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A functional analysis approach to the static replication of European options

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The replication of any European contingent claim by a static portfolio of calls and puts with strikes forming a continuum, formally proven by Carr and Madan [Towards a theory of volatility trading. In *Volatility: New Estimation Techniques for Pricing Derivatives*, edited by R.A. Jarrow, Vol. 29, pp. 417–427, 1998 (Risk books)], is part of the more general theory of integral equations. We use spectral decomposition techniques to show that exact payoff replication may be achieved with a discrete portfolio of special options. We discuss applications for fast pricing of vanilla options that may be suitable for large option books or high frequency option trading, and for model pricing when the characteristic function of the underlying asset price is known.

Keywords: Derivatives; Options; Static replication; Payoff; Integral equation; Functional analysis; Spectral theorem; Breeden-Litzenberger formula; Implied distribution

1. Introduction

We consider the general problem of replicating a target European option† with a static portfolio of cash, the underlying asset and a selection of 'replicant' European options. Replication problems arise in many areas of finance, such as in asset pricing theory where an asset is replicated with a finite number of other assets (e.g. Černý 2016, ch. 1, 2) using the techniques of finite-dimensional linear algebra, or option pricing theory, where Carr and Madan (1998) formally proved that any European option may be replicated with a portfolio of cash, forward contracts, and European call and/or put options with a continuum of strike prices. A key consequence of payoff replication is that if the prices of the replicating options are known, then the price of the target European option is also known and enforced by no-arbitrage considerations.

Specifically, given a target European option's payoff $F(x)$ to be replicated, where $x \in \mathcal{X} \subseteq \mathbb{R}_+$ is the terminal price of the option's underlying asset, and a family of replicating European options' payoffs $G(x,y)$ indexed by $y \in \mathcal{Y} \subseteq \mathbb{R}$, we are looking for portfolio quantities or weights such that, for all $x \in \mathcal{X}$,

$$F(x) = c + q\,x + \int_{y\in\mathcal{Y}} G(x,y)\phi(y)\,\mathrm{d}\mu(y), \qquad (1)$$

where c, q and $\phi(y)$ are the respective quantities of cash, underlying asset and replicating option with index y, and μ is an appropriate measure. In particular, if $\mathcal{Y}$ is discrete and μ is the counting measure, the above equation becomes $F(x) = c + q\,x + \sum_{y\in\mathcal{Y}} G(x,y)\phi(y)$ or, with the more habitual subscript notation for discrete sums,

$$F(x) = c + q\,x + \sum_{n\in\mathcal{Y}} \phi_n G_n(x).$$

We will especially focus on the case where both variables x, y belong to a continuous interval such as $[a,b]$ or (a,b) where $a,b \in \mathbb{R}\cup\{-\infty,\infty\}$ may be infinite, and μ is the Lebesgue measure, so that we may write equation (1) as

$$f(x) = \int_a^b G(x,y)\phi(y)\,\mathrm{d}y, \qquad (2)$$

where $f(x) := F(x) - c - q\,x$ is the target payoff function $F(x)$ up to affine terms. Observe that the second and higher derivatives of f and F coincide.

The origin of the Carr-Madan replication formula may be traced back to the seminal paper of Breeden and Litzenberger (1978) who showed that the terminal distribution of the underlying asset implicit in option prices, also known as the *implied distribution*, could be recovered by differentiating call prices twice with respect to the strike price. This elegant theoretical result allowed pricing any other European option payoff consistently with existing vanilla options.

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† We use the term 'option' to designate any derivative contract, also known as 'contingent claim', on a single underlying asset

However, it was not until the 1990s that practitioners and researchers became particularly interested in replication and hedging strategies for non-vanilla option payoffs, on the back of the expansion of option markets and the search for option contract innovation. Evidence of such interest can be found in the work of Dupire (1993), as well as Derman *et al.* (1994) who discuss especially static replication of barrier options.

Much research (e.g. Demeterfi *et al.* 1999) has been devoted to the static replication of the log-contract first introduced by Neuberger (1990), leading to the development of volatility and variance swap markets. In this context, Carr and Madan (1998) offered a general replication result that did not solely apply to the log-contract and was also probability- and model-free. To this day, option practitioners refer to the idea that any European option payoff can be replicated with a continuous portfolio of vanilla calls and puts as the 'Carr–Madan result'. Its most visible impact may be seen in the new calculation methodology of the VIX (see The CBOE volatility index–VIX 2009), which was adopted in 2002 by the Chicago Board Options Exchange.

In other related literature, Carr and Wu (2013) consider the static hedging of a longer-dated vanilla option using a continuum of shorter-term options. Balder and Mahayni (2006) expand on this work and explore various discretization strategies when the strikes are pre-specified and the underlying price dynamics are unknown, and recently Wu and Zhu (2017) propose a model-free strategy of statically hedging options with nearby options in strike and maturity dimensions. Madan and Milne (1994) price options under a Gaussian measure using Hermite polynomials as a basis. Carton de Wiart and Dempster (2011) use wavelet theory for partial differential equations used in derivatives pricing. Papanicolaou (2018) expresses a consistency condition between SPX Stochastic Volatility and VIX Market Models as an integral equation and solves it using an eigen series decomposition. Di Tella *et al.* (2019) find a sparse set of tradeable assets for semi-static hedging under a variance-optimal loss criterion.

Our ambition for this paper is to show the relevance and usefulness of functional analysis tools and concepts in the context of payoff replication. We establish that perfect replication can be achieved with a discrete portfolio of special options forming an orthogonal eigensystem, rather than a continuous portfolio of vanilla options with overlapping payoffs. In practice, a satisfactory approximation may be achieved with a smaller number of these special options compared with integral discretization schemes, and for some target options including vanillas our approach is more accurate than existing Fourier series methods.

The remainder of our paper is organized as follows: In section 2, we show that the Carr–Madan result is part of the general theory of integral equations. In section 3, we present key results of the theory about the existence and uniqueness of solutions, with particular focus on spectral decomposition within Hilbert spaces. In section 4, we proceed with the spectral decomposition of the 'straddle kernel', and we interpret our results in terms of option replication in section 5. In section 6, we propose a numerical application for fast pricing of vanilla options. In section 7, we propose a theoretical application to derive pricing formulas when the characteristic function of the underlying asset price is known. Finally, in section 8 we consider the case of the 'butterfly kernel' and derive equations for its eigensystem that may be solved numerically. Section 9 concludes.

2. Carr–Madan as part of the theory of integral equations

In functional analysis, equation (2) is known as a *Fredholm linear integral equation of the first kind*, and $G(x, y)$ is called the *integral kernel* or, with slight abuse of terminology, the *integral operator*. A shorthand notation for the equation is often $f = \langle G, \phi\rangle$ or simply $f = G\phi$. When $f(x)$ is identically zero the equation is called *homogeneous*; otherwise it is called *inhomogeneous*. Many authors further categorize an integral equation as *singular* when it has a convergent improper integral, as in equation (2) when either bound a, b is infinite.

Many integral kernels that are relevant to finance vanish for $y \geq x$ or $y \leq x$, in which case equation (2) respectively simplifies to

$$f(x) = \int_a^x G(x, y)\phi(y)\,\mathrm{d}y, \quad \text{or} \quad f(x) = \int_x^b G(x, y)\phi(y)\,\mathrm{d}y.$$

These equations are known as a *Volterra integral equations of the first kind* and they have special properties and methods (e.g. Polyanin and Manzhirov 2008, ch. 10, 11).

We will see in section 3 that solving equation (2) is considerably easier when the integral kernel $G(x, y)$ is *symmetric* and *injective*, as defined later. Table 1 lists several examples of kernels that are relevant to quantitative finance and indicates whether they are symmetric and/or injective. Note that, to a degree, log contracts and options trade on derivatives markets as options, futures and swaps on VIX and realized variance. Note also that, thanks to the development of electronic option markets, many option strategies combining vanilla options, such as straddles or butterfly spreads, quote and trade directly on dedicated platforms usually known as *complex order books*.

2.1. Carr–Madan kernel

The kernel $G(x, y) := (x - y)^+$ corresponds to the payoff replication problem with call options of various strike prices $y \in \mathcal{Y}$. When all strike prices form the continuum $\mathcal{Y} = \mathbb{R}_+$, the solution to equation (2) is then $\phi(y) = f''(y)$ as shown by Carr and Madan (1998) using standard calculus techniques. In fact, this solution can be viewed as a corollary of Taylor's theorem with remainder in integral form,

$$F(x) = F(0) + F'(0)x + \int_0^x (x - t)F''(t)\,\mathrm{d}t.$$

Substituting $(x - t)^+$ which is identically zero for $t > x$ yields the Carr–Madan formula at origin:

$$F(x) = F(0) + F'(0)x + \int_0^\infty (x - t)^+F''(t)\,\mathrm{d}t.$$

Table 1. Examples of integral kernels†

Kernel	European option payoff kernels $G(x,y)$, $x,y>0$	Symmetric	Injective
Forward contracts	$x-y$	No	No
Calls and puts	$(x-y)^+, (y-x)^+$	No	Yes
Straddles	$\lvert x-y\rvert$	Yes	Yes
Powers of the above	$G(x,y)^c$		
Strangles	$(\lvert x-y\rvert-c)^+$	Yes	Yes ($1/c \in 2\mathbb{N}$)
Butterfly spreads	$(c-\lvert x-y\rvert)^+$	Yes	Yes
Binary options	$H(x-y), H(y-x)$	No	Yes
Risk reversals	$(x-y-c)^+ - (y-x-c)^+$	No	Yes ($1/c \in 2\mathbb{N}$)
Log contracts	$\ln x/y$	No	No
Log calls and puts	$(\ln x/y)^+, (\ln y/x)^+$	No	Yes
Kernel	**Mathematical kernels $G(x,y)$, $x,y\in\mathbb{R}$**	**Symmetric**	**Injective**
Power	$x^y,\ x,y>0$	No	Yes
Gaussian	$\frac{1}{\sqrt{2\pi}}e^{-(x-y)^2/2}$	Yes	Yes
Laplace transform	e^{-xy}	Yes	Yes
Fourier transform	$e^{-2i\pi xy}$	Yes	Yes

† $c>0$ is a constant parameter, $H(\cdot)$ is Heaviside's step function, and i is the imaginary number.

The general Carr–Madan formula involves both call and put options whose strike prices are respectively above or below an arbitrary split level $x_0 \geq 0$:

$$F(x) = F(x_0) + F'(x_0)(x-x_0) + \int_0^{x_0} F''(y)(y-x)^+\,\mathrm{d}y + \int_{x_0}^{\infty} F''(y)(x-y)^+\,\mathrm{d}y. \quad (3)$$

Observe how the second term $F'(x_0)(x-x_0)$ corresponds to a long or short position in forward contracts with delivery price x_0. A convenient choice for practical applications is to set x_0 to the underlying asset's current forward price (respectively its current spot price), in which case all call and put options are out-of-the-money-forward (respectively out-of-the-money-spot).

The Carr–Madan formula (3) may be viewed as the solution $\phi(y) = f''(y)$ to the integral equation (2) with target function $f(x) := F(x) - F(x_0) - F'(x_0)(x-x_0)$ and *Carr-Madan kernel*

$$G(x,y;x_0) := (x-y)^+H(y-x_0) + (y-x)^+H(x_0-y), \quad (4)$$

where $H(\cdot)$ is Heaviside's step function. An alternative proof to Taylor's theorem is to carefully differentiate both sides of equation (2) twice, either with the help of Dirac's delta functions or by invoking Leibniz's integral rule.

2.2. Alternative expression

It is worth noting that the Carr–Madan kernel (4) may be rewritten as

$$G(x,y;x_0) = (x-y)\left[H(x-y) - H(x_0-y)\right],$$

by substituting $H(y-x_0) = 1 - H(x_0-y)$ and then $(x-y)^+ - (y-x)^+ = x-y$ into equation (4). Substituting the above into (2), we obtain the Volterra equation of the first kind,

$$f(x) = \int_{x_0}^{x} (x-y)\phi(y)\,\mathrm{d}y,$$

which is forward for $x > x_0$ and backward for $x < x_0$.

2.3. Limitations

The Carr–Madan formula has two major limitations:

(1) In practice, only a finite number of vanilla option strikes are available and the formula must be discretized accordingly. Hedging is imperfect and approximation errors get in the way.
(2) In the theory of integral equations, the Carr–Madan kernel $G(x,y;x_0)$ (equation (4)) is not symmetric and does not have an orthonormal decomposition.

In this paper we address the above limitations by substituting the 'better' *straddle kernel* $G(x,y) := |x-y|$ which is symmetric and therefore admits an orthonormal decomposition. This kernel remains tractable in terms of practical applications as it corresponds to the family of all straddles with a continuum of strikes $y \in \mathbb{R}_+$. Moreover, the following identity shows that the straddle kernel has a one-to-one correspondence with the Carr–Madan kernel:

$$G(x,y;x_0) = \frac{|x-y|}{2} + \frac{x-y}{2}\left[H(y-x_0) - H(x_0-y)\right].$$

This identity is straightforwardly established by substituting $(\pm u)^+ = (|u| \pm u)/2$ into equation (4).

