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# MULTI-ASSET STOCHASTIC LOCAL VARIANCE CONTRACTS

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Variance swaps now trade actively over-the-counter (OTC) on both stocks and stock indices. Also trading OTC are variations on variance swaps which localize the payoff in time, in the underlying asset price, or both. Given that the price of the underlying asset evolves continuously over time, it is well known that there exists a semirobust hedge for these localized variance contracts. Remarkably, the hedge succeeds even though the stochastic process describing the instantaneous variance is never specified. In this paper, we present a generalization of these results to the case of two or more underlying assets.

KEY WORDS: variance swap, basket option, stochastic volatility.

## 1. INTRODUCTION

Variance swaps now trade actively over-the-counter (OTC) on both stocks and stock indices. In this contract, one party agrees to pay the other the realized variance of returns of a specified underlying asset over a specified future period. In return, the party providing this positive payoff receives a fixed positive amount at expiry. As with any swap, the fixed positive amount is agreed upon at inception and chosen so that the swap is costless to enter.

Under certain conditions, the payoff to a variance swap can be replicated by either counterparty. The now standard approach requires assuming that the underlying asset price is positive and evolves continuously over time. Further assuming continuous path monitoring, deterministic interest rates and dividend yields, and the availability of European options with a continuum of positive strikes, the replicating strategy combines a static position in these options with dynamic trading in the underlying asset. Remarkably, this hedge succeeds even though the stochastic process describing the instantaneous variance is never specified. This recipe for replicating the payoff to variance swaps has become so well known that the definition of the VIX was revised in 2003 to emulate it.

Following upon the successful introduction of variance swaps into the marketplace, several institutions have offered variations on variance swaps whose payoffs are attuned to increasingly sophisticated views. For example, forward variance swaps have been offered

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to investors who wish to speculate or hedge on the variance realized over a period that both starts and ends at a future date. Similarly, corridor and conditional variance swaps have been offered to investors who wish to speculate or hedge on the variance realized while the underlying asset is inside or outside some corridor. These localizations of the variance swap payoff in time and space can also be combined, resulting in a contract that pays the realized variance while the space time process is in some region. As first shown in Dupire (1996), the payoffs to these localized variance contracts can be robustly replicated under the same conditions that lead to the replication of variance swaps.

So far these variations on variance swaps have all referenced a single underlying asset in defining their payoff. However, it is not hard to imagine yet further extensions which reference multiple assets in specifying their payoff. Indeed, the liquid market in basket options presently available over the counter suggests a healthy appetite on the part of investors for products which allow them to trade the variance of a portfolio of assets. The recognition of this appetite has led to the introduction of a plethora of exotic equity derivatives written on the price path of several underlying assets. Nowadays, one sees products such as Napoleons, Himalaya options, and correlation swaps stocking the portfolios of sophisticated investors who wish to either speculate on or lay off the risk that arises when options are written on portfolios of assets.

Lost in the rush to market of these multi-asset path-dependent products seems to be any notion of a robust hedging strategy or pricing model for these sophisticated products. While one can always combine a particular valuation model with Monte Carlo to provide the necessary numbers, one has to wonder if there does not exist a generalization of the single asset robust hedges to the case of multiple underlying assets. If it exists, such a generalization should be of interest both to traders charged with the challenging task of mitigating fluctuations in P&L without complete knowledge of the underlying stochastic processes involved, and to risk managers charged with the daunting task of objectively marking a structured product, without an outright market for the asset in question.

In this paper, we propose a generalization of Dupire's univariate results on local variance to the case of multiple underlying assets. The generalization arises from a re-interpretation of the assumptions on dynamics that are made in the univariate case. Recall that the replication of local variance contracts assumes that the underlying price process is a positive continuous stochastic process. No arbitrage further implies that the forward price of the underlying asset is a martingale under forward measure. Using the well-known result of Dambis (1965) and Dubins and Schwarz (1965), it can be shown that the class of processes with these properties can be obtained by running a driftless geometric Brownian motion (GBM) on an unspecified stochastic clock.

In the multi-asset case, no arbitrage requires that the vector forward price process $F_t = (F_{1t}, F_{2t}, \dots, F_{nt})$ be a martingale under forward measure. Extending our interpretation of the univariate results, we assume that each element of the vector process is obtained by running a scalar driftless GBM on an unspecified stochastic clock. As usual, the diffusion coefficient of each scalar GBM will depend on the asset. However, the stochastic clock that each GBM runs on is assumed to be common to all $n$ assets. The presumption that business time is common to all assets causes asset volatilities to rise and fall in tandem, while still permitting volatility levels to differ across assets.

In this multi-asset setting, we will show how to price and replicate local variance claims while never specifying the business time process. The variance in question is that of a specified portfolio of two or more assets. The localization again requires specifying a region in space–time for the space–time process. So long as the space–time process is in the specified region, the long side of the local variance claim receives the squared return of the specified portfolio. This paper determines the arbitrage-free premium that such an investor should initially pay in return for this stream of positive cash flows which terminate at a fixed maturity.

In order to replicate this stream of cash flows, we assume that the common stochastic clock is absolutely continuous w.r.t. calendar time and hence the composite vector process can never jump. Just as the univariate results employ a continuum of standard options in the hedge, our multivariate results employ a continuum of basket options in the hedge. As the region in which payoffs are received is allowed to be arbitrary, the replication requires holdings in multiple basket options which differ by the fixed weights in the underlying assets.

We show that the assumed continuity of asset prices over time permits a decomposition of the desired payoff into the sum of a path-independent component and a stochastic integral with respect to a vector martingale. The integrand just depends on the observed asset price vector and hence the stochastic integral can be created at zero cost. Using Radon transforms, we show how the path-independent component can be created out of a static position in basket options. This latter replication is completely model-free and hence of interest in its own right. It represents a generalization to $n$ assets of the well-known single asset result of Breeden and Litzenberger (1978) (henceforth BL).

To summarize, there are two major sets of new results in this paper. The first set of results involve extending the insights of BL to the multi-asset case. For these results, price continuity is not needed, i.e., the only assumption is frictionless markets in a wide variety of options. A straightforward application of BL's results to basket calls implies that the second strike derivative of the forward price of a European basket call is just the risk-neutral probability density function (RNPDF) of the terminal level of the underlying basket. Now suppose that one can observe forward prices of all basket calls on $n$ assets. Then we show that one can also obtain the *joint* RNPDF of the $n$ underlying asset prices from this information. In fact, working with $n = 2$, Lipton (2001) observes that the second strike derivative of the forward price of a basket call is just the Radon transform of the joint RNPDF of the two underlying asset prices. In this paper, we show that these observations extend without change to the $n \geq 2$ asset case, where $n$ is an arbitrary positive integer. This finding has two related implications. First, by inverting the Radon transform of the RNPDF, the joint RNPDF of the $n$ terminal asset prices can be extracted from the initial prices of basket calls. Second, one can determine the static position in basket calls needed to replicate the terminal payoff to any claim written on (just) the final level of the $n$ underlying asset prices. Although the results mentioned above are new, they are a straightforward extension of Lipton's work.

The second set of results is more novel. We introduce contingent claims that naturally generalize corridor variance swaps to two or more assets. These claims pay out the realized variance of a fixed portfolio while the prices of the underlying assets are in a specified region. We show how to price the payoffs of these claims relative to the given prices of basket options. We obtain a unique price despite the fact that the process describing the stochastic clock is unspecified. Our results require obtaining quotes on basket options for many underlying baskets and strikes. Fortunately, all major banks stand ready to provide OTC quotes on customized baskets.

From a mathematical perspective, our second set of results highlight the importance of fundamental solutions of second order *elliptic* partial differential equations (PDEs) for pricing local variance in $n \geq 1$ dimensions, when the underlying asset price paths are continuous over time. As is well known, fundamental solutions of parabolic PDEs play an important role in both probability and continuous time finance. Elliptic PDEs have also been used for pricing perpetual claims or claims that mature at an exponentially distributed random time. However, to our knowledge, no one has proposed using fundamental solutions of elliptic PDEs as the basis for pricing claims on local variance that mature at a fixed time. Our use of fundamental solutions to elliptic PDEs in extending the univariate results to the $n$-asset case is the main contribution of this paper.

An outline of this paper is as follows. Section 2 presents an extension of the BL (1978) univariate result to $n \geq 2$ underlying assets. The following section reviews the Dupire (1996) univariate result and its extension to the case of multiple underlying assets. In this section, we first consider the simpler case of a time-changed Bachelier process and then consider the more complicated time-changed Black–Scholes model, in order to enforce the reality of positive asset prices. The final section summarizes the paper and suggests directions for future research. Several appendices are devoted to the proofs of key results.

## 2. STATIC OPTION REPLICATION

In this section, we review and extend an important result due to BL (1978) on replicating path-independent payoffs. As the replicating strategy always just involves taking a static position in the appropriate options, the replication is robust in comparison to most work on derivative security valuation. In particular, volatilities and correlations are unrestricted and asset prices can jump.

### 2.1. Multi-Asset Extension of BL

In this section, we generalize the original work of Lipton (2001), who proposed that static positions in a set of European options on baskets of two stocks could be used to replicate the payoff from any European style-claim written on the terminal prices of the two underlying assets. Lipton (2001) observes that the second strike derivative of the forward price of a basket call is just the Radon transform of the joint RNPDF of the two underlying asset prices.

**Definition and inversion of the Radon transform.** We now define the Radon transform of a real-valued function $f$ of $n$ real variables for $n \geq 2$. Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $d\mathbf{x} = dx_1, \ldots, dx_n$, $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ and let $\bar{\mathbf{w}}$ be a unit vector in $\mathbb{R}^n$. Two good sources for the following mathematical results are Helgason (1980) and Deans (1983). The Radon transform of a function $f(\mathbf{x})$ is given by

$$R[f](\bar{\mathbf{w}}, k) = \int f(\mathbf{x})\delta(\bar{\mathbf{w}} \cdot \mathbf{x} - k)\, d\mathbf{x}.$$

Notice that the delta function is only nonzero on the set $\bar{\mathbf{w}} \cdot \mathbf{x} = k$. This is the equation of a hyperplane in $\mathbb{R}^n$. The distance from the hyperplane to the origin is exactly $k$. For fixed $(\bar{\mathbf{w}}, k)$, the Radon transform of $f$ is the integral of $f$ on the hyperplane defined by $\bar{\mathbf{w}} \cdot \mathbf{x} = k$. As we vary $\bar{\mathbf{w}}$ and $k$, we define the Radon transform of $f$.

The function $f$ is recovered from its Radon transform $R[f](\bar{\mathbf{w}}, k)$, using the *inverse Radon transform* as

$$f(\mathbf{x}) = \int_{|\bar{\mathbf{w}}|=\mathbf{1}} h(\bar{\mathbf{w}}, \bar{\mathbf{w}} \cdot \mathbf{x})\, d\bar{\mathbf{w}}, \tag{2.1}$$

where the definition of the function $h$ depends on whether $n$ is odd or even. If $n$ is odd, then

$$h(\bar{\mathbf{w}}, t) \equiv \frac{(i)^{n-1}}{2(2\pi)^{n-1}} \frac{\partial^{n-1}}{\partial t^{n-1}} R[f](\bar{\mathbf{w}}, t). \tag{2.2}$$

If $n$ is even, then

$$h(\bar{\mathbf{w}}, t) \equiv \frac{(i)^n}{2(2\pi)^{n-1}} \mathcal{H}\left[\frac{\partial^{n-1}}{\partial p^{n-1}} R[f](\bar{\mathbf{w}}, p)\right](t), \tag{2.3}$$

where $\mathcal{H}[g(p)](t)$ denotes the Hilbert transform of the function $g(p)$

$$\mathcal{H}[g(p)](t) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(p)}{p-t} dp, \tag{2.4}$$

where the integral is a Cauchy principle value. As a consequence, the discounted joint RNPDF can be determined from the prices of basket calls, as explained below.

**Application to Basket Calls.** To apply the above mathematical results to financial markets, we assume price transparency and liquidity in basket options, which are European-style options written on a basket of stocks. While both puts and calls trade liquidly OTC, put call parity implies that we need only use basket calls. Recall that the payoff $I(\mathbf{S})$ for $\mathbf{S} = (S_1, \ldots, S_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$, of a basket call is given by

$$I(\mathbf{S}) = (\mathbf{w} \cdot \mathbf{S} - K)^+,$$

where "·" is the scalar product.

To apply Radon transform technology to basket call prices, we first scale the weight vector $\mathbf{w}$ so that it lies on the unit sphere. Let $\bar{\boldsymbol{w}} = \frac{\mathbf{w}}{|\mathbf{w}|}$ and $\tilde{K} = \frac{K}{|\mathbf{w}|}$. Assuming that the risk-neutral joint distribution of the assets $\mathbf{S}_T$ at time $T$ permits a density function $f(\mathbf{S}, \mathbf{T})$, the time 0 price of a basket call with maturity $T$ and strike $K$ is

$$\begin{aligned} C(K, T) &= B(T) \int_{\mathbf{S} \in \mathbb{R}_+^n} (\mathbf{S} \cdot \mathbf{w} - K)^+ f(\mathbf{S}, T)\, dS_1 \ldots dS_n \\ &= |\mathbf{w}| B(T) \int_{\mathbf{S} \in \mathbb{R}_+^n} (\mathbf{S} \cdot \bar{\boldsymbol{w}} - \tilde{K})^+ f(\mathbf{S}, T)\, dS_1 \ldots dS_n, \quad \bar{\boldsymbol{w}} \in \mathcal{S}^{n-1}, \end{aligned} \tag{2.5}$$

where $B(T)$ is the time 0 price of a zero coupon bond maturing at time $T$ and where $\mathcal{S}^{n-1}$ is the boundary of the unit sphere in $\mathbb{R}^n$. In (2.5), each $dS_i$ in $dS_1 \ldots dS_n$ denotes a nonrandom integrator.

