Going with the flow

Contingent cashflows that depend on one or more stochastic variables are a ubiquitous part of the financial landscape. As is well known, the options implicit in such cashflows can be hedged either dynamically or statically, but each technique has its drawbacks. Here, Peter Carr, Alex Lipton and Dilip Madan propose a useful hybrid hedging technique

Many valuation problems arising in modern finance involve variable cashflows paid frequently over time. These cashflows are often a known function of the price of some traded asset. For example, projects modelled as real options may involve revenue and/or cost streams that are a function of the price of a commodity. Alternatively, we may consider credit-sensitive bonds that have a coupon rate that explicitly depends on the issuer's credit rating. If this rating is modelled as a known function of the firm's asset value or stock price, then the interest payments depend on a traded asset's price as required. Similarly, if a firm's equity is modelled as a call option on the firm's assets, then over long horizons, the dividends on the stock may be approximated as occurring continuously over time, and will probably depend on the firm's asset value or stock price. In particular, if either variable drops below some threshold, dividend payments are likely to be suspended.

There are many other examples in which the cashflows of the annuity vanish when the underlying's price is in some range. For example, the discount to apply to a corporate bond or preferred stock due to suspension of payments when the firm is in jeopardy can be modelled as the value of the cashflows lost when the asset value or stock price is below a boundary. Similarly, the early exercise premium of an American-style put capitalises the excess of an interest flow over a dividend flow when the underlying is below the early exercise boundary. Conversely, exchange rate restrictions may induce firms to repatriate funds on a daily basis, so long as the rate exceeds a floor. Finally, corridor notes sold over-the-counter pay a fixed amount for each day an underlying asset spends in a corridor.

The classical Black-Scholes (1973)/Merton (1973) approach for hedging such path-dependent securities involves dynamic trading in the underlying asset. Ross (1976) and Breeden & Litzenberger (1978) initiated an alternative approach that considers static positions in standard options as a second approach for valuing derivatives. When feasible, static hedging has the advantage of generating explicit valuation formulas in terms of option prices, and does not require specifying the stochastic process for the underlying. However, to replicate a variable annuity with continuous cashflows up to some fixed maturity, static hedging requires using options of all strikes and of all maturities up to that of the annuity.

This paper aims to explore a third approach for hedging contingent claims with continuous cashflows. We call this approach enhanced delta hedging (EDH), as standard dynamic trading in the underlying asset is enhanced by static positions in options. When compared with static hedging, EDH requires options of only a single maturity, but also requires specifying the stochastic process of the underlying. Compared with dynamic hedging, EDH has the advantage of permitting explicit valuation formulas and hedging strategies for fairly general specifications on the volatility and carrying costs of the underlying. More specifically, we show that when the payout on the variable annuity is deferred without interest to maturity, explicit solutions are possible whenever volatility is a known positive function of the underlying price. When payouts are not deferred, cashflows linked to the spot price can still be valued explicitly for arbitrary volatility functions, so long as dividends are modelled as constant over time. In contrast, non-deferred payouts linked to the futures price cannot be valued explicitly in closed form for an arbitrary volatility function. However, many commonly used volatility specifications do yield closed-form solutions.

Model set-up

We assume frictionless markets and consider a claim maturing at T with continuous cashflows, which are a given function $g(\cdot)$ of the price of some underlying asset. This price may be a spot price or it may be a futures or forward price for delivery at date $T' \geq T$. To deal with all three cases jointly, we let U_t denote the price of the underlying asset. We assume the risk-free rate is constant at r, thus, claims linked to forward prices are identical to the corresponding claim linked to futures prices. If the cashflows are paid out continuously and invested at r, then the random cumulative value at the maturity date T is $\int_0^T e^{r(T-t)} g(U_t) dt$. Since it is costly in practice to pay out cashflows continuously, we allow for the possibility that the cashflows on the contract are paid out in a single lump sum at the maturity date T. In this case, we let r_c denote the constant contracted rate at which cashflows earn interest between the time they are realised and the single payout date T. Thus, the only payment on these contracts is $\int_0^T e^{r_c(T-t)} g(U_t) dt$, which is paid at the maturity date T. Note that by setting $r_c = r$, we capture the possibility that cashflows are paid out continuously. However, since many contracts, such as corridor notes, defer the payout to maturity without interest, we set $r_c = 0$ to cover this possibility.

We assume that the cost of carrying the underlying asset over time is a function $r_u(\cdot)$ of the underlying asset price U. If the cashflows are a function of a futures price F_t, then $r_u(F) = 0$, since futures contracts are cost-free. If the cashflows are a function of the spot price S_t, and if dividends on the stock are paid continuously, then the cost of carrying the underlying is the risk-free rate less the dividend yield, where the latter can be an arbitrary function $\delta(S)$ of the spot price.

