

Semi-Closed Form Prices of Barrier Options in the Time-Dependent CEV and CIR Models

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KEY FINDINGS

- This article extends this methodology to the CIR model for zero-coupon bonds and to the CEV model for stocks, which are used as the corresponding underlying for the barrier options.
- One approach is a generalization of the method of heat potentials (for the heat equation) to the Bessel process, so we call it “the method of Bessel potentials.” The second approach is the method of generalized integral transform, which is also extended to the Bessel process.
- These methods do not duplicate, but instead they complement each other, as one provides very accurate results at shorter maturities and the other provides such results at longer maturities.

ABSTRACT: *The authors continue a series of articles where prices of the barrier options written on the underlying, whose dynamics follow a one factor stochastic model with time-dependent coefficients and the barrier, are obtained in semi-closed form, see Carr and Itkin (2020), Itkin and Muravey (2020). This article extends this methodology to the CIR model for zero-coupon bonds and to the CEV model for stocks, which are used as the corresponding underlying for the barrier options. The authors describe two approaches. One is a generalization of the method of heat potentials (for the heat equation) to the Bessel process, so it is named “the method of Bessel potentials.” The authors also propose a general scheme for constructing the potential method for any linear differential operator with time-independent coefficients. The second approach is the method of generalized integral transform, which is also extended to the Bessel process. In all cases, a semi-closed solution means that first, we*

need to numerically solve a linear Volterra equation of the second kind, and then the option price is represented as a one-dimensional integral. The authors demonstrate that their method is computationally more efficient than both the backward and forward finite difference methods, while providing better accuracy and stability. Also, it is shown that these methods do not duplicate, but instead they complement each other, as one provides very accurate results at shorter maturities and the other provides such results at longer maturities.

TOPICS: *Options, statistical methods**

This article continues a series of papers where the prices of barrier options written on the underlying, whose dynamics follows a one factor stochastic model with time-dependent coefficients and the barrier, are constructed

in semi-closed form, see Carr and Itkin (2020); Itkin and Muravey (2020). Here we extend our approach for two additional models: the Cox–Ingersoll–Ross (CIR) model, (Cox et al. 1985), and the time-dependent constant elasticity of variance (CEV) model, (Cox 1975). Both models are very popular among practitioners and are used for pricing various derivatives in asset classes such as equities, fixed income, commodities, FX, and so forth.

For pricing the time-dependent barrier options, we develop two analytic methods in parallel; both are based on the notion of generalized integral transform, (Carr and Itkin 2020; Itkin and Muravey 2020). The first method is the method of heat potentials which, as applied to the models considered in this article, is discussed in detail. We extend this method, since the partial differential equation (PDE) we need to solve cannot be transformed to the heat equation, but rather to the Bessel PDE. This approach is new and has not been developed yet in the literature. However, once this is done, the same method could be used for solving some other problems implicitly related to pricing of barrier options, e.g., pricing American options, (Carr and Itkin 2020), analyzing the stability of a single bank and a group of banks in the structural default framework, (Kaushansky et al. 2018), calculating the hitting time density, (Alil et al. 2005; Lipton and Kaushansky 2020a), and finding an optimal strategy for pairs trading, (Lipton and de Prado 2020). The method could also be used for solving various problems in physics, where it was originally developed for the heat equation, (Kartashov 2001; Friedman 1964; and references therein).

The other method is the method of generalized integral transform, which was actively developed by the Russian mathematical school to solve parabolic equations at the domain with moving boundaries, see e.g., Kartashov (1999) and references therein. However, again for the Bessel PDE this approach has not been developed yet in the literature in full (i.e., up to the final formula), although some comments on possible ways of achieving this can be found in Kartashov (1999). It is also worth mentioning that so far the only known problem solved by using this method is the heat equation with the boundary $\gamma(t)$ moving in time t , so the solution is defined at the domain $[0, \gamma(t)]$. In Itkin and Muravey (2020), this approach was extended to the domain $[\gamma(t), \infty)$. In addition, the problem at the domain $[\gamma(t), z(t)]$, which emerges, e.g., for the time-dependent double barrier

options, where the underlying follows a time-dependent Hull-White model, was also constructed in Itkin and Muravey (2020).

Going back to the CIR and CEV models with time-dependent coefficients, the prices of the barrier options in these models are not known in closed form yet. Instead, various numerical methods are used to compute them. This obviously produces a computational burden that could be excessive when these numerical procedures are used as a part of a calibration process. Therefore, our closed form solutions could be of importance for practitioners. By a semi-closed solution, we mean that first, one needs to solve numerically a linear Volterra equation of the second kind, and then the option price is represented as a one-dimensional integral of that solution. We demonstrate that our method is more efficient computationally than either the backward or forward finite difference methods, while providing better accuracy and stability.

Overall, our contribution to the existing literature is twofold. First, we solve the problem of pricing barrier options in the CIR and CEV models in semi-closed form, and provide resulting expressions that are not known yet in the literature. Second, we solve these problems by two methods that extend the existing methods and are developed by the authors.

The rest of the article is organized as follows. The next section briefly describes the CEV model and shows how to transform the pricing equation to the Bessel PDE. Then we do same for the CIR model. Note that we use the CEV model to price barrier options written on equities, while the CIR model is used to price barrier options written on zero-coupon bonds. Nevertheless, the transformed PDE is same for both models, although the solution could be defined for the same or different domains. After that, we develop a method of Bessel potentials that is an extension of the method of heat potentials, and we obtain the solution to our problems using this approach. The same program is then used for the method of generalized integral transform. Finally, the results of several numerical experiments are presented, which compare the performance and accuracy of our method with the finite-difference method (FDM) used to solve the forward Kolmogorov equation. (The FDM is currently the standard way to price barrier options in the time-dependent CIR and CEV models). The last section concludes.

We discovered that the potential method could be constructed for any PDE where the space operator is a linear differential operator with constant coefficients. We propose and discuss this generalization of the heat potential method in the Appendix.

THE CEV MODEL

The time-dependent constant elasticity of variance (CEV) model is a one-dimensional diffusion process that solves a stochastic differential equation

$$dS_t = \mu(t)S_t dt + \sigma(t)S_t^{\beta+1} dW_t, \quad S_{t=0} = S_0. \quad (1)$$

Here $t \geq 0$ is the time, S_t is the stochastic stock price, $\mu(t)$ is the drift, $\sigma(t)$ is the volatility, β is the elasticity parameter such that $\beta < 1$, $\beta \neq \{0, -1\}$,¹ and W_t is the standard Brownian motion. It is known that under the risk-neutral measure, $\mu(t) = r(t) - q(t)$, where $r(t)$ is the deterministic short interest rate and $q(t)$ is the continuous dividend. We assume that all parameters of the model are known either as continuous functions of time $t \in [0, \infty)$, or as a discrete set of N values for some moments t_i , $i = 1, \dots, N$.

The CEV model with constant coefficients has been introduced in Cox (1975) as an alternative to using geometric Brownian motion for modeling asset prices. Despite some level of sophistication, as compared to the Black-Scholes model, the model is still analytically tractable, and the prices of European options can be obtained in closed form. This is because the CEV process with constant coefficients is related to the Bessel process (Linetsky and Mendoza 2010). Also, as mentioned in that reference, the elasticity parameter β controls the steepness of the skew (the larger the $|\beta|$, the steeper the skew), while the volatility (scale) parameter σ fixes the at-the-money volatility level. This ability to capture the skew has made the CEV model popular in equity options markets.

For the standard CEV process with constant parameters, it is known that the change of variable $z_t = 1/(\sigma|\beta|)S_t^{-\beta}$ reduces the CEV process without drift ($\mu = 0$) to the standard Bessel process of order $1/2\beta$ (see Revuz and Yor 1999; Davydov and Linetsky 2001). Then the continuous part of the risk-neutral density of

S_t , conditional on $S_0 = S$, is obtained from the well-known expression for transition density of the Bessel process. If $\mu \neq 0$, using the result of Goldenberg (1991), this CEV process could be obtained from the process without drift via a scale and time change

$$S_t^\mu = e^{\mu\tau} S_{\tau(t)}^0, \quad \tau(t) = \frac{1}{2\mu\beta}(e^{2\mu\beta t} - 1).$$

Also, as shown in Davydov and Linetsky (2001), at $\beta > 0$, according to Feller's classification, the origin $S_t = 0$ is a natural boundary and infinity is an entrance boundary.

Below we show that a similar connection with the Bessel process can be established for the time-dependent version of the model in Eq. (1). Let us consider an up-and-out barrier call option, $C(t, S)$, written on the underlying process S_t . By the Feynman-Kac formula, this price solves the following partial differential equation (PDE)

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2(t)S^{2\beta+2}\frac{\partial^2 C}{\partial S^2} + [r(t) - q(t)]S\frac{\partial C}{\partial S} = r(t)C. \quad (2)$$

This equation should be solved subject to the terminal condition at the option maturity $t = T$

$$C(T, S) = (S - K)^+, \quad (3)$$

where K is the option strike, and the boundary conditions are

$$C(t, 0) = 0, \quad C(t, H(t)) = 0, \quad (4)$$

where $H(t)$ is the upper barrier, perhaps being time-dependent. Then, the following Proposition holds.

Proposition 1.1. *The PDE in Eq. (2) can be transformed to*

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\frac{\partial^2 u}{\partial z^2} + \frac{b}{z}\frac{\partial u}{\partial z}, \quad (5)$$

where b is some constant, $u = u(\tau, z)$ is the new dependent variable, and (τ, z) are the new independent variables. Equation (5) is the PDE associated with the one-dimensional Bessel process, (Revuz and Yor 1999)

$$dX_t = dW_t + \frac{b}{X_t} dt. \quad (6)$$

¹In the case $\beta = 0$, this model is the Black-Scholes model, while for $\beta = -1$, this is the Bachelier, or time-dependent Ornstein-Uhlenbeck (OU) model.

Proof. This transformation can be done in two steps. First, we make a change of variables

$$\begin{aligned} S &= (-x\beta)^{-1/\beta}, & C(t, S) &\rightarrow u(t, x)e^{\int_0^t r(k)dk}, \\ \phi &= \int_t^T \sigma^2(k)dk. \end{aligned} \quad (7)$$

This reduces the PDE in Eq. (2) to the form

$$\begin{aligned} \frac{\partial u}{\partial \phi} &= \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \left(xf(t) + \frac{b}{x} \right) \frac{\partial u}{\partial x}, \\ f(t) &= \beta \frac{r(t) - q(t)}{\sigma^2(t)}, & b &= \frac{\beta + 1}{2\beta}, & t &= t(\phi). \end{aligned} \quad (8)$$

The function $t(\phi)$ is the inverse map of the last term in Eq. (7). It can be computed for any $t \in [0, T]$ by substituting it into the definition of ϕ , then finding the corresponding value of $\phi(t)$, and finally inverting.

Equation (8) is a known type of PDE. Therefore, following Polyanin (2002), at the second step we make a new change of variables

$$z = xF(\phi), \quad \tau = \int_0^\phi F^2(k)dk, \quad F(\phi) = e^{\int_0^\phi f(k)dk}. \quad (9)$$

After this change, the final PDE takes the form of Eq. (5), which finalizes the proof. \square

Note that Carr and Linetsky (2006) extend the time-dependent CEV model considered in this article by allowing a jump to zero. They also reduce their stock price process to a time homogeneous Bessel process. We will discuss this extension as applied to the barrier options further in this article.

As far as the terminal (now the initial) condition in Eq. (3) and the boundary conditions in Eq. (4) in the new variables are concerned, we must distinguish two cases that are determined by the sign of β . If $-1 < \beta < 0$, the domain of definition for z is $z \in [0, \gamma(\tau)]$, where

$$\gamma(\tau) = -\frac{1}{\beta} H^{-\beta}(t(\tau))F(\phi(\tau)) > 0. \quad (10)$$

Accordingly, the initial condition now reads

$$u(0, z) = e^{-\int_0^T r(k)dk} \left[\left(-\frac{\beta z}{F(\phi(0))} \right)^{-1/\beta} - K \right]^+, \quad (11)$$

and the boundary conditions are

$$u(\tau, 0) = u(\tau, \gamma(\tau)) = 0. \quad (12)$$

However, if $0 < \beta < 1$, the left boundary goes to $-\infty$. Therefore, in this case it is convenient to redefine $x \rightarrow \bar{x} = -x$. This also redefines $z \rightarrow \bar{z} = -z$. Then the domain of definition for \bar{z} becomes $\bar{z} \in [\gamma(\tau), \infty)$ where

$$\gamma(\tau) = \frac{1}{\beta} H^{-\beta}(t(\tau))F(\phi(\tau)) > 0. \quad (13)$$

The initial condition now reads

$$u(0, \bar{z}) = e^{-\int_0^T r(k)dk} \left[\left(\frac{\beta \bar{z}}{F(\phi(0))} \right)^{-1/\beta} - K \right]^+, \quad (14)$$

and the boundary conditions are

$$u(\tau, \bar{z}) \Big|_{\bar{z} \rightarrow \infty} = u(\tau, \gamma(\tau)) = 0. \quad (15)$$

Also, in the case $0 < \beta < 1$, the PDE in Eq. (5) keeps the same form in the \bar{z} variables. It can be seen that in this case, the up-and-out barrier options transforms to the down-and-out counterpart.

