Covariance contracting for commodities

EPRM presents a solution to the problems in developing the replication of variance and covariance swaps **By Peter Carr and Anthony Corso**

A variance swap is a contract that pays the realised variance of the returns of a specified underlying asset over a specified period of time. It has been shown that under certain conditions, the payouts to a continuously monitored variance swap can be synthesised by combining continuous trading in the underlying with static positions in standard options maturing with the swap (Neuberger (see bibliography reference 12), Dupire (see bibliography reference 7), and Carr & Madan (see bibliography reference 5)). It has also been shown that the covariance of returns between two currencies can be replicated in this manner (Carr & Madan (see bibliography reference 4)).

But there are three problems with this proposed replication strategy, which are that it:

- replicates perfectly only if there are no jumps in the underlying;
- assumes the variance swap is continuously monitored, even though all variance swaps are discretely monitored in practice; and
- requires continuous trading in the underlying, which is problematic in the presence of transaction costs and market closings.

SOLUTION

Here we propose a solution to the drawbacks. We suggest changing the definition of variance in the swap from the realised variance of returns to the realised variance of price changes. This shows that the new payout can be perfectly replicated in the presence of jumps, discrete monitoring and discrete trading opportunities. It further shows that a contract paying the realised covariance of price changes can also be synthesised in this setting.

We will illustrate this in the context of commodity options. This is because the markets for commodity option structures have many of the features we require. In particular, to synthesise covariance swaps, we will use spread options, which represent one of the few options written on two assets and listed on an organised exchange.

Previous articles on the valuation of spread options include Broadie and Detemple (see bibliography reference 2), Grabbe (see reference 8),Heenk, Kemna & Vorst (see reference 10), Pearson (see reference 13), Ravindran (see reference 14), and Shimko (see reference 13).

All these papers assume the covariance between the two commmodities in the spread is constant. In contrast, this article assumes the covariance between the two commodities is random, and furthermore that the stochastic process governing covariance is unknown.

Rather than pricing spread options in terms of a fixed covariance, we turn the problem around. We show how the covariance between the price changes in two commodity futures can be traded, given the ability to trade dynamically in the futures and to take static positions in spread options and in options written on each component of the spread.

REVIEW OF STATIC HEDGING USING OPTIONS

Consider a single period setting in which investments are made at time t_0 with all payouts received at time t_n. In contrast to the standard intertemporal model, we assume there are no trading opportunities other than at times t_0 and t_n. We assume there exists a futures market in a commodity for delivery at some date $T \geq t_n$.

We also assume that markets exist for European-style futures options[1] of all strikes. While the assumption of a continuum of strikes is far from standard, it is essentially the analogue of the standard assumption of continuous trading.

Continuous trading is generally regarded as a reasonable approximation of an environment where investors can trade frequently, but not infinitely often. By analogy, we regard static positioning in a continuum of strikes as a reasonable approximation of a market environment with a large but finite number of option strikes.

This market structure allows investors to create any smooth function $f(F_n)$ of the terminal futures price F_n by taking a static position at time 0 in options.[2] When this theory is used to generate desired volatility exposures, here we show that it is only the second derivative of the payout that governs such exposures.

Consequently, we will always choose f so that its value and slope vanish at the initial futures price F_0 (that is to say, $f(F_0) = f'(F_0) = 0$). In this case, the results of Carr & Madan (see bibliography reference 3) imply that any twice differentiable payout can be spanned by the following position in out-of-the-money options:

$$f(F_n) = \int_0^{F_0-} f''(K)(K - F_T)^+ dK + \int_{F_0}^{\infty} f''(K)(F_T - K)^+ dK \qquad (1)$$

That is, to create a twice differentiable payout with value and slope vanishing at F_0, buy $f''(K)dK$ puts at all strikes less than F_0 and buy $f''(K)dK$ calls at all strikes greater than F_0.

In the absence of arbitrage, a decomposition similar to that in equation (1) must prevail among the initial values. Specifically, if we let V_0^f, $P_0(K)$ and $C_0(K)$ denote the initial prices of the payout $f(\cdot)$, the put and the call respectively, then the no-arbitrage condition requires that:

$$V_0^f = \int_0^{F_0-} f''(K)\ P_0(K)dK + \int_{F_0+}^{\infty} f''(K)C_0(K)dK \qquad (2)$$

Thus, the value of an arbitrary payout can be obtained from the option prices. Given the spectrum of European-style options on the spread of two futures prices $S = F_1 - F_2$, one can also create any smooth function of this spread $g(S)$.