3. Existence and uniqueness of solutions

3.1. Solving first-kind Fredholm equations

Early theory for integral equations was developed by Volterra (1896), Fredholm (1903), Hilbert (1904), Schmidt (1907), Riesz (1916). It turns out that first-kind Fredholm equations are very much related to second-kind equations,

$$f(x) = \lambda\phi(x) - \int_a^b G(x,y)\phi(y)\,\mathrm{d}y,$$

where λ is a nonzero complex parameter† .

Much of the literature about integral equations is dedicated to the theoretical and numerical resolution of second-kind equations with a continuous kernel operating on continuous or square-integrable functions. Famously, the *Fredholm alternative* states that, for any $\lambda \neq 0$, either the homogeneous Fredholm integral equation of the second kind,

$$0 = \lambda\phi(x) - \int_a^b G(x,y)\phi(y)\,\mathrm{d}y,$$

has a nontrivial solution $\phi \not\equiv 0$ and λ is called an *eigenvalue*, or the inhomogeneous equation,

$$f(x) = \lambda\phi(x) - \int_a^b G(x,y)\phi(y)\,\mathrm{d}y,$$

always has a unique solution for any $f(x)$ and λ is called a *regular value*. Note that when λ is an eigenvalue, the second-kind inhomogeneous equation has either no solution or infinitely many solutions.

First-kind equations can be significantly more challenging to solve. It is worth emphasizing that there may be no solution at all, and that the theory about the existence and uniqueness of solutions is very limited compared to the Fredholm alternative available for second-kind equations. Fundamentally, the difficulty for finding a solution results from the smoothing property of integration. To illustrate this point, consider a well-behaved continuous kernel $G(x,y)$ and an input function $\phi(y)$ that is only piecewise continuous. The resulting output $\int_a^b G(x,y)\phi(y)\,\mathrm{d}y$ will be smoother than $\phi(y)$. Therefore, if $f(x)$ is a continuous target function, it is very possible that solutions $\phi(y)$ are all discontinuous, and that no solution exists within the class of continuous functions (e.g. section 8 and figure 6.

This observation is relevant to our payoff replication problem wherein a continuous solution is neither required nor expected; in fact, we will be mostly interested in square-integrable solutions.

3.2. Formal framework

Let E denote the infinite-dimensional vector space of the payoff functions under consideration, such as $C([a,b])$ or $L^2([a,b])$. Define the linear operator:

$$\begin{array}{llll} \mathcal{G}: & E & \to & E \\ & \phi & \mapsto & \mathcal{G}\phi : x \mapsto \int_a^b G(x,y)\phi(y)\,\mathrm{d}y. \end{array}$$

With these notations, the first-kind linear integral equation (2) may be written as $\mathcal{G}\phi = f$. The existence of solutions for all $f \in E$ then corresponds to $\mathcal{G}$ being a surjective operator, i.e. $\mathcal{G}(E) = E$, while the uniqueness of any solution corresponds to $\mathcal{G}$ being an injective operator, i.e. $\mathcal{G}^{-1}(0_E) = \{0_E\}$ where 0_E is the null function of E.

A standard theoretical requirement is for $\mathcal{G}$ to be a *compact operator* (see Kress 2014, pp. 25—6, for a formal definition). It turns out that compact operators are never surjective (Kress 2014, pp. 297—8), and thus there always are infinitely many target functions $f \in E$ for which the first-kind equation has no solution at all. In contrast, the identity operator $I : E \to E, \phi \mapsto \phi$ is trivially surjective and thus never compact (Kress 2014, p. 27), and it can be shown that the second-kind operator $\lambda I - \mathcal{G}, \lambda \neq 0$ is surjective if and only if it is injective (Kress 2014, p. 38). Within this framework, the Fredholm alternative translates into a discussion whether $\lambda I - \mathcal{G}$ is injective.

On the topic of eigenvalues, it is worth noting that three classic important properties from finite-dimensional linear algebra extend to infinite-dimensional Hilbert spaces E:

(1) For a large class of integral operators, the series of eigenvalues $\sum \lambda_n$ converges to the operators's trace $\int_a^b G(x,x)\,\mathrm{d}x$ (Lax 2002, p. 329).
(2) *Perron-Frobenius theorem*: if the integral operator $\mathcal{G}$ is positive‡ , it has a positive eigenvalue which is the largest in absolute value among all eigenvalues, and its eigenfunction is positive (Lax 2002, p. 253).
(3) *Mercer's theorem*: if the integral operator $\mathcal{G}$ is symmetric and satisfies $\int_a^b \int_a^b \phi(x)G(x,y)\phi(y)\,\mathrm{d}x\,\mathrm{d}y \geq 0$ then it is a positive-semidefinite operator and all its eigenvalues are nonnegative (Lax 2002, p. 343).

3.3. Spectral decomposition of continuous symmetric kernels

When the vector space of payoff functions is the Hilbert space of square-integrable functions on a finite segment $E = L^2([a,b])$, the linear map $\mathcal{G}$ corresponding to the square-integrable kernel $G \in L^2([a,b] \times [a,b])$ is called a *Hilbert-Schmidt integral operator*. If the kernel $G(x,y)$ is continuous, the operator $\mathcal{G}$ is always compact and therefore never surjective, i.e. there always are target functions $f \in L^2([a,b])$ for which the first-kind integral equation $\mathcal{G}\phi = f$ has no solution at all.

By Hilbert-Schmidt theory, when the kernel $G(x,y)$ is continuous and symmetric, all eigenvalues of $\mathcal{G}$ are real and form a finite or countable subset of $\mathbb{R}$ and there is an orthonormal system of eigenfunctions (ϕ_n). In practical applications, we can find all nonzero eigenvalues λ_n of $\mathcal{G}$ and their associated eigenfunctions ϕ_n by solving the homogeneous integral equation of the second kind $(\lambda_n I - \mathcal{G})\phi_n \equiv 0$, for which

† Observe that when $\lambda = 0$ we have a first-kind equation.

‡ Here, an operator is positive when the function $\mathcal{G}\phi$ is positive for any nonnegative and nonnull function ϕ.

numerous methods exist. Moreover, we have the spectral decomposition (Eidelman *et al.* 2004, p. 94),

$$G(x,y) = \sum_n \lambda_n \phi_n(x)\phi_n(y), \tag{5}$$

where the convergence of the series is understood in the sense of $L^2([a,b]\times[a,b])$. As a corollary, $\sum_n \lambda_n^2 = \int_a^b \int_a^b G(x,y)^2 \,\mathrm{d}x\,\mathrm{d}y$. Substituting the above spectral decomposition identity (5) into equation (2) we obtain that, when a solution ϕ exists, the target function f is attained by a linear combination of all eigenfunctions ϕ_n,

$$f(x) = \sum_n \lambda_n \phi_n(x) \int_a^b \phi_n(y)\phi(y)\,\mathrm{d}y.$$

The financial interpretation of the above equation is that the target option payoff $F(x)$ discussed in section 1 is perfectly replicated by a combination of cash and underlying asset together with a *discrete* portfolio of independent 'spectroreplicant' options, i.e.

$$F(x) = c + q\,x + \sum_n w_n \phi_n(x), \tag{6}$$

where c, q are the quantities of cash and underlying asset, and $w_n := \lambda_n \int_a^b \phi_n(y)\phi(y)\,\mathrm{d}y$ is the weight or quantity of the n^{th} spectroreplicant option paying off $\phi_n(x)$.

3.4. Unique square-integrable solution for continuous, symmetric and injective kernels

In some cases an explicit solution $\phi(y)$ to a first-kind equation with symmetric kernel may be obtained using non-spectral techniques, such as the convolution method for difference kernels (e.g. Srivastava and Buschman 2013, ch. 3). However, many equations do not solve in this manner. Fortunately, theory provides for a criterion about the existence of a unique solution when the continuous and symmetric kernel $G(x,y)$ induces an injective integral operator $\mathcal{G}$ on the Hilbert space of square-integrable functions $E = L^2([a,b])$ or $E = L^2((a,b))$.

Indeed, when $\mathcal{G}$ is symmetric and injective the orthonormal eigensystem (ϕ_n) is complete and therefore a basis of E, and all eigenvalues are real. Denoting $f_n := \int_a^b f(x)\phi_n(x)\,\mathrm{d}x$ the coordinates of any target function $f \in E$ in the basis, it is then easy to see that the function

$$\phi(y) := \sum_n \frac{f_n}{\lambda_n}\phi_n(y)$$

is a well-defined element of E if and only if the series $\sum f_n^2/\lambda_n^2$ converges, in which case it is the unique solution to the first-kind integral equation $f = \mathcal{G}\phi$.

Note that if $\mathcal{G}$ is symmetric but *not injective*, solutions exist if and only if the series $\sum_{\lambda_n\neq 0} f_n^2/\lambda_n^2$ converges. The solution set is then the affine space $\hat{\phi} + \mathcal{G}^{-1}(0_E)$ where $\hat{\phi} := \sum_{\lambda_n\neq 0} f_n\phi_n/\lambda_n$ is unique. In the context of payoff replication it is worth emphasizing that the nullspace portfolios $\phi \in \mathcal{G}^{-1}(0_E)$ replicate the null payoff and thus always have zero cost. As such, they do not change the economics of replicating the target payoff and may be ignored. For ease of exposition we only consider injective kernels.

4. Spectral decomposition of the straddle kernel

In this section and the following three, we focus on payoff replication with straddles as replicant options. The corresponding straddle kernel $G(x,y) := |x-y|$, where y is the strike price, is continuous and symmetric and thus admits a spectral decomposition over any finite segment $[a,b]$. Moreover, there must be at least one negative and one positive eigenvalue since the kernel trace vanishes: $\int_a^b |x-x|\,\mathrm{d}x = 0$. In fact, since the straddle kernel induces a positive operator, it must have a positive eigenvalue which is the largest among all absolute eigenvalues.

For ease of exposure, and without loss of generality, we first derive the spectral decomposition of the straddle kernel on the unit interval $[a,b]=[0,1]$ with corresponding integral equation

$$f(x) = \int_0^1 |x-y|\phi(y)\,\mathrm{d}y, \quad 0 \le x \le 1.$$

The decomposition for an arbitrary interval $[a,b]$ is then straightforwardly obtained through the affine map $x \mapsto a + (b-a)x$ and similarly for y. Note that differentiating the above integral equation twice against x yields the solution $\phi(x) = \frac{1}{2}f''(x)$ which is unique† . In particular, the homogeneous equation only has the trivial solution and thus the kernel is injective. Furthermore, we can see that when $f(x)\equiv 0$, i.e. the target payoff function $F(x)$ is purely affine, the integral equation only has the trivial solution.

To find the eigensystem we must solve the homogeneous second-kind equation

$$\lambda\phi(x) = \int_0^1 |x-y|\phi(y)\,\mathrm{d}y, \tag{7}$$

for $\lambda \neq 0$. Again, differentiating twice against x yields that eigenfunctions must satisfy the homogeneous second-order linear differential equation

$$\lambda\phi''(x) = 2\phi(x), \quad 0 \le x \le 1,$$

whose general solution is of the form

$$\begin{cases} \phi(x) = \alpha e^{2\omega x} + \beta e^{-2\omega x} & \text{if } \lambda > 0, \quad \text{(8a)} \\ \alpha\cos 2\omega x + \beta \sin 2\omega x & \text{if } \lambda < 0, \quad \text{(8b)} \end{cases}$$

where α, β are real coefficients and $\omega := 1/\sqrt{2|\lambda|}$ is the semi-angular frequency associated with λ.

Following the notations of section 3.3 we index eigenelements by nonnegative integers $n \in \mathbb{N}$ from largest to smallest

† For a formal proof of uniqueness, suppose that $\tilde{\phi}$ is another solution; then $\int_0^1 |x-y|(\phi(y)-\tilde{\phi}(y))\,\mathrm{d}y = 0$ and differentiating twice against x yields $\tilde{\phi} \equiv \phi$.

absolute eigenvalue $|\lambda_n|$. In the next section 4.1 we will see that there is only one positive eigenvalue λ_0 which is the largest among all absolute eigenvalues.