Here we will interrupt our discussion of the time-dependent CEV model and postpone construction of the solution of the problems in Eq. (5), Eq. (11), Eq. (12) (or Eq. (5), Eq. (14), Eq. (15) until a later section. Instead, in the next section we consider the time-dependent CIR model and barrier options written on a zero-coupon bond (ZCB), which follows the CIR model. This is done, because, as we show below, the corresponding PDE could also be transformed to Eq. (5). Thus, our proposed method could be applied uniformly to both models.

THE CIR MODEL

The Cox–Ingersoll–Ross (CIR) model was invented by Cox et al. (1985) for modeling interest rates. In this model, the instantaneous interest rate r_t is a stochastic variable that follows the stochastic differential equation (SDE), also named the CIR process. For the time-dependent version of the model, this SDE reads

$$dr_t = \kappa(t)[\theta(t) - r_t]dt + \sigma(t)\sqrt{r_t}dW_t, \quad r_{t=0} = r. \quad (16)$$

Here $\kappa(t) > 0$ is the constant speed of mean-reversion and $\theta(t)$ is the mean-reversion level. The CIR model is an extension of the Hull-White model that we analyzed in Itkin and Muravey (2020) by making the volatility proportional to $\sqrt{r_t}$. This, on the one hand, allows us to avoid the possibility of negative interest rates when the Feller condition $2\kappa(t)\theta(t)/\sigma^2(t) > 1$ is satisfied. On the other hand, it still preserves the tractability of the model, see, for example, Andersen and Piterbarg (2010) and references therein.

Since the CIR model belongs to the class of exponentially affine models, the price of the ZCB $F(r, t, S)$ for this model is known in closed form. Here S is the bond expiration time. It is known that $F(r, t, S)$, under a risk-neutral measure, solves a linear parabolic partial differential equation (PDE) (Privault 2012)

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2(t)r \frac{\partial^2 F}{\partial r^2} + \kappa(t)[\theta(t) - r] \frac{\partial F}{\partial r} = rF. \quad (17)$$

It should be solved subject to the terminal condition

$$F(r, S, S) = 1, \quad (18)$$

and the boundary condition

$$F(r, t, S) \Big|_{r \rightarrow \infty} = 0. \quad (19)$$

The second boundary condition is necessary in case the Feller condition is violated, so the interest rate r_t can hit zero. Otherwise, the PDE in Eq. (17), itself at $r = 0$, serves as the second boundary condition.

The ZCB price can be obtained from Eq. (17), assuming that the solution is of the form

$$F(r, t, S) = A(t, S)e^{B(t, S)r}. \quad (20)$$

Substituting this expression into Eq. (2) and separating the terms proportional to r , we obtain two equations to determine $A(t, S)$, $B(t, S)$

$$\begin{aligned} \frac{\partial B(t, S)}{\partial t} &= 1 + \kappa(t)B(t, S) - \frac{1}{2}\sigma^2(t)B^2(t, S), \\ \frac{\partial A(t, S)}{\partial t} &= -A(t, S)B(t, S)\theta(t)\kappa(t). \end{aligned} \quad (21)$$

To obey the terminal condition Eq. (18), the first PDE in Eq. (21) should be solved subject to the terminal condition $B(S, S) = 0$, and the second one—to $A(S, S) = 1$.

The first equation in Eq. (21) is the Riccati equation. In this general form it cannot be solved analytically for arbitrary functions $\kappa(t)$, $\sigma(t)$, but it can be efficiently solved numerically. Also, in some cases, it can be solved approximately (asymptotically), see an example in Carr and Itkin (2020). Once the solution is obtained, the second equation in Eq. (21) can be solved analytically to yield

$$A(t, S) = e^{-\int_t^S B(m)\theta(m)\kappa(m)dm}. \quad (22)$$

When coefficients $\kappa(t)$, $\theta(t)$, $\sigma(t)$ are constants, it is known that the solution $B(t, S)$ can be obtained in closed form (Andersen and Piterbarg 2010) and reads

$$B(t, S) = -\frac{2[\exp((S-t)h) - 1]}{2h + (\theta + h)[\exp((S-t)h) - 1]}, \quad h = \sqrt{\theta^2 + 2\sigma^2}. \quad (23)$$

Thus, $B(t, S) < 0$, if $t < S$. Therefore, $F(r, t, S) \rightarrow 0$ when $r \rightarrow \infty$. In other words, the solution in Eq. (20) satisfies the boundary condition at $r \rightarrow \infty$. In cases where all the parameters of the model are deterministic functions of time and $B(t, S)$ solves the first equation in Eq. (21), this also remains true. This can be checked as follows. Since $\kappa(t) > 0$, $\sigma(t) > 0$ and from Eq. (20)

$$B(t, S) = \frac{\kappa(t)}{\sigma(t)^2} - \frac{1}{\sigma^2(t)} \sqrt{\kappa(t)^2 + 2[1 - B'(t, S)]}.$$

Therefore, $B(t, S) < 0$, if $B'(t, S) < 1$. On the other hand, if $B(t, S) < 0$, then from Eq. (20), $B'(t, S) < 1$. This finalizes the proof.

Down-and-Out Barrier Option

Let us consider a down-and-out barrier call option written on a ZCB. Under a risk-neutral measure, the option price $C(t, r)$ solves the same PDE as in Eq. (17), see Andersen and Piterbarg (2010).

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2(t)r \frac{\partial^2 C}{\partial r^2} + \kappa(t)[\theta(t) - r] \frac{\partial C}{\partial r} = rC. \quad (24)$$

The terminal condition at the option maturity $T \leq S$ for this PDE reads

$$C(T, r) = (F(r, T, S) - K)^+, \quad (25)$$

where K is the option strike.

By a standard contract, the lower barrier $L_F(t)$ (which we assume to be time-dependent as well) is set on the ZCB price, and not on the underlying interest rate r . This means that it can be written in the form

$$C(t, r) = 0 \quad \text{if } F(r, t, S) = L_F(t). \quad (26)$$

However, since the ZCB price $F(r, t, S)$ can be expressed in closed form in Eq. (20), this condition can be translated into the r domain by solving the equation

$$F(r, t, S) = A(t, S)e^{B(t, S)r} = L_F(t),$$

with respect to r . Denoting the solution of this equation as $L(t)$, we find

$$L(t) = \frac{1}{B(t, S)} \log \left(\frac{L_F(t)}{A(t, S)} \right) > 0, \quad (27)$$

where it is assumed that $L_F > A(t, S)$. Accordingly, in the r domain, the boundary condition for Eq. (24) reads

$$C(t, L(t)) = 0. \quad (28)$$

The second boundary can be naturally set at $r \rightarrow \infty$. As $r \rightarrow \infty$, the ZCB price tends to zero, see Eq. (20), and the call option price also vanishes in this limit. This yields

$$C(t, r) \Big|_{r \rightarrow \infty} = 0. \quad (29)$$

The PDE in Eq. (24) can also be transformed to that for the Bessel process in Eq. (6).

Proposition 2.1. *Eq. (24) can be transformed to*

$$\frac{\partial u}{\partial \tau} = \frac{1}{2} \frac{\partial^2 u}{\partial z^2} + \frac{b}{z} \frac{\partial u}{\partial z}, \quad (30)$$

where b is some constant, $u = u(\tau, z)$ is the new dependent variable, and (τ, z) are the new independent variables, if

$$\frac{\kappa(t)\theta(t)}{\sigma^2(t)} = \frac{m}{2}, \quad (31)$$

where $m \in [0, \infty)$ is some constant. The Eq. (30) is the PDE associated with the one-dimensional Bessel process in Eq. (6).

Proof. First make a change of variables

$$C(t, r) = u(t, z) e^{a(t)r + \int_0^t a(s)\kappa(s)\theta(s)ds}, \quad z = g(t)\sqrt{r}, \quad (32)$$

$$g(t) = \exp \left[\frac{1}{2} \int_0^t (\kappa(s) - a(s)\sigma^2(s))ds \right] \quad (33)$$

where $a(t)$ solves the Riccati equation

$$\frac{da(t)}{dt} = -\frac{\sigma^2(t)a^2(t)}{2} + \kappa(t)a(t) + 1. \quad (34)$$

This reduces the PDE in Eq. (24) to the form

$$\frac{4k(t)\theta(t) - \sigma^2(t)}{2z} \frac{\partial u}{\partial z} + \frac{1}{2} \sigma(t)^2 \frac{\partial^2 u}{\partial z^2} + \frac{4}{g^2(t)} \frac{\partial u}{\partial t} = 0. \quad (35)$$

Now make a change of time

$$\tau(t) = \frac{1}{4} \int_t^T g^2(s)\sigma^2(s)ds, \quad (36)$$

which transforms Eq. (35) to

$$\left(2 \frac{k(t)\theta(t)}{\sigma^2(t)} - \frac{1}{2} \right) \frac{1}{z} \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial z^2} = \frac{\partial u}{\partial \tau}, \quad t = t(\tau). \quad (37)$$

The function $t(\tau)$ is the inverse map of Eq. (36). It can be computed for any $t \in [0, T]$ by substituting it into the definition of τ , then finding the corresponding value of $\tau(t)$, and inverting.

Finally, as by assumption $k(t)\theta(t)/\sigma^2(t) = m - \text{const}$, we set $b = m - 1/2$. Thus, the final PDE takes the form of Eq. (5), which finalizes the proof. \square

Following from Proposition 2.1, for the time-dependent CIR model, the transformation from Eq. (24) to Eq. (30) cannot be done unconditionally. However, from a practitioner's points of view, the condition Eq. (31) seems not to be too restrictive. Indeed, the model parameters already contain the independent mean-reversion rate $\kappa(t)$ and volatility $\sigma(t)$. Since m is an arbitrary constant, it could be calibrated to the market data together with $\kappa(t)$ and $\sigma(t)$. Therefore, in this form, the model should be capable of calibration to the term-structure of interest rates.

Also, according to the change of variables made in Proposition 2.1, the terminal condition Eq. (25) in new variables reads

$$u(0, z) = e^{-a(t(0))z^2/g^2(t(0)) - \int_0^T a(s)\kappa(s)\theta(s)ds} (A(T, S)e^{B(T, S)z^2/g^2(T)} - K)^+. \quad (38)$$

And the boundary conditions in Eq. (28) and Eq. (29) transform to

$$\begin{aligned} C(\tau, \gamma(\tau)) &= 0, & C(\tau, z) \Big|_{z \rightarrow \infty} &= 0, \\ \gamma(\tau) &= \frac{1}{B(t(\tau), S(\tau))} \log \left(\frac{L_F(t(\tau))}{A(t(\tau), S(\tau))} \right). \end{aligned} \quad (39)$$

THE METHOD OF BESSEL POTENTIALS

For convenience of notation, let us call as *the CEV problem* the PDE in Eq. (5), which has to be solved subject to the initial condition in Eq. (14) and the boundary conditions in Eq. (15). Also, we call *the CIR problem* the PDE in Eq. (30), solved subject to the initial condition Eq. (38) and the boundary conditions in Eq. (39). Both problems can be considered simultaneously, as the PDE in Eq. (5) coincides with that in Eq. (30), and the boundary conditions in Eq. (15) coincide with those in Eq. (39). Accordingly, both solutions are defined at the domain $z \in [\gamma(\tau), \infty]$, though the definitions of $\gamma(\tau)$ in Eq. (13) and Eq. (39) differ. We will describe our method for this domain the next section. Another type of the CEV problem, where $\bar{z} \in [0, \gamma^-(\tau)]$, will be considered separately after that.

In Carr and Itkin (2020) and Itkin and Muravey (2020), a similar problem involving the heat equation was solved by using two approaches. The first one is a method of generalized integral transform, actively elaborated on by the Russian mathematical school to solve parabolic equations in a domain with moving boundaries, see Kartashov (1999, 2001) and references therein. These kinds of problems have been known in physics for a long time and arise in the fields of nuclear power engineering and the safety of nuclear reactors; in studying combustion in solid-propellant rocket engines; in the theory of phase transitions (the Stefan problem and the Verigin problem (in hydromechanics)); in the processes of sublimation in freezing and melting; in the kinetic theory of crystal growth; and so on. In Carr and Itkin

(2020) and Kartashov (1999), this method was applied to the domain $z \in [0, \gamma(\tau)]$, and in Itkin and Muravey (2020), the solution was obtained for the semi-infinite domain $z \in [\gamma(\tau), \infty]$ for the first time. We will further develop this method to solve the CEV and CIR problems.

The second method used to solve the same problems in Carr and Itkin (2020) and Itkin and Muravey (2020) is the method of heat potentials, see, e.g., Tikhonov and Samarskii (1963); Friedman (1964); Kartashov (2001), and references therein. The first use of this method in mathematical finance is attributed to Lipton (2002) for pricing path-dependent options with curvilinear barriers, and more recently in Lipton and de Prado (2020) and references therein. However, the CEV and CIR problems that we are dealing with in this article cannot be reduced to the heat equation, but rather to the Bessel PDE. Therefore, in this section we propose a generalization of the method for this type of equation. Accordingly, we call this generalization the method of Bessel potentials.