If we assume that the value and slope vanish at the initial spread $S_0 \in \Re$, that is to say, $g(S_0) = g'(S_0) = 0$, then the analogous expression for the value of this payout is:

$$V_0^g = \int_{-\infty}^{S_0-} f''(K) P_0^s(K)\ dK + \int_{S_0+}^{\infty} f''(K) C_0^s(K)dK \qquad (3)$$

where $P_0^s(K)$ and $C_0^s(K)$ are the initial prices of European-style put and call spread options struck at K. Note that no assumptions were made regarding the stochastic processes governing the futures prices.

CREATING A CONTRACT PAYING THE VARIANCE OF A COMMODITY

Consider a finite set of discrete times $\{t_0, t_1, \ldots, t_n\}$ at which one can trade futures contracts. For simplicity, we will take these times to be closing times of trading days, but this restriction is not necessary. F_i denotes the price traded at on day i, for i = 0,1,…,n. By day n, a standard estimator of the realised annualised variance of price changes will be:

$$\text{Var}(\Delta F) = \frac{N}{n}\sum_{i=1}^{n}\left(F_i - F_{i-1}\right)^2$$

where N is the number of trading days in a year.

The purpose of this section is to demonstrate a strategy whose terminal payout matches the above estimator of variance.

By Taylor's series, we note that:

$$F_i^2 = F_{i-1}^2 + 2F_{i-1}\left(F_i - F_{i-1}\right) + \left(F_i - F_{i-1}\right)^2, \quad i = 1,\ldots,n$$

Rearranging and summing implies:

$$\sum_{i=1}^{n}\left(F_i^2 - F_{i-1}^2\right) - \sum_{i=1}^{n} 2F_{i-1}\left(F_i - F_{i-1}\right) = \sum_{i=1}^{n}\left(F_i - F_{i-1}\right)^2, \quad i = 1,\ldots,n$$

The first sum on the left telescopes to $F_n^2 - F_0^2$. So, multiplying both sides by $\frac{N}{n}$ implies:

$$\text{Var}(\Delta F) = \frac{N}{n}\left(F_n^2 - F_0^2\right) - \sum_{i=1}^{n} 2\frac{N}{n}F_{i-1}\left(F_i - F_{i-1}\right) \tag{4}$$

The first term on the right-hand side can be regarded as a function $\phi(\cdot)$ of F_n, where:

$$\phi(F) = \frac{N}{n}\left(F - F_0\right)^2 \tag{5}$$

The first derivative is given by:

$$\phi'(F) = \frac{N}{n}2\left(F - F_0\right) \tag{6}$$

Thus the value and slope both vanish at $F = F_0$. It follows from equation (1) that the payout $\phi(F_n)$ can be replicated using options. The number of options held at each strike is proportional to the second derivative of ϕ, which is simply:

$$\phi''(F) = \frac{N}{n}2 \tag{7}$$

Substituting equation (1) into (4) results in:

$$\text{Var}(\Delta F) = \frac{N}{n}2\left[\int_0^{F_0-}\left(K - F_n\right)^+ dK + \int_{F_0+}^{\infty}\left(F_n - K\right)^+ dK\right] - \sum_{i-1}^{n} 2\frac{N}{n}F_{i-1}\left(F_i - F_{i-1}\right) \tag{8}$$

The initial cost of creating the first term on the right-hand side of equation (8) is:

$$V_0 = 2\frac{N}{n}\int_0^{F_0-}P_0(K,T)\,dK + 2\frac{N}{n}\int_{F_0+}^{\infty}C_0(K,T)\,dK \tag{9}$$

If we assume interest rates are constant at r, then the second term on the right-hand side of equation (8) can be regarded as the cumulative marking-to-market proceeds arising from holding:

$$-e^{-r(t_n - t_i)}2\frac{N}{n}F_{i-1}$$

futures contracts from time t_{i-1} to time t_i. Since futures positions are costless, the theoretically fair price to charge for this variance contract is V_0 given in (9).