4.1. Eigenfunction associated with the positive eigenvalue

Substituting (8) into equation (7) and simplifying, the straddle integral operator maps an eigenfunction ϕ_0 with positive eigenvalue $\lambda_0 > 0$ to

$$\begin{aligned}\int_0^1 |x-y|\phi_0(y)\,\mathrm{d}y \\ = \lambda_0\Big[\phi_0(x) + \left(\beta - \alpha e^{2\omega}\right)\left(1+e^{-2\omega}\right)\omega x \\ - \frac{\alpha}{2}e^{2\omega}\left(1-2\omega+e^{-2\omega}\right) - \frac{\beta}{2}e^{-2\omega}\left(1+2\omega+e^{2\omega}\right)\Big].\end{aligned} \tag{9}$$

For the remainder terms which are affine in x to vanish we must have $\beta = \alpha e^{2\omega}$. After substitution into equation (9) and simplifications, we obtain that ω must be the only fixed point of the hyperbolic cotangent $\omega_0 \approx 1.19968$; equivalently, the only positive eigenvalue of the straddle kernel is

$$\lambda_0 = \frac{1}{2\omega_0^2} \approx 0.34741.$$

Finally, solving $\int_0^1 \phi_0^2(y)\,\mathrm{d}y = 1$ for α we obtain the normalized eigenfunction

$$\begin{aligned}\phi_0(x) &= \frac{\sqrt{2}}{\cosh\omega_0}\cosh\omega_0(1-2x) \\ &\approx 0.78126 \times \cosh[1.19968 \times (1-2x)],\end{aligned} \tag{10}$$

which is a positive function as expected from the Perron-Frobenius theorem.

4.2. Eigenfunctions associated with negative eigenvalues

Substituting (8b) into equation (7) and simplifying through trigonometric identities, the straddle integral operator maps an eigenfunction $\phi_n, n \geq 1$ with negative eigenvalue $\lambda_n < 0$ to

$$\begin{aligned}\int_0^1 |x-y|\phi_n(y)\,\mathrm{d}y \\ = \lambda_n\Big[\phi_n(x) + 2\omega\cos\omega\,(\alpha\sin\omega - \beta\cos\omega)\,x \\ + \left(\beta\omega - \frac{\alpha}{2}\right)\cos 2\omega - \left(\alpha\omega + \frac{\beta}{2}\right)\sin 2\omega - \frac{\alpha}{2}\Big].\end{aligned} \tag{11}$$

The remainder terms affine in x vanish when either

(a) $\beta = 0$ and $\omega = (\pi/2) + k\pi, k \in \mathbb{Z}$; or
(b) $\beta = \alpha\tan\omega$, where $\omega \neq (\pi/2) + k\pi, k \in \mathbb{Z}$ satisfies $\cos\omega + \omega\sin\omega = 0$, i.e. it is an opposite fixed point of the cotangent function.

Solving $\int_0^1 \phi_n^2(y)\,\mathrm{d}y = 1$ for α and simplifying through trigonometric identities, we obtain the alternating system of normalized eigenfunctions

$$\phi_n(x) = \begin{cases} \sqrt{2}\cos n\pi x & \text{if } n \geq 1 \text{ is odd}, \\ \dfrac{\sqrt{2}}{\cos\omega_n}\cos\omega_n(1-2x) & \text{if } n \geq 2 \text{ is even}, \end{cases} \tag{12}$$

where ω_n is the only opposite fixed point of the cotangent function in the interval $((n-1)\pi/2, n\pi/2)$ when $n \geq 2$ is even. With the convention $\omega_n := n\pi/2$ when $n \geq 1$ is odd, the negative eigenvalues λ_n are indexed from largest to smallest in absolute value:

$$\lambda_n = -\frac{1}{2\omega_n^2}, \quad n \geq 1.$$

4.3. Remarks about the straddle eigensystem

The straddle eigensystem derived in sections 4.1 and 4.2 may be viewed as a modified Fourier basis of the Hilbert space $L^2([0,1])$, with the benefit that the basis is ordered by magnitude of eigenvalues. Additionally, we have the following properties:

(a) The eigenfunctions $\phi_n, n \geq 1$ take positive and negative values. This may have a numerical benefit when replicating a target payoff $f(x)$ which is small in absolute value.
(b) The eigensystem is consistent with the spectral decomposition of linear and symmetric Toeplitz matrices (Bünger 2014) which are a discrete version of the straddle kernel.
(c) All eigenfunctions satisfy $\phi_n(0) = \sqrt{2}$ and $\phi_n(1) = (-1)^n\sqrt{2}$.
(d) Since the kernel trace vanishes, we have:

$$\begin{aligned}\sum_{k=0}^{\infty}\lambda_{2k+2} &= \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} - \lambda_0 \\ &= \frac{2}{\pi^2}\frac{\pi^2}{8} - \lambda_0 \approx -0.09741.\end{aligned}$$

(e) By definition, for each eigenfunction we have $\phi_n'' = \frac{2}{\lambda_n}\phi_n = -4\omega_n^2\phi_n$ for $n \geq 1$ and $\phi_0'' = 4\omega_0^2\phi_0$.
(f) Asymptotically, when $n \geq 2$ is even, we have $\omega_n \sim n\pi/2$ as $n \to \infty$. Indeed, inverting the opposite fixed point equation $\cot\omega_n = -\omega_n$ produces $\omega_n = -\operatorname{arccot}\omega_n + (n\pi/2)$, and the inverse cotangent function is bounded. Therefore, for large n, we have $\omega_n = n\pi/2$ if n is odd and $\omega_n \sim n\pi/2$ if n is even.

4.4. Spectral decomposition on the unit interval

Substituting the normalized eigenfunction expressions of equations (10) and (12) into the spectral decomposition equation (5), and then simplifying, the spectral decomposition

Table 2. Top 20 eigenvalues and related coefficients of the straddle kernel

n	λ_n ($\times 10^{-3}$)	ω_n	c_n	Err. Norm
0	+ 347.4082690	1.199678640	+ 0.212046516	0.214416
1	− 202.6423673	1.570796327	− 0.405284735	0.070073
2	− 63.84909579	2.798386046	− 0.144005020	0.028871
3	− 22.51581859	4.712388980	− 0.045031637	0.018071
4	− 13.34411279	6.121250467	− 0.027400487	0.012186
5	− 8.105694691	7.853981634	− 0.016211389	0.009099
6	− 5.758866886	9.317866462	− 0.011650392	0.007045
7	− 4.135558516	10.99557429	− 0.008271117	0.005703
8	− 3.206946639	12.48645440	− 0.006455031	0.004716
9	− 2.501757621	14.13716694	− 0.005003515	0.003998
10	− 2.042994806	15.64412837	− 0.004102685	0.003437
11	− 1.674730308	17.27875959	− 0.003349461	0.003001
12	− 1.415208556	18.79640437	− 0.002838428	0.002646
13	− 1.199067262	20.42035225	− 0.002398135	0.002359
14	− 1.038184585	21.94561288	− 0.002080680	0.002118
15	− 0.900632744	23.56194490	− 0.001801265	0.001917
16	− 0.794086718	25.09291041	− 0.001590696	0.001745
17	− 0.701184662	26.70353756	− 0.001402369	0.001598
18	− 0.627008356	28.23893658	− 0.001255589	0.001470
19	− 0.561336198	29.84513021	− 0.001122672	0.001359

of the straddle kernel on the unit interval [0, 1] is

$$|x-y| = c_0 \cosh\omega_0(1-2x)\cdot\cosh\omega_0(1-2y) + \sum_{k=0}^{\infty} c_{2k+1}\cos[(2k+1)\pi x]\cdot\cos[(2k+1)\pi y] + \sum_{k=0}^{\infty} c_{2k+2}\cos\omega_{2k+2}(1-2x)\cdot\cos\omega_{2k+2}(1-2x), \tag{13}$$

where c_n are the scaling coefficients:

$$c_n := \begin{cases} 1/(\omega_0\cosh\omega_0)^2 & \text{if } n=0,\\ -4/(n\pi)^2 & \text{if } n\geq 1 \text{ is odd},\\ -1/(\omega_n\cos\omega_n)^2 & \text{if } n\geq 2 \text{ is even}.\end{cases}$$

In table 2, we report numerical estimates of λ_n, ω_n, c_n together with the L^2 norm of the running spectral decomposition error† $|x-y| - \sum_{k=0}^{n-1}\lambda_k\phi_k(x)\phi_k(y)$. Figure 1 illustrates the goodness of fit using 1, 2 and 6 eigenfunctions associated with top eigenvalues. As predicted by the rapidly decaying error norm, we can see that few eigenfunctions are needed to obtain a visually excellent fit.

4.5. *Spectral decomposition on a finite segment* $[a, b]$

Using affine transformations, it is easy to show that an orthonormal eigensystem for the straddle kernel defined over an arbitrary finite segment $[a, b]$ is simply

$$\left(\frac{1}{\sqrt{b-a}}\phi_n\left(\frac{x-a}{b-a}\right)\right), \quad a\leq x\leq b, n\geq 0,$$

with associated eigenvalues $(b-a)^2\lambda_n$, where ϕ_n, λ_n are defined in sections 4.1 and 4.2. The corresponding spectral decomposition is then given as

$$\begin{aligned}|x-y| &= (b-a)c_0\cosh\omega_0\left(1-2\frac{x-a}{b-a}\right)\\ &\quad\cdot\cosh\omega_0\left(1-2\frac{y-a}{b-a}\right)\\ &\quad+(b-a)\sum_{k=0}^{\infty}c_{2k+1}\cos\left[(2k+1)\pi\frac{x-a}{b-a}\right]\\ &\quad\cdot\cos\left[(2k+1)\pi\frac{y-a}{b-a}\right]\\ &\quad+(b-a)\sum_{k=0}^{\infty}c_{2k+2}\cos\omega_{2k+2}\left(1-2\frac{x-a}{b-a}\right)\\ &\quad\cdot\cos\omega_{2k+2}\left(1-2\frac{y-a}{b-a}\right),\end{aligned}$$

where the coefficients c_n are given in equation (13).

5. Consequences for option replication and pricing

Because equation (2) with straddle kernel has the unique solution $\phi(y) = \frac{1}{2}f''(y) = \frac{1}{2}F''(y)$ when it exists, the weights of the spectroreplicant options in equation (6) may be further specified as

$$w_n = \frac{b-a}{2}\lambda_n\int_a^b \phi_n\left(\frac{x-a}{b-a}\right)F''(x)\,\mathrm{d}x. \tag{14}$$

A proxy of order $n\geq 1$ for the target payoff is then simply obtained by the truncation

$$\hat{F}_n(x) := c + qx + \sum_{k=0}^{n-1} w_k\phi_k(x)$$

and the L^2 norm of the replication error is then given as

$$||F-\hat{F}_n||_2 := \left(\int_a^b [F(x)-\hat{F}_n(x)]^2\,\mathrm{d}x\right)^{1/2} = \sqrt{\sum_{k=n}^{\infty} w_k^2}.$$

To determine the cash and underlying asset quantities c, q we need two independent conditions. For instance, integrating the right-hand side of equation (1) by parts and evaluating at the boundaries $x = a$, b with straddle kernel $G(x, y) := |x-y|$,

† In the orthonormal eigensystem the error norm is $\|\sum_{k=n}^{\infty}\lambda_k\phi_k\| = \sqrt{\sum_{k=n}^{\infty}\lambda_k^2}$