Differentiating twice with respect to $\tilde{K}$

$$\begin{aligned} \frac{\partial^2}{\partial \tilde{K}^2}(C(|\mathbf{w}|\tilde{K}, T)) &= |\mathbf{w}| B(T) \int \delta(\mathbf{S} \cdot \bar{\boldsymbol{w}} - \tilde{K}) f(\mathbf{S}, T)\, dS_1 \ldots S_n \\ &= |\mathbf{w}| B(T) R[f](\bar{\boldsymbol{w}}, \tilde{K}). \end{aligned} \tag{2.6}$$

Thus, solving for the Radon transform, and using $\frac{\partial^2}{\partial \tilde{K}^2} = |\mathbf{w}|^2 \frac{\partial^2}{\partial K^2}$, we have

$$R[f]\left(\frac{\mathbf{w}}{|\mathbf{w}|}, \frac{K}{|\mathbf{w}|}\right) = \frac{|\mathbf{w}|}{B(T)} \frac{\partial^2 C}{\partial K^2}(K, T), \tag{2.7}$$

which can be rewritten as

$$R[f](\bar{w}, t) = \frac{|\mathbf{w}|}{B(T)} \frac{\partial^2 C}{\partial^2 K}(|\mathbf{w}|t, T).$$

Given the market prices of basket calls of all strikes and all weights, (2.7) implies that one can recover the risk-neutral density $f$ using the inverse Radon transform (2.1)

$$f(\mathbf{x}, T) = \mathcal{R}^{-1}\left(\frac{|\mathbf{w}|}{B(T)} \frac{\partial^2 C}{\partial K^2}(|\mathbf{w}|t, T)\right), \tag{2.8}$$

where $\mathcal{R}^{-1}$ is the inverse Radon transform described by the two step process starting with (2.3) or (2.2) and ending with (2.1). Note the $\mathbf{x}$ is the transform variable, as explained by formula (2.1) and is related to $t$ by $t = \bar{w} \cdot \mathbf{x}$, so that $\frac{\partial^2 C}{\partial K^2}(|\mathbf{w}|t, T)$ in (2.8) above may be expressed as $\frac{\partial^2 C}{\partial K^2}(\mathbf{w} \cdot \mathbf{x}, T)$. While the market for exchange-traded basket options is too thin to permit the right-hand side of (2.8) to be observed, most major banks stand ready to quote on OTC basket options with any set of weights and strikes. As a result, one can in principle determine the multi-asset risk-neutral density function $f$.

## 3. ALLOWING SEMI-STATIC STOCK TRADING

While the work of BL is completely robust to the dynamics of the underlying asset, its scope is limited to the replication of path-independent payoffs. This is due to the fact that the replicating strategy is restricted to only hold static positions in options. When one can furthermore dynamically trade the underlying asset, then certain path-dependent claims can be replicated (see Carr, Lewis, and Madan 2000 for a characterization). For example, in the univariate case, one can create a claim paying the quadratic variation of the price by combining a static position in options with dynamic trading in the underlying, even though jumps in price are allowed. By furthermore requiring that the price of the underlying asset evolves continuously over time, the set of attainable payoffs grows larger yet. For example, in the univariate case, a claim paying the quadratic variation of the log price becomes attainable if one assumes that the price is strictly positive and continuous over time.

In this section, we first review Dupire's result which assumes that there is a single underlying asset whose price evolves continuously over time. We then extend his results to two or more assets. We first cover the case of time-changed Bachelier dynamics for simplicity and then we consider the more complicated but more realistic case of time-changed Black–Scholes dynamics. As is well known, the Bachelier and Black–Scholes model both assume that the underlying asset prices follow time homogeneous diffusions. The Bachelier model is characterized by constant coefficients for the risk-neutral forward price process, while the Black–Scholes model is characterized by constant coefficients for the risk-neutral log price process.

### 3.1. Review of Dupire

The heuristic presentation below follows Dupire's ideas in his 1996 paper (1996) (see also Derman, Kani, and Kamal 1997). We present it here for the reader's convenience because that paper is hard to obtain and because it is the starting point for our generalizations in later sections. Also, rigorous proofs in the multivariate case follow from appropriate variations and extensions of his argument that will be presented later. For a more mathematically rigorous exposition of the treatment in the *single* asset case, see the paper by Klebaner (2002). We assume that the underlying's spot price process is continuous over time and that the value of a straddle $V_0(K, T)$ is at least once differentiable in $T$. While not necessary, we also assume zero interest rates and dividends for simplicity.[1] Under measure $\mathbb{Q}$, the spot price process can be written as

$$dS_t = \beta_t dW_t, \tag{3.1}$$

where $\beta_t$ is the stochastic (normal) volatility process. This latter process can depend on the paths of $S$ or $W$ up to $t$, but need not be determined by them. A sufficient condition for the stochastic integral implicit in (3.1) to be well defined is that

$$\mathbb{E}_0^{\mathbb{Q}} \int_0^t \beta_s^s ds < \infty. \tag{3.2}$$

The dynamics assumed in (3.1) imply that the stock price is a $\mathbb{Q}$ local martingale

$$\mathbb{E}_t^{\mathbb{Q}} dS_t = 0, \quad t \in [0, T], \tag{3.3}$$

and that quadratic variation increases continuously over time

$$d\langle S\rangle_t = \beta_t^2 dt, \quad t \in [0, T]. \tag{3.4}$$

Dupire showed that the *local variance* $\mathbb{E}_0^{\mathbb{Q}}[\beta_T^2 | S_T \in dK]$ can be determined at time 0 from the initial market prices of straddles struck around $K$ and maturing around $T$. To obtain his result, note that the Tanaka Meyer formula implies that

$$|S_T - K| = |S_0 - K| + \int_0^T \operatorname{sgn}(S_t - K)\, dS_t + \int_0^T \delta(S_t - K)\beta_t^2 dt. \tag{3.5}$$

Taking risk-neutral expectations, (3.5) implies that

$$V_0(K, T) = |S_0 - K| + \int_0^T \mathbb{E}_0^{\mathbb{Q}} \delta(S_t - K)\beta_t^2 dt \tag{3.6}$$

from (3.3) and Fubini. Differentiating w.r.t. $T$, the fundamental theorem of calculus implies that

$$\frac{\partial}{\partial T} V_0(K, T) = \mathbb{E}_0^{\mathbb{Q}}\left[\delta(S_T - K)\beta_T^2\right]. \tag{3.7}$$

Multiplying and dividing by $\mathbb{E}_0^{\mathbb{Q}} \delta(S_T - K)$

$$\frac{\partial}{\partial T} V_0(K, T) = \frac{1}{2}\frac{\partial^2}{\partial K^2} V_0(K, T)\mathbb{E}_0^{\mathbb{Q}}\left[\beta_T^2 | S_T \in dK\right],$$

[1]The results easily extend to deterministic interest rates and dividend yields, where the latter can be paid continuously and/or discretely.

from (3.6) and the definition of conditional expectation. Solving for the desired conditional expectation

$$\mathbb{E}_0^{\mathbb{Q}}\left[\beta_T^2 | S_T \in dK\right] = \frac{2\frac{\partial}{\partial T} V_0(K, T)}{\frac{\partial^2}{\partial K^2} V_0(K, T)}.$$

In words, the initial expectation of the terminal local variance given that $S_T = K$ is just twice the ratio of a calendar spread to a butterfly spread. Remarkably, the only assumption made on the nature of the stochastic process governing instantaneous volatility in this (not completely rigorous) argument is the technical condition (3.2). In particular, this volatility can depend on much more than the underlying asset price and time, and it can jump. The rigorous proof requires additional technical assumptions on $\beta_t$. Sufficient conditions in the continuous case are given in theorem 4 in Klebaner (2002).

Dupire's results also imply that one can robustly price and hedge any payoff of the form $\int_0^T \beta_t^2 f(S_t, t)\, dt$ for any $f(\cdot)$ without further assumptions. Gamma swaps, variance swaps, forward variance swaps, and corridor variance swaps are special cases that have arisen in practice.

## 4. THE EXTENSION OF DUPIRE'S RESULTS TO HIGHER DIMENSION

### 4.1. The Analytic Backdrop for Such a Generalization

Consideration of the results outlined in the previous section suggests that the principal ingredient in Dupire's heuristic derivation of formula (3.7) is the Tanaka–Meyer formula. Dupire did not attempt to give a rigorous proof of formula (3.7). A proof was achieved later by Klebaner (2002). Again, the principal ingredient is the Tanaka–Meyer formula. Therefore, a natural question that arises when attempting to extend Dupire's formula to $n$ assets is, in turn, the identification of the key ingredient in the proof of the Tanaka–Meyer formula (3.5). In our view, this key ingredient in the univariate case is the following formula, to be understood in the sense of the theory of distributions (see Schwartz 1966), for every $K \in \mathbb{R}$

$$\frac{1}{2}\frac{\partial^2}{\partial S^2}|S - K| = \delta(S - K). \tag{4.1}$$

In words, when the 1D Laplacian acts on one half of the absolute value of $S - K$, then the result is a delta function with support at $K$. From the theory of PDEs (see Garabedian 1998), we know that (4.1) can be reformulated as the following statement. The *fundamental solution* $f(S, K)$ of the 1-D elliptic operator $\frac{1}{2}\frac{\partial^2}{\partial S^2}$ is $|S - K|$. The fundamental solution has the property that the elliptic operator applied to it, *localizes* the outcome to an idealized point mass, of unit weight, at $K$.

To generalize such a localization property to $n \geq 2$ dimensions, it is natural to replace the 1D Laplacian with a second order linear differential operator in $n$ variables, of the form

$$\mathcal{L} = \frac{1}{2} a_{ij} \frac{\partial^2}{\partial S_i \partial S_j} + \mu_i \frac{\partial}{\partial S_i}, \tag{4.2}$$

where we have used the Einstein convention of summation over repeated indices, and where we assume that the matrix $a_{ij}$, $1 \leq i, j \leq n$ is positive-definite. A fundamental solution $f$ associated to the elliptic differential operator (4.2) is a function depending on $2n$ variables $(S_1, \ldots, S_n, \xi_1, \ldots, \xi_n)$ with the sifting property

$$-\mathcal{L}_{\mathbf{x}} f(x_1, x_2, \ldots, x_n, \xi_1, \ldots, \xi_n) = \delta(x_1 - \xi_1)\delta(x_2 - \xi_2) \ldots \delta(x_n - \xi_n). \tag{4.3}$$

The minus sign in front of the elliptic operator is there for convenience and clearly, by linearity the solution of $\mathcal{L}_{\mathbf{x}} u = \delta(x_1 - \xi_1)\delta(x_2 - \xi_2) \ldots \delta(x_n - \xi_n)$ is simply $-f$, where $f$ solves (4.3). In the sequel, whenever possible, for economy of notation, we will use vector notation and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$. The difference between the one-dimensional case and the multidimensional case is intimately connected with the following fact: unlike the one-dimensional case, the fundamental solution in the higher dimensional case is *unbounded*. More precisely the fundamental solution in dimension 2 has a pole when $\mathbf{x} = \boldsymbol{\xi}$ and behaves at the pole like $\log(|\mathbf{x} - \boldsymbol{\xi}|)$. Likewise, the fundamental solution of order $n$ has a pole of order $|\mathbf{x} - \boldsymbol{\xi}|^{2-n}$. The difference between the univariate and the multivariate case is not a technical difference, but rather a *fundamental* difference. It explains why there is no straightforward generalization of local time to higher dimension. In Stroock and Varadhan (2006, p. 117), they express the matter thus: "Although it is somewhat obscure in the present proof, what *underlies*[2] the existence of a local time for a 1-dimensional Itô process is the fact that points have a positive capacity in one dimension. The way in which this fact is used here is hidden in the boundedness at its pole of the fundamental solution for $\frac{1}{2}\frac{d^2}{dx^2}$."

In generalizing Dupire's argument to $n$ dimensions, we therefore must deal with the unboundedness of the fundamental solution. It makes no sense financially to have an unbounded payoff function $f(\mathbf{S}, \boldsymbol{\xi})$, so our procedure below will be to re-define $f$ close to the pole, in such a way that it is sufficiently regular for us to apply Itô's formula in $n$ dimensions in a manner analogous to that in which Dupire uses the Tanaka–Meyer formula. As a consequence, we will show that we are able to price local variance contracts when the underlying asset prices are in certain subsets of $\mathbb{R}^n$. What we cannot do in $n \geq 2$ dimensions is to consider a region localized *at a point*, whereas this was possible in one dimension.

## 4.2. The Financial Motivation for Generalizing Local Variance to Multiple Assets

A multi-asset local variance contract is a contract that gives its holder the possibility of obtaining the realized variance of an index, when the underlying stocks in the index are in certain target zones ("pockets"). The monitoring of the stock price paths can either begin the day the contract is signed or there can be a forward start. To our knowledge, such contracts are not yet sold on today's financial markets. However, just as corridor variance swaps did not trade when they were first proposed and yet were later introduced, one could envision the future introduction of multi-asset local variance contracts.

There are several reasons why an investor might have an interest in such a contract. For example, an investor might wish to receive the realized variance of a portfolio if and only if all of the assets in the portfolio drop by more than a fixed percentage amount relative to their current levels, as they would in a crash. This contract amounts to a multi-asset generalization of a protective put. A similar example arises just after a takeover is

[2]The italics are ours.

announced, but before the deal is consummated. Suppose that an investor believes that the stock prices of the acquirer and the target would both fall if the deal fails, and that this event is more likely than the market is reflecting in prices. Then a structure that pays realized variance if and only if both prices fall would be attractive to such an investor.

One can also imagine local variance contracts which pay out the realized variance of a portfolio when all of the constituents rise in price rather than fall. For example, an investor may wish to generate premium income by selling realized variance paid if and only if all assets in the portfolio rise by more than a fixed percentage amount. This contract amounts to a multi-asset generalization of a buy-write strategy.

Given the current interest in correlation swaps and dispersion trading, one can also imagine local variance structures that are tuned to proxies for correlation. For example, suppose that an investor is concerned only about the realized variance of a portfolio of two assets if and only if the correlation between the two assets' returns is very positive. This could be captured by specifying two regions in which variance is received. The first region would be where both assets are up by more than a fixed percentage and the second would be where both assets are down by more than a fixed percentage. One can also extend this line of thinking to three or more assets. For all of these examples, the investor might specify that the date at which they start receiving variance is in the future.