Suppose that the price process of the underlying is given by the following stochastic differential equation (SDE)[1]:

$$\frac{dU_t}{U_t} = \alpha(t, U.)dt + \sigma(t, U.)dW_t, \qquad t \in [0, T] \tag{1}$$

where W is a standard Brownian motion on the line. Thus, under the original probability measure, the expected growth rate α and volatility σ can depend on the time t and on the price path $U.$ up to this time. If the origin is attainable and regular, we impose an absorbing boundary condition there.

The continuity of the SDE allows the claim described above to be perfectly hedged by dynamically trading in the underlying. In an effort to obtain closed-form solutions for the theoretical value that this strategy engenders, we now impose the restrictive assumption that the volatility at time t depends only on the contemporaneous price:

[1] *Letting* $b(t, x.) \equiv x\alpha(t, x.)$ *and* $a(t, x.) \equiv x\sigma(t, x.)$, *the existence of a strong solution requires that* b *and* a *be Lipschitz, ie, there exists some* $K < \infty$ *such that:*

$$\left|b(t, x.) - b(t, y.)\right| \leq K\left|x - y\right|_t^* \qquad \left|a(t, x.) - a(t, y.)\right| \leq K\left|x - y\right|_t^*$$

for every $t \in [0, T]$ *and* $x \in \Re$, $y \in \Re$, *where* $f_t^* \equiv \sup\{|f(s)| : s \leq t\}$. *It also requires that for each constant* $T > 0$, *there is some* C_T *such that* $|a(s, 0)| + |b(s, 0)| \leq C_T$ *for all* $s \leq T$ *(see Rogers & Williams, 1987, page 136, for an excellent exposition)*

$$\sigma(t, U_\cdot) = \sigma(U_t)$$

Then by a minor generalisation of the results in Merton (1974), the claim value $V(U, t)$ solves the following partial differential equation (PDE):

$$\frac{\sigma^2(U)U^2}{2}\frac{\partial^2 V}{\partial U^2}(U,t) + r_u(U)U\frac{\partial V}{\partial U}(U,t) - rV(U,t)$$
$$+\frac{\partial V}{\partial t}(U,t) + e^{(r_c - r)(T-t)}g(U) = 0, \qquad U > 0, t \in [0,T]$$

subject to the following terminal condition $V(U, T) = 0$:

It is also well known that by the Feynman-Kac theorem, the solution[2] has the following risk-neutral representation:

$$V(U,t) = e^{-r(T-t)}\int_t^T e^{r_c(T-u)}E\left[g(U_u)\middle|U_t = U\right]du \tag{2}$$

where expectations are evaluated under the risk-neutral process:

$$\frac{dU_t}{U_t} = r_u(U_t)dt + \sigma(U_t)dW_t, \qquad t \in [0,T]$$

Unfortunately, there is no general solution for this expectation for an arbitrary carrying cost $r_u(U)$ or volatility function $\sigma(U)$.

One solution to this problem is to assume the existence of a complete term and strike structure of European-style options. It is well known (see Breeden & Litzenberger, 1978, Green & Jarrow, 1987, Nachman, 1988, and Ross, 1976) that a complete strike structure of options maturing at t allows one to statically replicate a continuous payout $g(U_t)$ occurring at t. In particular, Carr & Madan (2000) show that any C^2 payout[3] decomposes into the payouts from static positions in bonds and options maturing at t:

$$g(U_t) = g(\kappa_t) + g'(\kappa_t)\left[(U_t - \kappa_t)^+ - (\kappa_t - U_t)^+\right]$$
$$+\int_0^{\kappa_t -} g''(K)(K - U_t)^+ dK + \int_{\kappa_t +}^{\infty} g''(K)(U_t - K)^+ dK \tag{3}$$

where κ_t is an arbitrary deterministic process. In particular, the investor holds $g(\kappa_t)$ bonds, $g'(\kappa_t)$ synthetic forwards, $g''(K)dK$ puts for all strikes $K < \kappa_t$ and $g''(K)dK$ calls for all strikes $K > \kappa_t$.