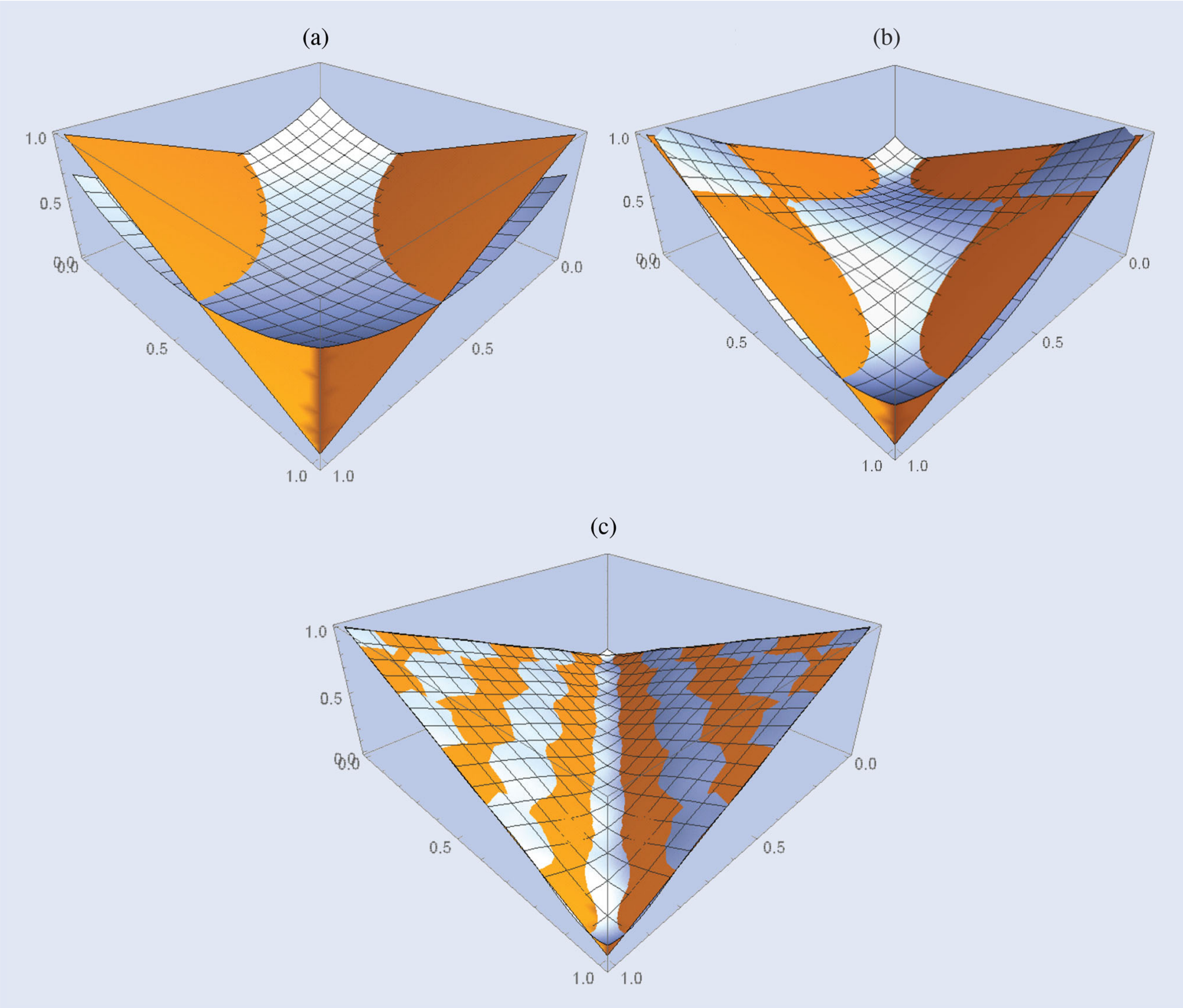

Figure 1. Straddle kernel fit with top eigenfunctions. (a) Top eigenfunction. (b) Top two eigenfunctions and (c) Top six eigenfunctions

we obtain

$$\begin{aligned} F(a) &= c + q\,a + \frac{1}{2}\int_a^b (y-a)F''(y)\,\mathrm{d}y \\ &= c + q\,a + \frac{1}{2}\left[(b-a)F'(b) - (F(b)-F(a))\right], \\ F(b) &= c + q\,b + \frac{1}{2}\int_a^b (b-y)F''(y)\,\mathrm{d}y \\ &= c + q\,b - \frac{1}{2}\left[(b-a)F'(a) - (F(b)-F(a))\right]. \end{aligned}$$

Solving for c, q yields

$$\begin{aligned} c &= \frac{1}{2}\left[F(a) + F(b) - aF'(a) - bF'(b)\right], \\ q &= \frac{1}{2}\left[F'(a) + F'(b)\right]. \end{aligned}$$

5.1. Benefits for option replication and hedging

Given that the spectroreplicant options induced by the straddle kernel do not trade, the practical benefits of equation (6) in terms of hedging are limited in this case. However, our general framework is not confined to the straddle kernel and leaves the door open to other symmetric kernels $G(x, y)$ that might decompose into more practical spectroreplicant options.

Nevertheless, in terms of approximation accuracy, we found that our spectroreplication approach tends to perform better than a classical Fourier series decomposition such as the COS method of Fang and Oosterlee (2009), for at least two European target payoffs of high practical relevance: the log contract and vanilla calls. This is a useful property in view of the pricing applications discussed in the following sections. For the log contract $F(x) := \ln x$, figure 2(a) shows that the L^2 norm of the error decays more rapidly in our method than the Fourier series method as the number of terms n grows, and is more accurate for $n \geq 16$ terms. For vanilla calls $F(x) := (x-K)^+$, figure 2(b) shows that the error norm of our method is about 42% lower across all strikes with only $n = 10$ terms, while table 3 below reports a consistently lower error norm at various truncation orders n.

We provide below some arguments analyzing the comparative performance of our spectroreplication method against the Fourier COS series expansion over the unit interval [0, 1]. Let

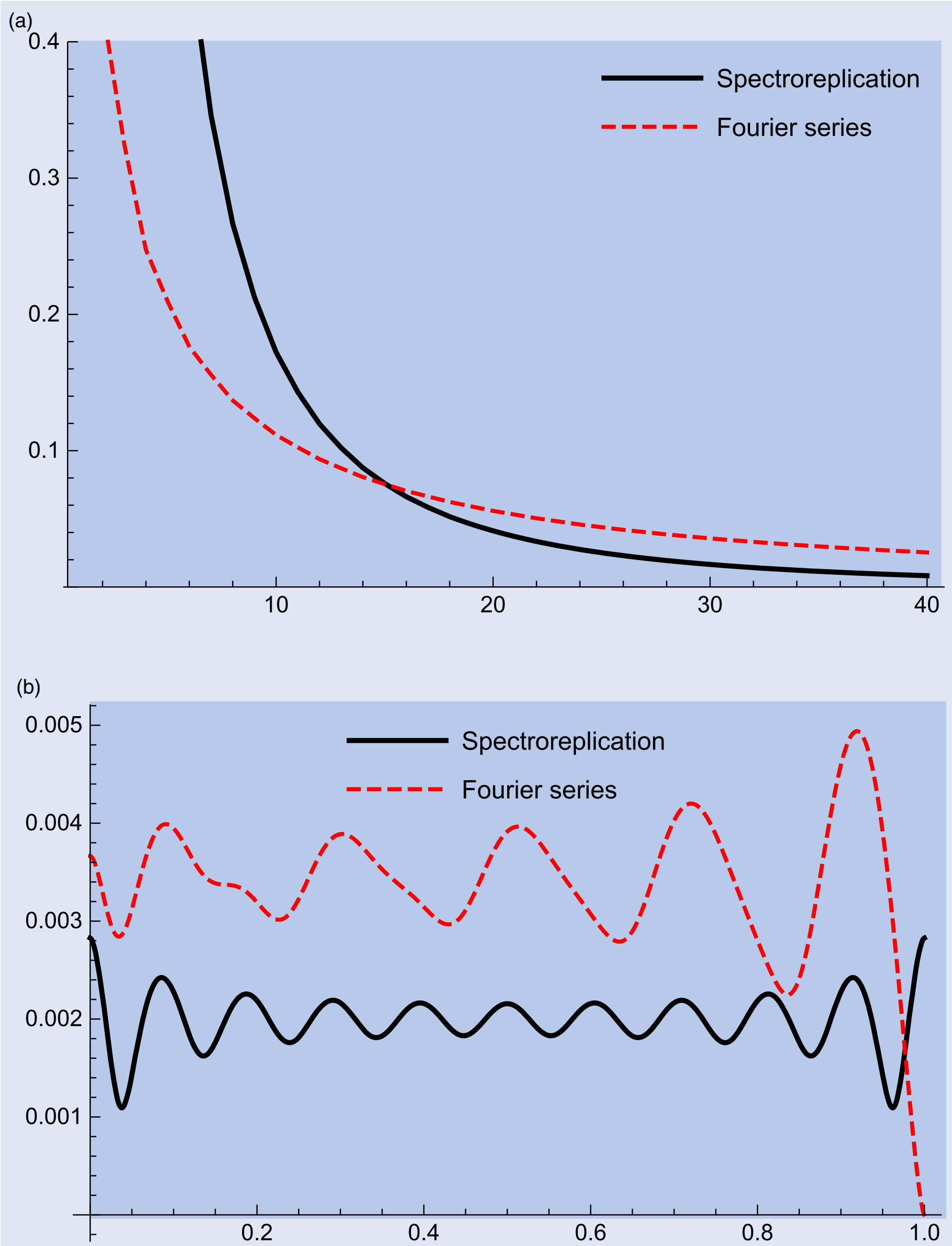

Figure 2. Comparison of proxy error norms between the spectroreplication and Fourier basis methods. (a) Error norm decay for the log contract as the truncation order n increases, using $[a, b] = [0.01, 1.01]$ and (b) Error norm at order $n = 10$ for vanilla calls as a function of strike $0 \le K \le 1$, using $[a, b] = [0, 1]$.

$\psi_n(x) := \cos(n\pi x)$ for $n \ge 0$, and denote the scalar product of two functions by $\langle f, g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$. The proxies at order n are

$$\hat{F}_n(x) := c + qx + \sum_{k=0}^{n-1} \frac{\lambda_k}{2}\langle F'', \phi_k\rangle \phi_k(x),$$

$$\check{F}_n(x) := \langle F, \psi_0\rangle + 2\sum_{k=1}^{n-1}\langle F, \psi_k\rangle \psi_k(x).$$

By orthogonality of Fourier basis functions we have $\langle F, \check{F}_n\rangle = \|\check{F}_n\|^2$ and thus $\|F - \check{F}_n\|^2 = \|F\|^2 - \langle F, \check{F}_n\rangle = \langle F, F - \check{F}_n\rangle$. Similarly, $\|F - \hat{F}_n\|^2 = \langle F(x) - c - qx, F(x) - \hat{F}_n(x)\rangle$, so that

$$\begin{aligned}
&\|F - \hat{F}_n\|^2 - \|F - \check{F}_n\|^2 \\
&\quad = \langle F(x) - c - qx, F(x) - \hat{F}_n(x)\rangle - \langle F, F - \check{F}_n\rangle \\
&\quad = \langle F(x) - c - qx, \check{F}_n(x) - \hat{F}_n(x)\rangle \\
&\qquad - \langle c + qx, F(x) - \check{F}_n(x)\rangle \\
&\quad = \langle F(x) - c - qx, \check{F}_n(x) - \hat{F}_n(x)\rangle \\
&\qquad - q\langle x, F(x) - \check{F}_n(x)\rangle,
\end{aligned} \tag{15}$$

Table 3. Comparison of proxy error norms for vanilla calls on the domain $[a,b]=[0,1]$, averaged across all strikes $0\le K\le 1$ spaced 0.01 apart, at various truncation orders n

n	Spectroreplication	Fourier series	Ratio
5	0.0061372	0.010485	0.59
10	0.0018960	0.003265	0.58
15	0.0010996	0.0020171	0.55
20	0.00071673	0.0011074	0.65
25	0.00051208	0.00061407	0.83
30	0.00039795	0.00052633	0.76
35	0.00031651	0.00041700	0.76
40	0.00026859	0.00031527	0.85

whose sign will depend on the particular choice of target payoff F. In Appendix we show that, for large enough n,

$$\begin{aligned}\check{F}_n(x)-\hat{F}_n(x)&\approx\langle F,\psi_0\rangle-c-qx\\&\quad-\int_0^1\left[\frac{1}{3}+\frac{x^2-(x+y)+y^2}{2}\right]F''(y)\,\mathrm{d}y\\&\quad+2\sum_{k=1}^{n-1}\frac{(-1)^kF'(1)-F'(0)}{(k\pi)^2}\psi_k(x).\end{aligned}$$

Substituting into equation (15) and rearranging we get $\|F-\hat{F}_n\|^2-\|F-\check{F}_n\|^2\approx A+B_n$ where

$$\begin{aligned}A&:=\int_0^1(F(x)-c-qx)\left(\langle F,\psi_0\rangle-c-qx\right)\,\mathrm{d}x\\&\quad-\int_0^1\int_0^1(F(x)-c-qx)\\&\quad\times\left[\frac{1}{3}+\frac{x^2-(x+y)+y^2}{2}\right]F''(y)\,\mathrm{d}x\,\mathrm{d}y,\\B_n&:=2\sum_{k=1}^{n-1}\frac{(-1)^kF'(1)-F'(0)}{(k\pi)^2}\\&\quad\times\int_0^1(F(x)-c-qx)\,\psi_k(x)\,\mathrm{d}x\\&\quad-q\int_0^1x\left(F(x)-\check{F}_n(x)\right)\,\mathrm{d}x.\end{aligned}$$

Note that the approximate criterion $A+B_n$ is easy to calculate for any n as it only involves the Fourier proxy $\check{F}_n$ and basis functions $\psi_k(x):=\cos(k\pi x)$ and does *not* require to calculate the spectroreplicant proxy $\hat{F}_n(x)$.

For the log-contract $F(x):=\ln(x+0.01)$ we find $A\approx 199.63027$ and $B_{30}\approx -199.63170$ giving $\|F-\hat{F}_{30}\|^2-\|F-\check{F}_{30}\|^2\approx A+B_{30}\approx -0.00143$. This further suggests that, for large enough n, our spectroreplication approach is more accurate than the Fourier cosine series.

