T, K_p) = K_p\left[e^{-rT_D}N(-d_2(0, Y_0(1-q), T_D, L)) - e^{-r(T-T_D)}N(-d_2(0, Y_0(1-q), T, L))\right]$$

$$- \frac{K_p}{r + \frac{\sigma^2}{2}}\frac{rY_0(1-q)}{L}[N(-d_1(0, Y_0(1-q), T_D, L)) - N(-d_1(0, Y_0(1-q), T, L))]$$

$$+ \frac{K_p}{r + \frac{\sigma^2}{2}}\frac{\sigma^2}{2}\left(\frac{Y_0(1-q)}{L}\right)^{\frac{-2r}{\sigma^2}}\left[N\left(d_2(0, Y_0(1-q), T_D, L) - \frac{2r}{\sigma}\sqrt{T_D}\right)\right.$$

$$\left. - N\left(d_2(0, Y_0(1-q), T, L) - \frac{2r}{\sigma}\sqrt{T}\right)\right],$$

so the price at time $t = 0$ for the hybrid Put is given by:

$$P^H(0, Y_0, T, K_p; L) = p(0, Y_0, T, K_p) + EEP^H(0, T, K_p). \quad \text{(A11)}$$

Step 2

As an improved approximation for the early exercise premium and the price of the hybrid American option introduced above, we let the exponential growth coefficient g be non-zero in (11). The boundary is again assumed to be flat at 0 before the dividend date T_D.

Several methods are available to obtain the constants L, g specifying the approximating exponential function. Here, we obtain these constants by imposing *value matching* conditions at times T_D and T, according to the outcomes in Ju (1998).

We extend the approach used in Step 2 and define a function $\tilde{f}$ which represents the payoff flow of the EEP in the time span $(T_D, T]$.

Define the auxiliary process $\tilde{Y}_t := Y_t e^{-gt}$. Under the risk-neutral probability measure Q, the price changes of $\tilde{Y}$ are described by:

$$d\tilde{Y}_t = (r-g)\tilde{Y}_t dt + \sigma\tilde{Y}_t dW_t - \delta(t-T_D)D_t dt.$$

We can use a similar approach to that of a constant boundary and find a function $\tilde{f}$ such that

$$\mathbb{E}_t^Q\left[rK_p\int_t^T e^{-r(u-t)}\mathbf{1}_{\{\tilde{Y}_u<L\}}du\right] = \tilde{f}(\tilde{Y}_t) - e^{-r(T-t)}\mathbb{E}_t^Q\left[\tilde{f}(\tilde{Y}_T)\right], \tag{A12}$$

which is obtained by solving:

$$\frac{\tilde{f}''(\tilde{Y})}{2}\sigma^2\tilde{Y}^2 + (r-g)\tilde{Y}\tilde{f}'(\tilde{Y}) - r\tilde{f}(\tilde{Y}) = -rK\mathbf{1}_{\{Y<L\}}. \tag{A13}$$

The solution is:

$$\tilde{f}(\tilde{Y};L,g) = \frac{2rK_p}{\sigma^2(p_+-p_-)}\left[\mathbf{1}_{\{\tilde{Y}<L\}}\left(\frac{1}{p_+}-\frac{1}{p_-}-\frac{1}{p_+}\left(\frac{\tilde{Y}}{L}\right)^{p_+}\right) - \mathbf{1}_{\{\tilde{Y}>L\}}\frac{1}{p_-}\left(\frac{\tilde{Y}}{L}\right)^{p_-}\right], \tag{A14}$$

where p_+, p_- are the roots of $\mathcal{P}(p) = \frac{\sigma^2}{2}p^2 + (r-g-\frac{\sigma^2}{2})p - r$.

Note that once this new function is computed, the case of an exponential boundary for the underlying stock Y is equivalent to that of a constant boundary for the process $\tilde{Y}$.

Hence, as in the previous subsection, we get

$$EEP^H(T_D,T,K) = \mathbb{E}_{T_D}^Q\left[\tilde{f}(\tilde{Y}_{T_D}) - e^{-r(T-T_D)}\tilde{f}(\tilde{Y}_T)\right] =$$

$$\frac{2rK_p}{\sigma^2(p_+-p_-)}\left[\left(\frac{1}{p_+}-\frac{1}{p_-}\right)\left[\mathbf{1}_{\{\tilde{Y}_{T_D}<L\}} - e^{-r(T-T_D)}N\left(-d_2\left(T_D,\tilde{Y}_{T_D},T,L\right)\right)\right]\right.$$

$$\left.-\frac{1}{p_+}\left(\frac{\tilde{Y}_{T_D}}{L}\right)^{p_+}\left[\mathbf{1}_{\{\tilde{Y}_{T_D}<L\}} - N\left(-d_2\left(T_D,\tilde{Y}_{T_D},T,L\right) - p_+\sigma\sqrt{T-T_D}\right)\right]\right]$$

$$-\frac{2rK_p}{\sigma^2(p_+-p_-)}\frac{1}{p_-}\left(\frac{\tilde{Y}_{T_D}}{L}\right)^{p_-}\left[\mathbf{1}_{\{\tilde{Y}_{T_D}>L\}} - N\left(d_2\left(T_D,\tilde{Y}_{T_D},T,L\right) + p_-\sigma\sqrt{T-T_D}\right)\right],$$

and similarly we obtain the early premium at time $t = 0$ by computing its discounted value and the corresponding hybrid option price.

Step 3 We finally consider the case of an American option which can be exercised throughout its lifetime $[0,T]$. This can be done by applying the outcomes obtained in Step 2 for different exponential approximating functions before and after the dividend date T_D, as defined in (11), and considering the cum-dividend and the ex-dividend price processes respectively. The challenge here is to obtain the constant values C, h for which we cannot apply the same method as above. By looking at Figure 5 and Figure 6, it is evident that the exercise boundary is essentially linear before time T_D and vanishes at time T_D. We want the exponential approximation before time T_D to emulate the linear behaviour displayed by the true boundary computed numerically; since $T_D < T \approx 1/12$, this is easily achieved by setting the constant h in (11) to a small value. Furthermore, we surmise from the pictures that the intercept is approximately at the level $\frac{r}{q}KT_D$. A linear boundary approximation with the above properties is given by $y = \frac{r}{q}K(T_D - t)$, which can be in turn approximated by an exponential function with $C = \frac{r}{q}KT_D$ and $h = \frac{1}{T_D}$. The linear boundary may also be calibrated by matching the average slope of the area under the linear and exponential boundary approximations; the results are qualitative analogous. Once the values for C, h are computed, then the overall early exercise premium for an American Put with strike K_p and expiration date T, can be computed by summing the two early exercise premiums for the first and second part of the approximated boundary. Precisely,

$$\begin{aligned} EEP(0,T,K) = &\tilde{f}(Y_0^c; C, h) - e^{-rT_D}\mathbb{E}^Q\left[\tilde{f}(Y_{T_D}^c e^{hT_D}; C, h)\right] \\ &+ e^{-rT_D}\mathbb{E}^Q\left[\tilde{f}(Y_{T_D}^x e^{-gT_D}; L, g)\right] \\ &- e^{-rT}\mathbb{E}^Q\left[\tilde{f}(Y_T^x e^{-gT}; L, g)\right], \end{aligned} \tag{A15}$$

where the expected values are calculated using the cum-dividend initial price for the first two terms and the ex-dividend price for the others. This finally proves Theorem 4.1.