It follows that if investors can also take positions in all option maturities from zero to T, then they can statically create the payout $\int_0^T e^{r_c(T-t)}g(U_t)dt$ occurring at T. In particular, multiplying equation (3) by $e^{r_c(T-t)}$ and integrating over t gives:

$$\int_0^T e^{r_c(T-t)}g(U_t)dt = \int_0^T e^{r_c(T-t)}g(\kappa_t)dt$$
$$+\int_0^T e^{r_c(T-t)}g'(\kappa_t)\left[(U_t - \kappa_t)^+ - (\kappa_t - U_t)^+\right]dt$$
$$+\int_0^T e^{r_c(T-t)}\left[\int_0^{\kappa_t -} g''(K)(K - U_t)^+ dK + \int_{\kappa_t +}^{\infty} g''(K)(U_t - K)^+ dK\right]dt$$

Recognising that the options mature at t rather than T, one can value the cashflow by:

$$V_0 = e^{-rT}\int_0^T e^{r_c(T-t)}g(\kappa_t)dt$$
$$+\int_0^T e^{(r_c - r)(T-t)}g'(\kappa_t)\left[C_0(\kappa_t, t) - P_0(\kappa_t, t)\right]dt$$
$$+\int_0^T\int_0^{\kappa_t -} e^{(r_c - r)(T-t)}g''(K)P_0(K,t)dKdt \tag{4}$$
$$+\int_0^T\int_{\kappa_t +}^{\infty} e^{(r_c - r)(T-t)}g''(K)C_0(K,t)dKdt$$

where $C_0(K, t)$ and $P_0(K, t)$ denote the respective initial prices of European-style calls and puts of strike K and maturity t. In contrast to the results for dynamic replication, this valuation formula is always explicit (in terms of option prices), and holds for an arbitrary underlying price process.

Unfortunately, the term structure we find in listed options markets is far from continuous. The next section in this article develops an EDH strategy that also permits explicit valuation formulas, but only assumes that options of maturity T are available. To obtain our results, we assume the continuous price process described by equation (1) and use dynamic trading in the underlying to supplement the strike structure of the single maturity T.

Enhanced delta hedging

To develop the enhanced delta hedge of a claim with the single payout $\int_0^T e^{r_c(T-t)}g(U_t)dt$ occurring at T, apply Itô's lemma to the product of a twice differentiable function $f(U_t)$ and the function of time $e^{r_c(T-t)}$:

$$f(U_T) = f(U_0)e^{r_cT} + \int_0^T e^{r_c(T-t)}f'(U_t)dU_t$$
$$+\int_0^T e^{r_c(T-t)}\left[f''(U_t)\frac{\sigma^2(U_t)U_t^2}{2} - r_cf(U_t)\right]dt \tag{5}$$

Now subtract the carrying cost $r_u(U_t)U_tdt$ from the stochastic integrator:

$$f(U_T) = f(U_0)e^{r_cT} + \int_0^T e^{r_c(T-t)}f'(U_t)\left[dU_t - r_u(U_t)U_tdt\right]$$
$$+\int_0^T e^{r_c(T-t)}\left\{\frac{\sigma^2U_t^2}{2}f''(U_t) + r_u(U_t)U_tf'(U_t) - r_cf(U_t)\right\}dt \tag{6}$$

Suppose we choose $f(S)$ to solve the following ordinary differential equation (ODE):

$$\frac{\sigma^2(U)U^2}{2}f''(U) + r_u(U)Uf'(U) - r_cf(U) = g(U) \tag{7}$$

Then, substituting equation (7) in equation (6) and rearranging gives:

$$\int_0^T e^{r_c(T-t)}g(U_t)dt = f(U_T) - f(U_0)e^{r_cT}$$
$$-\int_0^T e^{r_c(T-t)}f'(U_t)\left[dU_t - r_u(U_t)U_tdt\right] \tag{8}$$

Recall the static decomposition (3) applied to the function $f(U)$:

$$f(U_T) = f(\kappa) + f'(\kappa)\left[(U_T - \kappa)^+ - (\kappa - U_T)^+\right]$$
$$+\int_0^{\kappa-} f''(K)(K - U_T)^+ dK + \int_{\kappa+}^{\infty} f''(K)(U_T - K)^+ dK$$

where $\kappa = \kappa_T$. Suppose we also require that f has zero value and slope at κ:

$$f(\kappa) = f'(\kappa) = 0 \tag{9}$$

Then the decomposition simplifies into the following static position in options:

$$f(U_T) = \int_0^{\kappa-} f''(K)(K - U_T)^+ dK + \int_{\kappa+}^{\infty} f''(K)(U_T - K)^+ dK \tag{10}$$

Substituting equation (10) in equation (8) gives the following representation of the desired payout:

$$\int_0^T e^{r_c(T-t)}g(U_t)dt = -f(U_0)e^{r_cT} + \int_0^{\kappa-} f''(K)(K - U_T)^+ dK$$
$$+\int_{\kappa+}^{\infty} f''(K)(U_T - K)^+ dK - \int_0^T e^{r_c(T-t)}f'(U_t)\left[dU_t - r_u(U_t)U_tdt\right] \tag{11}$$