### 4.3. Multi-Asset Stochastic Local Variance: Normal Dynamics

This subsection is divided into two parts. The first sub-subsection presents a localized multidimensional analog of Dupire's results. The second sub-subsection presents a multidimensional generalization of a corridor variance swap (see Carr and Lewis 1994).

*4.3.1. The Localized Claims.* In this subsection, we present an $n$-dimensional extension of Dupire's local results (3.7) in the setting of assets driven by $n$ correlated Brownian motions. We assume zero interest rates and dividends[3] over the time interval $[0, T]$ for simplicity. We also assume a stochastic multivariate spot price process $\mathbf{S}_t \equiv \{S_{1t}, \ldots, S_{nt}: t \in [0, T]\}$. We assume a continuous stochastic process for the spot prices $\mathbf{S}_t$ under the statistical measure $\mathbb{P}$, and assume the existence of a risk-neutral measure under which the stock dynamics is given by

$$S_{it} = S_{i0} + \int_0^t \sigma_i \beta_s \, dW_{is}, \quad t \in [0, T], \tag{4.4}$$

where $S_{i0}$, $i = 1, \ldots, n$ are the initial values, $\sigma_i$, $i = 1, \ldots, n$ are constants, $\{W_{it}: t \in [0, T]\}$ are standard Brownian motions, with constant correlation coefficients $\rho_{ij}$, i.e.:

$$\langle W_i, W_j \rangle_t = \rho_{ij} t,$$

and where $\beta_t$ is a stochastic process assumed independent of the filtration generated by $(W_{1t}, \ldots, W_{nt})$. Under the dynamics of this section, spot prices can go negative, which means that our analysis should not be applied to limited liability assets. We deal with the complications induced by nonnegative prices in section 4.4. Note that the normal volatility of stock $i$ is the product of a constant idiosyncratic component $\sigma_i$ and a stochastic systematic component $\beta_t$, for $i = 1, \ldots, n$. The assumption that the

[3]Once again, our results easily generalize to deterministic interest rates and dividend yields.

idiosyncratic component is constant is restrictive since it implies that volatility ratios are constant. It also implies that the slope coefficient of a regression of one stock on any index is constant.

Returning to the stochastic differential equations (4.4)

$$dS_{it} = \beta_t \sigma_i dW_{it},\ i = 1, \ldots, n, \tag{4.5}$$

the instantaneous variance of $\sum w_i S_i$ at time $t$ is $\beta_t^2 w_i w_j \sigma_i \sigma_j \rho_{ij} dt$ and it will be useful below to separate out the idiosyncratic part of this instantaneous variance

$$V_I \equiv w_i w_j \sigma_i \sigma_j \rho_{ij}. \tag{4.6}$$

Our objective is to synthesize an approximating version of *multivariate local variance* $\beta_T^2 V_I dt \delta(\mathbf{S}_T - \mathbf{K})$, where $\delta(\cdot)$ is the multivariate Dirac delta function, whose payoff is zero unless, for each $i = 1, \ldots, n$, $S_{iT} \in dK_i$ and in this event, $\beta_T^2 V_I dt$ is the increment in the quadratic variation of $\sum_{i=1}^n w_i S_i$. Consider a function $g(\mathbf{S})$ which solves the following canonical version of the $n$-dimensional constant coefficient elliptic equation

$$-\frac{1}{2} a_{ij} g_{S_i S_j} = V_I \delta(\mathbf{S} - \mathbf{K}), \tag{4.7}$$

where[4] $\mathbf{a} = \{a_{ij}\}_{i,j=1}^n$ and where $a_{ij} = \sigma_i \sigma_j \rho_{ij}$. Such a function is called a *fundamental solution*. When $n = 1$, a solution of (4.7) is a straddle payoff, which Dupire used to observe the conditional mean of the terminal variance rate of the underlying asset. A natural conjecture is that solutions of (4.7) for $n \geq 2$ produce a payoff which allow one to observe the conditional mean of the terminal variance rate of the underlying basket.

To pursue this conjecture, first note that for $n \geq 1$, the function $g$ is known to be analytic everywhere except at the point $\mathbf{S} = \mathbf{K}$ and a solution to (4.7) is known to be $V_I F_n$, where (see Shimakura 2002)

$$F_2(x, \boldsymbol{\xi}) = -\frac{1}{\pi\sqrt{\Delta}} \ln \frac{1}{\sqrt{\sum_{i,j=1}^{2} A_{ij}(x_i - \xi_i)(x_j - \xi_j)}}, \tag{4.8}$$

for $n = 2$, and:

$$F_n(x, \boldsymbol{\xi}) = \frac{2}{\sigma_{n-1}\sqrt{\Delta}\left[\sum_{i,j=1}^{n} A_{ij}(x_i - \xi_i)(x_j - \xi_k)\right]^{\frac{n-2}{2}}}, \tag{4.9}$$

for $n > 2$, where $\mathbf{A} = \{A_{ij}\}_{i,j=1}^n$ is the inverse of the matrix $\mathbf{a}$, $\Delta$ is the determinant of $\mathbf{a}$, and where $\sigma_{n-1} = (n-2)\omega_n$, where $\omega_n$ is the area of the unit sphere in $\mathbb{R}^n$. Note that both $F_2$ and $F_n$ are positive when $|\mathbf{x} - \boldsymbol{\xi}|$ is small.

We intend to use $g$ in a way analogous to that in which Dupire used the straddle payoff in the last section. Thus, we need to apply an appropriate generalization of Itô's formula and of the Meyer–Tanaka formula to the function $g(\mathbf{S})$. Unfortunately, to our knowledge,

[4]For an intuitive motivation and discussion of how this equation arises naturally in our setting, see the discussion below. In a nutshell, it is the payoff which makes equation (4.10) hold true, after the singular $g$ is replaced by the finite $g^{\epsilon}$.

there is no such generalization which can handle the singularity of $g$ at $\mathbf{S} = \mathbf{K}$. It makes no sense financially to design a derivative security with an unbounded payoff at a point that can be reached, since the short party could face unbounded losses on his position. Also, on the mathematical side, the singularity of $g$ becomes compounded when we use it in conjunction with Itô's rule, since differentiating the function $g$ increases the order of the pole at $\mathbf{K}$. Thus, in lieu of Dupire's straddle function, we propose to design a family of derivatives whose payoffs $g^\epsilon(\mathbf{S})$ are such that

$$g^\epsilon(\mathbf{S}_0, \boldsymbol{\xi}) - \mathbb{E}^{\mathbb{Q}}[g^\epsilon(\mathbf{S}_T, \boldsymbol{\xi})] = V_I \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \beta_t^2 \mathbf{1}_{\mathbf{S}_t \in \mathcal{E}(\mathbf{K},\epsilon)}\, dt\right], \tag{4.10}$$

where $\mathcal{E}(\mathbf{K}, \epsilon)$ is the ellipsoid of "radius" $\epsilon$ around the point $\mathbf{K}$, i.e., the set $\{\mathbf{x} : (\mathbf{x} - \mathbf{K})^t \mathbf{A}(\mathbf{x} - \mathbf{K}) \le \epsilon\}$.

In order to motivate our main results, we first illustrate the main idea by proceeding at a *heuristic* level. *Were it* legitimate to apply Itô's formula to the process $G_t$ defined by $G_t = g(S_{1t}, S_{2t}, \dots S_{nt}, \boldsymbol{\xi})$, we would get

$$g(\mathbf{S}_T, \boldsymbol{\xi}) = g(\mathbf{S}_0, \boldsymbol{\xi}) + \sum_{i=1}^n \int_0^T \frac{\partial g(\mathbf{S}_t \boldsymbol{\xi})}{\partial S_i} dS_{it} - \frac{1}{2}\int_0^T \beta_t^2 \sum_{i=1}^n \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 g(\mathbf{S}_t, \boldsymbol{\xi})}{\partial S_i \partial S_j} dt. \tag{4.11}$$

Since $g$ satisfies (4.7), the last term in (4.11) simplifies and re-arranging implies

$$V_I \int_0^T \beta_t^2 \delta(\mathbf{S}_t - \mathbf{K}_t)\, dt = g(\mathbf{S}_0, \boldsymbol{\xi}) - g(\mathbf{S}_T, \boldsymbol{\xi}) + \sum_{i=1}^n \int_0^T \frac{\partial g(\mathbf{S}_t, \boldsymbol{\xi})}{\partial S_i} dS_{it}. \tag{4.12}$$

Although the above formula for $g$ fails to hold, due to the pole of $g$ and of $g$'s partial derivatives, we show below that after taking expectations, it does hold for the strongly localizing payoffs $g^\epsilon$ mentioned above. Indeed, for these payoffs, the following version of (4.12) will be shown to hold:

THEOREM 4.1. *Suppose that $\beta_t$, $t \in [0, T)$, has continuous sample paths, is independent of the filtration generated by the Brownian motions driving $S_{it}$, $t \in [0, T]$, and verifies the integrability condition $\mathbb{E}^{\mathbb{Q}}[\int_0^T \beta_s^2 ds] < \infty$. Suppose that $\mathbf{S}_t$ satisfies the SDEs (4.4). Let $\mathcal{E}_\epsilon^\xi = \{\mathbf{S} : A_n(\mathbf{S}, \boldsymbol{\xi}) := (\mathbf{S} - \boldsymbol{\xi})^t \mathbf{A}(\mathbf{S} - \boldsymbol{\xi}) \le \epsilon\}$, where $\mathbf{A} = \mathbf{a}^{-1}$ and let $A_n(\mathbf{S}, \boldsymbol{\xi})$ denote the quadratic form associated to the $n \times n$ matrix $\mathbf{A}$. Define a family of functions $g^\epsilon(\mathbf{x}, \boldsymbol{\xi})$ as follows.*

*In $\mathcal{E}_\epsilon^\xi$*

$$g^\epsilon(\mathbf{S}, \xi) = \begin{cases} -\dfrac{V_I}{\pi\sqrt{\Delta}\epsilon^2} A_2(\mathbf{S}, \xi) + \dfrac{V_I}{\pi\sqrt{\Delta}}(1 - \ln(\epsilon^2)), & \text{for } n = 2 \\[2ex] V_I \dfrac{2-n}{\sigma_{n-1}\sqrt{\Delta}\epsilon^n} A_n(\mathbf{S}, \boldsymbol{\xi}) + V_I \dfrac{n}{\sigma_{n-1}\sqrt{\Delta}\,\epsilon^{n-2}}, & \text{for } n > 2, \end{cases} \tag{4.13}$$

*and in $\mathbb{R}^n \setminus \mathcal{E}_\epsilon^\xi$, let*

$$g^\epsilon(\mathbf{S}, \boldsymbol{\xi}) = V_I F_n(\mathbf{S}, \boldsymbol{\xi}), \tag{4.14}$$

*where $F_2$ is defined in (4.8), and $F_n$, $n \geq 3$ in (4.9). In this formula, $\triangle$ denotes the determinant of matrix* $\mathbf{a}$. *Then, $g^\epsilon$ satisfies the equation*

$$\frac{V_I}{b(n)\sqrt{\triangle}\epsilon^n}\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \beta_t^2 \mathbf{1}_{\mathbf{S}_t\in\mathcal{E}_\epsilon^\xi} dt\right] = g^\epsilon(\mathbf{S}_0, \boldsymbol{\xi}) - \mathbb{E}^{\mathbb{Q}}[g^\epsilon(\mathbf{S}_T, \boldsymbol{\xi})], \tag{4.15}$$

*where $b_n = -\frac{1}{2\pi}$ for $n = 2$ and $b_n = -\frac{n(n-2)}{\sigma_{n-1}}$ for $n \geq 3$.*

*Proof*. The full proof of this theorem is given in Appendix A.1.2.

REMARK 4.2. The payoff $g^\epsilon$ can be replicated using our results related to Radon transforms, which allow us to back out the joint probability distribution. Given the joint probability distribution, we can trivially price any known payoff such as $g^\epsilon$ written on the asset's spot price $\mathbf{S}_T$ at time $T$.

Having established (4.15), we next show that it is possible to establish a differentiated form. More precisely, we have:

THEOREM 4.3. *Under the same conditions and with the same notation as in Theorem 4.1, we have for almost any $t_0 \in (0, T)$ and for $n \geq 2$ that*

$$\frac{V_I}{b(n)\sqrt{\triangle}\epsilon^n}\mathbb{E}^{\mathbb{Q}}\big[\beta_{t_0}^2 \mathbf{1}_{\mathbf{S}_{t_0}\in\mathcal{E}_\epsilon^\xi}\big] = -\frac{\partial}{\partial t}\mathbb{E}^{\mathbb{Q}}[g^\epsilon(\mathbf{S}_{t_0}, \boldsymbol{\xi})]. \tag{4.16}$$

*Proof*. Clearly

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \beta_s^2 \mathbf{1}_{\mathbf{S}_s\in\mathcal{E}_\epsilon^\xi} ds\right] \leq \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \beta_s^2 ds\right].$$

By Fubini's theorem since, $E^{\mathbb{Q}}[\int_0^T \beta_s^2 dt] < +\infty$, we have

$$\frac{1}{t-t_0}E^{\mathbb{Q}}\left[\int_{t_0}^t \beta_s^2 ds \mathbf{1}_{\mathbf{S}_s\in\mathcal{E}_\epsilon^\xi}\right] = \frac{1}{t-t_0}\int_{t_0}^t E^{\mathbb{Q}}\big[\beta_s^2 \mathbf{1}_{\mathbf{S}_s\in\mathcal{E}_\epsilon^\xi}\big]\, ds$$

and that $s \to \mathbb{E}^{\mathbb{Q}}[\beta_s^2 \mathbf{1}_{\mathbf{S}_s\in\mathcal{E}_\epsilon^\xi}]$ is integrable for almost every $s \in [0, T]$. Therefore, since the integral of an integrable function of one variable is absolutely continuous, we have

$$\lim_{t\to t_0}\frac{1}{t-t_0}\mathbb{E}^{\mathbb{Q}}\left[\int_{t_0}^t \beta_s^2 \mathbf{1}_{\mathbf{S}_s\in\mathcal{E}_\epsilon^\xi} ds\right] = -\mathbb{E}^{\mathbb{Q}}\big[\beta_{t_0}^2 \mathbf{1}_{\mathbf{S}_{t_0}\in\mathcal{E}_\epsilon^\xi}\big], \quad \text{a.e. } t_0 \in (0, T).$$

Thus, as a byproduct, we have also demonstrated that the function $t \to \mathbb{E}^{\mathbb{Q}}[g^\epsilon(\mathbf{S}_t, \boldsymbol{\xi})]$ is differentiable almost everywhere and the theorem follows. □

REMARK 4.4. Since the derivative of the expectation of $g^\epsilon$ can be approximated by a difference quotient, we see that we can effectively go long/short the left-hand side of (4.16), by going long/short a *calendar spread* of contracts on $g^\epsilon$.