Note that the potential method could be constructed for any PDE where the space operator is a linear differential operator with time-independent coefficients. We propose and discuss this generalization of the heat potential method in the Appendix. Thus, the heat and Bessel potentials are just two particular cases of this general scheme.

Domain $\gamma(\tau) \leq z < \infty$

Since both the CEV and CIR problems have the inhomogeneous initial condition, our first step is to reduce them to the alternative problems with a homogeneous initial condition. Since Green's function of the Bessel equation at the infinite domain is known in closed form (Polyanin 2002), this can be achieved by representing $u(\tau, z)$ in the form

$$u(\tau, z) = q(\tau, z) + \int_{\gamma(0)}^{\infty} u(0, \xi) q_{\tau}(z, \xi, b) d\xi. \quad (40)$$

Here $q_{\tau}(z, \xi, b)$ is the fundamental solution (or the transition density, or Green's function) of Eq. (30) at the domain $z \in [0, \infty)$. This density can be obtained, assuming that the Bessel process stops when it reaches the origin. But since the domain of definition of z is $z \in [\gamma(\tau), \infty)$, we moved the left boundary from 0 to $\gamma(0)$.

By the definition of b in Eq. (8), for the CEV model, $b = (1 + \beta)/(2\beta)$. Since in this case $0 < \beta < 1$, we get $b > 1$. It is known that in case $b \geq 1/2$ the density $q_\tau(z, \zeta, b)$ is a good density with no defect of mass, i.e., it integrates into 1 (Lawler 2018; Linetsky and Mendoza 2010). The explicit representation reads (Cox 1975; Emanuel and Macbeth 1982)

$$q_\tau(z, \zeta, b) = \frac{\sqrt{z\zeta}}{\tau} \left(\frac{\zeta}{z}\right)^b e^{-\frac{z^2 + \zeta^2}{2\tau}} I_{b-1/2}\left(\frac{z\zeta}{\tau}\right). \quad (41)$$

Here $I_\nu(x)$ is the modified Bessel function of the first kind (Abramowitz and Stegun 1964).

For the CIR model, $b = m - 1/2$, where $m \in [0, \infty)$ can be found by calibration. Therefore, if $m > 1$ (i.e., if the Feller condition is satisfied, and the process never hits the origin), Green's function $q_\tau(z, \zeta, b)$ is given by Eq. (41). Otherwise, $0 < m < 1$ and $-1/2 < b < 1/2$. Then, by another change of variables, (Polyanin 2002)

$$w(\tau, z) = z^{2(1-m)} u(\tau, z),$$

the Eq. (30) transforms to the same equation with respect to $w(\tau, z)$, but now with $b = (3 - 2m)/2$. Accordingly, since $0 < m < 1$, we have $b > 1/2$. Therefore, again Green's function is represented by Eq. (41).

The function $q(\tau, z)$ solves the problem

$$\begin{aligned} \frac{\partial q(\tau, z)}{\partial \tau} &= \frac{1}{2} \frac{\partial^2 q(\tau, z)}{\partial z^2} + \frac{b}{z} \frac{\partial q(\tau, z)}{\partial \tau}, \\ q(0, z) &= 0, \quad \gamma(0) < z < \infty, \\ q(\tau, z) \Big|_{z \rightarrow \infty} &= 0, \quad q(\tau, \gamma(\tau)) = \zeta(\tau), \\ \zeta(\tau) &= - \int_{\gamma(0)}^{\infty} u(0, \zeta) q_\tau(\gamma(\tau), \zeta, b) d\zeta. \end{aligned} \quad (42)$$

This problem is like that in Eq. (30), Eq. (14), Eq. (15) Eq. (38), Eq. (39)), but now with a homogeneous initial condition. Therefore, following the general idea of the method of heat potentials, we represent the solution in the form of a generalized potential for the Bessel PDE

$$q(\tau, z) = \int_0^\tau \Psi(k) \frac{\partial}{\partial \xi} \left[\frac{\sqrt{z\xi}}{\tau - k} \left(\frac{\xi}{z}\right)^b e^{-\frac{z^2 + \xi^2}{2(\tau - k)}} I_{b-1/2}\left(\frac{z\xi}{\tau - k}\right) \right] \Big|_{\xi \rightarrow \gamma(k)} dk, \quad (43)$$

where $\Psi(k)$ is the potential density. It can be seen that $q(\tau, z)$ solves Eq. (30), as the derivative of the integral on the upper limit is proportional to the delta function, which vanishes due to $z \neq \gamma(\tau)$. The solution in Eq. (43) also satisfies the initial condition at $\tau = 0$, and the vanishing condition at $z \rightarrow \infty$. For the large values of argument $z\xi/(\tau - k)$, we propose to use the following approximation:

$$q(\tau, z) \approx \frac{1}{\sqrt{2\pi}} \int_0^\tau \Psi(k) \frac{\partial}{\partial \xi} \left[\frac{1}{\sqrt{\tau - k}} \left(\frac{\xi}{z}\right)^b e^{-\frac{(z - \xi)^2}{2(\tau - k)}} \right] \Big|_{\xi \rightarrow \gamma(k)} dk \quad (44)$$

At the barrier $z = \gamma(\tau)$, function $q(\tau, z)$ is discontinuous. Following a similar approach for the heat potentials method, (Tikhonov and Samarskii 1963), it can be shown that the limiting value of $q(\tau, z)$ at $z = \gamma(\tau) + 0$ is equal to $\zeta(\tau)$:

$$\begin{aligned} \zeta(\tau) &= \Psi(\tau) + \int_0^\tau \Psi(k) \\ &\frac{\partial}{\partial \gamma(k)} \left[\frac{\sqrt{\gamma(\tau)\gamma(k)}}{\tau - k} \left(\frac{\gamma(k)}{\gamma(\tau)}\right)^b e^{-\frac{\gamma^2(\tau) + \gamma^2(k)}{2(\tau - k)}} I_{b-1/2}\left(\frac{\gamma(\tau)\gamma(k)}{\tau - k}\right) \right] dk. \end{aligned} \quad (45)$$

Equation (45) is a linear Volterra equations of the second kind (Polyanin and Manzhirov 2008). Since $\zeta(\tau)$ is a continuously differentiable function, Eq. (45) has a unique continuous solution for $\Psi(\tau)$. The Volterra equation can be efficiently solved numerically, see Itkin and Muravey (2020) for a discussion on various approaches to the numerical solution for this type of equation. In brief, for instance, the integral in the RHS is approximated using some quadrature rule with N nodes in k space, and the solution is obtained at N nodes in the τ space. Thus, the matrix equation obtained can be solved with the complexity $O(N^2)$, since the matrix is lower triangular. Since N could be small ($N \approx 20 - 30$), the solution is fast. We discuss numerical aspects of the solution in more detail towards the end of this article.

Once Eq. (45) is solved and the function $\Psi(\tau)$ is found, the final solution reads

$$\begin{aligned} u(\tau, z) &= \int_0^\tau \Psi(k) \\ &\frac{\partial}{\partial \gamma(k)} \left[\frac{\sqrt{z\gamma(k)}}{\tau - k} \left(\frac{\gamma(k)}{z}\right)^b e^{-\frac{z^2 + \gamma^2(k)}{2(\tau - k)}} I_{b-1/2}\left(\frac{z\gamma(k)}{\tau - k}\right) \right] \\ &+ \int_{\gamma(0)}^\infty u(0, \xi) q_\tau(z, \xi, b) d\xi. \end{aligned} \quad (46)$$

Domain $0 < z < y(\tau)$

The construction of the solution in this case is similar to that described in the previous section. Again, to obtain a PDE with a homogeneous initial condition, we represent the solution in the form

$$\begin{aligned} u(\tau, z) &= q(\tau, z) - \varsigma_0(\tau) + \int_0^{y(0)} u(0, \xi) q_\tau(z, \xi, b) d\xi, \\ \varsigma_0(\tau) &= - \int_0^{y(0)} u(0, \xi) q_\tau(0, \xi, b) d\xi, \end{aligned} \quad (47)$$

where, (Abramowitz and Stegun 1964)

$$q_\tau(0, \xi, b) = \frac{2^{1/2-b} \xi^{2b}}{\tau^{b+1/2} \Gamma\left(b + \frac{1}{2}\right)} e^{-\frac{\xi^2}{2\tau}},$$

and $\Gamma(x)$ is the Euler gamma function.

Then the function $q(x, \tau)$ solves the problem

$$\begin{aligned} \frac{\partial q(\tau, z)}{\partial \tau} &= \frac{1}{2} \frac{\partial^2 q(\tau, z)}{\partial z^2} + \frac{b}{z} \frac{\partial q(\tau, z)}{\partial z}, \\ q(0, z) &= 0, \quad 0 < z < y(0), \\ q(\tau, 0) &= 0, \quad q(\tau, y(\tau)) = \varsigma(\tau) + \varsigma_0(\tau). \end{aligned} \quad (48)$$

We search for the solution in the form of the Bessel potential in Eq. (43). The potential density $\Psi(\tau)$ solves the following Volterra equation of the second kind

$$\begin{aligned} \varsigma(\tau) + \varsigma_0(\tau) &= \Psi(\tau) + \int_0^\tau \Psi(k) \\ \frac{\partial}{\partial y(k)} &\left[\frac{\sqrt{y(\tau)y(k)}}{\tau - k} \left(\frac{y(k)}{y(\tau)} \right)^b e^{-\frac{y^2(\tau) + y^2(k)}{2(\tau - k)}} I_{b-1/2} \left(\frac{y(\tau)y(k)}{\tau - k} \right) \right] dk. \end{aligned} \quad (49)$$

Once this function is found, the final solution reads

$$\begin{aligned} u(\tau, z) &= \int_0^\tau \Psi(k) \frac{\partial}{\partial y(k)} \left[\frac{\sqrt{zy(k)}}{\tau - k} \left(\frac{y(k)}{z} \right)^b \right. \\ &\quad \left. e^{-\frac{z^2 + y^2(k)}{2(\tau - k)}} I_{b-1/2} \left(\frac{zy(k)}{\tau - k} \right) \right] dk \\ &\quad + \int_0^{y(0)} u(0, \xi) q_\tau(z, \xi, b) d\xi. \end{aligned} \quad (50)$$

Double Barrier Options

Double barrier options for time-dependent models can be also priced by using the method of potentials. In particular, this is demonstrated in Itkin and Muravey (2020) for the time-dependent Hull-White model. Here we use a similar approach and apply the idea proposed in Itkin and Muravey (2020) to construction of the semi-closed form solutions for double barrier options for the CIR and CEV models.

Let us provide the explicit formulae for the CEV model alone, as this can be done for the CIR model in exactly the same way. Suppose we need the price of a double barrier call option with the lower barrier $L(t)$ and the upper barrier $H(t) > L(t)$. After doing the transformation to the Bessel PDE as described previously, this implies solving the following problem

$$\begin{aligned} \frac{\partial u}{\partial \tau} &= \frac{1}{2} \frac{\partial^2 u}{\partial z^2} + \frac{b}{z} \frac{\partial u}{\partial z}, \\ u(\tau = 0, z) &= u(0, z), \quad y(0) < x < h(0), \\ u(y(\tau), \tau) &= u(h(\tau), \tau) = 0, \end{aligned} \quad (51)$$

where for $-1 < \beta < 0$

$$y(\tau) = -\frac{1}{\beta} L(t(\tau))^{-\beta} F(\phi(\tau)), \quad h(\tau) = -\frac{1}{\beta} H(t(\tau))^{-\beta} F(\phi(\tau)), \quad (52)$$

and for $0 < \beta < 1$

$$y(\tau) = \frac{1}{\beta} L(t(\tau))^{-\beta} F(\phi(\tau)), \quad h(\tau) = \frac{1}{\beta} H(t(\tau))^{-\beta} F(\phi(\tau)). \quad (53)$$

Thus, in this case, the solution is defined at the z -domain with two moving (time-dependent) boundaries.