For the vanilla call $F(x):=(x-K)^+$ we find $A=-(K^4/24)+(K^2/12)-\frac{1}{48}$ and

$$\begin{aligned}B_{10}&=-\frac{K^3}{12}+\frac{K^2}{8}-\frac{1}{24}+\frac{2.16387}{\pi^4}\\&\quad+\frac{4}{\pi^4}\left[\cos(K\pi)+\frac{\cos(3K\pi)}{81}+\frac{\cos(5K\pi)}{625}\right.\\&\quad\left.+\frac{\cos(7K\pi)}{2401}+\frac{\cos(9K\pi)}{6561}\right]\\&\quad-\frac{1}{\pi^4}\left[\frac{\cos(2K\pi)}{8}+\frac{\cos(4K\pi)}{128}\right.\\&\quad\left.+\frac{\cos(6K\pi)}{648}+\frac{\cos(8K\pi)}{2048}\right].\end{aligned}$$

Figure 3 gives the signed square root of the relative squared error $\|F-\hat{F}_{10}\|^2-\|F-\check{F}_{10}\|^2$ versus its proxy $A+B_{10}$ as functions of strike $0\le K\le 1$. We can see that both measures are negative, suggesting again that the spectroreplication approach is more accurate than the Fourier cosine series.

5.2. *Benefits for option pricing*

When equation (6) holds and all relevant quantities converge in L^2, the price of the target option is simply given as

$$F=c+qX+\sum_{n=0}^{\infty}w_n\Phi_n,\tag{16}$$

where F, X are the respective prices of the target option and underlying asset, Φ_n is the price of the nth spectroreplicant option, and all prices are forward (i.e. paid on the common maturity date.) The above pricing equation can be established using classical arbitrage arguments under the assumptions that short-selling and the instant trading of infinitely many securities are both feasible. In practice, just as with the Carr–Madan formula, the latter assumption is not realistic and must be mitigated by selecting a finite number of replicant options. For example, a proxy of order $n\ge 1$ based on the largest absolute eigenvalues would be

$$\hat{F}_n:=c+qX+\sum_{k=0}^{n-1}w_k\Phi_k.\tag{17}$$

The key benefit of equation (17) compared to a discretization of the Carr–Madan formula is that spectroreplicant options are orthogonal in the sense of the scalar product $\langle f,g\rangle=\int f(x)g(x)\,\mathrm{d}x$. In contrast, the continuum of call and put replicants in Carr–Madan are very codependent due to their overlapping payoff functions. This suggests that, for non-pathological target payoff $F(x)$, a limited number n of spectroreplicant options is enough to achieve satisfactory pricing accuracy.

An obvious practical disadvantage of equation (17) is that the fair prices $(\Phi_k)_{0\le k\le n}$ of spectroreplicant options must be discovered by another method. In this regard, we may distinguish between:

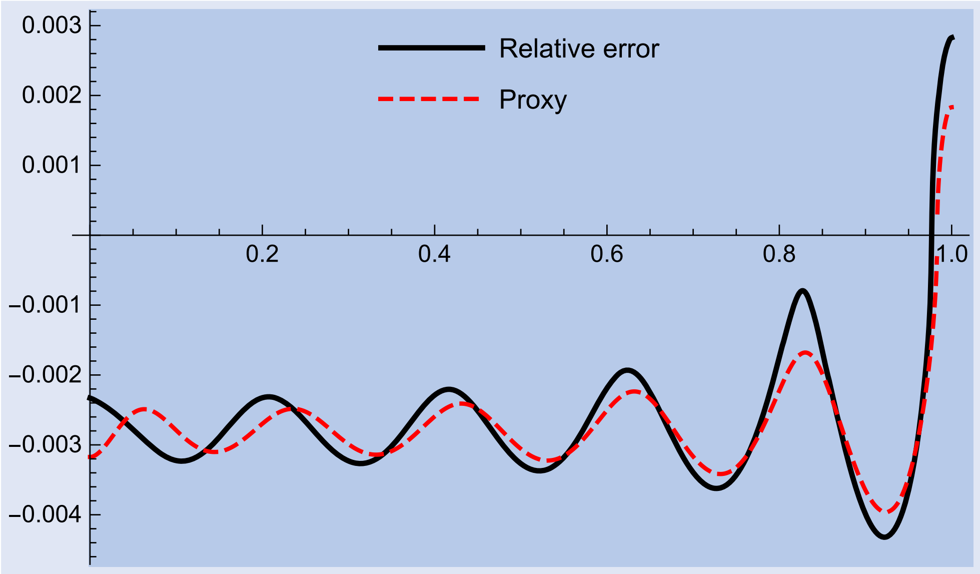

Figure 3. Comparison between exact and proxy relative squared error at order $n = 10$ for the vanilla call $F(x) := (x - K)^+$, as functions of strike $0 \le K \le 1$

(1) A model-based option pricing method, as shown in section 7;
(2) A model-free option pricing method, such as the Carr–Madan formula for option prices, either discretized along listed option strikes as shown in section 6 or using a numerical integration scheme together with the Black and Scholes (1973) formula and a model of the implied volatility smile.

Here, we must pause to dissipate any concern of circularity within the second approach: if we price spectroreplicant options using the Carr–Madan formula, do we not end up where we started and effectively price the target option using the Carr–Madan kernel? This would be true if we used infinitely many spectroreplicants, but the key benefit of equation (17) is that only a small number n of spectroreplicants is required to achieve satisfactory accuracy. In a practical implementation, the fixed weights w_k and the fair spectroreplicant prices Φ_k need only be precomputed once; then, for each target option in the book with specific payoff $F(x)$, computing the proxy price $\hat{F}_n$ only requires $n + 3$ multiplications and additions.

As an illustration, consider a vanilla option market with 200 listed strikes, and an option book of 1000 exotic European options. Pricing each exotic option individually costs 200 operations using the discretized Carr–Madan formula, for a total of 200,000 operations for the entire book. In contrast, precomputing 20 spectroreplicant option prices costs 4000 operations, and then pricing all exotic options in the book using equation (17) only costs 20,000 operations, for a grand total of 24,000 operations — an 88% gain in efficiency.

This gain of speed is likely to be very relevant for electronic market-making, risk management of large portfolios of options or high frequency option trading. Moreover, the computational cost of refreshing the prices Φ_k throughout the trading day can be mitigated using Greek sensitivities.

6. Numerical application: fast vanilla option pricing

6.1. Proxy formula for vanilla option prices

For the vanilla call target payoff $F(x) := (x - K)^+$ where K is the strike price, the second-order derivative is Dirac's delta function $F''(x) = \delta(x - K)$; substituting into equation (14) we obtain the proxy formula for the call price

$$\hat{c}_n(K) = \frac{X - K}{2} + \sum_{k=0}^{n-1} w_k(K)\Phi_k, \tag{18}$$

with weights

$$w_k(K) = \frac{b-a}{2}\lambda_k \phi_k\left(\frac{K-a}{b-a}\right).$$

Similarly, the put proxy formula is given as

$$\hat{p}_n(K) = \frac{K - X}{2} + \sum_{k=0}^{n-1} w_k(K)\Phi_k$$

with the same weights $w_k(K)$.

Table 4. Spectroreplicant option prices for the S&P 500 option market as of 20 November 2018

n	Φ_n	n	Φ_n
0	0.9636258443	10	− 0.2156408037
1	− 1.010953323	11	0.2308846696
2	− 0.08605859769	12	− 0.1771790305
3	0.562828561	13	0.09907369714
4	− 0.8733519895	14	− 0.02579248218
5	0.682335075	15	− 0.03158846323
6	− 0.3531713684	16	0.05740449608
7	0.08321391368	17	− 0.06505514540
8	0.1384173306	18	0.05272523143
9	− 0.2156408037	19	− 0.03139028403

6.2. Numerical results

We repriced 30-day out-of-the-money options on the S&P 500 index using the top 20 spectroreplicant options, based on sample bid and offer data as of 20 November 2018. We report the spectroreplicant option prices Φ_k in table 4 obtained with a VIX-style discretization of the Carr–Madan formula (3). Then, we compute the proxy option prices for strike prices ranging from $a = 1225$ to $b = 3075$ using the formulas above.

In figure 4(a) we plot our results for listed strikes between 1225 and 3075 on a scale from 0 to 1, where 0 corresponds to the market bid and 1 corresponds to the market offer price. Remarkably, all but two proxy option prices lie within the bid-offer range.

A valuable additional benefit of the spectral decomposition method is to provide a natural 'fit' of the implied volatility smile for arbitrary strikes $a \leq K \leq b$. In figure 4(b), we show our results in the slightly extended range [1000, 3300]. We can see that the fit is visually pleasing and the extrapolated values on the left and right regions of the chart look plausible.

6.3. Arbitrage considerations

It is worth emphasizing that the proxy formula of equation (18) is not theoretically free of arbitrage due to the oscillatory nature of the spectroreplicant options. Indeed, the tails of the corresponding implied distribution $h_{n+1}(K) := \hat{c}''_{n+1}(K)$ can become negative as shown in figure 5, indicating

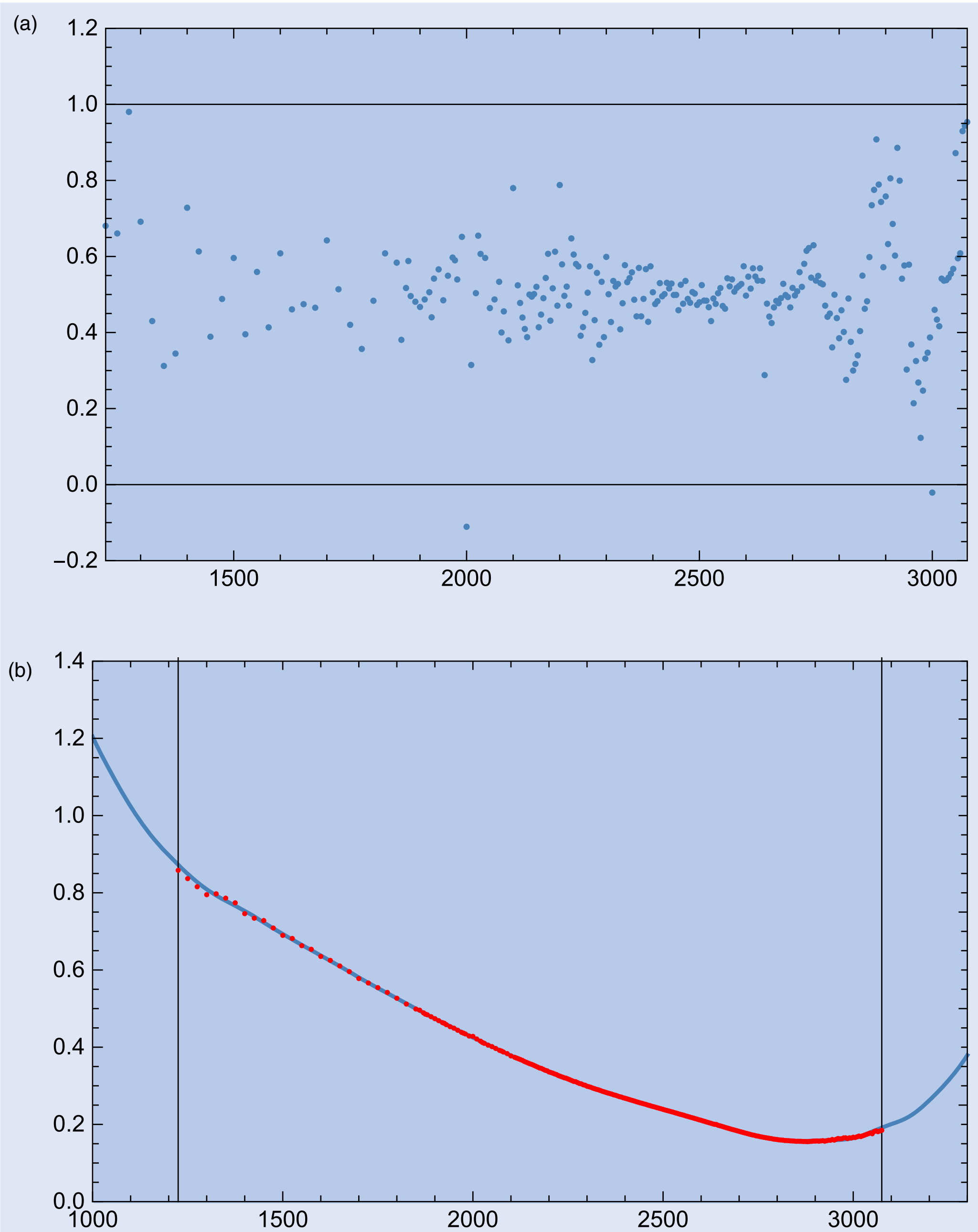

Figure 4. Proxy OTM option prices and corresponding proxy implied volatility smile. (a) Proxy prices on a market bid-offer 0-1 scale and (b) Proxy implied volatility smile