REMARK 4.5 (Conditional version). Instead of (4.16), we can derive a conditional form, using

$$\frac{V_I}{b(n)\sqrt{\triangle}\epsilon^n}\mathbb{E}^{\mathbb{Q}}\big[\beta_{t_0}^2 \,\big|\, \mathbf{S}_{t_0} \in \mathcal{E}_\epsilon^{\boldsymbol{\xi}}\big] = \frac{V_I}{b(n)\sqrt{\triangle}\epsilon^n}\frac{\mathbb{E}^{\mathbb{Q}}\big[\beta_{t_0}^2 \mathbf{1}_{\mathbf{S}_{t_0}\in\mathcal{E}_\epsilon^{\xi}}\big]}{\mathbb{Q}\big(\mathbf{S}_{t_0} \in \mathcal{E}_\epsilon^{\xi}\big)}$$

$$= \frac{-\frac{\partial}{\partial t}\mathbb{E}^{\mathbb{Q}}\big[g^\epsilon(\mathbf{S}_t, \boldsymbol{\xi})\big]\big|_{t=t_0}}{\mathbb{Q}\big(\mathbf{S}_{t_0} \in \mathcal{E}_\epsilon^{\xi}\big)}.$$

Note that using the results in Section II on backing out the risk-neutral probability density $\mathbb{Q}$ from call option prices with all weights and strikes, we can then determine the denominator $\mathbb{Q}\left(\mathbf{S}_{t_0} \in \mathcal{E}_\epsilon^{\xi}\right)$ in the above expression.

*4.3.2. Multi-Asset Corridor Variance Swaps.* In this section, we introduce **multi-asset corridor variance swaps**. A multi-asset corridor variance swap is a payoff $g^C(\mathbf{S})$ with the following property

$$\mathbb{E}^{\mathbb{Q}}\left[g^C(\mathbf{S}_T)\right] = g^C(\mathbf{S}_0) - \frac{1}{2}V_I\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \beta_t^2 \mathbf{1}_{C(\mathbf{A},\mathbf{B})}(\mathbf{S}_t)\,dt\right], \tag{4.17}$$

where $C(\mathbf{A}, \mathbf{B})$, the $n$-dimensional corridor, is defined by

$$C(\mathbf{A}, \mathbf{B}) = \{A_1 \le s_1 \le B_1, \ldots, A_n \le s_n \le B_n\}.$$

This means that the contract $g^C$'s expected value is the time integrated local variance of the index option whose local variance at time $t$ is $V_I\beta_t^2$. To construct the claim $g^C$, we solve the following Poisson equation instead of (4.7)

$$-\frac{1}{2}a_{ij}g^C_{S_iS_j} = V_I\,\mathbf{1}_{C(\mathbf{A},\mathbf{B})}(\mathbf{S}).$$

As is well known, the solution of the inhomogeneous equation (4.18) is obtained by convolution with the fundamental solution $F_n$, which was defined earlier (see (4.8) and (4.9)). Thus, noting that $F_n(\mathbf{S}, \boldsymbol{\xi}) = \bar{F}_n(\mathbf{S} - \boldsymbol{\xi})$, we have

$$\begin{aligned} g^c(\mathbf{S}) &= V_I\int_{\mathbb{R}^n} \mathbf{1}_{C(\mathbf{A},\mathbf{B})}(\mathbf{s})\bar{F}_n(\mathbf{S}-\mathbf{s})\,d\mathbf{s}, \quad \mathbf{S} \in \mathbb{R}^n_+ \\ &= V_I\int_{\mathbf{s}\in C(\mathbf{A},\mathbf{B})} \bar{F}_n(\mathbf{S}-\mathbf{s})\,d\mathbf{s} \\ &= V_I\int_{a_n}^{b_n}\int_{a_{n-1}}^{b_{n-1}}\ldots\int_{a_1}^{b_1} \bar{F}_n(\mathbf{S}-\mathbf{s})\,ds_1ds_2\ldots ds_n. \end{aligned} \tag{4.18}$$

As a consequence, we have the following theorem:

THEOREM 4.6. *Suppose that the process $\beta_t$, $t \in [0, T]$ has continuous sample paths, is independent of the filtration generated by the Brownian motions driving $S_{it}$, $t \in [0, T]$, and verifies the integrability condition $\mathbb{E}^{\mathbb{Q}}[\int_0^T \beta_s^2 ds] < \infty$. Let $g^c$ be an $\epsilon$-approximating payoff,* *defined for all dimensions $n \geq 2$ by (4.18). Then we have*

$$\mathbb{E}^{\mathbb{Q}}[g^c(\mathbf{S}_t)] = g^c(\mathbf{S}_0) - \mathbb{E}^{\mathbb{Q}}\left[ V_I \int_0^T \beta_t^2 \mathbf{1}_{C(\mathbf{A},\mathbf{B})}(\mathbf{S}_t)\, dt \right]. \tag{4.19}$$

*Localizing in time, this result has the form*

$$\frac{\partial}{\partial t}\mathbb{E}^{\mathbb{Q}}[g^c(\mathbf{S}_t)]|_{t=t_0} = -\mathbb{E}^{\mathbb{Q}}\left[ V_I \beta_{t_0}^2 \mathbf{1}_{C(\mathbf{A},\mathbf{B})}(\mathbf{S}_{t_0}) \right], \quad \text{for a.e } t_0 \in [0, T). \tag{4.20}$$

*Proof*. Since the right-hand side of the Poisson equation (4.18) is in $L^\infty(\mathbb{R}^n)$, by well-known results (see Gilbarg and Trudinger 1983, sections 9.4 and 9.5), we have that the solution $g^c$ is in $W^{2,p}(\mathbb{R}^n)$, for all $p > 1$. The solution of the Poisson equation, when restricted to the set $\mathbb{R}^n_+$ is not unique, since we can add any solution of the homogeneous equation on that set. The representation (4.18), which uses the fundamental solution $F_n$ (see (4.8) and (4.9)) picks out a *particular solution* of the inhomogeneous equation. The standard Calderon-Zygmund potential estimates cited above then imply that the solution is in the required space. Since $g^c$ is in $W^{2,p}$ for all $p > 1$, we can repeat verbatim the argument used in the proof of Theorem 4.1 (see Proposition A.2 in the Appendix) to conclude that (4.19) holds. Then, to establish (4.20), we proceed exactly as in Theorem 4.3, Section 4.3.1. □

## 4.4. Multi-Asset Stochastic Local Variance: Lognormal Dynamics

In this subsection, we present the more realistic but more complicated version of the previous results, when the idiosyncratic component of the assets' price behavior is captured by a multivariate GBM.

*4.4.1. Local Results*. In the previous section, the spot prices could go negative, which is an undesirable property if the assets have limited liability. In this subsection, we present the lognormal version of the asset dynamics, which have the property that all spot prices are always positive. Natural logarithms of prices will have the same martingale component as in the last section, but the usual convexity correction forces the complications of nonzero drift upon us. The fundamental solution is different as a result of this drift and is more complicated than in the last section. However, it is still explicit and equally useful.

In this subsection, we suppose that there are $n \geq 2$ correlated stock prices $\mathbf{S}_t = S_{1t}, \ldots, S_{nt}$ whose risk-neutral dynamics are governed by the following stochastic differential equations[5]

$$dS_{it} = S_{it}\beta_t\sigma_i dW_{it} \quad i = 1, \ldots, n, \tag{4.21}$$

where $W_{it}$ are independent standard Brownian motions and $\beta_t$ is a common stochastic factor assumed independent of the filtration generated by the underlying Brownian motions. The stochastic process $\beta$ induces macro-economic shocks to all of the assets' volatilities and allows dependence on sources of uncertainty not captured by the filtration associated to the Brownian motions. The $\sigma_{ij}, i = 1, \ldots, n, j = 1, \ldots, n$ appearing in (4.21) are assumed to be constant.

[5]Here and below we use the Einstein summation convention to sum over repeated indices.

Let $\nu_i \equiv -\sum_{i=1}^n \sigma_{ij}^2/2, i = 1, \ldots, n$ be the constant idiosyncratic component of the convexity correction of $x_i \equiv \ln S_i$ and let:

$$V_I^{\ln} \equiv \rho_{ij}\sigma_i\sigma_j \tag{4.22}$$

be the constant component of the instantaneous variance of the sum of the log returns $\sum_{i=1}^n \ln S_i$. Similarly, let

$$V_I^n = \rho_{ij}\sigma_i\sigma_j S_i S_j \tag{4.23}$$

be the instantaneous normal variance of the basket. Ideally, our objective is again to synthesize the multi-asset local variance $\beta_T^2 V_I^\alpha dt\delta(S_{1T} - K_1, \ldots, S_{nT} - K_n) = \beta_T^2 V_I^\alpha dt\delta(x_{1T} - \ln K_1, \ldots, x_{nT} - \ln K_n)$, where $\alpha = \ln$ or $\alpha = n$. Once again, this payoff is zero unless $(S_{1T}, \ldots, S_{nT}) \in d(K_1, \ldots K_n)$ and in this event, $\beta_T^2 V_I^\alpha dt$ is the increment in the quadratic variation of $S_1 + \cdots + S_n$ for $\alpha = n$, and $\ln S_1 + \cdots + \ln S_n$ for $\alpha = \ln$. Alternative definitions for $V_I^\alpha$ would allow us to synthesize the increment of quadratic variation of other processes $f(S_{1t}, \ldots, S_{nt})$, which are also of interest in finance. For instance, if $n = 2$ and $f(S_1, S_2) = w_1\ln(S_1) + w_2\ln(S_2)$, where $w_i$, $i = 1, 2$ are fixed weights, then we let $V_I = w_1^2\sigma_1^2 + 2\sigma_1\sigma_2\rho w_1 w_2 + \sigma_2^2 w_2^2$.

Consider the elliptic PDE

$$\frac{1}{2}a_{ij}u_{x_i x_j} + \nu_i u_{x_i} = -V_I^\alpha \delta(\mathbf{x} - \mathbf{k}), \quad \alpha = n \text{ or } \alpha = \ln, \tag{4.24}$$

where $\mathbf{x} = \ln(\mathbf{S})$, $\mathbf{k} = \ln(\mathbf{K})$ and where the diffusion matrix $\Sigma = \{a_{ij}\}_{i,j=1}^n$ is given by $\sigma\sigma^t$. For $i = 1, \ldots, n$, $\nu_i \equiv -\frac{\sigma_i^2}{2}$ is the idiosyncratic component of the drift in $x_i$ and $V^\alpha$ was defined above. We will suppose throughout that the rank of the matrix $\sigma$ is $n$. This implies that the rank of $\sigma\sigma^{\mathrm{T}}$ is also $n$ and so our matrix $a$ is positive and invertible. The left-hand side of equation (4.24) is an elliptic counterpart of the $n$-dimensional Black–Scholes equation in log price coordinates. A fundamental solution can be found in explicit form. As shown in Appendix A.2, it is given by $V_I^\alpha \mathcal{F}_n^{\log}$, where

$$\begin{aligned} \mathcal{F}_n^{\log}(\mathbf{x}, \mathbf{y}, \mathbf{k}) = &\frac{V_I^\alpha}{\pi\sqrt{\triangle}} \left(\frac{Q}{2\pi\sqrt{(\mathbf{x}-\mathbf{k})^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})}}\right)^{\frac{n-2}{2}} \\ &\times \exp\left[-\tfrac{1}{2}\tilde{\boldsymbol{\nu}}^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})\right] K_{\frac{n}{2}-1}\left[Q\sqrt{(\mathbf{x}-\mathbf{k})^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})}\right], \end{aligned} \tag{4.25}$$

where $\triangle$ again denotes the determinant of the variance-covariance matrix $\Sigma$ with $a_{ij} = \rho_{ij}\sigma_i\sigma_j$, and $\tilde{\boldsymbol{\nu}}$ equals twice the vector of drifts

$$\tilde{\boldsymbol{\nu}} \equiv \begin{pmatrix} 2\nu_1 \\ 2\nu_2 \\ \cdot \\ \cdot \\ 2\nu_n \end{pmatrix}, \tag{4.26}$$

the matrix $\mathbf{A} \equiv \Sigma^{-1}$ is the inverse of $\Sigma$, $K_{n/2-1}$ is the modified Bessel function of the second kind and degree zero (either fractional or integer order, depending on whether $n$ is odd or even), and where $Q$ is the constant $\sqrt{\tilde{\mathbf{v}}^{\mathrm{T}}\mathbf{A}\tilde{\mathbf{v}}}$. Hence, the argument of the modified Bessel function is proportional to

$$(\mathbf{x}-\mathbf{k})^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k}) = \sum a^{ij}(x_i-k_i)(x_j-k_j). \tag{4.27}$$

For instance, when $n = 2$, this argument may be expressed in the familiar form

$$\left(\frac{x_1-k_1}{\sigma_1}\right)^2 - 2\rho\frac{x_1-k_1}{\sigma_1}\frac{x_2-k_2}{\sigma_2} + \left(\frac{x_2-k_2}{\sigma_2}\right)^2. \tag{4.28}$$

Notice that the function $g$ has a singularity at $(x_1,\ldots,x_n) = (k_1,\ldots,k_n)$. Once again, we introduce a family of functions, $g^\epsilon$ parametrized by $\epsilon$, with the property that, if an investor believes that the multi-asset dynamics are lognormal and wishes to purchase or sell protection against the level of volatility of an index written on these assets, when the assets are in an $\epsilon$ neighborhood of a target level (for instance at-the-money), then the derivatives with payoff $g^\epsilon$ allow him to do so.