Since this problem has an inhomogeneous initial condition, the method of potentials cannot be directly applied. Therefore, similar to Eq. (40), we represent the solution in the form

$$u(\tau, z) = q(\tau, z) + \int_{y(0)}^{h(0)} u(0, \xi) q_\tau(z, \xi, b) d\xi. \quad (54)$$

Now the function $q(x, \tau)$ solves a similar problem, but with the homogeneous initial condition

$$\begin{aligned}
\frac{\partial q}{\partial \tau} &= \frac{1}{2} \frac{\partial^2 q}{\partial z^2} + \frac{b}{z} \frac{\partial q}{\partial z}, \\
q(0, z) &= 0, \quad \gamma(0) < x < h(0), \\
q(\tau, \gamma(\tau)) &= -\phi_2(\tau), \quad q(\tau, h(\tau)) = -\psi_2(\tau), \\
\phi_2(\tau) &= -\int_{\gamma(0)}^{h(0)} u(0, \xi) q_\tau(\gamma(\tau), \xi, b) d\xi, \\
\psi_2(\tau) &= -\int_{\gamma(0)}^{h(0)} u(0, \xi) q_\tau(h(\tau), \xi, b) d\xi. \quad (55)
\end{aligned}$$

Based on the method of Itkin and Muravey (2020), we construct the solution of Eq. (55) in the form of a generalized Bessel potential

$$\begin{aligned}
q(\tau, z) &= \int_0^\tau \left\{ \Psi(k) \frac{\partial}{\partial \xi} \left[\frac{\sqrt{z\xi}}{\tau-k} \left(\frac{\xi}{z} \right)^b e^{-\frac{z^2+\xi^2}{2(\tau-k)}} I_{b-1/2} \left(\frac{z\xi}{\tau-k} \right) \right] \right|_{\xi \rightarrow \gamma(k)} \\
&\quad + \Phi(k) \frac{\partial}{\partial \xi} \left[\frac{\sqrt{z\xi}}{\tau-k} \left(\frac{\xi}{z} \right)^b e^{-\frac{z^2+\xi^2}{2(\tau-k)}} I_{b-1/2} \left(\frac{z\xi}{\tau-k} \right) \right] \right|_{\xi \rightarrow h(k)} \Bigg\} dk. \quad (56)
\end{aligned}$$

Here $\Psi(k)$, $\Phi(k)$ are the Bessel potential densities to be determined. Using the boundary conditions in Eq. (55), and the fact that the expression in square brackets at $\tau = k$ is the Dirac delta function, one can find that they solve a system of two Volterra equations of the second kind

$$\begin{aligned}
\phi_2(\tau) &= \Psi(\tau) + \int_0^\tau \left\{ \Psi(k) \frac{\partial}{\partial \xi} \left[\frac{\sqrt{\gamma(\tau)\xi}}{\tau-k} \left(\frac{\xi}{\gamma(\tau)} \right)^b e^{-\frac{\gamma^2(\tau)+\xi^2}{2(\tau-k)}} I_{b-1/2} \left(\frac{\gamma(\tau)\xi}{\tau-k} \right) \right] \right|_{\xi \rightarrow \gamma(k)} \\
&\quad + \Phi(k) \frac{\partial}{\partial \xi} \left[\frac{\sqrt{\gamma(\tau)\xi}}{\tau-k} \left(\frac{\xi}{\gamma(\tau)} \right)^b e^{-\frac{\gamma^2(\tau)+\xi^2}{2(\tau-k)}} I_{b-1/2} \left(\frac{\gamma(\tau)\xi}{\tau-k} \right) \right] \right|_{\xi \rightarrow h(k)} \Bigg\} dk. \quad (57)
\end{aligned}$$

$$\begin{aligned}
\psi_2(\tau) &= \Phi(\tau) + \int_0^\tau \left\{ \Psi(k) \frac{\partial}{\partial \xi} \left[\frac{\sqrt{h(\tau)\xi}}{\tau-k} \left(\frac{\xi}{h(\tau)} \right)^b e^{-\frac{h^2(\tau)+\xi^2}{2(\tau-k)}} I_{b-1/2} \left(\frac{h(\tau)\xi}{\tau-k} \right) \right] \right|_{\xi \rightarrow \gamma(k)} \\
&\quad + \Phi(k) \frac{\partial}{\partial \xi} \left[\frac{\sqrt{h(\tau)\xi}}{\tau-k} \left(\frac{\xi}{h(\tau)} \right)^b e^{-\frac{h^2(\tau)+\xi^2}{2(\tau-k)}} I_{b-1/2} \left(\frac{h(\tau)\xi}{\tau-k} \right) \right] \right|_{\xi \rightarrow h(k)} \Bigg\} dk. \quad (58)
\end{aligned}$$

This system can be solved by various numerical methods with complexity $O(N^2)$ (see the discussion after Eq. (45)). Once this is done, the solution of the double barrier problem is found.

THE METHOD OF GENERALIZED INTEGRAL TRANSFORM

In this section, we solve the same problem, but use the method of generalized integral transform. As applied to finance, this method was successfully used in Carr and Itkin (2020) and Itkin and Muravey (2020) to price barrier options in the time-dependent OU model and American options for equities, and the Hull-White model for interest rates. The method was borrowed from physics, where it was used to solve the Stefan problem and other heat and mass transfer problems with a moving boundary (or moving interphase boundary), see Kartashov (1999, 2001) and references therein. In particular, in Itkin and Muravey (2020), the authors extended this approach to an infinite domain where the solution was not known yet. Below we extend this approach and apply it to the CEV and CIR problems.

Note that so far, the method was elaborated on and used simply for obtaining a semi-closed form solution of the heat equation. However, in this article we are dealing with the Bessel PDE. In Kartashov (1999), the author proposes to construct the direct integral transform for this equation by using Bessel functions. However, aside from this recommendation, no explicit solution has been presented. Moreover, our analysis shows that using the form of the transform proposed in Kartashov (1999) doesn't give rise to the solution, as construction of the inverse transform faces various technical problems.

Therefore, our method presented in this section is i) original, and ii) gives rise to the closed form solution of the problem. In other words, we solve the CIR and CEV problems by using the method of generalized integral transform to the very end, and, to the best of the authors' knowledge, this is done for the first time in the literature. As such, this approach could be also very useful in physics for solving various problems. As mentioned in Kartashov (1999) and previously here, those problems appear in (but are not limited to) the field of nuclear power engineering and safety of nuclear reactors; in studying combustion in solid-propellant rocket engines; in laser action on solids; in the theory of phase transitions (the Stefan problem and the Verigin problem

(in hydromechanics)); in the processes of sublimation in freezing and melting; in the kinetic theory of crystal growth; and so forth, see Kartashov (1999) and references therein.

Domain $0 < z < y(\tau)$

Recall, that based on the description at the beginning of this article, this problem emerges when we consider the CEV problem with $\beta < 0$. Since the Laplace transform of Eq. (30) gives rise to the Bessel ODE, (Abramowitz and Stegun 1964), it would be natural to search for the general integral transform in the class of Bessel functions. Therefore, by analogy with Kartashov (2001) and Carr and Itkin (2020), we introduce the generalized integral transform of the form

$$\bar{u}(\tau, p) = \int_0^{y(\tau)} z^{v+1} u(\tau, z) J_{|v|}(zp) dz, \quad (59)$$

where $p = a + iw$ is a complex number, $J_v(x)$ is the Bessel function of the first kind, and $v = 1/(2\beta) < 0$, since $\beta < 0$. Next, let us multiply both parts of Eq. (30) by $z^{v+1} J_{|v|}(zp)$ and integrate on z from 0 to $y(\tau)$. For the LHS, this yields

$$\begin{aligned} \int_0^{y(\tau)} z^{v+1} \frac{\partial u}{\partial \tau} J_{|v|}(zp) dz \\ = \frac{\partial \bar{u}}{\partial \tau} - y'(\tau) [\gamma(\tau)]^{v+1} u(\tau, y(\tau)) J_{|v|}(\gamma(\tau)p). \end{aligned} \quad (60)$$

The last term in the RHS of Eq. (60) vanishes due to the boundary condition in Eq. (12).

Accordingly, for the RHS of Eq. (30), we have $b = v + 1/2$, and

$$\begin{aligned} J_1 &= \frac{1}{2} \int_0^{y(\tau)} z^{v+1} \frac{\partial^2 u}{\partial z^2} J_{|v|}(zp) dz = \frac{1}{2} z^{v+1} \frac{\partial u}{\partial z} J_{|v|}(zp) \Big|_0^{y(\tau)} \\ &\quad - \frac{1}{2} u(\tau, z) \frac{\partial}{\partial z} (z^{v+1} J_{|v|}(zp)) \Big|_0^{y(\tau)} \\ &\quad + \frac{1}{2} \int_0^{y(\tau)} u(\tau, z) \frac{\partial^2}{\partial z^2} (z^{v+1} J_{|v|}(zp)) dz, \\ J_2 &= \int_0^{y(\tau)} z^{v+1} \frac{v+1/2}{z} \frac{\partial u}{\partial z} J_{|v|}(zp) dz \\ &= (v+1/2) z^v J_{|v|}(zp) u(\tau, z) \Big|_0^{y(\tau)} \\ &\quad - (v+1/2) \int_0^{y(\tau)} u(\tau, z) \frac{\partial}{\partial z} (z^v J_{|v|}(zp)) dz. \end{aligned} \quad (61)$$

Due to the boundary conditions in Eq. (12), the sum $J_1 + J_2$ can be represented as

$$\begin{aligned} J_1 + J_2 &= y^{v+1}(\tau) J_{|v|}(\gamma(\tau)p) \Psi(\tau) + \frac{1}{2} \int_0^{y(\tau)} u(\tau, z) z \\ &\quad \left[\frac{1-2v}{z} \frac{\partial}{\partial z} (z^v J_{|v|}(zp)) + \frac{\partial^2}{\partial z^2} (z^v J_{|v|}(zp)) \right] dz, \\ \Psi(\tau) &= \frac{\partial u}{\partial z} \Big|_{z=y(\tau)}. \end{aligned} \quad (62)$$

It can be checked by using the theory of cylinder functions (Bateman and Erdélyi 1953) that the Bessel function $J_{|v|}(zp)$ also solves the following ordinary differential equation (ODE)

$$\begin{aligned} \frac{d^2 Z}{dz^2} + \frac{1-2v}{z} \frac{dZ}{dz} + p^2 Z &= 0, \\ Z(z) &= C_1 z^v J_{|v|}(zp) + C_2 z^v Y_{|v|}(pz). \end{aligned} \quad (63)$$

Here $Y_{|v|}(z)$ denotes the Bessel function of the second kind (also known as the Neumann or Weber function), which is linearly independent of $J_{|v|}(z)$. Assuming $C_1 = 1$, $C_2 = 0$, from Eq. (60), Eq. (61) we obtain the following Cauchy problem for \bar{u}

$$\begin{aligned} \frac{d\bar{u}(\tau, p)}{d\tau} &= \frac{1}{2} \left[-p^2 \bar{u}(\tau, p) + y^{v+1}(\tau) J_{|v|}(\gamma(\tau)p) \Psi(\tau) \right], \\ \bar{u}(p, 0) &= \int_0^{y(0)} z^{v+1} J_{|v|}(zp) u(0, z) dz. \end{aligned} \quad (64)$$

The solution of this problem reads

$$\bar{u} = e^{-p^2 \tau/2} \left[\bar{u}(0, p) + \frac{1}{2} \int_0^\tau e^{p^2 s/2} \Psi(s) y^{v+1}(s) J_{|v|}(\gamma(s)p) ds \right]. \quad (65)$$

By analogy with Carr and Itkin (2020), we can obtain the Fredholm equation of the first type for the function $\Psi(\tau)$. For doing so, let us set $p = i\lambda$, $\lambda \in \mathbb{R}$, tend τ to infinity, and apply the formula $J_{|v|}(ix) = e^{i\pi/2} I_v(x)$, which connects the Bessel function $J_v(x)$ with the modified Bessel function $I_v(x)$. This yields

$$\begin{aligned} \int_0^\infty e^{-\lambda^2 s/2} \Psi(s) y^{v+1}(s) I_v(\gamma(s)\lambda) ds \\ = -2 \int_0^{y(0)} q^{v+1} I_v(q\lambda) u(0, q) dq. \end{aligned} \quad (66)$$

The solution of this integral equation $\Psi(\tau)$ can be found numerically on a grid by solving a system of linear equations, see e.g., Carr and Itkin (2020) for a discussion on this subject and a numerical example. Once the function $\Psi(\tau)$ is found, it has to be substituted into Eq. (64) to obtain the generalized transform of $u(\tau, z)$ in the explicit form. Then, if this transform can be inverted back, we have solved the problem of pricing up-and-out barrier call options for the CEV model with $\beta < 0$.

The Inverse Transform

As already mentioned, it is reasonable to seek the solution of the CEV problem in the class of the Bessel functions. Therefore, we represent the solution in the form

$$u(\tau, z) = z^{-\nu} \sum_{n=1}^{\infty} \alpha_n(\tau) J_{|\nu|} \left(\frac{\mu_n z}{\gamma(\tau)} \right) \quad (67)$$

Here μ_n is an ordered sequence of the positive zeros of the Bessel function $J_{|\nu|}(\mu)$:

$$J_{|\nu|}(\mu_n) = J_{|\nu|}(\mu_m) = 0, \quad \mu_n > \mu_m > 0, \quad n > m.$$

Note that the definition in Eq. (67) automatically respects the vanishing boundary conditions for $u(\tau, z)$. We assume that this series converges absolutely and uniformly $\forall z \in [0, \gamma(\tau)]$ for any $\tau > 0$.