The first term on the right-hand side of equation (11) is the payment required to cover a short position in $f(U_0)e^{r_cT}$ bonds, each paying a dollar at T. The next two terms on the right-hand side add up to the payout from being long $f''(K)dK$ puts for all strikes $K < \kappa$ and long $f''(K)dK$ calls for all strikes $K > \kappa$, with all options maturing at T. The last term is the cumulative gains from a dynamic strategy in $e^{(r_c - r)(T-t)}f'(U_t)$ units of the underlying asset at each $t \in [0, T)$, where all purchases are financed by borrowing at the risk-free rate r and all sales earn interest at r. Thus, the desired payout is the sum of a static position in bonds and options of all strikes and a dynamic strategy in the underlying asset. Since the cost of financing the dynamic strategy has already been accounted for, the initial value of the payout $\int_0^T e^{r_c(T-t)}g(U_t)dt$ at T is given by:

$$V_0 = -f(U_0)e^{(r_c - r)T} + \int_0^{\kappa-} f''(K)P_0(K,T)dK + \int_{\kappa+}^{\infty} f''(K)C_0(K,T)dK \tag{12}$$

where f solves the ODE (7).

The decomposition (12) underlying EDH is important, even if options

[2] *We thank a referee for pointing out that this solution can also be expressed in terms of the resolvent operator G_{r_c} associated with the risk-neutral diffusion process for U. See Rogers & Williams (1984), chapter three, for an excellent introduction to resolvents*

[3] *The smoothness requirements on g in equation (3) can be weakened by using generalised functions*

are not used in the hedge. The decomposition can be used to transfer intuition on options to the variable annuity and to reduce the valuation problem for the annuity down to determining the expected value of a terminal payout. Thus, our decomposition makes the annuity valuation problem equivalent to the problem of valuing a standard option, which is in turn equivalent to solving for the transition density. For many valuation environments, the valuation of these claims is quite likely to have already been implemented. These benefits accrue because the decomposition in equation (12) eliminates the time integral that arises in the valuation formulas (2) and (4), derived via dynamic and static replication respectively. Thus, a problem which *a priori* appears to require an integration across both time and space reduces to a single integration across space. Hence, this result is the financial equivalent of Green's theorem, which analogously reduces a surface integral to a line integral.

Deferred payouts

Our valuation equation (12) only becomes explicit once we solve the ODE (7) for f. This section shows that this ODE can always be solved whenever the payout on the annuity is deferred without interest to maturity, ie, $r_c = 0$. The next section presents the corresponding analysis for non-deferred cashflows, ie, $r_c = r$.

□ **Deferred payout on futures.** Suppose the cashflow at t is a given function $g(\cdot)$ of the futures price F_t for delivery at $T' \geq T$. Thus we assume the diffusion process (1) for the futures price and that there exists an entire strike structure of European-style futures options maturing at T. Since listed futures options are usually American-style, the latter assumption is a potential obstacle to EDH. One approach is to ignore the American-style feature, since futures options are typically only optimally exercised near maturity. A better approach available to the issuer of the variable annuity is to set $T' = T$ in the definition of the variable annuity and hedge with listed European-style options on spot instead. Another alternative is to link the cashflows to the spot price, as is done in the next subsection of this article. Since futures contracts are cost-free, the cost of carrying the underlying is $r_u(F) = 0$. Since this section assumes that the cashflows are deferred to maturity without interest, we also set $r_c = 0$. Replacing U_t with F_t and substituting $r_u = r_c = 0$ in equations (11) and (7) gives:

$$\int_0^T g(F_t)dt = -f(F_0) + \int_0^{\kappa-} f''(K)(K - F_T)^+ dK + \int_{\kappa+}^{\infty} f''(K)(F_T - K)^+ dK - \int_0^T f'(F_t)dF_t \tag{13}$$

where $f(F)$ solves the one-dimensional Poisson equation:

$$f''(F) = \frac{2g(F)}{\sigma^2(F)F^2} \tag{14}$$

Integrating once gives the first derivative:

$$f'(F) = \int_\kappa^F \frac{2g(x)}{\sigma^2(x)x^2}dx \tag{15}$$

where for $b > a$, $\int_b^a h(x)dx \equiv -\int_a^b h(x)dx$. The lower limit of the integral has been chosen to be consistent with equation (9). Similarly, integrating once more gives:

$$f(F) = \int_\kappa^F \int_\kappa^y \frac{2g(x)}{\sigma^2(x)x^2}dxdy \tag{16}$$

Substituting equation (14) to (16) in (13) gives an explicit representation of the payout:

$$\int_0^T g(F_t)dt = -\int_\kappa^{F_0}\int_\kappa^y \frac{2g(x)}{\sigma^2(x)x^2}dxdy + \int_0^{\kappa-}\frac{2g(K)}{\sigma^2(K)K^2}(K - F_T)^+ dK + \int_{\kappa+}^{\infty}\frac{2g(K)}{\sigma^2(K)K^2}(F_T - K)^+ dK - \int_0^T \int_\kappa^{F_t}\frac{2g(x)}{\sigma^2(x)x^2}dxdF_t \tag{17}$$

Although this decomposition appears complicated, we note that the integrals determining f and f' in equations (16) and (15) can often be found in closed form, since the payout rate g is likely to be a simple function.