THEOREM 4.7. *Let $(S_{1t},\ldots,S_{nt})$ be a solution of (4.21), $\mathbf{x}_t = (\ln S_{1t},\ldots,\ln S_{nt})$, where we assume that $\{\beta_s, 0\le s\le T\}$ has continuous sample paths, and is independent of the filtration generated by $(W_{1s},\ldots,W_{ns})$, $0\le s\le t$ and satisfies the condition $\mathbb{E}^{\mathbb{Q}}[\int_0^T \beta_s^2 ds] < +\infty$. Let $\rho^{\mathbf{x},\mathbf{k}} = \sqrt{(\mathbf{x}-\mathbf{k})^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})}$ and given $\epsilon$, let $g^\epsilon$ be defined by*[6]

$$g^\epsilon(\mathbf{x},\mathbf{k}) = \begin{cases} \dfrac{V_I}{\pi\sqrt{\triangle}}\left(\dfrac{\mathbb{Q}}{2\pi\rho^{\mathbf{x},\mathbf{k}}}\right)^{\frac{n-2}{2}} \exp\left[-\dfrac{1}{2}\tilde{\mathbf{v}}^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})\right] K_{\frac{n}{2}-1}[\mathbb{Q}\rho^{\mathbf{x},\mathbf{k}}] & \text{for } \rho^{\mathbf{x},\mathbf{k}} > \epsilon \\ V_I\left[a_1(\epsilon)(\rho^{x,k})^2 + a_2(\epsilon)\right]\exp\left[-\dfrac{1}{2}\tilde{\mathbf{v}}^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})\right] & \text{for } \rho^{\mathbf{x},\mathbf{k}} \le \epsilon \end{cases} \tag{4.29}$$

*where $\triangle$ denotes the determinant of the variance-covariance matrix $\Sigma = \sigma\sigma^{\mathrm{T}}$ (of the logarithms of the returns), $A = \Sigma^{-1}$ and where the coefficients $a_1(\epsilon)$, $a_2(\epsilon)$ have the property that*

$$a_1(\epsilon) = \begin{cases} \dfrac{n-2}{2\pi}\dfrac{1}{\epsilon^2}, & n = 2 \\ \dfrac{2-n}{4\pi}\dfrac{1}{\epsilon^3}, & n = 3 \\ (2-n)\dfrac{\left(\frac{n}{2}-1\right)!}{4\pi^{\frac{n}{2}}}\dfrac{1}{\epsilon^n}, & n > 2, \quad n \text{ even} \\ \dfrac{(2-n)}{2}(1.3.5..(n-4))\left(\dfrac{1}{2\pi}\right)^{\frac{n-1}{2}}\dfrac{1}{\epsilon^n} & n \ge 5, \quad n \text{ odd} \end{cases} \tag{4.30}$$

[6]Note that if $n = 1$ there is no need for a cutoff for $|x-k|$ small. A solution generalizing (4.1) can be easily obtained for all $x$.

*and*

$$a_2(\epsilon) = \frac{n}{2-n}\epsilon^2 a_1(\epsilon).$$

*Then we have the identity*

$$V_I^\alpha \frac{d_n(\epsilon)}{\epsilon^n}\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \beta_t^2 \mathbf{1}_{\mathbf{x}_t \in \mathcal{E}_\epsilon^k} dt\right] + O(\epsilon) = g^\epsilon(\mathbf{x}_0) - \mathbb{E}^{\mathbb{Q}}[g^\epsilon(\mathbf{x}_T)] \quad \text{for} \quad \epsilon = o(1), \tag{4.31}$$

*where, in formula (4.31), $d_n$ is defined by $d_n = \epsilon^n a_1(\epsilon) = O(1)$.*

*Proof*. Note that when compared with (4.15), the main difference is the extra term $O(\epsilon)$ as $\epsilon$ tends to zero. Again, we refer to Appendix A.2.3 for construction of the family $g^\epsilon$ and to the end of that section for the conclusion of the proof of Theorem 4.7. □

THEOREM 4.8. *Under the same conditions as those in Theorem 4.7, we have, for any $t_0 \in [0, T)$*

$$V_I^\alpha \frac{2nd(\epsilon)}{\epsilon^n}\mathbb{E}^{\mathbb{Q}}\left[\beta_{t_0}^2 \mathbf{1}_{\mathbf{x}_{t_0} \in \mathcal{E}_\epsilon^k} dt\right] + O(\epsilon) = \frac{\partial}{\partial t}\mathbb{E}^{\mathbb{Q}}[g^\epsilon(\mathbf{x}_t)]|_{t=t_0}.$$

*Proof*. The proof is the same as that of Theorem 4.3 in the preceding section. □

*4.4.2. Corridor Multi-Asset Variance Swaps.* In this section, we present the analog of the results in section 4.3.2 for the lognormal case. As the reader may expect by now, we again solve a Poisson equation, namely

$$\frac{1}{2}a_{ij}u_{x_i x_j} + v_i u_{x_i} = -V_I^\alpha \mathbf{1}_{C(\mathbf{A},\mathbf{B})}(\mathbf{x}).$$

This is an inhomogeneous linear elliptic equation with constant coefficients. A particular solution is once again given by convolution with the fundamental solution $\mathcal{F}_n^{\log}$ (see (4.25)), $u(\mathbf{x}) = V_I^\alpha \mathcal{F}_n^{\log} * \mathbf{1}_{C(\mathbf{A},\mathbf{B})}$. By elliptic regularity theory, we again have that the solution is in $W^{2,p}(\mathbb{R}^n)$, for $p \geq 1$ and therefore, with the same argument as in the proof of Theorem 4.6, we can establish the following theorem:

THEOREM 4.9. *Under the same conditions as those in Theorem 4.6, but assuming now that $\mathbf{S}_t$ satisfies (4.21), we have (4.19) and (4.20).*

## 4.5. Illustration: Multi-Asset Stochastic Volatility Model with Common Stochastic Factor

In the previous sections of this paper, we have not specified the process driving the systematic component of the volatility. While we regard the generality of these results to be of interest, we also think that it may be of interest to apply these results in a familiar, albeit restricted, context. As a consequence, in this section, we suppose that the model is specified as follows

$$
\begin{aligned}
&dF_{1t} = \sigma_1\sqrt{v}F_{1t}dW_{1t} \\
&dF_{2t} = \sigma_2\sqrt{v}F_{2t}dW_{2t} \\
&dv_t = \iota(\theta - v)dt + \alpha\sqrt{v_t}dW_{3t}, \qquad F_{10} = \bar{F}_1^0,\ F_{20} = \bar{F}_2^0;\ v_0 = \bar{v}_0
\end{aligned} \tag{4.32}
$$

where $F_i$, $i = 1, 2$ are the forward prices of the stocks and where we assume that

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[dW_{1t}, dW_{2t}] = \rho dt \\
&\mathbb{E}^{\mathbb{Q}}[dW_{it}, dW_{3t}] = 0, \quad i = 1, 2,
\end{aligned}
$$

and $W_{it}$, $i = 1, 2, 3$ are standard Brownian motions. Note that this model is a special case of the family of models introduced in Section 4.4.1, the "Lognormal dynamics" case. Indeed, for the model considered in *this* section, we take $n = 2$, in (4.21) and still in the notation of that section, the independent stochastic process $\beta_s$, $s \in [0, T]$, is given here by $\sqrt{v_s}$, $s \in [0, T]$. We recall from Section 4.4.1 and 4.4.2 that in the construction of the family of payoffs $g^\epsilon$, the process driving the stochastic factor $\beta_t$ does not play a role. This is because the construction of $g^\epsilon$ is based only on the idiosyncratic part of the process driving the asset, i.e., $g^\epsilon$ is constructed with reference to the operator

$$
\mathcal{L}u = -\left[\frac{1}{2}\sigma_1^2 u_{x_1x_1} + \rho\sigma_1\sigma_2 u_{x_1x_2} + \frac{1}{2}\sigma_2^2 u_{x_2x_2} - \frac{1}{2}\sigma_1^2 u_{x_1} - \frac{1}{2}\sigma_2^2 u_{x_2}\right].
$$

For our $\epsilon$ approximating claim $g^\epsilon$ defined by (4.29), we set $n = 2$. Since in the present setting, the dynamics of $\beta_t$ (here $\sqrt{v_t}$) are specified by (4.32), we can make formula (4.16) more explicit. For this, let $B_t^v = \int_0^t v_s^2 ds$ and let $p_t(B)$ denote the probability transition density for the time average of the squared $v$ process, ie, $p_t(B) = \text{Prob}(B_t \in (B - dB, B + dB))$. Suppose we are trying to synthesize the local variance a.e. of a basket option with payoff $(F_1 + F_2 - K)^+$. In that case, letting $\mathbf{F} = (F_1, F_2)$, and recalling (4.22) and (4.23), let $V_I^n$, $V_I^{ln}$ denote respectively the instantaneous variance at $\mathbf{F} = \boldsymbol{\xi}$ of the basket and that of the log returns, where $V_I^n = \sigma_1^2\xi_1^2 + 2\sigma_1\sigma_2\rho\xi_1\xi_2 + \sigma_2^2\xi_2^2$, and $V^{ln} = \sigma_1^2 + 2\sigma_1\sigma_2\rho + \sigma_2^2$. In these formulas $\sigma_i$, $i = 1, 2$ are the idiosyncratic part of the coefficients in the $F_{it}$ dynamics. Also, conditional on a path of $v_t$, the distribution of $(F_{1t}, F_{2t})$ is lognormal, i.e., $(F_{1,t}, F_{2t})$ can be expressed in the form $(e^{x_{1t}}, e^{x_{2t}})$, where $(x_{1t}, x_{2t})$ is normally distributed $\mathcal{N}^{\int_0^t v_s^2 ds}(\Lambda, M)$ with mean vector $(-\frac{1}{2}\sigma_1^2\int_0^t v_s^2, -\frac{1}{2}\sigma_2^2\int_0^t v_s^2)$ and covariance matrix $\Lambda \equiv \begin{pmatrix} \sigma_1^2\int_0^t v_s^2 ds & \sigma_1\sigma_2\rho\int_0^t v_s^2 ds \\ \sigma_1\sigma_2\rho\int_0^t v_s^2 ds & \sigma_2^2\int_0^t v_s^2 ds \end{pmatrix}$. Also, we use the definition of the $\epsilon$ approximating payoff $g^\epsilon(\mathbf{F}, \boldsymbol{\xi})$ defined in (4.29), in which we take $n = 2$. Then we have, recalling that the process $B_t$, $t \in [0, T]$ is independent of $\mathbf{F}_t$, $t \in [0, T]$

$$
\begin{aligned}
&\frac{V_I}{b(n)\sqrt{\triangle}\epsilon^n}\mathbb{E}^{\mathbb{Q}}\big[\beta_T^2 \mathbf{1}_{\mathbf{F}_T \in \mathcal{E}_\epsilon^\xi}\big] \\
&= -\frac{\partial}{\partial t}\{\mathbb{E}^{\mathbb{Q}}[g^\epsilon(\mathbf{F}_t, \boldsymbol{\xi})]\}|_{t=T}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\partial}{\partial t}\Bigg(\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)B}} \\
&\qquad \times g^\epsilon(e^{x_1}, e^{x_2}, \xi) e^{-\frac{(x_1-\log\bar{F}_1^0+\sigma_1^2 B/2)^2}{2\sigma_1^2 B}+\rho\frac{(x_1-\log\bar{F}_1^0+\sigma_1^2 B/2)(x_2-\log\bar{F}_2^0\sigma_2^2 B/2)}{\sigma_1\sigma_2 B}-\frac{(x_2-\log\bar{F}_2^0+\sigma_2^2 B/2)^2}{2\sigma_2^2 B}} \\
&\qquad \times p_t(B)dx_1dx_2dB\Bigg)\Bigg|_{t=T} \\
&= -\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)B}} \\
&\quad \times g^\epsilon(e^{x_1}, e^{x_2}, \xi) e^{-\frac{(x_1-\log\bar{F}_1^0+\sigma_1^2 B/2)^2}{2\sigma_1^2 B^2}+\rho\frac{(x_1-\log\bar{F}_1^0+\sigma_1^2 B/2)(x_2-\log\bar{F}_2^0+\sigma_2^2 B/2)}{\sigma_1\sigma_2 B}-\frac{(x_2-\log\bar{F}_2^0+\sigma_2^2 B/2)^2}{2\sigma_2^2 B}} \\
&\quad \times \frac{\partial p_t(B)}{\partial t}\bigg|_{t=T} dx_1dx_2dB.
\end{aligned}
$$

## 5. SUMMARY AND FUTURE RESEARCH

We have shown how to extend the fundamental results of BL (1978) and Dupire (1996) to the multi-asset case. In particular, the joint risk-neutral PDF of asset prices can be extracted from basket option prices using Radon transforms, as first pointed out by Lipton (2001) for the two asset case. Assuming only that price processes are continuous, the conditional expectation of the variance of a basket can also be extracted from basket option prices. The recipe for constructing this conditional expectation depends on the nature of the idiosyncratic component of each asset's variance. We explicitly displayed the construction when this idiosyncratic component displayed constant normal volatility and constant lognormal volatility. One can also consider the general case where the idiosyncratic component of the volatilities depends on the assets' prices, which leads to elliptic equations with variable coefficients. These can be tackled using a method of Hadamard (see Shimakura 1992).

In the single asset case, Dupire's work has been extended by adding jumps or by replicating payoffs which are a nonlinear function of the total variance. It follows that one could consider these extensions in the multi-asset case. In the interests of brevity, these extensions are best left for future research.

## APPENDIX

### A.1. The *n-D* Normal Case

In this section, we show how to construct the family of functions $g^\epsilon$ in the $n$-$D$ normal case. The calculations in all other multidimensional cases have many points in common with this case, but are more involved. We begin with its construction.