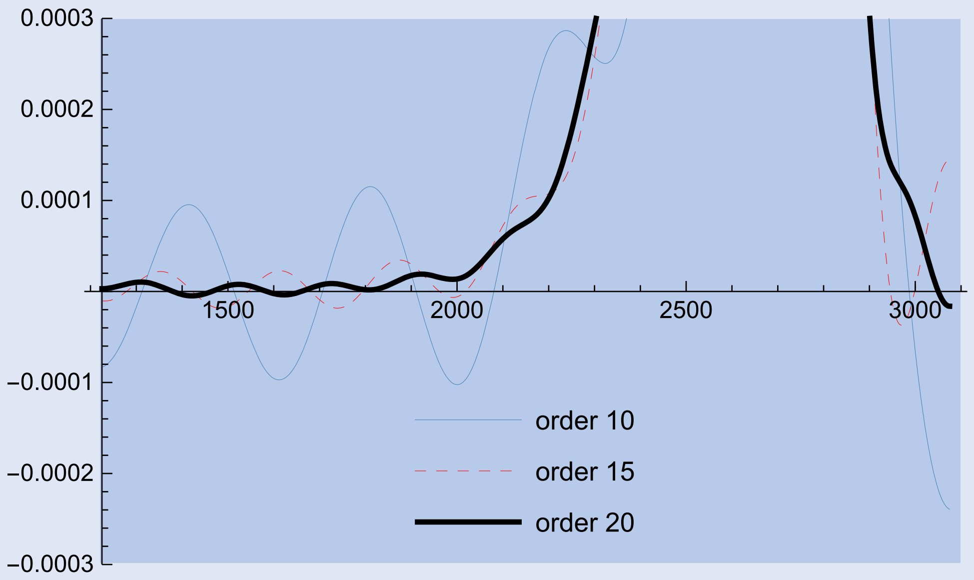

Figure 5. Implied distribution tails corresponding to proxy call prices

the theoretical existence of butterfly arbitrages. However, our empirical results shown in figure 4(a) suggest that such arbitrages are unlikely to have any practical relevance once bid-offer spreads are taken into account.

As expected, in the limit as $n \to \infty$, the proxy formula is arbitrage-free as long as all spectroreplicant prices Φ_n are known and priced off a valid implied distribution $h(K)$. This can be verified by substituting $\Phi_n = \int_a^b \phi_n((x-a)/(b-a))h(x)\,\mathrm{d}x$ into (18) to get

$$\begin{aligned} c(K) &:= \frac{X-K}{2} + \sum_{n=0}^{\infty} w_n(K)\Phi_n \\ &= \frac{X-K}{2} + \int_a^b h(x)\,\mathrm{d}x \sum_{n=0}^{\infty} w_n(K)\phi_n\left(\frac{x-a}{b-a}\right). \end{aligned}$$

Substituting into the above the expression for $w_n(K)$, recognizing the spectral decomposition (5) of the straddle kernel $|x-K|$, and differentiating both sides twice against K we recover the implied distribution $c''(K) = h(K)$. In addition, note that the truncated implied distribution $h_{n+1}(K)$ has the simple expression

$$h_{n+1}(K) := \hat{c}''_{n+1}(K) = \sum_{k=0}^{n} \Phi_k \phi_k\left(\frac{K-a}{b-a}\right),$$

which is obtained by differentiating (18) twice against K using $\phi''_n(y) = (2/\lambda_n)\phi_n(y)$.

7. Theoretical application: formulas for spectroreplicant option prices when the characteristic function is known

Consider an option pricing model where the characteristic function $\varphi(z) := \mathbb{E}(e^{izX_T}), z \in \mathbb{C}$ of the terminal underlying value X_T at time T is known in closed form. The characteristic function for $\tilde{X}_T := (X_T - a)/(b-a)$ is then

$$\tilde{\varphi}(z) = e^{-i(a/(b-a))z}\varphi\left(\frac{z}{b-a}\right),$$

and we may recover pricing formulas for spectroreplicant options through the identities

$$\mathbb{E}\cosh\omega(1-2\tilde{X}_T) = \frac{1}{2}e^{\omega}[\tilde{\varphi}(2i\omega) + \tilde{\varphi}(-2i\omega)], \tag{19a}$$

$$\mathbb{E}\cos\omega\tilde{X}_T = \frac{1}{2}[\tilde{\varphi}(\omega) + \tilde{\varphi}(-\omega)], \tag{19b}$$

$$\mathbb{E}\cos\omega(1-2\tilde{X}_T) = \frac{1}{2}e^{i\omega}\tilde{\varphi}(-2\omega) + \frac{1}{2}e^{-i\omega}\tilde{\varphi}(2\omega). \tag{19c}$$

At this point, we are faced with two technical issues. In many classical pricing models such as Black and Scholes (1973), Heston (1993), Merton (1976), the price X_T belongs to the entire real half-line rather than a finite interval $[a, b]$. In addition, the characteristic function is only known for the *log-price*. The second issue may be resolved by rewriting the target payoff in log-price space as $\bar{F}(x) := F(e^x)$, so that x now corresponds to $\ln X_T$, without changing the kernel $G(x, y) := |x-y|$ which now corresponds to log straddles. The coefficients w_n in equation (14) then become

$$\begin{aligned} w_n &= \frac{b-a}{2}\lambda_n \int_a^b \phi_n\left(\frac{x-a}{b-a}\right)\bar{F}''(x)\,\mathrm{d}x \\ &= \frac{b-a}{2}\lambda_n \int_a^b \phi_n\left(\frac{x-a}{b-a}\right)[e^x F'(e^x) + e^{2x}F''(e^x)]\,\mathrm{d}x \end{aligned}$$

and may be used in the pricing equations (16) and (17).

The finite domain issue is harder to address. Fang and Oosterlee (2009) propose a somewhat arbitrary formula (eq.(49)) for the range $[a, b]$ based on the *cumulants* of X_T. We derive an alternative approach in the following paragraphs. Note that once a suitable range $[a, b]$ has been found, both the spectroreplicant prices Φ_n obtainable from equations (19) and the resulting target option price F of equation (16) will have the required numerical accuracy for the chosen model.

Let $\breve{F}$ denote the model price of the target option, while F remains its price over the restricted domain $[a, b]$:

$$\breve{F} := \mathbb{E}[F(X_T)] = \int_{-\infty}^{\infty} F(e^x)p(x)\,\mathrm{d}x$$
$$\approx \int_a^b F(e^x)p(x)\,\mathrm{d}x =: F,$$

where $p(x)$ is the probability density function of $\ln X_T$. An upper bound for the absolute error is then

$$|\breve{F} - F| \leq \int_{\mathbb{R}\setminus[a,b]} |F(e^x)|\, p(x)\,\mathrm{d}x.$$

If the payoff function $F(e^x)$ is bounded we have the trivial upper bound

$$|\breve{F} - F| \leq [1 - (P(b) - P(a))] \sup_{x\in\mathbb{R}\setminus[a,b]} |F(e^x)|,$$

where $P(x)$ is the cumulative distribution function of $\ln X_T$. Otherwise, if the payoff function is unbounded but square-integrable under measure P, we may apply Cauchy-Schwartz's inequality to the product $F(e^x)\sqrt{p(x)} \times \sqrt{p(x)}$ to write

$$|\breve{F} - F|^2 \leq \int_{\mathbb{R}\setminus[a,b]} F(e^x)^2 p(x)\,\mathrm{d}x \int_{\mathbb{R}\setminus[a,b]} p(x)\,\mathrm{d}x$$
$$\leq [1 - (P(b) - P(a))] \int_{\mathbb{R}\setminus[a,b]} F(e^x)^2 p(x)\,\mathrm{d}x$$
$$\leq [1 - (P(b) - P(a))]\, \mathbb{E}\big[F(X_T)^2\big],$$

where the last bound above is loose. Typically the value of $\mathbb{E}[F(X_T)^2]$ is not known but in practice we may use an estimate obtained from an analytically tractable model such as Black and Scholes (1973).

Finally, if the probability range $P(b) - P(a)$ is not available in closed form, we may use the classical Bienaymé-Chebyshev's inequality

$$1 - (P(m_1 + L) - P(m_1 - L)) \leq \frac{m_2 - m_1^2}{L^2}$$

where $L > 0$ is arbitrary and $m_k := \mathbb{E}(\ln^k X_T), k \in \{1, 2\}$ are moments recoverable from the characteristic function. Alternatively, a tighter bound could be obtained from the formula (e.g. Durrett 2019, p.115)

$$1 - (P(L) - P(-L)) \leq \frac{L}{2} \int_{-2/L}^{2/L} [1 - \varphi(x)]\,\mathrm{d}x.$$

As an illustration we show how $[a, b]$ may be determined within the Black and Scholes (1973) model, for which the characteristic function of the log-price $x \equiv \ln X_T$ is known to be

$$\varphi_{\mathrm{BS}}(x) = e^{ix[\ln X_0 + (r-\sigma^2/2)T] - x^2\sigma^2 T/2},$$

where X_0 is the underlying asset spot price, r is the continuous interest rate and σ is the volatility parameter. The probability range $P(b) - P(a)$ is also well known in closed form as

$$P(b) - P(a) = \mathbb{P}(a \leq \ln X_T \leq b)$$
$$= N\left(\frac{b - m_1}{\sigma\sqrt{T}}\right) - N\left(\frac{a - m_1}{\sigma\sqrt{T}}\right),$$

where $m_1 = \ln X_0 + (r - \sigma^2/2)T$. Choosing $b = m_1 + L\sigma\sqrt{T}, a = m_1 - L\sigma\sqrt{T}, L > 0$, the above simplifies to

$$P(b) - P(a) = N(L) - N(-L) = 1 - 2N(-L).$$

For a bounded target payoff such as a call spread with strikes spaced ΔK apart, a given precision target ε is achieved by choosing L such that $2N(-L)\Delta K \leq \varepsilon$, i.e. $L \geq N^{-1}(\varepsilon/2\Delta K)$. Using $X_0 = 1, r = 0.05, T = 1, \sigma = 0.2, \Delta K = 1$ and a target precision $\varepsilon = 0.01$ we find $L \approx 2.58$ giving $[a, b] = [-0.4851, 0.5452]$.

For an unbounded payoff such as the log contract $F(X_T) := \ln(X_T) \equiv x$, we have $\mathbb{E}[F(X_T)^2] = \int_{-\infty}^{\infty} x^2 p(x)\,\mathrm{d}x = m_2 = m_1^2 + \sigma^2 T$ and a given precision target ε is achieved by choosing $L \geq -N^{-1}(\varepsilon^2/2m_2)$. Using the same parameters as above we find $L \approx 3.03$ and $[a, b] = [-0.5760, 0.6361]$.

8. Spectral decomposition of the butterfly kernel

To further underscore the generality of our approach, in this final section we consider the butterfly kernel

$$G(x, y; c) := (c - |x - y|)^+$$
$$= (x - y + c)^+ - 2(x - y)^+ + (x - y - c)^+$$

for finite domain $[a, b]$ and fixed call spread parameter $0 < c \leq \frac{1}{3}(b - a)$. As stated in table 1, this kernel is symmetric and injective. Indeed, we can write $G(x, y; c) = cK(x - y)$ where $K(z) := (1 - |z|/c)^+$, and G is a positive-definite kernel if and only if K is a positive-definite *function*. By Bochner's theorem (Lax 2002, p. 144) a function is positive-definite if and only if it is the Fourier transform of a probability density, and it is easy to verify that

$$\hat{K}(u) := \frac{1}{c\pi} \cdot \frac{1 - \cos cu}{u^2}$$

is such a density. Indeed, by Fubini and then the property that the real number $\phi(y)$ is equal to its conjugate, we may write

$$\int_a^b \int_a^b \phi(x)K(x - y)\phi(y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_a^b \int_a^b \phi(x)\phi(y) \int_{-\infty}^{\infty} e^{i(x-y)u}\hat{K}(u)\,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{-\infty}^{\infty} \hat{K}(u)\,\mathrm{d}u \int_a^b \phi(x)e^{iux}\,\mathrm{d}x \int_a^b \overline{\phi(y)e^{iuy}}\,\mathrm{d}y \geq 0,$$

and equality implies $\phi \equiv 0$. Therefore, the butterfly kernel only has strictly positive eigenvalues and it is injective.