Applying the direct integral transform in Eq. (59) to both parts of Eq. (67), and using a change of variables $z \rightarrow \hat{z} = z\gamma(\tau)$ yields

$$\frac{\bar{u}(\tau, p)}{\gamma^2(\tau)} = \sum_{n=1}^{\infty} \alpha_n(\tau) \int_0^1 \hat{z} J_{|\nu|}(\mu_n \hat{z}) J_{|\nu|}(p\gamma(\tau)\hat{z}) d\hat{z}. \quad (68)$$

The set of functions $J_{|\nu|}(\alpha\hat{z})$ with $\alpha \in \mu_n, n = 1, \dots$, forms an orthogonal basis in the space $C[0, 1]$, with the scalar product

$$\begin{aligned} \langle J_{|\nu|}(\alpha z), J_{|\nu|}(\beta z) \rangle &= 2 \int_0^1 \frac{z J_{|\nu|}(\alpha z) J_{|\nu|}(\beta z) dz}{J_{|\nu|+1}(\alpha) J_{|\nu|+1}(\beta)} \\ &= \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha \neq \beta \end{cases} \end{aligned} \quad (69)$$

Therefore, the explicit formula for each coefficient $\alpha_n(\tau)$ is straightforward

$$\alpha_n(\tau) = 2 \frac{\bar{u}(\tau, \mu_n / \gamma(\tau))}{\gamma^2(\tau) J_{|\nu|+1}^2(\mu_n)}. \quad (70)$$

Thus, the final solution for $u(\tau, z)$ reads

$$\begin{aligned} u(\tau, z) &= 2 \frac{z^{-\nu}}{\gamma^2(\tau)} \sum_{n=1}^{\infty} \left[\int_0^{\gamma(0)} u(0, s) s^{\nu+1} e^{-\frac{\mu_n^2}{2\gamma^2(\tau)} \tau} \right. \\ &\quad \frac{J_{|\nu|}(\mu_n s / \gamma(\tau)) J_{|\nu|}(\mu_n z / \gamma(\tau))}{J_{|\nu|+1}^2(\mu_n)} ds \\ &\quad \left. + \frac{1}{2} \int_0^{\tau} \gamma^{\nu+1}(s) \Psi(s) e^{-\frac{\mu_n^2}{2\gamma^2(\tau)} (\tau-s)} \right. \\ &\quad \left. \frac{J_{|\nu|}(\mu_n \gamma(s) / \gamma(\tau)) J_{|\nu|}(\mu_n z / \gamma(\tau))}{J_{|\nu|+1}^2(\mu_n)} ds \right]. \end{aligned} \quad (71)$$

This expression can be also rewritten in the form

$$\begin{aligned} u(\tau, z) &= 2 \frac{z^{-2\nu}}{\gamma^2(\tau)} \left[\int_0^{\gamma(0)} s u(0, s) \Theta_{|\nu|} \left(\frac{\sqrt{\tau}}{\gamma(\tau)}, \frac{s}{\gamma(\tau)}, \frac{z}{\gamma(\tau)} \right) ds \right. \\ &\quad \left. + \frac{1}{2} \int_0^{\tau} \gamma(\tau) \Psi(s) \Theta_{|\nu|} \left(\frac{\sqrt{\tau-s}}{\gamma(\tau)}, \frac{\gamma(s)}{\gamma(\tau)}, \frac{z}{\gamma(\tau)} \right) ds \right], \end{aligned} \quad (72)$$

where we introduced a new function

$$\Theta_{|\nu|}(\theta, x_1, x_2) = \sum_{n=1}^{\infty} e^{-\frac{\mu_n^2 \theta^2}{2}} (x_1 x_2)^{\nu} \frac{J_{|\nu|}(\mu_n x_1) J_{|\nu|}(\mu_n x_2)}{J_{|\nu|+1}^2(\mu_n)}. \quad (73)$$

The function $\Theta_{|\nu|}(\theta, x_1, x_2)$ is an analog of the Jacobi theta function, which is a periodic solution of the heat equation. Indeed, in Carr and Itkin (2020), the solution of a similar problem for the time-dependent OU model (so $\beta = -1, \nu = -1/2$ and $|\nu| = 1/2$), with moving boundaries but for the heat equation has been obtained in terms of the theta functions. It can be checked that if $\nu = 1/2$, we have

$$\begin{aligned} \Theta_{1/2} \left(\frac{\sqrt{\tau}}{\gamma(\tau)}, \frac{s}{\gamma(\tau)}, \frac{z}{\gamma(\tau)} \right) &= \frac{\gamma(\tau)}{2} \left[\theta_3 \left(e^{-\frac{\pi^2 \theta^2}{8}}, \frac{n\pi(s-z)}{4\gamma(\tau)} \right) - \theta_3 \left(e^{-\frac{\pi^2 \theta^2}{8}}, \frac{n\pi(s+z)}{4\gamma(\tau)} \right) \right], \end{aligned} \quad (74)$$

where $\theta_3(\omega, x)$ is the Jacobi theta function, (Mumford et al. 1983). Accordingly, function $\theta_{|\nu|}(\theta, x_1, x_2)$ is a periodic solution of the Bessel equation.

As an alternative to the Fredholm equation of the first kind in Eq. (66), which is ill-posed and requires special methods to solve it (see Carr and Itkin 2020), we can use a trick proposed in Itkin and Muravey (2020) and instead derive the Volterra equation of the second kind for the function $\Psi(\tau)$. For doing that, one needs to differentiate Eq. (71) on z , and then let $z = \gamma(\tau)$. This yields

$$\begin{aligned} \Psi(\tau) = & -\frac{2}{\gamma^{3+\nu}(\tau)} \sum_{n=1}^{\infty} (\mu_n + \nu) \left[\int_0^{\gamma(0)} u(0, s) s^{\nu+1} \right. \\ & e^{-\frac{\mu_n^2 \tau}{2\gamma^2(\tau)}} \frac{J_{|\nu|}(\mu_n s / \gamma(\tau))}{J_{|\nu|+1}(\mu_n)} ds \\ & \left. + \frac{1}{2} \int_0^{\tau} \gamma^{\nu+1}(s) \Psi(s) e^{-\frac{\mu_n^2(\tau-s)}{2\gamma^2(\tau)}} \frac{J_{|\nu|}(\mu_n \gamma(s) / \gamma(\tau))}{J_{|\nu|+1}(\mu_n)} ds \right]. \end{aligned} \quad (75)$$

This equation has to be solved numerically, again see Itkin and Muravey (2020) for a discussion and examples.

Some Approximations

In some cases, Eq. (75) can be solved asymptotically. For instance, one can apply the following approximations

$$\begin{aligned} J_{|\nu|}^2(z) + J_{|\nu|+1}^2(z) & \approx \frac{2}{\pi z}, \quad x \gg \nu, \\ J_{|\nu|+1}^2(\mu_n) & \approx \frac{2}{\pi \mu_n}, \quad \mu_n \gg \nu \\ J_{|\nu|}(z) & \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{2|\nu|+1}{4}\pi\right), \quad z \rightarrow \infty \\ \mu_n & \approx \pi\left(n + \frac{2|\nu|+1}{4}\right), \quad n \rightarrow \infty. \end{aligned} \quad (76)$$

Then the infinite sum in Eq. (71) can be truncated up to keep first N terms. The reminder (the error of this method) reads

$$\begin{aligned} R(N, \tau, z) = & \frac{1}{z^{\nu+1/2} \gamma(\tau)} \sum_{n=N+1}^{\infty} \\ & \left\{ \int_0^{\gamma(0)} s^{\nu+1/2} u(0, s) e^{-\frac{\pi^2 \lambda_n^2}{2\gamma^2(\tau)} \tau} \left[\cos\left(\frac{\pi \lambda_n(s+z)}{\gamma(\tau)} - 2\pi \lambda_n\right) \right. \right. \\ & \left. \left. + \cos\left(\frac{\pi \lambda_n(s-z)}{\gamma(\tau)}\right) \right] ds + \frac{1}{2} \int_0^{\tau} \gamma^{\nu+1/2}(s) \Psi(s) e^{-\frac{\pi^2 \lambda_n^2}{2\gamma^2(\tau)}(\tau-s)} \right. \\ & \left. \left[\cos\left(\frac{\pi \lambda_n(\gamma(s)+z)}{\gamma(\tau)} - 2\pi \lambda_n\right) + \cos\left(\frac{\pi \lambda_n(\gamma(s)-z)}{\gamma(\tau)}\right) \right] ds \right\}. \end{aligned} \quad (77)$$

Here $\lambda_n = n + (2\nu - 1)/4$

A simple assessment shows that, since $-1 < \beta < 0$, from Eq. (8) at typical values of the model parameters we have $f(t) \approx f_1 \gg 1$. Assume that $H = \text{const}$. Hence, in Eq. (9)

$$\phi = \frac{\log(2\beta f_1 \tau + 1)}{2\beta f_1}, \quad \gamma(\tau) = -\frac{H^{-\beta} \sqrt{2\beta f_1 \tau + 1}}{\beta}.$$

Then it can be checked that the expression

$$-\frac{\mu_n^2 \tau}{2\gamma^2(\tau)}$$

rapidly drops down for $-1 < \beta < 0$, $\tau > 0$, unless $\tau \ll 1$ and $H \gg 1$. Therefore, in this case just first few terms in the sum in Eq. (75) would be a good approximation.

Another approximation can be proposed to compute function $\theta_{|\nu|}(\theta, x_1, x_2)$ defined in Eq. (73). The idea is that, as mentioned above, if $\tau \ll 1$ and $H = O(1)$, the first argument θ of this function is small, $\theta \ll 1$. Then the function $c_0 = e^{-\frac{\mu_n^2 \theta^2}{2}}$ is small at large n , and $\frac{\mu_n^2 \theta^2}{2}$ is small at small n . Therefore, for small n we represent c_0 by using the Padé approximation $(k, 1)$, and for large n we replace the small values of c_0 with the same Padé approximation. To estimate how accurate this trick is, let us pick, for instance, $k = 2$, so

$$c_0 \approx \frac{1}{1 + \frac{x}{2}} \left(1 - \frac{x}{2}\right) + O(x^2), \quad x = \frac{\mu_n^2 \theta^2}{2}. \quad (78)$$

Thus, expanding the parenthesis into two terms, the function $(x_1 x_2)^{-\nu} \theta_{|\nu|}(\theta, x_1, x_2)$ can be represented as

$$(x_1 x_2)^{-\nu} \Theta_{|\nu|}(\theta, x_1, x_2) = A_1(\theta, x_1, x_2) + A_2(\theta, x_1, x_2). \quad (79)$$

Now observe that, (Bateman and Erdélyi 1953)

$$\begin{aligned} A_1(\theta, x_1, x_2) &= \sum_{n=1}^{\infty} \frac{J_{|\nu|}(\mu_n x_1) J_{|\nu|}(\mu_n x_2)}{J_{|\nu|+1}^2(\mu_n)} \frac{1}{1 + \frac{\mu_n^2 \theta^2}{4}} \\ &= -\frac{4}{\theta^2} \sum_{n=1}^{\infty} \frac{J_{|\nu|}(\mu_n x_1) J_{|\nu|}(\mu_n x_2)}{J_{|\nu|+1}^2(\mu_n) (\gamma^2 - \mu_n^2)} \\ &= \frac{\pi J_{|\nu|}(x_2 \bar{\gamma})}{\theta^2 J_{|\nu|}(\bar{\gamma})} \left[J_{|\nu|}(x_1 \gamma) Y_{|\nu|}(\gamma) - J_{|\nu|}(\gamma) Y_{|\nu|}(x_1 \gamma) \right] \\ &= \frac{2 I_{|\nu|}(x_2 \bar{\gamma})}{\theta^2 I_{|\nu|}(\bar{\gamma})} \left[K_{|\nu|}(x_1 \bar{\gamma}) I_{|\nu|}(\bar{\gamma}) - K_{|\nu|}(\bar{\gamma}) I_{|\nu|}(x_1 \bar{\gamma}) \right], \\ \gamma &= 2i/\theta, \quad \bar{\gamma} = 2/\theta. \end{aligned} \quad (80)$$

Here we again use the formulae

$$\begin{aligned} J_{\nu}(ix) &= e^{i\nu\pi/2} I_{\nu}(x), \\ Y_{\nu}(ix) &= e^{(i\nu+1)\pi/2} I_{\nu}(x) - 2/\pi e^{-i\nu\pi/2} K_{\nu}(x), \end{aligned}$$

which connects the Bessel functions $J_{\nu}(x)$ and $Y_{\nu}(x)$ with the modified Bessel functions $I_{\nu}(x)$ and $K_{\nu}(x)$.