A.1.1. *The Construction of the $\epsilon$-Family $g^\epsilon$.* In this section, we show how to modify the fundamental solution (4.7) in a neighborhood of the pole $\boldsymbol{\xi}$ so that the modified solution $u_\epsilon(\mathbf{x}, \xi)$ is analytic except at the point $\boldsymbol{\xi}$ and $C^1$ for all $\mathbf{x} \in \mathbb{R}^n$, and so that the partial derivatives $\partial_{x_i}\partial_{x_j} u_\epsilon(x, \xi)$ are all locally bounded in a neighborhood[7] of $\mathbf{x} = \boldsymbol{\xi}$.

[7]The second order partial derivatives taken in the sense of distributions are then globally bounded functions, since $u$ vanishes at infinity.

Let

$$A_n(\mathbf{x}, \boldsymbol{\xi}) = \sum_{i,j=1}^{n} A_{ij}(x_i - \xi_i)(x_j - \xi_j).$$

For $\epsilon > 0$ and $\boldsymbol{\xi}$, consider the ellipsoids defined by

$$\mathcal{E}_\epsilon^\xi(n) = \{\mathbf{x} : \sqrt{A_n(\mathbf{x}, \boldsymbol{\xi})} < \epsilon\}.$$

We have

$$F_n(\mathbf{x}, \boldsymbol{\xi}) = \begin{cases} -\dfrac{1}{\pi\sqrt{\Delta}} \ln(A_2(\mathbf{x}, \boldsymbol{\xi})) & \text{for } n = 2 \\ \dfrac{2}{\sigma_{n-1}\sqrt{\Delta}} A_n^{\frac{2-n}{2}}(\mathbf{x}, \boldsymbol{\xi}) & \text{for } n > 2. \end{cases} \tag{A.1}$$

The first order partials are

$$\partial_{x_i} F_n(\mathbf{x}, \boldsymbol{\xi}) = \begin{cases} -\dfrac{1}{\pi\sqrt{\Delta}} A_2^{-1}\, \partial_{x_i} A_2, & \text{for } n = 2 \\ \dfrac{2-n}{\sigma_{n-1}\sqrt{\Delta}} A_n^{-n/2}\, \partial_{x_i} A_n, & \text{for } n > 2. \end{cases} \tag{A.2}$$

To define a continuation into the interior of the ellipsoid $\mathcal{G}_\epsilon^\xi$, we make an ansatz of the following form

$$\bar{g}(\mathbf{x}, \boldsymbol{\xi}) = \begin{cases} \dfrac{a_2}{\epsilon^2} A_2(\mathbf{x}, \boldsymbol{\xi}) + b_2, & \text{for } n = 2 \\ \dfrac{a_n}{\epsilon^n} A_n(\mathbf{x}, \boldsymbol{\xi}) + b_n, & \text{for } n > 2, \end{cases} \tag{A.3}$$

where the coefficients $a_n, b_n, n \geq 2$ will be determined in such a way that the fundamental solution along with all its first partial derivatives match along the boundary of $\mathcal{E}_\epsilon$ those of the function $\hat{g}$. The construction also guarantees that all second order partial derivatives of $\hat{g}$ are bounded in $\mathcal{E}_\epsilon$.

For the continuity of the first partial derivatives, we need

$$\begin{cases} -\dfrac{1}{\pi\sqrt{\Delta}} \dfrac{1}{\epsilon^2} \partial_{x_i} A_2|_{x\in\partial\mathcal{E}_\epsilon} = \dfrac{a_2}{\epsilon^2} \partial_{x_i} A_2|_{x\in\partial\mathcal{E}_\epsilon^\xi} & \text{for } n = 2 \\ \dfrac{2-n}{\sigma_{n-1}\sqrt{\Delta}} \dfrac{1}{\epsilon^n} \partial_{x_i} A_n|_{x\in\partial\mathcal{E}_\epsilon}^\xi = \dfrac{a_n}{\epsilon^n} \partial_{x_i} A_n|_{x\in\partial\mathcal{E}_\epsilon^\xi} & \text{for } n > 2. \end{cases}$$

From this, we read off the coefficients

$$a_2 = -\frac{1}{\pi\sqrt{\Delta}} \tag{A.4}$$

$$a_n = \frac{2-n}{\sigma_{n-1}\sqrt{\Delta}} \quad \text{for } n > 2. \tag{A.5}$$

Next, we determine $b_i$, $i = 1, 2$ from the requirement that the function itself be continuous. This leads to

$$-\frac{1}{\pi\sqrt{\Delta}}\ln(\epsilon^2) = -\frac{1}{\pi\sqrt{\Delta}} + b_2(\epsilon)$$
$$\frac{2}{\sigma_{n-1}\sqrt{\Delta}}\epsilon^{2-n} = \frac{2-n}{\sigma_{n-1}\sqrt{\Delta}}\epsilon^{2-n} + b_n(\epsilon),$$

so that

$$b_2(\epsilon) = \frac{1}{\pi\sqrt{\Delta}}\left(1 - \ln(\epsilon^2)\right) \tag{A.6}$$

$$b_n(\epsilon) = \frac{n}{\sigma_{n-1}\sqrt{\Delta}}\epsilon^{2-n}. \tag{A.7}$$

In summary, we have that

$$\bar{g}^\epsilon(\mathbf{x}, \boldsymbol{\xi}) = \begin{cases} -\dfrac{1}{\pi\sqrt{\Delta}\epsilon^2}A_2(x,\xi) + \dfrac{1}{\pi\sqrt{\Delta}}(1 - \ln(\epsilon^2)) & \text{for } n = 2 \\ \dfrac{2-n}{\sigma_{n-1}\sqrt{\Delta}\,\epsilon^n}A_n(\mathbf{x},\boldsymbol{\xi}) + \dfrac{n}{\sigma_{n-1}\sqrt{\Delta}\,\epsilon^{n-2}} & \text{for } n > 2, \end{cases} \tag{A.8}$$

and therefore, switching to the $\mathbf{S}$ variable as in Section 4.3.1, define

$$g^\epsilon(\mathbf{S}, \boldsymbol{\xi}) = \begin{cases} V_I F_n(\mathbf{S}, \boldsymbol{\xi}) & \text{for } \mathbf{S} \in (\mathcal{E}^\xi_\epsilon)^c \\ V_I\,\bar{g}(\mathbf{S}, \boldsymbol{\xi}) & \text{for } S \in \mathcal{E}^\xi_\epsilon, \end{cases} \tag{A.9}$$

where $V_I$ was defined in (4.6) as the idiosyncratic component of the instantaneous normal variance of $\sum_{i=1}^n w_i S_i$ at $\mathbf{S} = \boldsymbol{\xi}$ and $F_n$ was defined in (A.1).

A.1.2. *Justification of Itô's Formula Applied to the Family* $g^\epsilon$. In this section, we complete the proof of Theorem 4.1, by establishing formula (4.15). The formula is a straightforward but not direct consequence of known extensions of Itô's formula that appear in the literature, for instance in Bensoussan and Lions (1982), Krylov (1980), and Foellmer and Protter (2000). It is not direct in that we need to apply an Itô type formula in a context where the stochastic factor $\beta_s$ enters only"path-by-path" in the definition of the elliptic operator $A$, appearing in the statement of Proposition A.1. The reason for this distinction can be traced back to the basic setting of the assumed stochastic dynamics for the asset $S_{it}$ (see (4.4)), wherein we do not constrain $\beta_s$ by imposing any dynamics on it or any conditions other than the integrability condition and the continuity and independence condition. The way we intend to use the generalized Itô's lemma is to apply it to $\mathbf{S}_t$ dynamics *conditional on* a realization of the path $\{\beta_s, s \in [0, T]\}$. When we condition of such a realization, $\mathbf{S}_t$ becomes a Gaussian process with *time-dependent variance*. In addition our function $g^\epsilon$ is $C^{1,1}$ (all first order partial derivatives are Lipschitz functions) and vanishes at infinity. Therefore $g^\epsilon$ belongs to the Sobolev class $W^{2,p}$ for every $p \geq 1$, and we have enough regularity to apply the Bensoussan–Lions version (see below) of Itô's formula. The last step in the proof is to take expectations over all possible realizations of the process $\beta_t$. We now give exact statements and proofs.

The following lemma is established in Bensoussan and Lions (1980), on page 183, and its proof relies on properties of the generator stated on page 156 of that treatise. Their version is more general than the one we need here, since they consider a bounded domain and processes, stopped prior to exiting the domain. We state only the simpler version that suffices for our purposes.

PROPOSITION A.1. *Let* $\mathbf{y}(t) \in \mathbb{R}^n$ *be a solution of the system*

$$y_i(t) = y_i(0) + \int_0^t h_i(\mathbf{y}(s), s)\, ds + \int_0^t \sigma_{ij}(\mathbf{y}(s), s)\, dW_{sj}, \quad i = 1, \ldots, n$$

*with infinitesimal generator* $A(t) = Trace(\frac{1}{2}\sigma\sigma^{\mathrm{T}} D^2 u) + \mathbf{h} \cdot Du$. *Let* $\Phi(x, t)$ *be a continuous function on* $\mathcal{Q} = \mathbb{R}^n \times [0, T]$, *such that* $(\frac{\partial}{\partial t} - A(t))\Phi \in L^p(0, T; L^p(\mathbb{R}^n))$,[8] $p > \frac{n}{2} + 1$, *and assume that the transition probability (fundamental solution)* $p(\mathbf{x}, t_1, \boldsymbol{\xi}, t_2)$ *satisfies the estimate*

$$p(\mathbf{x}, t_1, \boldsymbol{\xi}, t_2) \le M(t_2 - t_1)^{-n/2} \exp^{\frac{-\alpha|\mathbf{x}-\boldsymbol{\xi}|^2}{t_2 - t_1}}, \tag{A.10}$$

*for some positive constant* $\alpha$. *Then, for* $0 \le \theta \le \theta' \le T$ *we have that*

$$\mathbb{E}^{\mathbb{Q}}[\Phi(\theta')] = \mathbb{E}^{\mathbb{Q}}[\Phi(\theta)] + E^{\mathbb{Q}} \int_\theta^{\theta'} \left[ \left( \frac{\partial \Phi}{\partial t} - A(t) \right) \Phi(y(s), s)\, ds \right]. \tag{A.11}$$

Note that in our setting $g^\epsilon = \Phi$ does not depend explicitly on $t$. To apply the proposition, the operator $A^\omega(t)$ we will consider, for each *fixed* $\omega$, *is*

$$A^\omega(t) = \beta_t(\omega) a_{ij} \frac{\partial^2}{\partial S_i \partial S_j}.$$

Note that, since $g^\epsilon$ does not depend explicitly on $t$ we have $(\frac{\partial}{\partial t} - A^\omega(t))g^\epsilon(\mathbf{S}) = \beta_t(\omega) a_{ij} \frac{\partial^2}{\partial S_i \partial S_j} g^\epsilon = Cst\beta_t(\omega)$. Since $\beta_s$ is for each $\omega$ a continuous function of $t$, $g^\epsilon(\mathbf{S})$ verifies the conditions required by $\Phi$ in Proposition A.1. The exact form of the constant ("Cst," above) on the right-hand side of the equation for $g^\epsilon$ is given in the sequel (see equation (A14)). Since, by construction $g^\epsilon$ is in $C^{1,1}(\mathbb{R}^n)$, it verifies all the conditions that $\Phi$ must verify in the theorem.[9] In addition, for fixed $\omega$, the operators $A^\omega(t)$ trivially verify the estimate (A.10), since the transition probability matrix associated to the generator $A^\omega(t)$ is $\frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}(t)|^{1/2}} e^{-\frac{1}{2}\mathbf{x}'\boldsymbol{\Sigma}(\mathbf{t})^{-1}\mathbf{x}}$ where

$$\Sigma(t) = \left\{ \int_0^t \beta_s^2(\omega)\, ds \right\} \mathbf{a}.$$

PROPOSITION A.2 (Conclusion of the proof of Theorem 4.1). *Let* $\mathbf{S}_t = (S_{1t}, \ldots, S_{nt})$, $t \in [0, T]$ *be a stock process that follows the process (4.5) (multivariate version) and assume in addition that* $\beta_t$, $t \in [0, T]$ *is a stochastic factor, with continuous sample paths, and is*

[8] As Bensoussan and Lions point out, $L^p(0, T; L^p_{loc}(\mathbb{R}^n))$ suffices, provided we stop the process when it exits a sufficiently large ball.

[9] Bensoussan and Lions' statement also assumes that $\Phi$ is $C_0$, i.e., vanishes on the boundary of the domain, but this is not necessary in our setting, since we work on all of $\mathbb{R}^n$.