The dynamic strategy in futures can be interpreted as an attempt to hedge the payout $\phi(F_n)$ made at t_n, conducted under the false assumption of zero volatility. Given this ridiculous assumption, the value function is:

$$V_{i-1}^{\phi}\left(F_{i-1}, t_{i-1}\right) = e^{-r(t_n - t_{i-1})}\frac{N}{n}\left(F_{i-1}^2 - F_0^2\right)$$

for i = 1,…, n.

Recognising that the marking-to-market proceeds are realised one trading day after the position is put on, the zero volatility hedger holds:

$$e^{r(t_i - t_{i-1})}\frac{\partial V_{i-1}^{\phi}\left(F_{i-1}, t_{i-1}\right)}{\partial F} = e^{-r(t_n - t_i)}\frac{N}{n}2F_{i-1}^2$$

futures contracts from time t_{i-1} to time t_i. This is exactly the dynamic strategy needed to create the last term in equation (8). Since realised volatility will in fact be positive, an error arises, and the magnitude of this error is given by the left side of equation (8).

CONTRACT PAYING THE COVARIANCE OF TWO FUTURES PRICES

Recall that:

$$S_i = F_{1,i} - F_{2,i}, \qquad i = 0,1,\ldots,n \tag{10}$$

denotes the spread on day t_i, where $F_{1,i}$ and $F_{2,i}$ denote the contemporaneous futures prices of the two components of the spread.

This section shows how to create a contract paying:

$$\text{Cov}\left(\Delta F_1, \Delta F_2\right) = \frac{N}{n}\sum_{i=1}^{n}\left(F_{1,i} - F_{1,i-1}\right)\left(F_{2,i} - F_{2,i-1}\right)$$

at time t_n by combining static positions in options with dynamic trading in the underlying futures.

We begin by recalling the well known[3] result that:

$$\text{Var}(\Delta S) = \text{Var}\left(\Delta\left(F_1 - F_2\right)\right)$$
$$= \text{Var}\left(\Delta F_1\right) - 2\text{Cov}\left(\Delta F_1, \Delta F_2\right) + \text{Var}\left(\Delta F_2\right)$$

Rearranging this expression gives:

$$\text{Cov}\left(\Delta F_1, \Delta F_2\right) = -\frac{1}{2}\text{Var}(\Delta S) + \frac{1}{2}\text{Var}\Delta\left(F_1\right) + \frac{1}{2}\text{Var}\left(\Delta F_2\right) \tag{11}$$

Thus, one can synthesise a covariance swap by selling half a variance swap on the spread and buying half a variance swap on each of the spread components. As these variance swaps are unlikely to be explicitly available, they can be synthesised. Substituting equation (8) in each term on the right-hand side of equation (11) implies:

$$\begin{aligned}\text{Cov}\left(\Delta F_1, \Delta F_2\right) = &-\frac{N}{n}\left[\int_0^{S_0-}\left(K - S_n\right)^+ dK + \int_{S_0+}^{\infty}\left(S_n - K\right)^+ dK\right] \\ &+ \sum_{i=1}^{n}\frac{N}{n}S_{i-1}\left(S_i - S_{i-1}\right) + \frac{N}{n}\left[\int_0^{F_0^{(1)}-}\left(K - F_n^{(1)}\right)^+ dK + \int_{F_0^{(1)}+}^{\infty}\left(F_n^{(1)} - K\right)^+ dK\right] \\ &- \sum_{i=1}^{n}\frac{N}{n}F_{i-1}^{(1)}\left(F_i^{(1)} - F_{i-1}^{(1)}\right) + \frac{N}{n}\left[\int_0^{F_0^{(2)}-}\left(K - F_n^{(2)}\right)^+ dK + \int_{F_0^{(2)}+}^{\infty}\left(F_n^{(2)} - K\right)^+ dK\right] \\ &- \sum_{i=1}^{n}\frac{N}{n}F_{i-1}^{(2)}\left(F_i^{(2)} - F_{i-1}^{(2)}\right)\end{aligned} \tag{12}$$