To compute the next term in Eq. (78)

$$A_2(\theta, x_1, x_2) = -\frac{\theta^2}{4} \sum_{n=1}^{\infty} \frac{J_{|\nu|}(\mu_n x_1) J_{|\nu|}(\mu_n x_2)}{J_{|\nu|+1}^2(\mu_n)} \frac{\mu_n^2}{1 + \frac{\mu_n^2 \theta^2}{4}} \quad (81)$$

let us differentiate the LHS of Eq. (80) by θ , so

$$\begin{aligned} &\frac{\partial}{\partial \theta} \sum_{n=1}^{\infty} \frac{J_{|\nu|}(\mu_n x_1) J_{|\nu|}(\mu_n x_2)}{J_{|\nu|+1}^2(\mu_n)} \frac{1}{1 + \frac{\mu_n^2 \theta^2}{4}} \\ &= -\frac{\theta}{2} \sum_{n=1}^{\infty} \frac{J_{|\nu|}(\mu_n x_1) J_{|\nu|}(\mu_n x_2)}{J_{|\nu|+1}^2(\mu_n)} \frac{\mu_n^2}{\left(1 + \frac{\mu_n^2 \theta^2}{4}\right)^2} \\ &\approx -\frac{\theta}{2} \sum_{n=1}^{\infty} \frac{J_{|\nu|}(\mu_n x_1) J_{|\nu|}(\mu_n x_2)}{J_{|\nu|+1}^2(\mu_n)} \frac{\mu_n^2}{1 + \frac{\mu_n^2 \theta^2}{2}}. \end{aligned} \quad (82)$$

Therefore, the second term in Eq. (78) takes the form

$$\begin{aligned} A_2(\theta, x_1, x_2) &= -\frac{\bar{\gamma}}{4} \frac{\partial}{\partial \bar{\gamma}} \left\{ \frac{\bar{\gamma}^2 I_{|\nu|}(x_2 \bar{\gamma})}{I_{|\nu|}(\bar{\gamma})} \left[K_{|\nu|}(x_1 \bar{\gamma}) I_{|\nu|}(\bar{\gamma}) - K_{|\nu|}(\bar{\gamma}) I_{|\nu|}(x_1 \bar{\gamma}) \right] \right\}, \\ \bar{\gamma} &= 2\sqrt{2}/\theta, \quad \frac{\partial I_{|\nu|}(\gamma)}{\partial \gamma} = \frac{1}{2} (I_{|\nu|-1}(\gamma) + I_{|\nu|+1}(\gamma)), \\ \frac{\partial K_{|\nu|}(\gamma)}{\partial \gamma} &= \frac{1}{2} (K_{|\nu|-1}(\gamma) + K_{|\nu|+1}(\gamma)). \end{aligned} \quad (83)$$

A similar approximation can be developed for Eq. (75) by using the identity, (Bateman and Erdélyi 1953)

$$\sum_{n=1}^{\infty} \frac{\mu_n J_{|\nu|}(\mu_n x)}{(\mu_n^2 - k^2) J_{|\nu|+1}(\mu_n)} = \frac{J_{|\nu|}(kx)}{2 J_{|\nu|}(k)}. \quad (84)$$

As an example, let us consider the CEV model with constant parameters given in Exhibit 1. In Exhibit 2, we present the difference of two values of the function $G_{\nu}(x_1, x_2) = (x_1 x_2)^{-\nu} \Theta_{|\nu|}(\theta, x_1, x_2)$: one computed by the definition in Eq. (73) using the first N terms in the sum, and the other computed by using the approximation in Eq. (79). It can be seen that the latter approximation provides an accuracy of about 25%, except for the area where both τ and β are simultaneously rather large, and hence, the value of $G_{\nu}(x_1, x_2)$ is very small.

This approach can be further developed by increasing k in the Padé approximations $(k, 1)$, and computing the consecutive terms in the expansion by using the same trick as shown above. In other words, to obtain the next term in the numerator of the Padé approximation, we can differentiate the LHS of Eq. (80) twice on θ , and then express this derivative via the RHS of Eq. (80), and so forth.

Connection to the First Passage Time Problem

Let us consider the following stopping moment

$$T_{x_0}^{\gamma} = \inf \{ \tau \geq 0, X_{\tau} \geq \gamma(\tau) \},$$

where X_{τ} is the Bessel process defined by Eq. (6) and originated from $X_0 = x_0$. From standard results in probability theory, the p.d.f. $\rho_{x_0}^{\gamma}(\tau)$ of the moment $T_{x_0}^{\gamma}$ can be found via the Fokker–Planck–Kolmogorov equation associated with the process X_{τ} :

EXHIBIT 1

Parameters of the Test

r	q	σ	H	N	$x_1/y(\tau)$	$x_2/y(\tau)$
0.02	0.01	0.5	0.2	100	0.5	0.5

$$\frac{1}{2} \frac{\partial^2 F_{x_0}(\tau, x)}{\partial x^2} - \frac{b}{x} \frac{\partial F_{x_0}(\tau, x)}{\partial x} + \frac{b}{x^2} F_{x_0}(\tau, x) = \frac{\partial F_{x_0}(\tau, x)}{\partial \tau}$$

$$F_{x_0}(\tau, \gamma(\tau)) = 0, \quad F_{x_0}(\tau, 0) = \delta(x - x_0) \quad (85)$$

here $\delta(x - x_0)$ is the Dirac measure at the point x_0 . The density $\rho_{x_0}^y(\tau)$ has the following representation:

$$\rho_{x_0}^y(\tau) = \frac{1}{2} \frac{\partial F_{x_0}(\tau, x)}{\partial x} \Big|_{x=\gamma(\tau)}. \quad (86)$$

The solution of the problem Eq. (30) and the function $\Psi(\tau)$ can be represented in terms of the function F :

$$u(z, \tau) = z^{-2b} \int_0^{\gamma(0)} s^{2b} u(0, s) F_s(\tau, z) ds,$$

$$\Psi(\tau) = 2\gamma(\tau)^{-2b} \int_0^{\gamma(0)} s^{2b} u(0, s) \rho_s^y(\tau) ds \quad (87)$$

For the boundary moving linearly in time $\gamma(\tau) \approx \alpha + \beta \tau$, it is possible to propose the following approximation for the function $\Psi(\tau)$

$$\Psi(\tau) \approx 2\gamma(\tau)^{-(2v+1)} \int_0^{\gamma(0)} s^{2v+1} u(s, 0) \frac{e^{\frac{\beta}{2\alpha}(\alpha^2 - s^2) + \frac{b^2}{2}\tau}}{(\alpha + \beta\tau)^{v+2}} ds$$

$$\sum_{n=1}^{\infty} \frac{s^{-v} \mu_n J_{|v|}(\mu_n z / \alpha)}{\alpha^{-2v} J_{|v|+1}(\mu_n)} e^{-\mu_n^2 \frac{\tau}{2\alpha(\alpha + \beta\tau)}} ds. \quad (88)$$

This is due to the fact that the pdf $\rho_{x_0}^y(\tau)$ of the first hitting time of the line $a\tau + b$ for the Bessel process is known explicitly (Alili and Patie 2010).

Note, that a similar problem for the OU process with both constant and time-dependent coefficients has been studied in Lipton and Kaushansky (2020a). The authors considered the first hitting time density to a moving boundary for a diffusion process, which satisfies the Cherkasov condition, and hence, can be reduced to a standard Wiener process. They give two complementary (forward and backward) formulations of this problem

and provide semi-analytical solutions for both of them by using the method of heat potentials.

Domain $z > y(\tau)$

To recall the earlier discussion, this problem occurs in both the CEV model with $0 < \beta < 1$ and the CIR model. We will construct the solution to this problem by using the Weber–Orr transform

$$\bar{u}(\tau, p) = \int_{\gamma(\tau)}^{\infty} z^{v+1} W(\tau, p, z) u(\tau, z) dz$$

$$u(\tau, z) = z^{-v} \int_0^{\infty} \frac{p W(\tau, p, z)}{V(\tau, p)} \bar{u}(\tau, p) dp. \quad (89)$$

The kernel $W(a, b)$ and the function $V(p)$ are defined as follows, (Bateman and Erdélyi 1953)

$$W(\tau, a, b) = J_{|v|}(ab) Y_{|v|}(a\gamma(\tau)) - Y_{|v|}(ab) J_{|v|}(a\gamma(\tau)),$$

$$V(\tau, p) = J_{|v|}^2(p\gamma(\tau)) + Y_{|v|}^2(p\gamma(\tau)). \quad (90)$$

The definitions in Eq. (90) are generalizations of the Pythagorean and angle sum identities for trigonometric functions to the case of cylinder functions $J_{|v|}$ and $Y_{|v|}$. The functions $W(\tau, a, b)$, as the functions of the second argument a , also form an orthogonal basis in the space $C[\gamma(\tau), \infty)$ for all $\tau > 0$.

However, we cannot apply this transform directly to the Bessel equation due to fact that the kernel is time-dependent. Therefore, we propose to represent \bar{u} as the weighted sum of the following transforms

$$\bar{u}_J(p, \tau) = \int_{\gamma(\tau)}^{\infty} z^{v+1} J_{|v|}(zp) u(\tau, z) dz,$$

$$\bar{u}_Y(p, \tau) = \int_{\gamma(\tau)}^{\infty} z^{v+1} Y_{|v|}(zp) u(\tau, z) dz, \quad (91)$$

so

$$\bar{u}(\tau, p) = \bar{u}_J(\tau, p) Y_{|v|}(\gamma(\tau)p) - \bar{u}_Y(\tau, p) J_{|v|}(\gamma(\tau)p) \quad (92)$$

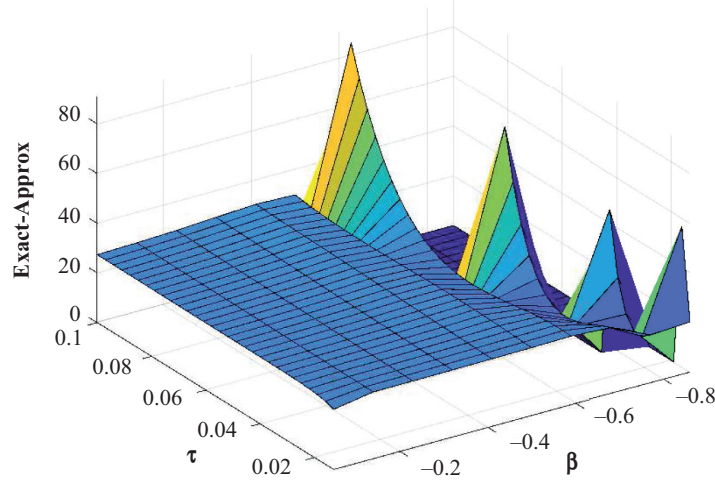
The explicit formulae for \bar{u}_J and \bar{u}_Y read

$$\bar{u}_J(\tau, p) = e^{-p^2 \tau / 2} \left[\bar{u}_J(0, p) + \frac{1}{2} \int_0^{\tau} e^{p^2 s / 2} \Psi(s) \gamma^{v+1}(s) J_{|v|}(\gamma(s)p) ds \right],$$

$$\bar{u}_Y(\tau, p) = e^{-p^2 \tau / 2} \left[\bar{u}_Y(0, p) + \frac{1}{2} \int_0^{\tau} e^{p^2 s / 2} \Psi(s) \gamma^{v+1}(s) Y_{|v|}(\gamma(s)p) ds \right], \quad (93)$$

EXHIBIT 2

Comparison of the Exact and Approximated Values of the Function $Gv(x1, x2)$



where $\Psi(\tau)$ is defined in Eq. (62). Using the inversion formula from Eq. (89), we immediately get the explicit formula for $u(\tau, z)$:

$$\begin{aligned} u(z, \tau) = & z^{-v} \int_0^\infty \int_{y(0)}^\infty s^{v+1} u(0, s) \\ & e^{-\frac{p^2 \tau}{2}} \frac{W(\tau, p, z) W(\tau, p, s)}{V(\tau, p)} p dp ds \\ & + \frac{z^{-v}}{2} \int_0^\infty \int_0^\tau \gamma^{v+1}(s) \Psi(s) \\ & e^{-\frac{p^2}{2}(\tau-s)} \frac{W(\tau, p, z) W(\tau, p, \gamma(s))}{V(\tau, p)} p dp ds. \end{aligned} \quad (94)$$

By analogy with the previous section, the function $\Psi(\tau)$ solves the Fredholm equation of the first kind

$$\begin{aligned} & \int_0^\infty e^{-\lambda^2 s/2} \Psi(s) \gamma^{v+1}(s) K_v(\gamma(s) \lambda) ds \\ & = -2 \int_{y(0)}^\infty q^{v+1} K_v(q \lambda) u(0, q) dq, \end{aligned} \quad (95)$$

or the Volterra equation of the second kind

$$\begin{aligned} \Psi(\tau) = & \gamma^{-v}(\tau) \left[\int_0^\infty \int_{y(0)}^\infty q^{v+1} u(0, q) e^{-\frac{p^2 \tau}{2}} \frac{Q(\tau, p) W(\tau, p, q)}{V(\tau, p)} dp dq \right] \\ & + \frac{1}{2} \int_0^\infty \int_0^\tau \gamma^{v+1}(s) \Psi(s) e^{-\frac{p^2}{2}(\tau-s)} \frac{Q(\tau, p) W(\tau, p, \gamma(s))}{V(\tau, p)} dp ds. \end{aligned} \quad (96)$$

Here

$$Q(\tau, p) = J_{v+1}(p\gamma(\tau)) Y_v(p\gamma(\tau)) - Y_{v+1}(p\gamma(\tau)) J_v(p\gamma(\tau)). \quad (97)$$

In some cases, Eq. (94) can be solved asymptotically. For instance, one can apply the following approximations

$$\begin{aligned} & J_{|v|}^2(z) + Y_{|v|}^2(z) \\ & \approx \frac{2}{\pi z} \sum_{k=0}^\infty \frac{(2k-1)!!}{2^k z^{2k}} \frac{\Gamma(|v|+k+1/2)}{k! \Gamma(|v|-k+1/2)}, \\ & J_{|v|}^2(z) + Y_{|v|}^2(z) \sim \frac{2}{\pi z}, \quad z \rightarrow \infty \\ & Y_{|v|}(z) \sim \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{2|v|+1}{4}\right), \quad z \rightarrow \infty, \\ & W(\tau, a, b) \sim \frac{2 \sin(a[b - \gamma(\tau)])}{\pi a \sqrt{b\gamma(\tau)}}, \quad a \rightarrow \infty. \end{aligned} \quad (98)$$