For ease of exposure, and without loss of generality, we assume $[a,b] = [0,1], 0 < c \leq \frac{1}{3}$ as we did in section 4. Differentiating the integral equation (2) with butterfly kernel $G(x,y;c)$ twice against x we obtain the linear recurrence equation for ϕ

$$f''(x) = \phi(x-c) - 2\phi(x) + \phi(x+c),$$

with the convention $\phi(x) \equiv 0$ for $x < 0$ or $x > 1$. When $c = 1/N, N \in \mathbb{N} \setminus \{0,1,2\}$, the solution is

$$\begin{aligned}\phi(x) &= -\frac{N-n}{N+1}\sum_{k=0}^{n}(n+1-k)f''(x-kc) \\ &\quad - \frac{n+1}{N+1}\sum_{k=1}^{N-n-1}(N-n-k)f''(x+kc), \\ &\quad x = nc + r, \quad 0 \leq r < c, \quad n \in \{0,1,\ldots,N-1\},\end{aligned} \tag{20}$$

wherein n is the Euclidean quotient of x by c with remainder r (i.e. x modulo c). In particular, the homogeneous equation with $f(x) \equiv 0$ only has the trivial solution $\phi(x) \equiv 0$, thereby confirming that the butterfly kernel is injective when $c = 1/N$.

It is worth noting that the solution (20) is typically discontinuous at every step c, and that the integral $\int_0^1 G(x,y;c)\phi(y)\,\mathrm{d}y$ matches $f(x)$ up to affine terms. Figure 6 shows the solution obtained for $F(x) = e^x$ and $c = 1/6$.

In the fashion of section 4, it is possible to identify the general form of eigenfunctions $\phi(x)$ satisfying

$$\lambda\phi''(x) = \begin{cases} -2\phi(x) + \phi(x+c) & \text{for } 0 \leq x < c, \\ \phi(x-c) - 2\phi(x) + \phi(x+c) & \text{for } c \leq x \leq 1-c, \\ \phi(x-c) - 2\phi(x) & \text{for } 1-c < x \leq 1, \end{cases}$$

for an eigenvalue $\lambda > 0$. This may be done by splitting the domain $[0,1]$ at every step c and solving the system of second-order ordinary linear differential equations

$$\lambda \mathbf{u}''(x) = -\mathbf{A}.\mathbf{u}(x), \quad 0 \leq x < c,$$

where $\mathbf{u}(x) := (\phi(x), \phi(x+c), \ldots, \phi(x+1-c))^T$ is a vector of length N and $\mathbf{A}$ is the familiar $N \times N$ tridiagonal matrix

$$\mathbf{A} := \begin{bmatrix} 2 & -1 & & (0) \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \\ (0) & & -1 & 2 \end{bmatrix},$$

which is positive-definite with principal square root $\mathbf{A}^{1/2}$; formulas for the spectral elements of $\mathbf{A}$ can be found in e.g.Smith (1985), pp. 55, 154–156. The homogeneous second-kind integral equation with butterfly kernel may then be written in terms of $\mathbf{u}$ as

$$\lambda \mathbf{u}(x) = \int_0^c \mathbf{G}(x,y).\mathbf{u}(y)\,\mathrm{d}y \tag{21}$$

where $\mathbf{G}(x,y) := ((c - |x - y + (n-p)c|)^+)_{0 \leq n,p \leq N-1}$ is a $N \times N$ matrix defined for $0 \leq x, y < c$.

The general solution to the system of second-order ordinary linear differential equations $\lambda \mathbf{u}'' = -\mathbf{A}.\mathbf{u}$ is known to be

$$\mathbf{u}(x) = \cos\left(\frac{x}{\sqrt{\lambda}}\mathbf{A}^{1/2}\right).\mathbf{k}_1 + \sin\left(\frac{x}{\sqrt{\lambda}}\mathbf{A}^{1/2}\right).\mathbf{k}_2,$$

where $\mathbf{k}_1, \mathbf{k}_2$ are two column vectors of N constant coefficients. Substituting into (21) and integrating by parts twice

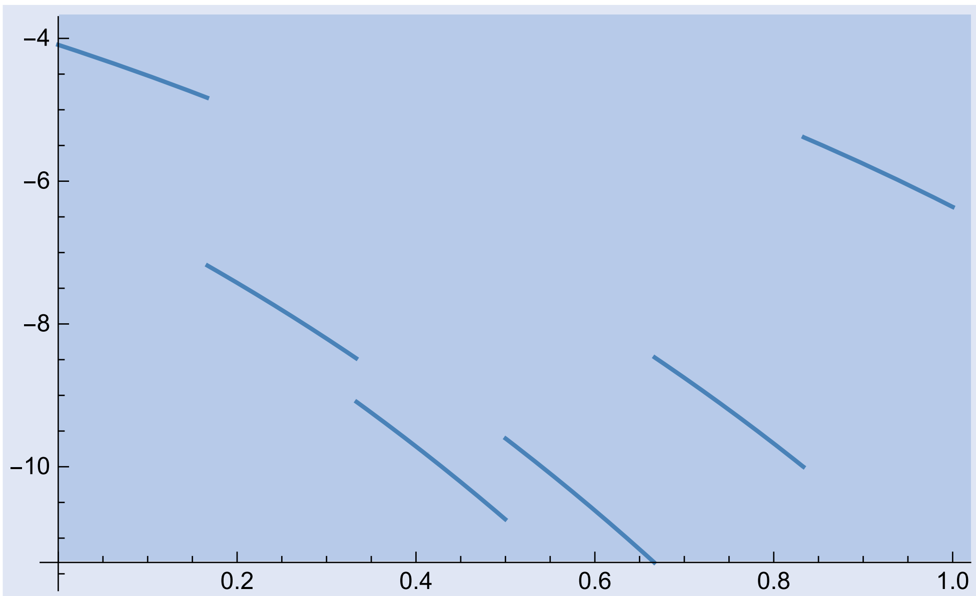

Figure 6. Solution to integral equation with butterfly kernel $G(x,y;c), c = \frac{1}{6}$, target function $F(x) = e^x$, domain $[a,b] = [0,1]$

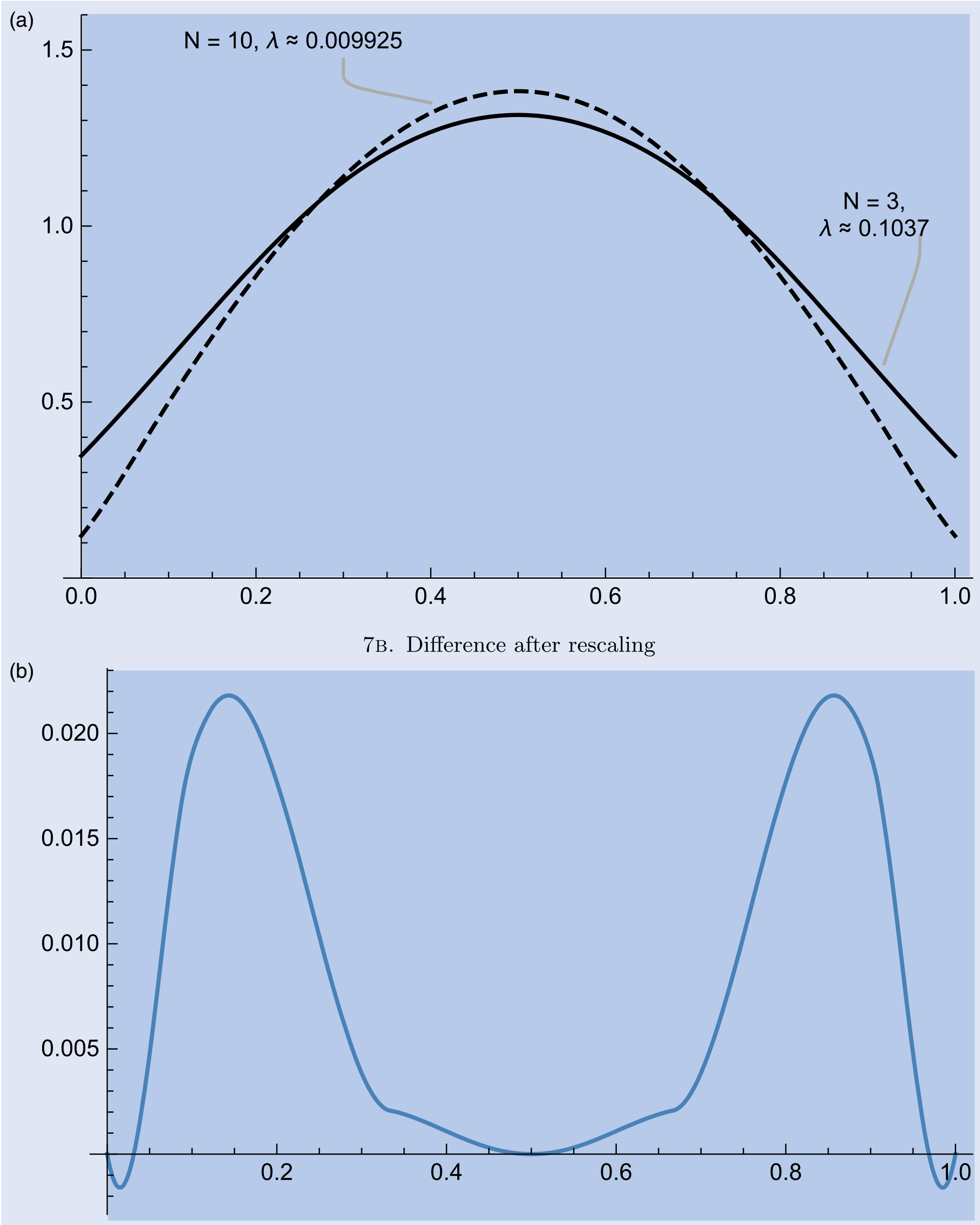

Figure 7. Comparison of top eigenfunctions of the butterfly kernel $G(x, y; c), c = 1/N$ over the domain $[a, b] = [0, 1]$. (a) Top normalized eigenfunction for $N = 3$ and $N = 10$. (b) Difference after rescaling

we obtain†

$$\lambda \mathbf{u}(x) = -\lambda \left[\mathbf{G}(x,y)\mathbf{A}^{-1}\mathbf{u}'(y) - \mathbf{G}_y(x,y)\mathbf{A}^{-1}\mathbf{u}(y)\right]_{y=0}^{y\to c^-} - \lambda \int_0^c \mathbf{G}_{yy}(x,y)\mathbf{A}^{-1}\mathbf{u}(y)\,\mathrm{d}y,$$

where $\mathbf{G}_y, \mathbf{G}_{yy}$ are the first- and second-order partial derivatives of $\mathbf{G}(x, y)$ against y. Substituting the identities $\mathbf{G}(x, c) = x\mathbf{I} + (c - x)\mathbf{L}$, $\mathbf{G}(x, 0) = (c - x)\mathbf{I} + x\mathbf{L}^T$, $\mathbf{G}_y(x, c) = -\mathbf{I} + \mathbf{L}$, $\mathbf{G}_y(x, 0) = \mathbf{I} - \mathbf{L}^T$, where $\mathbf{L}$ is the lower shift matrix with ones on the subdiagonal, we may rewrite the bracket in the above equation as the block matrix expression

$$\mathbf{b}_\lambda(x) := \left[\mathbf{I} - \mathbf{L} \quad x(\mathbf{I} - \mathbf{L}) + c\mathbf{L}\right] \times \begin{bmatrix} \mathbf{C}_\lambda & \mathbf{S}_\lambda \\ -\frac{1}{\sqrt{\lambda}}\mathbf{A}^{1/2}\mathbf{S}_\lambda & \frac{1}{\sqrt{\lambda}}\mathbf{A}^{1/2}\mathbf{C}_\lambda \end{bmatrix} \begin{bmatrix} \mathbf{A}^{-1}\mathbf{k}_1 \\ \mathbf{A}^{-1}\mathbf{k}_2 \end{bmatrix} + \left[\mathbf{I} - \mathbf{L}^T \quad \frac{x}{\sqrt{\lambda}}(\mathbf{I} - \mathbf{L}^T)\mathbf{A}^{1/2} - \frac{c}{\sqrt{\lambda}}\mathbf{A}^{1/2}\right] \times \begin{bmatrix} \mathbf{A}^{-1}\mathbf{k}_1 \\ \mathbf{A}^{-1}\mathbf{k}_2 \end{bmatrix}$$

where $\mathbf{C}_\lambda := \cos((c/\sqrt{\lambda})\mathbf{A}^{1/2})$ and $\mathbf{S}_\lambda := \sin((c/\sqrt{\lambda})\mathbf{A}^{1/2})$. The vector $\mathbf{b}_\lambda(x)$ is affine in x and will vanish if and only if the intercept and slope vectors $\mathbf{b}_\lambda(0), \mathbf{b}'_\lambda$ are zero, leading to

† Observe that the matrix versions of cos, sin commute with any power of the argument matrix, and that $-\lambda\mathbf{A}^{-1}\mathbf{u}'(x), -\lambda\mathbf{A}^{-1}\mathbf{u}(x)$ are respectively first- and second-order antiderivatives of $\mathbf{u}(x)$.

the homogeneous block matrix equation in $\mathbf{k}_1, \mathbf{k}_2$

$$\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix} = \left(\begin{bmatrix}\mathbf{I}-\mathbf{L} & c\mathbf{L}\\ \mathbf{O} & \mathbf{I}-\mathbf{L}\end{bmatrix} \begin{bmatrix}\mathbf{C}_\lambda & \mathbf{S}_\lambda\\ -\frac{1}{\sqrt{\lambda}}\mathbf{A}^{1/2}\mathbf{S}_\lambda & \frac{1}{\sqrt{\lambda}}\mathbf{A}^{1/2}\mathbf{C}_\lambda\end{bmatrix} + \begin{bmatrix}\mathbf{I}-\mathbf{L}^T & -\frac{c}{\sqrt{\lambda}}\mathbf{A}^{1/2}\\ \mathbf{O} & (\mathbf{I}-\mathbf{L}^T)\frac{1}{\sqrt{\lambda}}\mathbf{A}^{1/2}\end{bmatrix}\right) \begin{bmatrix}\mathbf{A}^{-1}\mathbf{k}_1\\ \mathbf{A}^{-1}\mathbf{k}_2\end{bmatrix},$$

where $\mathbf{0}$ is the null column vector of $\mathbb{R}^N$ and $\mathbf{O}$ is the null matrix of $\mathbf{R}^{N\times N}$. It is worth noting that solving the above equation is difficult: we need to find λ such that the $2N \times 2N$ block matrix between parentheses is singular, and then find the corresponding nullspace to identify non-trivial solutions $\mathbf{k}_1, \mathbf{k}_2$. However, with some algebra we can simplify this problem for some eigenvalues λ, as detailed below.