The decomposition (17) once again indicates that the desired payout is the sum of a static position in bonds and options of all strikes combined with a dynamic strategy in the underlying futures. Assuming that marking-to-market occurs continuously, the dynamic strategy involves holding:

$$-e^{-r(T-t)}\int_\kappa^{F_t}\frac{2g(x)}{\sigma^2(x)x^2}dx$$

futures contracts at each $t \in [0, T]$. Since futures contracts are cost-free, the initial value of the payout $\int_0^T g(F_t)dt$ at T is explicitly given by:

$$V_0 = -e^{-rT}\int_\kappa^{F_0}\int_\kappa^y \frac{2g(x)}{\sigma^2(x)x^2}dxdy + \int_0^{\kappa-}\frac{2g(K)}{\sigma^2(K)K^2}P_0(K,T)dK + \int_{\kappa+}^{\infty}\frac{2g(K)}{\sigma^2(K)K^2}C_0(K,T)dK \tag{18}$$

To illustrate further, consider valuing a corridor note paying coupons at a constant rate c for each instant the underlying futures price is inside a corridor bracketing the initial futures price, ie, $F_0 \in (L, H)$. Thus, the given payout rate is $g(F) = c1_{F \in (L, H)}$ where 1_A denotes the indicator function of the event A. To compare with known results, we assume a CEV process for the futures price, ie, $\sigma(F) = \sigma F^p$, where σ and p are constants. To obtain the enhanced delta hedge for the corridor note, observe that substituting g and σ in equation (14) yields:

$$f_c''(F) = \frac{2c1_{F\in(L,H)}}{\sigma^2 F^{2p+2}} \tag{19}$$

Setting $\kappa = F_0$, equation (18) implies that the sale of a corridor note can be hedged by initially buying:

$$\frac{2c}{\sigma^2 K^{2p+2}}dK$$

puts for all strikes K between L and F_0 and:

$$\frac{2c}{\sigma^2 K^{2p+2}}dK$$

calls of all strikes K between F_0 and H. To complete the hedge, define $\underline{\overline{F}}$ as the futures price capped at H and floored at L. Then, substituting g and σ in equation (15):

$$f_c'(F) = \int_{F_0}^F \frac{2c1_{x\in(L,H)}}{\sigma^2 x^{2p+2}}dx = \begin{cases} \dfrac{-2c}{\sigma^2(2p+1)}\left[F_0^{-2p-1} - \left(\underline{\overline{F}}\right)^{-2p-1}\right] & \text{if } p \neq -\dfrac{1}{2} \\ \dfrac{2c}{\sigma^2}\ln\left(\dfrac{\underline{\overline{F}}}{F_0}\right) & \text{if } p = -\dfrac{1}{2} \end{cases} \tag{20}$$

From (11), the number of futures held at each $t \in [0, T]$ is $-e^{-r(T-t)}f_c'(F_t)$. Note that in contrast to standard delta hedging, no dynamic trading is required to hedge the corridor note whenever $F_t < L$ or $F_t > H$. To obtain the corridor note value, simply substitute g and σ in equation (12):

$$v_0 = \frac{2C}{\sigma^2}\left[\int_L^{F_0-}\frac{1}{K^{2p+2}}P_0(K,T)dK + \int_{F_0+}^H \frac{1}{K^{2p+2}}C_0(K,T)dK\right] \tag{21}$$

Clearly, more complicated payout structures and volatility functions are easily handled.

Note from equation (14) that if the claim to be hedged has a non-negative payout (ie, $g \geq 0$), as is typically the case, then $f'' \geq 0$ and so from equation (12), the EDH strategy involves a long position in options. The positive gamma in the enhanced delta hedge portfolio reduces the transaction costs associated with trading in the underlying. It is also able to provide *ad hoc* protection against stochastic volatility and jumps, although further analysis is then required to insure optimally against these contingencies.