Then, the outer integral in Eq. (94) can be truncated from above and approximated by the integral over the domain $[0, P]$, $0 < P < \infty$. The reminder (the error of this method) reads

$$\begin{aligned}
R(P, \tau, z) = & z^{-v-1/2} \int_P^\infty \left\{ \int_{\gamma(0)}^\infty s^{v+1/2} u(0, s) e^{-\frac{p^2 \tau}{2}} \right. \\
& \sin(p[z - \gamma(\tau)]) \sin(p[s - \gamma(\tau)]) ds \\
& + \frac{1}{2} \int_0^\tau \gamma^{v+1/2}(s) \Psi(s) e^{-\frac{p^2}{2}(\tau-s)} \\
& \left. \sin(p[z - \gamma(\tau)]) \sin(p[\gamma(s) - \gamma(\tau)]) ds \right\} dp.
\end{aligned} \quad (99)$$

Introducing the new function $\Upsilon(P, t, \eta)$

$$\Upsilon(P, t, \eta) = e^{-\frac{\eta^2}{2t}} \left[\operatorname{erfc}\left(\frac{Pt + i\eta}{\sqrt{2t}}\right) + \operatorname{erfc}\left(\frac{Pt - i\eta}{\sqrt{2t}}\right) \right],$$

and taking into account the identity

$$\int_P^\infty e^{-\frac{p^2 t}{2}} \cos(\eta p) dp = \frac{1}{2} \sqrt{\frac{\pi}{2t}} \Upsilon(P, t, \eta),$$

we obtain the following explicit representation for $R(P, \tau, z)$

$$\begin{aligned}
R(P, \tau, z) = & \frac{z^{-v-1/2} \sqrt{\pi}}{4\sqrt{2}} \left\{ \int_{\gamma(0)}^\infty \frac{s^{v+1/2} u(0, s)}{\sqrt{\tau}} [\Upsilon(P, \tau, z - s) \right. \\
& - \Upsilon(P, \tau, z + s - 2\gamma(\tau))] ds \\
& + \frac{1}{2} \int_0^\tau \frac{\gamma^{v+1/2}(s) \Psi(s)}{\sqrt{\tau - s}} [\Upsilon(P, \tau - s, z - \gamma(s)) \\
& \left. - \Upsilon(P, \tau - s, z + \gamma(s) - 2\gamma(\tau))] ds \right\}. \quad (100)
\end{aligned}$$

Now we show that under the assumptions

$$\begin{aligned}
\int_{\gamma(0)}^\infty q^{v+1/2} u(0, q) dq &\leq M_1, \\
\int_{\gamma(0)}^\infty q^{2v+1} u(0, q) dq &\leq M_2, \quad M_1, M_2 - \text{const}, \quad (101)
\end{aligned}$$

we can set the upper limit of integration P such that $|R(P, \tau, z)| < \epsilon$ for any $\epsilon > 0$. Indeed, using the following inequalities (see Eq. (87)):

$$\begin{aligned}
\frac{1}{\sqrt{2t}} \Upsilon(P, t, \eta) &\leq \operatorname{erfc}(P), \\
\Psi(\tau) &\leq 2\gamma(\tau)^{-(1+2v)} \int_{\gamma(0)}^\infty q^{2v+1} u(0, q) dq, \quad (102)
\end{aligned}$$

yields

$$\begin{aligned}
|R(P, \tau, z)| &\leq \frac{z^{-v-1/2} \sqrt{\pi} \operatorname{erfc}(P)}{2} \\
&\left\{ \int_{\gamma(0)}^\infty q^{v+1/2} u(0, q) dq + \int_0^\tau \gamma^{-v-1/2}(s) ds \int_{\gamma(0)}^\infty q^{2v+1} u(0, q) dq \right\}. \quad (103)
\end{aligned}$$

Since the second integral is bounded for any τ , we obtain the following inequality:

$$\begin{aligned}
|R(P, \tau, z)| &\leq \frac{z^{-v-1/2} \sqrt{\pi} \operatorname{erfc}(P)}{2} \\
(M_1 + M_2 M_3(\tau)), \quad M_3(\tau) &= \int_0^\tau \gamma^{-v-1/2}(s) ds. \quad (104)
\end{aligned}$$

NUMERICAL EXPERIMENTS

Similar to Itkin and Muravey (2020), to check performance and accuracy of the proposed methods we construct the following test. We consider up-and-out barrier call option written on the underlying stock, which follows the CEV process with $0 < \beta < 1$. This case is described at the end of the first section. In that case, after the change of variables proposed there is completed, the problem is transformed into the solution of the Bessel PDE at the domain, $\bar{z} \in [\gamma(\tau), \infty)$ with the boundary and initial conditions given in Eq. (15) and Eq. (14).²

In this test, we use the explicit form of parameters $r(t)$, $q(t)$, $\sigma(t)$

$$r(t) = r_0 - r_k(a + t), \quad q(t) = q_0 - q_k(a + t), \quad \sigma(t) = \sigma_0 \sqrt{a + t}, \quad (105)$$

where r_0 , q_0 , σ_0 , r_k , q_k , and σ_k are constants. We also assume $r_0 = q_0$, and $H - \text{const}$. With these definitions one can find

$$\begin{aligned}
\phi(t) &= -\frac{1}{2} \sigma_0^2 (t - T)(2a + t + T), \quad \phi(\tau) \\
&= \frac{\sigma_0^2}{2\beta(q_k - r_k)} \log \left(\frac{\sigma_0^2 + 2\beta(q_k - r_k)\tau}{\sigma_0^2} \right), \\
F(\phi) &= \frac{\sqrt{2\beta\tau(q_k - r_k) + \sigma_0^2}}{\sigma_0}, \quad \gamma(\tau) = F(\phi) H^{-\beta} / \beta. \quad (106)
\end{aligned}$$

²Hence, in new variables the up-and-out option transforms to the down-and-out option.

We approach pricing the up-and-out barrier call option in the CEV model in a twofold manner. First, as a benchmark, we solve the PDE in Eq. (2) by using a finite-difference (FD) scheme of the second order in space and time. We use the Crank-Nicolson scheme with few first Rannacher steps on a non-uniform grid compressed close to the barrier level, see Itkin (2017). Accordingly, our domain in S space is $S \in [0, H]$.³

Alternatively, we apply the method of Bessel potentials (BP) developed earlier to solve the Bessel PDE in Eq. (5). For doing so, first we solve the Volterra equation in Eq. (45), where the kernel is approximated on a rectangular grid $M \times M$, and the integral is computed using the trapezoidal rule. This implies solving the following system of linear equations

$$\|\zeta\| = (I + P)\|\Psi\|. \quad (107)$$

Here $\|\Psi\|$ is the vector of discrete values of $\Psi(\tau)$, $\tau \in [0, \tau(t)|_{t=0}]$ on a grid with M nodes, $\|\zeta\|$ is a similar vector of $\zeta(\tau)$, I is the unit $M \times M$ matrix, and P is the $M \times M$ matrix of the kernel values on the same grid. Note that the matrix P is lower triangular. Therefore, the solution of Eq. (107) can be completed with complexity $O(M^2)$.

As the kernel (and so the matrix P) do not depend on strike K , but only on the function $\zeta(\tau)$, Eq. (107) can be solved simultaneously for all strikes by inverting the matrix $I + P$ with the complexity $O(M^2)$, and then multiplying it by vectors $\|\zeta\|_k$, $k = 1, \dots, \bar{k}$, \bar{k} is the total number of strikes. Therefore, the total complexity of this step remains $O(\bar{k}M^2)$, but this operation, however, can be vectorized in k . The Volterra equation could also be solved by iterative methods, but with almost the same complexity, see a discussion in Itkin and Muravey (2020).

The model parameters for this test are presented in Exhibit 3. We run the test for a set of maturities $T \in [1/12, 0.3, 0.5, 1]$ and strikes $K \in [59, 64, 69, 74, 79, 84]$. The up-and-out barrier call option prices computed in such an experiment are presented in Exhibit 4.

³ A similar approach can be developed for the down-and-out options, with $S \in [L, \infty)$. Then instead of truncating the infinite semi-interval, one can transform it to the fixed interval $[-1, 1]$, or to $[0, 1]$ and solve a modified PDE on the new interval. The boundary behavior of the solution can be obtained using Fichera theory and/or Green's integral formula, see Wilmott et al. (2014), where this was done in many cases and it was proven that this approach works well.

EXHIBIT 3 Parameters of the Test

r_0	q_0	σ_0	r_k	q_k	σ_k	α	H	S
0.01	0.01	0.3	0.01	0.005	0.2	1.0	100	70

Comparison with the BP Method

The same results computed by using the BP method are displayed in Exhibit 5 for the option prices. Also, in Exhibit 6, the percentage difference between the prices obtained by using the BP and FD methods is presented as a function of the option strike K and maturity T . Here to provide a comparable accuracy, we run the FD solver with 101 nodes in space S and 100 steps in time t . Otherwise, the quality of the FD solution is not sufficient.

It can be seen that the agreement of both methods is good (less than 1%) if the option price is not too small, which happens when the strike K is close to the barrier, or at long maturities. In this case, as this is seen from Exhibit 4, the relative difference becomes large, but the absolute difference of two methods is about one cent, which is almost insignificant. Obviously, such cases are a challenge for any FD method, as at $t = T$ there is a jump in the initial condition at the boundary, and the first derivative of the solution doesn't exist in this point.

As far as the performance of both methods is concerned, to decrease the elapsed time for the FD method, instead of Eq. (2), we solve the corresponding forward PDE. Therefore, the prices of all options for a given set of strikes and maturities could be obtained in one sweep. This also requires $\bar{m} \times \bar{k}$ integrations of the product of the density function with the payoff function, where \bar{m} is the total number of maturities, and \bar{k} is the total number of strikes. In this test, the elapsed time for the FD method is, on average, 140 msec.

For the BP method, since the expression for $\zeta(\tau)$ in Eq. (42) is not known in closed form, we compute this integral numerically by using the Simpson quadratures. Nevertheless, to make the results accurate, we need to increase the number of the grid nodes M . As compared with Itkin and Muravey (2020), where a similar expression for the Hull-White model could be computed in closed form and $M = 20$ provided a sufficient accuracy, here we need to take $M = 100$. Nevertheless, the elapsed time, on average, is 70 msec, i.e., twice as fast as the FD method for the forward equation.

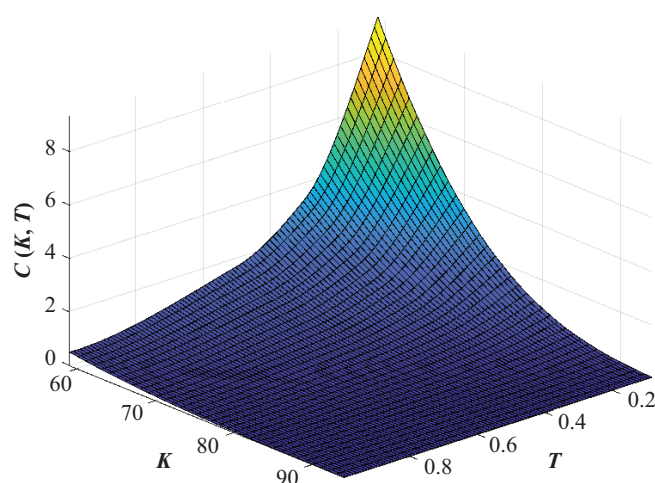
EXHIBIT 4

Up-and-Out Barrier Call Option Prices Computed by Using the BP and FD Methods

K/T	BP				FD				Difference %			
	0.0833	0.3	0.5	1.0	0.0833	0.3	0.5	1.0	0.0833	0.3	0.5	1.0
59	9.3192	3.3642	1.6845	0.4976	9.2924	3.3554	1.6884	0.5175	0.2876	0.2604	-0.2321	-3.9899
64	6.2167	2.1795	1.0671	0.3038	6.2025	2.1831	1.0793	0.3252	0.2286	-0.1654	-1.1438	-7.0291
69	3.8402	1.3219	0.6339	0.1731	3.8341	1.3319	0.6494	0.1931	0.1597	-0.7624	-2.4444	-11.5443
74	2.1608	0.7350	0.3450	0.0891	2.1605	0.7477	0.3606	0.1061	0.0118	-1.7293	-4.5240	-19.1029
79	1.0746	0.3612	0.1652	0.0388	1.0775	0.3736	0.1787	0.0522	-0.2700	-3.4102	-8.1779	-34.6241
84	0.4448	0.1462	0.0641	0.0121	0.4484	0.1561	0.0743	0.0216	-0.7971	-6.7649	-15.9277	-78.1360

EXHIBIT 5

Up-and-Out Barrier Call Option Price Computed by Using the BP Method



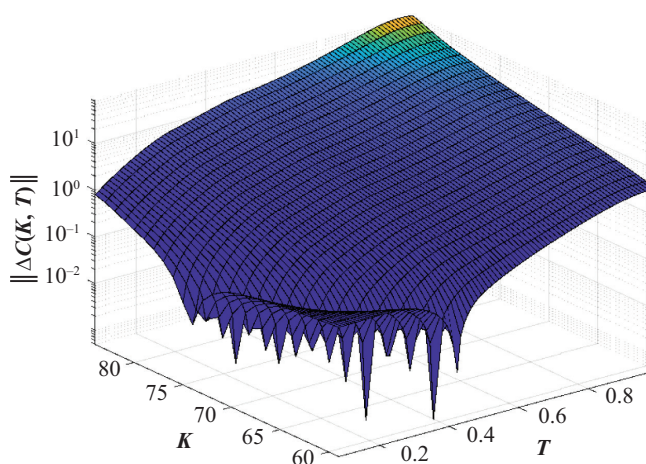
Decreasing M doesn't impact the accuracy of the method very much at large values of the options prices, while there is slight deterioration in the quality of the feed at large maturities and large strikes. Changing M from 100 to 70 drops the elapsed time down to 50 msec. However, for the FD scheme, decreasing the grid to 50×50 drops the elapsed time down to 30 msec, while increasing the error for small maturities by nearly 2x. Overall, we can conclude that the method of BP demonstrates, at least the same performance as the forward FD solver.