Left-multiplying both sides of the previous equation by $\begin{bmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \sqrt{\lambda}\mathbf{A}^{-1/2}\end{bmatrix}\begin{bmatrix}\mathbf{I}-\mathbf{L} & c\mathbf{L}\\ \mathbf{O} & \mathbf{I}-\mathbf{L}\end{bmatrix}^{-1}$, we obtain

$$\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix} = \left(\begin{bmatrix}\mathbf{C}_\lambda & \mathbf{S}_\lambda\\ -\mathbf{S}_\lambda & \mathbf{C}_\lambda\end{bmatrix} + \begin{bmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \sqrt{\lambda}\mathbf{A}^{-1/2}\end{bmatrix} \times \begin{bmatrix}\mathbf{I}-\mathbf{L} & c\mathbf{L}\\ \mathbf{O} & \mathbf{I}-\mathbf{L}\end{bmatrix}^{-1} \begin{bmatrix}\mathbf{I}-\mathbf{L}^T & -\frac{c}{\sqrt{\lambda}}\mathbf{A}^{1/2}\\ \mathbf{O} & (\mathbf{I}-\mathbf{L}^T)\frac{1}{\sqrt{\lambda}}\mathbf{A}^{1/2}\end{bmatrix}\right) \times \begin{bmatrix}\mathbf{A}^{-1}\mathbf{k}_1\\ \mathbf{A}^{-1}\mathbf{k}_2\end{bmatrix}. \tag{22}$$

It is easy to show that the second term above between parentheses simplifies to

$$\begin{bmatrix}\mathbf{e}\mathbf{v}^T-\mathbf{L}^T & \mathbf{O}\\ \mathbf{O} & \mathbf{A}^{-1/2}(\mathbf{e}\mathbf{v}^T-\mathbf{L}^T)\mathbf{A}^{1/2}\end{bmatrix} - \frac{c}{\sqrt{\lambda}}\begin{bmatrix}\mathbf{w}\\ \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \mathbf{A}^{1/2}\mathbf{v}\end{bmatrix}^T,$$

where $\mathbf{e} := (1,\dots,1)^T$ is the first diagonal vector of $\mathbb{R}^N$, $\mathbf{v} := (1,0,\dots,0)^T$ is the first coordinate vector, and $\mathbf{w} := (1,2,\dots,N)^T$. Equation (22) may thus be rewritten as

$$\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix} = \left(\mathbf{M}_\lambda - \frac{c}{\sqrt{\lambda}}\begin{bmatrix}\mathbf{w}\\ \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \mathbf{A}^{1/2}\mathbf{v}\end{bmatrix}^T\right)\begin{bmatrix}\mathbf{A}^{-1}\mathbf{k}_1\\ \mathbf{A}^{-1}\mathbf{k}_2\end{bmatrix},$$

where

$$\mathbf{M}_\lambda := \begin{bmatrix}\mathbf{C}_\lambda & \mathbf{S}_\lambda\\ -\mathbf{S}_\lambda & \mathbf{C}_\lambda\end{bmatrix} + \begin{bmatrix}\mathbf{e}\mathbf{v}^T-\mathbf{L}^T & \mathbf{O}\\ \mathbf{O} & \mathbf{A}^{-1/2}(\mathbf{e}\mathbf{v}^T-\mathbf{L}^T)\mathbf{A}^{1/2}\end{bmatrix}.$$

When $\mathbf{M}_\lambda$ is invertible, the Sherman-Morrison formula (e.g. Golub and Loan 1996, p. 51) states that

$$\mathbf{M}_\lambda - \frac{c}{\sqrt{\lambda}}\begin{bmatrix}\mathbf{w}\\ \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \mathbf{A}^{1/2}\mathbf{v}\end{bmatrix}^T$$

is singular if and only if λ satisfies the scalar equation

$$1 - \frac{c}{\sqrt{\lambda}}\begin{bmatrix}\mathbf{0}\\ \mathbf{A}^{1/2}\mathbf{v}\end{bmatrix}^T \mathbf{M}_\lambda^{-1}\begin{bmatrix}\mathbf{w}\\ \mathbf{0}\end{bmatrix} = 0, \tag{23}$$

and in this case $\mathbf{M}_\lambda^{-1}\begin{bmatrix}\mathbf{w}\\ \mathbf{0}\end{bmatrix}$ is in the nullspace, giving a nontrivial solution

$$\begin{bmatrix}\mathbf{k}_1\\ \mathbf{k}_2\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{O}\\ \mathbf{O} & \mathbf{A}\end{bmatrix}\mathbf{M}_\lambda^{-1}\begin{bmatrix}\mathbf{w}\\ \mathbf{0}\end{bmatrix}.$$

In figure 7 we plot the top eigenfunction that we obtained by numerically solving equation (23) for $N = 3$ and $N = 10$ and then computing $\mathbf{k}_1, \mathbf{k}_2$ as written above. As expected, the eigenfunctions are continuous and smooth. Note that there may be eigenvalues λ for which $\mathbf{M}_\lambda$ is not invertible, in which case equation (23) cannot be relied upon.

9. Summary and conclusions

Integral equations and theory from functional analysis can be used to generalize the formula of Carr and Madan (1998) and allow from some improvements for the replication of European contingent claims. The results in this paper show how replication can be achieved with a discrete portfolio of special options forming an orthogonal eigensystem, which is easier to manage numerically when compared to a continuous portfolio of vanilla options. Our approach considers a general class of symmetric kernels, including the straddle and butterfly kernels, which have spectral decompositions allowing for replication with a discrete portfolio in the form of a series of special option payoffs. Truncation of this series provides a numerical method for fast pricing of vanilla options. An advantage of this method is that it may be possible to use a smaller number of special options rather than integral discretization schemes. For kernels having eigenfunctions given by the Fourier basis elements, the replicating series can take advantage of probability densities with explicit Fourier transforms. Overall, the results presented in this paper are part of a greater potential generalization, which in future work will extend to multivariate European payoffs and some families of exotic payoffs.

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Appendix. Derivation of proxy formula for the difference between the truncated Fourier series and spectroreplicant methods

Keeping the notations of section 5.1 with $[a,b]=[0,1]$, a double integration by parts on $\langle F,\psi_n\rangle, n\geq 1$ yields

$$\langle F,\psi_n\rangle = -\frac{1}{(n\pi)^2}\langle F'',\psi_n\rangle - \frac{1}{(n\pi)^2}F'(0) + \frac{(-1)^n}{(n\pi)^2}F'(1),$$

so that

$$\begin{aligned}\breve{F}_n(x) &= \langle F,\psi_0\rangle - 2\sum_{k=1}^{n-1}\frac{\langle F'',\psi_k\rangle}{(k\pi)^2}\psi_k(x)\\ &\quad + 2\sum_{k=1}^{n-1}\left[\frac{(-1)^k}{(k\pi)^2}F'(1) - \frac{F'(0)}{(k\pi)^2}\right]\psi_k(x).\end{aligned}$$

The above decomposition gives us an insight into the type of integral equation solved by a Fourier series. Specifically, we may reconstruct the corresponding integral kernel at order $n\geq 1$ as

$$K_n(x,y) := -\sum_{k=1}^{n-1}\frac{1}{(k\pi)^2}[\sqrt{2}\psi_k(x)][\sqrt{2}\psi_k(y)],$$

so that

$$\begin{aligned}\breve{F}_n(x) &= \langle F,\psi_0\rangle + \int_0^1 K_n(x,y)F''(y)\,\mathrm{d}y\\ &\quad + 2\sum_{k=1}^{n-1}\left[\frac{(-1)^k}{(k\pi)^2}F'(1) - \frac{F'(0)}{(k\pi)^2}\right]\psi_k(x).\end{aligned}$$

As $n\to\infty$, the Fourier expansion kernel $K_n(x,y)$ converges pointwise to

$$K(x,y) := -\sum_{k=1}^{\infty}\frac{1}{(k\pi)^2}[\sqrt{2}\psi_k(x)][\sqrt{2}\psi_k(y)]$$

$$= -\frac{1}{2\pi^2}\left[\mathrm{Li}_2\left(e^{\imath\pi(x+y)}\right) + \mathrm{Li}_2\left(e^{-\imath\pi(x+y)}\right) + \mathrm{Li}_2\left(e^{\imath\pi(x-y)}\right) + \mathrm{Li}_2\left(e^{-\imath\pi(x-y)}\right)\right],$$

where $\mathrm{Li}_2(z) \coloneqq \sum_{k=1}^{\infty} z^k/k^2$ is the complex dilogarithm function. From the identity $\mathrm{Li}_2(z) + \mathrm{Li}_2(z^{-1}) = -(\pi^2/6) - \frac{1}{2}\ln^2(-z)$ (e.g. Zagier 2007, p.8) we further obtain

$$\begin{aligned} K(x,y) &= \frac{1}{6} + \frac{1}{4\pi^2}\left[\ln^2\left(-e^{\imath\pi(x+y)}\right) + \ln^2\left(-e^{\imath\pi(x-y)}\right)\right] \\ &= \frac{1}{6} - \frac{1}{4}\left[(x+y-1)^2 + \min\left((x-y-1)^2, (x-y+1)^2\right)\right] \\ &= -\frac{1}{3} - \frac{1}{2}\left(x^2 - 2\max(x,y) + y^2\right) \\ &= \frac{|x-y|}{2} - \left(\frac{1}{3} + \frac{x^2 - (x+y) + y^2}{2}\right). \end{aligned}$$

The above identity establishes that the straddle kernel $G(x,y) \coloneqq |x-y|$ is embedded within the Fourier expansion kernel together with a residual kernel:

$$K \equiv \frac{1}{2}G + R, \quad R(x,y) \coloneqq -\left(\frac{1}{3} + \frac{x^2 - (x+y) + y^2}{2}\right).$$

Writing $K_n = \frac{1}{2}G_n + (K_n - \frac{1}{2}G_n)$ where $G_n(x,y) \coloneqq \sum_{k=0}^{n-1}\lambda_k \phi_k(x)\phi_k(y)$ is the truncated straddle kernel at order n, and noting that $\int_0^1 G_n(x,y)(F''(y)/2)\,\mathrm{d}y = \hat{F}_n(x) - c - qx$, we have

$$\begin{aligned} \check{F}_n(x) &= \hat{F}_n(x) + (\langle F, \psi_0\rangle - c - qx) \\ &\quad + \int_0^1 \left[K_n(x,y) - \frac{1}{2}G_n(x,y)\right] F''(y)\,\mathrm{d}y \\ &\quad + 2\sum_{k=1}^{n-1}\left[\frac{(-1)^k}{(k\pi)^2}F'(1) - \frac{F'(0)}{(k\pi)^2}\right]\psi_k(x), \end{aligned}$$

with $K_n - \frac{1}{2}G_n \xrightarrow[n\to\infty]{} R$ pointwise, so that, for large n,

$$\begin{aligned} \check{F}_n(x) - \hat{F}_n(x) &\approx (\langle F, \psi_0\rangle - c - qx) \\ &\quad + \int_0^1 R(x,y)F''(y)\,\mathrm{d}y \\ &\quad + 2\sum_{k=1}^{n-1}\left[\frac{(-1)^k}{(k\pi)^2}F'(1) - \frac{F'(0)}{(k\pi)^2}\right]\psi_k(x). \end{aligned}$$