*independent of the filtration generated by $S_t$, with the property $\mathbb{E}^{\mathbb{Q}}[\int_0^T \beta_s^2 ds] < +\infty$. Let $g^\epsilon(\mathbf{S}, \xi)$ be defined by (A.9), then Dynkin's formula holds for $g^\epsilon(\mathbf{S}_t, \boldsymbol{\xi})$ i.e.,*

$$\mathbb{E}^{\mathbb{Q}}\left[g^\epsilon(\mathbf{S}_T, \boldsymbol{\xi})\right] = g^\epsilon(\mathbf{S}_0, \boldsymbol{\xi}) - \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{2}\int_0^T \beta_t^2 \text{Trace}(\mathbf{a}D^2 g^\epsilon)\, dt\right].$$

*Proof*. Conditional on a path of the independent stochastic process $\beta_t$, $\mathbf{S}_t = (S_{1t}, \ldots, S_{nt})$ is a Gaussian process with variance-covariance matrix whose $(i, j)$ entry is given by $\sigma_i \sigma_j \rho_{ij} \int_0^t \beta_s^2$, provided we define $\rho_{ij} = 1$ for $i = j$. Denote this Gaussian process by $\mathbf{S}_t^{\{\beta_s, s \le t\}}$. □

Next, take expectations over the distribution of the paths of the process $\{\beta_s\}_{0 \le s \le t}$, which was assumed independent of the filtration $\mathcal{F}_s$ generated by the Brownian motions $W_{1t}, \ldots, W_{nt}$. Note that it is permissible to do so, because by assumption, $\mathbb{E}^{\mathbb{Q}}[\int_0^T \beta_s^2 ds] < +\infty$ and since $\mathbf{a}D^2 g^\epsilon$ is everywhere bounded (given our construction of $g^\epsilon$), so that $\beta_s^2(\mathbf{a}D^2 f) < \text{Cst}\, \beta_s^2$. Thus

$$0 = \mathbb{E}^{\mathbb{Q}}\left[g^\epsilon(\mathbf{S}_t, \boldsymbol{\xi}) - f(\mathbf{S}_0, \boldsymbol{\xi}) - \frac{1}{2}\int_0^T \beta_t^2 \text{Trace}(\mathbf{a}D^2 g^\epsilon)\, dt\right], \tag{A.12}$$

as desired. Using the definition (A.9) of the latter and (A.3) and using formulas (A.5) for the coefficients, we find

$$D_S^2(g^\epsilon(\mathbf{S}, \boldsymbol{\xi})) = \begin{cases} V_I D_S^2 F_n & \text{for } (S_1, \ldots, S_n) \in \mathbb{R}_n^+ \setminus \mathcal{E}_\epsilon^\xi \\ -\dfrac{2}{\pi\sqrt{\Delta}\epsilon^2} V_I \mathbf{A} & \text{for } n = 2 \quad \in \mathcal{E}_\epsilon^\xi \\ -\dfrac{2(n-2)V_I}{\sigma_{n-1}\sqrt{\Delta}\epsilon^n}\mathbf{A} & \text{for } n \ge 3 \quad \in \mathcal{E}_\epsilon^\xi, \end{cases} \tag{A.13}$$

where we recall that $\mathbf{A}$ is the inverse matrix of $\mathbf{a}$. Since $\mathbf{a}$ and $\mathbf{A}$ are inverse matrices $\mathbf{aA} = I$, $\text{Trace}(I) = n$ and since $F_n$, $n \ge 2$ is a fundamental solution outside the ellipsoid, we have

$$\text{Trace}(\mathbf{a}D_S^2(g^\epsilon(\mathbf{S}, \boldsymbol{\xi})) = \begin{cases} 0, & \in \mathbb{R}_n^+ \setminus \mathcal{E}_\epsilon^\xi, n \ge 2, \\ -\dfrac{V_I}{\pi\sqrt{\Delta}\epsilon^2} & \text{for } n \ge 3 \quad \in \mathcal{E}_\epsilon^\xi, \\ -\dfrac{2n(n-2)V_I}{\sigma_{n-1}\sqrt{\Delta}\epsilon^n}, & \text{for } n \ge 3 \quad \in \mathcal{E}_\epsilon^\xi. \end{cases} \tag{A.14}$$

Now insert these values into (A.12) to obtain (4.15). Taking expectations we get

$$g^\epsilon(\mathbf{S}_0, \boldsymbol{\xi}) - \mathbb{E}[g^\epsilon(\mathbf{S}_t, \boldsymbol{\xi})] = \begin{cases} \dfrac{V_I}{2\pi\sqrt{\Delta}\,\epsilon^2}\mathbb{E}\left[\displaystyle\int_0^T \beta_t^2 \mathbf{1}_{\mathbf{S}_t \in \mathcal{E}_\epsilon^\xi}\, dt\right], & n = 2 \\ \dfrac{V_I n(n-2)}{\sigma_{n-1}\sqrt{\Delta}\,\epsilon^n}\mathbb{E}\left[\displaystyle\int_0^T \beta_t^2 \mathbf{1}_{\mathbf{S}_t \in \mathcal{E}_\epsilon^\xi}\, dt\right], & n \ge 3. \end{cases} \tag{A.15}$$

**Special Case:** $\beta_t = 1$. Note that

$$\int_0^T \mathbf{1}_{\mathbf{S}_t \in \mathcal{E}_\epsilon^\xi}\, dt = \left| t \in [0, T] : (S_{1t}(\omega), \ldots, S_{nt}(\omega)) \in \mathcal{E}_\epsilon^\xi(n) \right| . \tag{A.16}$$

We know that the transition probability density of the process $\mathbf{S}_t$ is given by

$$p\,(\mathbf{S}_0, 0, \mathbf{S}_t, t) = (2\pi t)^{-n/2} (\Delta)^{-1/2} e^{-\frac{1}{2t}(\mathbf{S}_t - \mathbf{S}_0)' \mathbf{A} (\mathbf{S}_t - \mathbf{S}_0)}, \tag{A.17}$$

so we have:

$$\mathbb{E}\left[\int_0^T \mathbf{1}_{\mathbf{S}_t \in \mathcal{E}_\epsilon^\xi}\, dt\right] = (\Delta)^{-\frac{1}{2}} \int_0^T \left\{ (2\pi t)^{-n/2} \int_{\mathbf{S}_t \in \mathcal{E}_\epsilon^\xi} e^{-\frac{1}{2t}(\mathbf{S}_t - \mathbf{S}_0)' \mathbf{A} (\mathbf{S}_t - \mathbf{S}_0)}\, d\mathbf{S}_t \right\} dt, \tag{A.18}$$

where we have used the Fubini theorem to interchange the order of integration. Letting $CC'$ be the Cholesky factorization of $\mathbf{a} = \mathbf{A}^{-1}$, and noting that $\mathbf{C}'\mathbf{A}\mathbf{C} = \mathbf{I}$, we see that by making the change of variables $S_t - S_0 = Cy$, the right-hand side of (A.18) becomes

$$\int_0^T \left\{ (2\pi t)^{-n/2} \int_{y \in B(\xi - S_0, \epsilon)} e^{-\frac{1}{2t}|y|^2} dy \right\} dt = \int_0^T \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{B_t \in B(\xi - S_0, \epsilon)}]\, dt. \tag{A.19}$$

Now, returning to (A.15) and taking $V_I = 1$, we see that:

$$\mathbb{E}[\tilde{g}^\epsilon(\mathbf{S}_T, \boldsymbol{\xi})] = \tilde{g}^\epsilon(\mathbf{S}_0, \boldsymbol{\xi}) - \frac{n(n-2)}{\sigma_{n-1}} \left\{ \frac{1}{\epsilon^n} \int_0^T \mathbb{E}^{\mathbb{Q}}[\mathbf{1}_{B_t \in B(\xi - S_0, \epsilon)}]\, dt \right\}. \tag{A.20}$$

Although local time does not exist for $n > 1$, a comparison of the last expression with that used in approximating local time in the one-dimensional setting suggests that the second term on the right-hand side of (A.20) can be thought of (up to a constant) as the occupation time that the standard $n$-dimensional Brownian motion $\mathbf{B}_s$ (starting at the origin) spends in a neighborhood of the point $\boldsymbol{\xi} - \mathbf{S}_0$.

## A.2. The Lognormal Case

Suppose that the stock price processes follow the lognormal dynamics,

$$dS_{it} = \beta_t S_{it} \sigma_{ij} dW_{jt}, \quad t \in [0, T], i = 1, \ldots, n.$$

Recall from section 4.4 that if we set $x_{it} = \ln S_{it}$, then

$$dx_{it} = -\frac{\beta_t^2}{2} \sum_{i=1}^n \sigma_{ij}^2 dt + \beta_t \sigma_{ij} dW_{jt}.$$

Consider the constant coefficient PDE

$$a_{ij} u_{x_i x_j} + \tilde{\nu}_i u_{x_i} = -2\delta(x - k), \tag{A.21}$$

where $\tilde{\nu}_i \equiv -\sum_{i=1}^n \sigma_{ij}^2$. We again conjecture that any solution of this PDE is a payoff which allows one to observe the conditional mean of the terminal variance rate of the underlying basket.

A.2.1. *A Fundamental Solution of the Elliptic PDE in the Lognormal Case.* The approach that we take in this section is the method of reduction as described for instance in Garabedian (1998). However, we have found no source in which all details are carried out and where it is shown that the final result can be put in the precise form (A.29), needed in Theorem 4.6. This precise form of the solution and the derived final form in (4.29) are needed to establish formula (4.31), i.e., to show that the remainder is of order $O(\epsilon)$ there. For this reason, we provide details in this section.

To begin, since $\mathbf{a}$ is a symmetric matrix, it can be diagonalized by letting

$$\mathbf{x} - \mathbf{k} = \mathbf{C}\mathbf{y}, \tag{A.22}$$

where the columns of $C$ are the eigenvectors of the matrix $\mathbf{a}$. In the new coordinates, our equation (A.21) becomes

$$\lambda_i \frac{\partial^2 V}{\partial y_i^2} + (\tilde{\boldsymbol{\nu}} \cdot C_{\cdot,i}) \frac{\partial V}{\partial y_i} = -2\delta(\mathbf{C}\mathbf{y}), \tag{A.23}$$

where $\lambda_i$, $i = 1, \ldots, n$ are the eigenvalues of $\mathbf{a}$ (possibly with multiplicity) and where $C_{\cdot,i}$ is the $i$-th column of the matrix $\mathbf{C}$, i.e., the $i$-th eigenvector of the matrix $\mathbf{a}$. Changing scale

$$y_i = \tilde{y}\sqrt{\lambda_i}, \tag{A.24}$$

this becomes

$$\Delta v + \tilde{\boldsymbol{\nu}}^{\mathrm{T}} \mathbf{C} \boldsymbol{\Lambda}^{-1/2} \nabla v = -2\delta(\mathbf{C}\boldsymbol{\Lambda}^{1/2}\tilde{\mathbf{y}}), \tag{A.25}$$

where

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

and $\lambda_i$, $i = 1, \ldots, n$ are the eigenvalues of the matrix $\mathbf{a}$, possibly with multiplicity. From the change of variables formula for multiple integrals for distributions (see for instance Kanwal 2004), we easily see that

$$\delta(\mathbf{C}\boldsymbol{\Lambda}^{1/2}\tilde{\mathbf{y}}) = \frac{1}{\det(\mathbf{C}\boldsymbol{\Lambda}^{1/2})} \delta(\tilde{\mathbf{y}}).$$

Note that $\sqrt{\det(\boldsymbol{\Lambda})}\,\det(\mathbf{C}) = \sqrt{\det(\mathbf{a})} = \sqrt{\Delta}$.

Therefore, (A.25) can be expressed as

$$\Delta v + \tilde{\boldsymbol{\nu}}^{\mathrm{T}} \mathbf{C} \boldsymbol{\Lambda}^{-1/2} \nabla v = -\frac{2}{\sqrt{\Delta}} \delta(\tilde{\mathbf{y}}).$$

For brevity, let

$$\mathbf{b}^{\mathrm{T}} \equiv \tilde{\boldsymbol{\nu}}^{\mathrm{T}} C \Lambda^{-1/2}.$$

Next, we make the additional change of variables

$$v(\tilde{\mathbf{y}}) = w(\tilde{\mathbf{y}}) \exp\left(-\frac{1}{2}\mathbf{b}^{\mathrm{T}}\tilde{y}\right).$$

Then the equation can be re-written as

$$\Delta w - \left(\frac{1}{4}|b|^2\right) w = -\frac{2}{\sqrt{\Delta}}\delta(\tilde{\mathbf{y}}) \exp\left(\frac{1}{2}\mathbf{b}^{\mathrm{T}} \tilde{\boldsymbol{y}}\right).$$

Hence we obtain

$$\Delta w - \frac{|b|^2}{4} w = \frac{2}{\sqrt{\Delta}}\delta(\tilde{\mathbf{y}}).$$

If we introduce

$$W = (\sqrt{\Delta})w,$$

then

$$\Delta W - \frac{|b|^2}{4} W = -2\delta(\tilde{\mathbf{y}}). \tag{A.26}$$

Setting $Q = \frac{|b|}{2}$, we see that we are seeking a fundamental solution for the Helmholtz equation

$$\Delta W - Q^2 W = -2\delta(\tilde{\mathbf{y}}). \tag{A.27}$$

We will show how to obtain such a solution below. Denote this solution by $F_Q$, where

$$F_Q = \frac{1}{\pi}\left(\frac{Q}{2\pi r}\right)^{\frac{n-2}{2}} K_{n/2-1}(Qr), \tag{A.28}$$

and $K_{n/2-1}$ is a modified Bessel function of the second kind, of integer order when $n$ is even and of fractional order when $n$ is odd. Retracing our steps, we find that

$$\begin{aligned}
w(\tilde{\boldsymbol{y}}) &= \frac{1}{\sqrt{\Delta}} F_Q(|\tilde{\boldsymbol{y}}|) \\
v(\tilde{\boldsymbol{y}}) &= \frac{1}{\sqrt{\Delta}} F_Q(|\tilde{\boldsymbol{y}}|) \exp\left(-\frac{1}{2}b^{\mathrm{T}}\tilde{\boldsymbol{y}}\right) \\
u(\mathbf{x}) &= \frac{1}{\sqrt{\Delta}} F_Q\left[|\Lambda^{-1/2}C^{\mathrm{T}}(\mathbf{x}-\mathbf{k})|\right] \exp\left[-\frac{1}{2}b^{\mathrm{T}}\ C\Lambda^{-1/2}(\mathbf{x}-\mathbf{k})\right] \\
&= \frac{1}{\sqrt{\Delta}} F_Q\left[\sqrt{\mathbf{x}-\mathbf{k})^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k}}\right] \exp\left[-\frac{1}{2}\tilde{\boldsymbol{\nu}}^{\mathrm{T}} C\Lambda^{-1}C^{\mathrm{T}}(\mathbf{x}-\mathbf{k})\right] \\
&= \frac{1}{\sqrt{\Delta}} F_Q\left[\sqrt{\mathbf{x}-\mathbf{k})^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k}}\right] \exp\left[-\frac{1}{2}\tilde{\boldsymbol{\nu}}^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})\right],
\end{aligned} \tag{A.29}$$

where

$$\begin{aligned}
Q^2 &= -\frac{|b|^2}{4} \\
&= \tilde{\boldsymbol{\nu}}^{\mathrm{T}} C\Lambda^{-1/2}\Lambda^{-1/2}C^{\mathrm{T}}\tilde{\boldsymbol{\nu}} \\
&= \tilde{\boldsymbol{\nu}}^{\mathrm{T}} C\Lambda^{-1}C^{\mathrm{T}}\tilde{\boldsymbol{\nu}} \\
&= \tilde{\boldsymbol{\nu}}^{\mathrm{T}}\mathbf{A}\tilde{\boldsymbol{\nu}}.
\end{aligned} \tag{A.30}$$