Note that when valuing a simple corridor note under geometric Brownian motion, the standard approach would require evaluating:

$$V_0^c = Be^{-rT}E_0^Q \int_0^T c1_{F_t \in (L,H)} dt = ce^{-rT} \int_0^T Q_0 \{F_t \in (L,H)\} dt$$

under the risk-neutral process:

$$\frac{dF_t}{F_t} = \sigma dW_t, \qquad t \in [0,T]$$

While one can evaluate the integral explicitly, the answer must be the same as evaluating $e^{-rT}E_0^Q f_c(F_T)$ where for $p = 0$, f_c is given in equation (16) as:

$$\frac{2c}{\sigma^2}\left[\ln\left(\frac{F_0}{\underline{\bar{F}}}\right) + F\left(\frac{1}{F_0} - \frac{1}{\underline{\bar{F}}}\right)\right]$$

This expectation is easily expressed in terms of the standard normal distribution and density functions.

□ **Deferred payout on spot.** The results of the last subsection required the ability to trade in futures on the underlying and in European-style futures options. For some assets, futures contracts may not be available, or European-style options may only be written on the spot price. For these reasons, this subsection considers EDH when the payout depends on the spot price path $\{S_t, t \in [0, T]\}$ of an asset. Without loss of generality, we take this asset to be a stock. We assume that dividends are paid continuously over time and thus the cost of carrying the stock is $r_u = r - \delta(S)$, where the dividend yield at t is assumed to be an arbitrary function $\delta(S_t)$ of the stock price. As in the last subsection, we assume that the payout is deferred without interest to maturity ($r_c = 0$), so that the final payout at T is $\int_0^T g(S_t)dt$. The next section considers the analysis when the contractual rate is the risk-free rate ($r_c = r$). Replacing U_t with S_t and substituting $r_u = r - \delta(S)$ and $r_c = 0$ in equations (11) and (7) gives:

$$\int_0^T g(S_t)dt = -f(S_0) + \int_0^{\kappa-} f''(K)(K - S_T)^+ dK + \int_{\kappa+}^{\infty} f''(K)(S_T - K)^+ dK - \int_0^T f'(S_t)\{dS_t - [r - \delta(S_t)]S_t dt\} \qquad (22)$$

where $f(S)$ solves the following ODE:

$$\frac{\sigma^2(S)S^2}{2} f''(S) + [r - \delta(S)]Sf'(S) = g(S) \qquad (23)$$

subject to equation (9). The first three terms on the right-hand side of equation (22) again comprise a static position in bonds, puts and calls respectively. The final term is now the value at T from being short $e^{-r(T-t)}f'(S_t)$ shares at each time $t \in [0, T]$. Since the integral accounts for the fact that all purchases and sales are financed with the risk-free asset, there is no cost associated with initialising or maintaining this strategy. Consequently, the initial value of the payout $\int_0^T g(S_t)dt$ at T is again given by:

$$V_0 = -f(S_0)e^{-rT} + \int_0^{\kappa-} f''(K)P_0(K,T)dK + \int_{\kappa+}^{\infty} f''(K)C_0(K,T)dK \qquad (24)$$

Thus, to complete the valuation, we must solve the ODE (23). This is a first-order ODE in f' that must be solved subject to (9). Introducing an integrating factor yields:

$$f'(S) = e^{-\beta(S)}\int_\kappa^S e^{\beta(y)} \frac{2g(y)}{\sigma^2(y)y^2} dy, \quad \text{with } \beta(x) \equiv \int^x \frac{2[r - \delta(z)]z}{\sigma^2(z)z^2} dz \qquad (25)$$

where recall that for $b > a$, $\int_b^a h(y)dy \equiv -\int_a^b h(y)dy$. If we set $\kappa = S_0$, then from equations (9) and (24), there is no need to integrate equation (25) for f. There is also no need to differentiate (25) for f'', since (23) implies that f'' is a simple function of f' and g:

$$f''(S) = -\frac{2[r - \delta(S)]S}{\sigma^2(S)S^2} f'(S) + \frac{2}{\sigma^2(S)S^2} g(S) \qquad (26)$$

The standard delta-hedging strategy involves solving a partial differential equation (PDE) for the value. In contrast, the EDH strategy only requires solving a first-order ODE for the delta of a final payout. Solutions to PDEs are only explicitly given for certain dividend yield and volatility functions. In contrast, for arbitrary dividend yield and volatility functions of the stock price, the ODE can always be solved, with the solution expressed as an integral on a bounded domain. Since the payout rate g is likely to be simple, this integral can often be found in closed form.