From equation (10), the second term on the right-hand side can be created by dynamic trading in futures on the spread components:

$$\begin{aligned}\sum_{i=1}^{n}\frac{N}{n}S_{i-1}\left(S_i - S_{i-1}\right) &= \sum_{i=1}^{n}\frac{N}{n}S_{i-1}\left[\left(F_i^{(1)} - F_i^{(2)}\right) - \left(F_{i-1}^{(1)} - F_{i-1}^{(2)}\right)\right] \\ &= \sum_{i=1}^{n}\frac{N}{n}S_{i-1}\left(F_i^{(1)} - F_{i-1}^{(1)}\right) - \sum_{i=1}^{n}\frac{N}{n}S_{i-1}\left(F_i^{(2)} - F_{i-1}^{(2)}\right)\end{aligned} \tag{13}$$

Substituting equations (10) and (13) in (12) and simplifying implies:

$$\begin{aligned}\text{Cov}\left(\Delta F_1, \Delta F_2\right) = &-\frac{N}{n}\left[\int_0^{S_0-}\left(K - S_n\right)^+ dK + \int_{S_0+}^{\infty}\left(S_n - K\right)^+ dK\right] \\ &+ \frac{N}{n}\left[\int_0^{F_0^{(1)}-}\left(K - F_n^{(1)}\right)^+ dK + \int_{F_0^{(1)}+}^{\infty}\left(F_n^{(1)} - K\right)^+ dK\right] - \sum_{i=1}^{n}\frac{N}{n}F_{i-1}^{(2)}\left(F_i^{(1)} - F_{i-1}^{(1)}\right) \\ &+ \frac{N}{n}\left[\int_0^{F_0^{(2)}-}\left(K - F_n^{(2)}\right)^+ dK + \int_{F_0^{(2)}+}^{\infty}\left(F_n^{(2)} - K\right)^+ dK\right] - \sum_{i=1}^{n}\frac{N}{n}F_{i-1}^{(1)}\left(F_i^{(2)} - F_{i-1}^{(2)}\right)\end{aligned} \tag{14}$$

Since futures positions are costless, the fair price to charge for the covariance swap is the cost of creating the static options position:

$$V_0 = -\frac{N}{n}\int_0^{S_0-} P_0^s(K,T)\,dK - \frac{N}{n}\int_{S_0+}^{\infty} C_0^s(K,T)\,dK$$
$$+\frac{N}{n}\int_0^{F_0^{(1)}-} P_0^{(1)}(K,T)\,dK + \frac{N}{n}\int_{F_0^{(1)}+}^{\infty} C_0^{(1)}(K,T)\,dK$$
$$+\frac{N}{n}\int_0^{F_0^{(2)}-} P_0^{(2)}(K,T)\,dK + \frac{N}{n}\int_{F_0^{(2)}+}^{\infty} C_0^{(2)}(K,T)\,dK$$

The investor must also trade futures on a daily basis, holding $-e^{-r(t_n-t_i)}F_{2,i-1}$ units of the first futures contract and $-e^{-r(t_n-t_i)}F_{1,i-1}$ units of the second from time t_{i-1} to time t_i.

FUTURE RESEARCH

We have shown that by combining static positions in options with dynamic trading in futures, investors can synthesise contracts paying the realised variance of a commodity or paying the realised covariance between two commodities. Importantly, these contracts were created without assuming anything about the underlying price.

It would be interesting to extend our results to other payouts besides variance and covariance. Indeed, Carr, Lewis & Madan (see bibliography reference 6) characterise the entire set of continuously paid cashflows that can be spanned in our structure.

We could perhaps consider non-linear functions of realised variance or covariance, such as options on these moments.

Finally, it would be interesting to develop contracts on other statistics of the sample path, such as the standard deviation, the Sharpe ratio, skewness or correlation. In the interests of brevity, such enquiries are best left for future research. ■

Notes

1 *Note that listed futures options are generally American-style. However, by setting $T = t_n$, the underlying futures will converge to the spot at t_n and so the assumption is that there exists European-style spot options in this special case*

2 *This observation was first noted in Breeden & Litzenberger (bibliography reference 1) and established formally in Green & Jarrow (bibliography reference 9) and Nachman (bibliography reference 11)*

3 *See Zerolis (bibliography reference 16) for a geometric derivation derived from the law of cosines*

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