A.2.2. *The Solution of the Helmholtz Equation.* This section recalls known results concerning the method of solution of the Helmholtz equation and deduces the asymptotic behavior of the fundamental solution (an ingredient in determining the asymptotic behavior of the family $g^\epsilon$) from the known asymptotic behavior of fractional order Bessel functions that are used to define these solutions. In the two-dimensional case, the solution of the Helmholtz equation (A.27) is given by

$$\frac{1}{\pi} K_0(Qr),$$

where $r = |\mathbf{y}|$ and $K_0(z)$ is the modified Bessel function of the second kind of order zero. The known asymptotic behavior of $I_0(z)$ as $z \to 0$ is (see Abramowitz and Stegun 1972, p. 375) $I_0(z) \sim 1$ as $z \to 0$, so that

$$K_0(z) \sim -\ln(z) \quad z \to 0.$$

Also in the case $n = 3$, we have the well-known solution to the Helmholtz equation

$$\frac{e^{-Q\rho}}{2\pi\rho}.$$

Here we say a few words about how the solution for all $n$ can be derived. In spherical coordinates, the Helmholtz equation (A.27) can be written in the form

$$\frac{d}{d\rho}\left(\rho^{n-1}\frac{dW}{d\rho}\right) - Q^2\rho^{n-1} = 0.$$

By making the substitution $v = W\rho^{1-\frac{n}{2}}$, we obtain

$$\frac{d}{d\rho}\left(\rho\frac{dv}{d\rho}\right) - \frac{v}{\rho}\left(1 - \frac{n}{2}\right)^2 - Q^2\rho v = 0.$$

This is *Bessel's equation* of order $\frac{n}{2} - 1$ and parameter $-Q^2$. The solution is known to be

$$\tilde{F}_Q(\rho, -Q^2) = \frac{1}{\pi}\left(\frac{Q}{2\pi\rho}\right)^{(n-2)/2} K_{n/2-1}(Q\rho), \tag{A.31}$$

where $K_{n/2-1}$ is a modified Bessel function of fractional order when $n$ is odd and integer order when $n$ is even (see Abramowitz and Stegun 1972, p. 375, when $n$ is even and page 443, when $n$ is odd). Given the way our fundamental solution is normalized ($2\delta$ on RHS), we see that we recover (A.28). The exact expressions for the $K_{n/2-1}$ are not needed for our purposes in the next section so we refer the interested reader to the above reference. But we will need the known asymptotic behavior for these functions given in the above reference, from which we then easily derive the following asymptotic behavior for the solution of the original elliptic PDE (4.24).

We get

$$
(\text{A.32})\qquad F_Q \sim \begin{cases} -\dfrac{1}{\pi}\ln z, & \text{for } n=2 \quad \text{as} \quad z\to 0\\ \dfrac{1}{2\pi z}, & \text{for } n=3 \quad \text{as} \quad z\to 0\\ \dfrac{\left(\frac{n}{2}-1\right)!}{2\pi^{\frac{n}{2}}}\dfrac{1}{z^{n-2}}, & \text{for } n\geq 4 \quad n\,\text{even} \quad \text{as} \quad z\to 0\\ \left(\dfrac{1}{2\pi}\right)^{\frac{n-2}{2}}[1\cdot 3\cdot 5\ldots(n-4)]\,\dfrac{1}{z^{n-2}}, & \text{for } n\geq 5 \quad n\text{ odd} \quad \text{as} \quad z\to 0 \end{cases}
$$

A.2.3. *Construction of an Approximating Sequence.* Recall from (A.28) and (A.29) that our fundamental solution $\mathcal{F}^{\text{logn}}$[10] in the original $x$ variables can be expressed as

$$
\begin{aligned}
\mathcal{F}^{\text{logn}} &= \frac{1}{\sqrt{\Delta}} F_Q\big[\sqrt{(\mathbf{x}-\mathbf{k})^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})}\big]\exp\left[-\frac{1}{2}\boldsymbol{\nu}^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})\right]\\
&= \frac{1}{\pi\sqrt{\Delta}}\left(\frac{Q}{2\pi\sqrt{(\mathbf{x}-\mathbf{k})^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})}}\right)^{(n-2)/2} K_{n/2-1}\big[Q\sqrt{(\mathbf{x}-\mathbf{k})^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})}\big]\\
&\quad\times \exp\left[-\frac{1}{2}\boldsymbol{\nu}^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})\right].
\end{aligned}
$$

Let

$$
\hat{E} = \frac{1}{\pi\sqrt{\Delta}}\left(\frac{Q}{2\pi}\right)^{\frac{n-2}{2}}.
$$

Since we will be matching derivatives only on the boundary of the ellipsoid $\mathcal{E}_\epsilon$, note that $A_{ij}(x_i-k_i))(x_j-k_j)$ is constant there, so we simplify notation below and let

$$
(\text{A.33})\qquad \bar{\rho} = \sqrt{(\mathbf{x}-\mathbf{k})^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})},
$$

and consequently $\bar{\rho}=\epsilon$ on $\partial\mathcal{E}_\epsilon$. Then the fundamental solution takes the more compact form

$$
\frac{1}{\pi\sqrt{\Delta}}\left(\frac{Q}{2\pi\bar{\rho}}\right)^{(n-2)/2} K_{n/2-1}\left[Q\bar{\rho}\right]\times\,\exp\left[-\frac{1}{2}\boldsymbol{\nu}^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})\right].
$$

The approach is the same as in Appendix A.1, i.e., we match the first order partial derivatives on the boundary of the ellipsoid $\bar{\rho}=\epsilon$. We make the following ansatz for the continuation of the fundamental solution into the interior of the ellipsoid

$$
(\text{A.34})\qquad g^\epsilon = \left[a_1(\epsilon)\bar{\rho}^2(\mathbf{x}) + a_2(\epsilon)\right]\exp\left(-\tilde{\boldsymbol{\nu}}^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})\right).
$$

The work now consists in determining the right form for $a_1(\epsilon)$ and $a_2(\epsilon)$, so that if we paste this solution on the boundary of the ellipsoid to the fundamental solution of the

[10] Note that our right-hand side is $-2\delta$ rather than $-\delta$ to accommodate the origin of our second order terms in the Itô calculus.

elliptic equation (4.24), the result is globally $C^{1,1}$. However, prior to the "pasting step," given at the end of this section, we derive the exact form for the result of applying the original elliptic operator to *any* expression of the form (A.34), since this will be needed when we insert $g^\epsilon$ into Itô's formula. We claim

$$a_{ij} g^\epsilon_{x_i x_j} + \tilde{\nu}_i g^\epsilon x_i = f(\mathbf{x} - \mathbf{k}). \tag{A.35}$$

The exact form of $f(\mathbf{x})$ is most easily arrived at by the following argument. Via the same changes of independent variables as before (see equations (A.22) and (A.24)), i.e., setting $\tilde{\boldsymbol{y}} = \Lambda^{-1/2} C^{\mathrm{T}}(\mathbf{x} - \mathbf{k})$, we have with $v(\tilde{\boldsymbol{y}}) = g^\epsilon(\mathbf{x})$ and $b^{\mathrm{T}} = \tilde{\boldsymbol{\nu}}^{\mathrm{T}} C \Lambda^{-1/2}$

$$\Delta_{\tilde{y}} v + b^{\mathrm{T}} \nabla_{\tilde{y}} v = f(C\sqrt{\Lambda}\,\tilde{\boldsymbol{y}}).$$

Making the change of dependent variable $v(\tilde{\boldsymbol{y}}) = w(\tilde{\boldsymbol{y}}) \exp(-\frac{1}{2} b^{\mathrm{T}} \tilde{\boldsymbol{y}})$, we have $w(\tilde{\boldsymbol{y}}) = (a_1(\epsilon)\bar{\rho}^2 \tilde{\boldsymbol{y}}) + a_2(\epsilon))$ and (A.35) can be written as

$$\Delta(a_1\bar{\rho}^2 + a_2) - \frac{\sum_{i=1}^n b_i^2}{4}(a_1\bar{\rho} + a_2) = f(C^{\mathrm{T}}\sqrt{\Lambda}\,\tilde{\boldsymbol{y}}) \exp\left(\frac{1}{2} b^{\mathrm{T}} \tilde{\boldsymbol{y}}\right).$$

It is clear that the dependence on $\tilde{\rho}$ on the $\tilde{\boldsymbol{y}}$ variable is on the modulus of $\tilde{\boldsymbol{y}}$ (by design), so that the last equation holds iff

$$2na_1 - \frac{\sum_{i=1}^n b_i^2}{4}\left(a_1\bar{\rho}^2 + a_2\right) = f(C^{\mathrm{T}}\sqrt{\Lambda}\,\tilde{\boldsymbol{y}}) \exp\left(\frac{1}{2} b^{\mathrm{T}} \tilde{\boldsymbol{y}}\right),$$

so that

(A.36)

$$\begin{aligned} f(\mathbf{x} - \mathbf{k}) &= \left(2na_1 - \frac{\tilde{\boldsymbol{\nu}}^{\mathrm{T}}\mathbf{A}\tilde{\boldsymbol{\nu}}}{4} a_2 - \frac{\tilde{\boldsymbol{\nu}}^{\mathrm{T}}\mathbf{A}\tilde{\boldsymbol{\nu}}}{4} a_1 \bar{\rho}^2\right) \exp\left(-\frac{1}{2}\tilde{\boldsymbol{\nu}}^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})\right) \\ &= \left(2na_1 - \frac{\tilde{\boldsymbol{\nu}}^{\mathrm{T}}\mathbf{A}\tilde{\boldsymbol{\nu}}}{4} a_2\right)\left(1 + \tilde{\boldsymbol{\nu}}^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k}) + O(|\mathbf{x}-\mathbf{k}|^2)\right) - \frac{\tilde{\boldsymbol{\nu}}^{\mathrm{T}}\mathbf{A}\tilde{\boldsymbol{\nu}}}{4} a_1 \bar{\rho}^2 \exp\left(-\frac{1}{2}\tilde{\boldsymbol{\nu}}^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})\right) \\ &= \left(2na_1 - \frac{\tilde{\boldsymbol{\nu}}^{\mathrm{T}}\mathbf{A}\tilde{\boldsymbol{\nu}}}{4} a_2\right)\left(1 + O(|\mathbf{x}-\mathbf{k}|)\right). \end{aligned}$$

Having successfully determined the function $f$, we now turn to the final step in our analysis, to choose $a_1(\epsilon)$ and $a_2(\epsilon)$ so that $g^\epsilon$ pastes smoothly to the fundamental solution of the elliptic equation for $\bar{\rho} = \epsilon$. For convenience, we simplify the notation by introducing the notation $c(n)$, $n \geq 2$ for the coefficients in the asymptotic expansion given earlier (A.32), e.g., let

$$c(n) = \begin{cases} -\dfrac{1}{\pi}, & n = 2 \\ \dfrac{1}{2\pi}, & n = 3 \\ \dfrac{\left(\frac{n}{2} - 1\right)!}{2\pi^{\frac{n}{2}}} \dfrac{1}{z^{n-2}}, & n \geq 4,\ n \text{ even} \\ \left(\dfrac{1}{2\pi}\right)^{\frac{n-1}{2}} [1 \cdot 3 \cdot 5 \ldots (n-4)], & \text{for } n \geq 5, n \text{ odd}. \end{cases} \tag{A.37}$$

Calculating the first order partial derivatives at $\bar{\rho} = \epsilon$:

$$\begin{aligned} g^{\epsilon}_{x_i}|_{\bar{\rho}=\epsilon} &= \left(\exp\left[-\frac{1}{2}\tilde{\mathbf{v}}^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})\right]\right)_{x_i}|_{\bar{\rho}=\epsilon}\ (a_1(\epsilon)\bar{\epsilon}^2 + a_2(\epsilon)) \\ &+ \left(\exp\left[-\frac{1}{2}\tilde{\mathbf{v}}^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})\right]\right) 2\epsilon a_1(\epsilon)(\bar{\rho})_{x_i}|_{\bar{\rho}=\epsilon}, \end{aligned} \tag{A.38}$$

and for the fundamental solution, using the notation above and the behavior summarized (A.32), we get

$$\begin{aligned} &\frac{\partial}{\partial x_i}\mathcal{F}^{\log}_n \\ &\sim \left(\exp\left[-\frac{1}{2}\tilde{\mathbf{v}}^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})\right]\right)_{x_i}|_{\bar{\rho}=\epsilon}\ c_n \\ &+ \exp\left[-\frac{1}{2}\tilde{\mathbf{v}}^{\mathrm{T}}\mathbf{A}(\mathbf{x}-\mathbf{k})\right]|_{\bar{\rho}=\epsilon}\ (2-n)\frac{c_n}{\epsilon}\bar{\rho}_{x_i}|_{\bar{\rho}=\epsilon}. \end{aligned} \tag{A.39}$$

Now, notice that an appropriate choice of $a_1(\epsilon)$ makes the second term in (A.38) match the second term in (A.39). Then an appropriate choice of $a_2(\epsilon)$ makes the first terms in (A.38) and (A.39) identical. The right choices for $a_1(\epsilon)$ and $a_2(\epsilon)$ are

$$\begin{aligned} a_1(\epsilon) &= \frac{(2-n)c_n}{2\epsilon^2} \\ a_2(\epsilon) &= \frac{n}{2}c_n. \end{aligned}$$

Using these formulas now in (A.36), we see that the leading order term comes from $a_1(\epsilon)$ as claimed.

**Conclusion of the proof of Theorem 4.7.** Since the preceding section established the required asymptotic behavior of the family $g^{\epsilon}$, all that remains to be proved is that it is justifiable to apply Itô's lemma to this family. Since the constructed family $g^{\epsilon}$ is $C^{1,1}$, we can use the exact same extension of Itô's lemma that was used in the normal case, see Proposition A.2, the Appendix.

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