To illustrate this EDH strategy, again consider valuing a corridor note paying c for each instant the underlying stock price is in a corridor bracketing the initial price, ie, $g(S) = c1_{S\in(L,H)}$ with $S_0 \in (L, H)$. To compare with known results, we assume constant proportional dividends, ie, $\delta(S) = \delta$ and a CEV process for the stock, ie, $\sigma(S) = \sigma S^p$, where δ, σ and p are constants. Then from equation (25), $\beta(x) = -c_0x^{-2p}$ with:

$$c_0 \equiv \frac{r - \delta}{\sigma^2 p}$$

for $p \neq 0$ and:

$$\beta(x) = 2\frac{r - \delta}{\sigma^2}\ln x$$

when $p = 0$. Substituting g and β in equation (25) and setting $\kappa = S_0$ gives:

$$f_c'(S) = \begin{cases} e^{c_0S^{-2p}}\int_{S_0}^S e^{-c_0y^{-2p}} \dfrac{2c1_{y\in(L,H)}}{\sigma^2 y^{2+2p}} dy & \text{if } p \neq 0 \\ e^{-2\frac{r-\delta}{\sigma^2}\ln S}\int_{S_0}^S e^{2\frac{r-\delta}{\sigma^2}\ln y} \dfrac{2c1_{y\in(L,H)}}{\sigma^2 y^2} dy & \text{if } p = 0 \end{cases} \qquad (27)$$

Defining $\underline{\bar{S}}$ as the stock price capped at H and floored at L gives:

$$f_c'(S) = \begin{cases} e^{c_0S^{-2p}} \dfrac{2c}{\sigma^2}\int_{S_0}^{\underline{\bar{S}}} e^{-c_0y^{-2p}} y^{-2-2p} dy & \text{if } p \neq 0 \\ S^{-2\frac{r-\delta}{\sigma^2}} \dfrac{2c}{\sigma^2}\int_{S_0}^{\underline{\bar{S}}} y^{2\frac{r-\delta}{\sigma^2}-2} dy & \text{if } p = 0 \end{cases} \qquad (28)$$

When $p \neq 0$, and for[4] $r \neq \delta$, the change of variables $t = c_0y^{-2p}$ implies that the integral can be expressed in terms of the incomplete gamma function $\gamma(a, z) \equiv \int_0^z t^{a-1}e^{-t}dt$:

$$f_c'(S) = e^{c_0S^{-2p}} \frac{cc_0^{-q}}{r - \delta}\left[\gamma\left(q, c_0S_0^{-2p}\right) - \gamma\left(q, c_0\underline{\bar{S}}^{-2p}\right)\right] \qquad (29)$$

where $q \equiv \frac{1}{2p} + 1$.

For $p = -1$ and $p = -\frac{1}{2}$, the integral in equation (28) can also be expressed in terms of other special functions. When $p = -1$, the process is Gaussian[5] and the integral can be expressed in terms of the standard normal distribution function. When $p = -\frac{1}{2}$, the process is a square root process and the integral can be expressed in terms of the exponential integral:

$$Ei(x) \equiv \int_{-\infty}^x \frac{e^t}{t} dt$$

When $p = 0$, the process is geometric Brownian motion and the integral can be solved explicitly:

$$f_c'(S) = S^{-2\frac{r-\delta}{\sigma^2}} \frac{c}{r - \delta - \sigma^2/2}\left[\underline{\bar{S}}^{2\frac{r-\delta}{\sigma^2}-1} - S_0^{2\frac{r-\delta}{\sigma^2}-1}\right] \qquad (30)$$

Recall from equation (22) that the replication involved a short position in $e^{-r(T-t)}f'(S_t)$ shares at each time $t \in [0, T]$. The static portion involves initially buying $f''(K)dK$ puts at all strikes below S_0 and $f''(K)dK$ calls at all strikes above S_0. The second derivative is obtained from substituting $g(S) = c1_{S\in(L,H)}$, $\delta(S) = \delta$ and $\sigma(S) = \sigma S^p$ in equation (26):

[4] If $r = \delta$, the spot price process matches the forward/futures price process covered in the previous section

[5] One should place an absorbing boundary at the origin to rule out negative stock prices

$$f_c''(K) = -\frac{2[r-\delta]K}{\sigma^2 K^{2p+2}} f_c'(K) + \frac{2}{\sigma^2 K^{2p+2}} 1_{K \in (L,H)} \quad (31)$$

Substituting this and $f(S_0) = 0$ in equation (24) gives the initial value of the corridor note.

Non-deferred payouts

Thus far, we have dealt with claims whose payout is deferred without interest to maturity. We now consider claims that pay out continuously over time, ie, pay-as-you-go claims. When the cashflows of the annuity are linked to the spot price, the annuity can be explicitly valued for arbitrary volatility functions, provided that dividends are constant. For an elaboration of this result, please download the associated working paper described at the end of this article. In contrast, when the cashflows of the annuity are linked to the futures price, explicit solutions are obtainable only by restricting volatility, although dividends are now unrestricted.

☐ **Pay-as-you-go on futures.** For some assets, eg, crude oil, the futures market is more liquid than the spot market. Furthermore, if the underlying used to determine the continuous cash payout of a swap is a futures price, then no assumption on dividends is required to hedge the claim. For these reasons, this subsection examines the pricing and hedging of pay-as-you-go claims written on the futures price with final payout $\int_0^T e^{r(T-t)} g(F_t)dt$. Replacing U_t with F_t and setting $r_u = 0$ and $r_c = r$ in equations (11) and (7) leads to the following decomposition of this payout:

$$\int_0^T e^{r(T-t)} g(F_t)dt = -f(F_0)e^{rT} + \int_0^{F_0-} f''(K)(K - F_T)^+ dK + \int_{F_0+}^{\infty} f''(K)(F_T - K)^+ dK - \int_0^T e^{r(T-t)} f'(F_t)dF_t \quad (32)$$

where f solves the following ODE:

$$\frac{\sigma^2(F)F^2}{2} f''(F) - rf(F) = g(F) \quad (33)$$

subject to equation (9). Thus from equation (32), the desired payout is the sum of a static position in bonds and options and a cost-free dynamic short position in $f'(F_t)$ futures at each time $t \in [0, T]$. The arbitrage-free value of the payout is the initial cost of creating the replicating portfolio:

$$V_0 = -f(F_0) + \int_0^{F_0-} f''(K)P_0(K,T)dK + \int_{F_0+}^{\infty} f''(K)C_0(K,T)dK \quad (34)$$

Fortunately, equation (33) has an explicit solution for a general payout function $g(F)$ when $\sigma(F) = \sigma F^p$, ie, for a CEV process. To obtain it, we first solve the homogeneous ODE:

$$\frac{\sigma^2 F^{2p}}{2} h''(F) - rh(F) = 0 \quad (35)$$

The two linearly independent solutions are functions of the modified Bessel functions:

$$h_1(F) = \sqrt{F} I_\nu\left(\wp F^{\frac{1}{2\nu}}\right) \quad (36) \quad \text{and} \quad h_2(F) = \sqrt{F} K_\nu\left(\wp F^{\frac{1}{2\nu}}\right) \quad (37)$$

where:

$$\nu = \frac{1}{2(1-p)} \quad \text{and} \quad \wp \equiv \frac{\sqrt{2r}}{\sigma(1-p)}$$

Using reduction of order, these homogeneous solutions can be used to find the Green's function $G(F; K)$ solving the inhomogeneous ODE:

$$\frac{\sigma^2 F^{2p}}{2} G''(F;K) - rG(F;K) = \delta(F-K) \quad (38)$$

and the homogeneous boundary conditions:

$$G(0;K) = 0 \quad (39) \quad \text{and} \quad G(\infty;K) = 0 \quad (40)$$

The function $h_1(F)$ in equation (36) clearly solves (39) while the function $h_2(F)$ in (37) solves (40). Accordingly, the Green's function is (after some algebra):

$$G(F;K) = -\frac{4\nu\sqrt{FK}^{-\frac{3}{2}+\frac{1}{\nu}}}{\sigma^2} I_\nu\left(\wp (F \wedge K)^{\frac{1}{2\nu}}\right) K_\nu\left(\wp (F \vee K)^{\frac{1}{2\nu}}\right) \quad (41)$$

Thus, the solution of:

$$\frac{\sigma^2 F^{2p}}{2} f''(F) - rf(F) = g(F) \quad (42)$$

subject to:

$$f(0) = 0 \quad (43) \quad \text{and} \quad f(\infty) = 0 \quad (44)$$

is:

$$\int_0^\infty g(K)G(F;K)dK \quad (45)$$

Summary and further extensions

We have shown how EDH can be used to value and hedge contracts with continuous cashflows over time. We considered both deferred and non-deferred payouts and we considered payouts linked to both futures prices and to spot prices. In many realistic cases, by valuing relative to option prices, explicit valuation and hedging results were obtained when volatility is an arbitrary function of price.

The foregoing results can be extended in many ways. For example, one could allow for some time-dependence in the cashflows. A corridor note linked to the level of a geometric Brownian motion is easily valued if the lower and upper barriers defining the corridor grow exponentially at the same rate as the expected price. One can also extend the results to cashflows that knock out when the underlying reaches a barrier by hedging with barrier options. One can also consider EDH of claims with terminal payouts of the form $f(U_T)\int_0^T g(U_t)dt$ by changing measure. Finally, some cashflows can be perfectly synthesised using EDH when the process is an arbitrary semi-martingale (see Carr, Lewis & Madan, 2000).

When compared with the alternatives of pure dynamic hedging and pure static hedging, EDH holds out the possibility of extracting the best features

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of both alternatives. As pointed out by a referee, it is also true that EDH might suffer from the worst features of both alternatives. Unfortunately, given the constraints of time and space, determining which of these outcomes arises in a given situation will have to be left for future research. ■

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