Comparison with the Method of General Integral Transform (GIT)

Here we solve the same problem using the GIT method developed previously. Our numerical scheme

EXHIBIT 6

Percentage Difference of Up-and-Out Barrier Call Option Prices Computed by Using the BP and FD Methods



is similar to that for the BP method: first we solve the Volterra equation in Eq. (96) and then compute the value of the integrals in Eq. (94) using the trapezoidal rule. The inner integrals in the first summand in Eq. (96) and Eq. (94) can be computed explicitly for the payoff in Eq. (11) via the following formulas (Gradshtein and Ryzhik 2007)

$$\int_0^1 x^{v+1} J_v(ax) dx = a^{-1} J_{v+1}(a), \quad \Re(v) > -1,$$

$$\int_0^1 x^{1-v} J_v(ax) dx = \frac{a^{v-2}}{2^{v-1} \Gamma(v)} - a^{-1} J_{v-1}(a), \quad \Re(v) < 1,$$

$$\int_0^1 x^{v+1} Y_v(ax) dx = a^{-1} Y_{v+1}(a) + 2^{v+1} a^{-v-2} \Gamma(v+1), \quad \Re(v) > -1,$$

$$\int_0^1 x^{1-v} Y_v(ax) dx = \frac{a^{v-2} \cot(v\pi)}{2^{v-1} \Gamma(v)} - a^{-1} Y_{v-1}(a), \quad \Re(v) < 1.$$

EXHIBIT 7

Up-and-Out Barrier Call Option Prices Computed by Using the GIT and FD Methods

K/T	GIT				FD				Difference %			
	0.0833	0.3	0.5	1.0	0.0833	0.3	0.5	1.0	0.0833	0.3	0.5	1.0
59	9.3617	3.3532	1.6865	0.5158	9.2924	3.3554	1.6884	0.5175	0.71108	-0.2191	-0.3634	-0.7888
64	6.2737	2.1817	1.0781	0.3241	6.2025	2.1831	1.0793	0.3252	1.1101	-0.2149	-0.3654	-0.7997
69	3.90483	1.3315	0.6488	0.1924	3.8341	1.3319	0.6494	0.1931	1.7729	-0.20803	-0.3556	-0.8132
74	2.2194	0.7475	0.3604	0.1057	2.1605	0.7477	0.3606	0.1061	2.6313	-0.1879	-0.3318	-0.8992
79	1.1171	0.3736	0.1786	0.0519	1.0775	0.3736	0.1787	0.0522	3.5572	-0.1327	-0.3129	-1.0134
84	0.4692	0.1563	0.0742	0.0214	0.4484	0.1561	0.0743	0.0216	4.4718	0.0044	-0.2848	-1.1541

However, for the second summands, we have to numerically compute two-dimensional integrals containing special functions, see Eq. (94) and the definition of W , V in Eq. (89).

We run the same test described above, and the results of this numerical experiment are presented in Exhibit 7 and also in Exhibit 8, which depicts the percentage difference between the prices obtained by using the GIT and FD methods. For this test, we use $M = 10$ steps in time. This algorithm was implemented in Python.

It can be seen that this method produces very accurate results at high strikes and maturities (i.e., where the option price is relatively small) in contrast to the BP method. This can be verified by looking at the exponents in Eq. (94), which are proportional to the time τ . In contrast, when the price is higher (shorter maturities, low strikes), the GIT method is slightly less accurate than the BP method, as in Eq. (45), the exponent is inversely proportional to τ . Obviously, the accuracy of the GIT method increases when M increases.

This situation is well investigated for the heat equation with constant coefficients. As applied to pricing double barrier options, it is described in Lipton (2002). There exist two representations of the solution: one is obtained by using the method of images, and the other one is obtained by the Fourier series. Despite the fact that both solutions are equal as the infinite series, their convergence properties are different. In particular, the Fourier method is superior when the difference between the upper H and lower L barriers is small and the time is relatively long. The image expansion should be used otherwise.

In this article, we come to a similar principle for the time-dependent problems, not only for the heat

equation, but also for the Bessel equation. Thus, it is important that both the BP and GIT methods don't duplicate, but rather complement each other.

The speed of our Python implementation is a bit slower than that for BP (approx. 0.158 sec). However, the latter method was implemented in Matlab. It is known that linear algebra in Python (numpy) is almost three times slower than that in Matlab. Therefore, the performance of both the BP and GIT methods is roughly same.

In addition, one can find that the main computational time is spent on many calls to the routine computing the values of the integrands. The integrand in Eq. (94) is a four-dimensional function of $z, p, \gamma(\tau)$, and $\gamma(s)$. Let us denote this functions as $Z(z, p, v, w)$, where $v = \gamma(\tau)$ and $w = \gamma(s)$. We can gain speed by the following trick: first, pre-compute the values of Z on a regular four-dimensional grid, and then design this computational routine as interpolation.

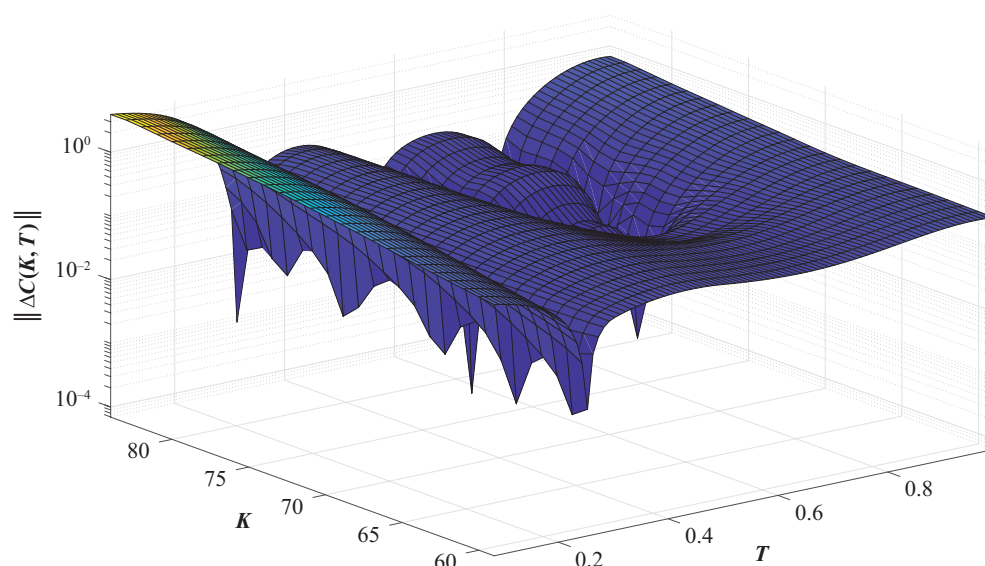
It is also worth mentioning that in many situations, the parameters of the model are such that the boundary $\gamma(\tau)$ changes slowly with time, i.e., $\gamma(\tau)$ is almost constant. Then the first integral in Eq. (94) is a good approximation of the price. Accordingly, we don't need to solve the Volterra equation Eq. (96), which makes the algorithm about 2.5 times faster.

CONNECTION TO PHYSICS

We have already mentioned that both the method of heat potentials (which, in our case, is extended to the method of Bessel potentials) and the method of generalized integral transforms were first developed in physics (see Kartashov 1999, 2001; Tikhonov and Samarskii 1963; Friedman 1964 and references therein). This was done to solve various problems of heat and mass transfer

EXHIBIT 8

Percentage Difference of Up-and-Out Barrier Call Option Prices Computed by Using the FD and GIT Methods



that are widely present in physics, chemistry, energetic, nuclear engineering, geology, and many other areas of science and engineering. It turns out that many of those problems can be formulated mathematically in terms of stationary and non-stationary heat transfer. This includes such problems as diffusion, sedimentation, viscosity flows accompanied by various kinetic processes, astronomy, atomic physics, absorption, combustion, phase transitions, and many others.

As an example, in this section we consider the Stefan problem, which is a particular kind of boundary value problem for the heat equation adapted to the case in which a phase boundary can move with time. It was introduced by I. Stefan in 1889 (see a detailed review in Lyubov 1978). The classical Stefan problem describes the temperature distribution in a homogeneous medium undergoing a phase change, ice becoming water, for example. This is accomplished by solving the heat equation, imposing the initial temperature distribution on the whole medium, and a particular boundary condition, the Stefan condition, on the evolving boundary between its two phases. Note that this evolving boundary is an unknown hyper-surface; hence, Stefan problems are examples of free boundary problems. However, a temperature gradient at this boundary is supposed to be known.

As such, treating it in financial terms, one can immediately recognize this as the pricing problem for an American option where the exercise boundary is also a free boundary, i.e., it is not known. However, the option delta $\frac{\partial u}{\partial z}$ at the boundary $z = \gamma(\tau)$ is known; it follows from the conditions $\frac{\partial C_A}{\partial S}|_{S=S_B(t)} = 1$ and $\frac{\partial P_A}{\partial S}|_{S=S_B(t)} = -1$. Also, the boundary conditions for the American call and put at the exercise boundary (the moving boundary) are set as $C_A(S_B(t), t) = S_B(t) - K$ for the call, and $P_A(S_B(t), t) = K - S_B(t)$ for the put. This problem was solved in Carr and Itkin (2020) by using the method of generalized integral transform for the time-dependent OU model, and in Lipton and Kaushansky (2020b) for the Black-Scholes model with constant coefficients by using the method of heat potentials.

However, we want to emphasize that many of the problems considered in this article have not been solved in physics, and are mentioned in Kartashov (2001) as yet-unsolved problems. Therefore, the results obtained in this article also make a contribution to physics, as they can be easily reformulated in terms of the above-mentioned physics problems.

Another connection of our results to physics concerns the first passage time (FPT) problem considered in the middle of this article. As mentioned in Ding and Rangarajan (2004) and references therein, the FPT

problem finds applications in many areas of science and engineering. A sampling of these applications includes, but is not limited to:

- Statistical physics (study of anomalous diffusion)
- Neuroscience (analysis of neuron firing models)
- Civil and mechanical engineering (analysis of structural failure)
- Chemical physics (study of noise assisted potential barrier crossings)
- Hydrology (optimal design of dams)
- Imaging (study of image blurring due to hand jitter)

In particular, in Redner (2001), the author analyses the fundamental connection between the first-passage properties of diffusion and electrostatics. Basic questions about the first passage include a) where is a diffusing particle absorbed on a boundary and b) when does this absorption event occur? These are time-integrated attributes, obtained by integration of a time-dependent observable over all time. For example, to determine when a particle is absorbed, we should compute the first-passage probability to the boundary and then integrate over all time to obtain the eventual hitting probability. However, it is more elegant to reverse the order of calculation and first integrate the equation of motion over time and then compute the outgoing flux at the boundary. This first step transforms the diffusion equation to the simpler Laplace equation. Then, in computing the flux, the exit probability is simply the electric field at the boundary point. Thus, there is a complete correspondence between a first-passage problem and an electrostatic problem in the same geometry. This mapping is simple yet powerful, and can be adapted to compute related time-integrated properties, such as the splitting probabilities and the moments of the exit time.

Other connections to physics problems such as kinetics of spin systems, first passage in composite and fluctuating systems, hydrodynamic transport, reaction-rate theory, and so forth can also be found in Redner (2001).

With the hope that we have managed to convince the reader about a strong connection between the subject of this article and physics, we will end this excursion into the wonderful world of physics here, leaving curious readers to extend it themselves.

DISCUSSION

In this paper, we have constructed semi-closed form solutions for the price of an up-and-out barrier call option C_{uao} . Despite the fact that the same could be done for down-and-out options, alternatively we can use the parity for barrier options (Hull 1997). Then the price of the down-and-out barrier call option C_{dao} can be found as $C_{dao} = C_v - C_{uao}$, where C_v is the price of the European vanilla call option. For the models considered in this article, the latter is known in closed form (Andersen and Piterbarg 2010). The double barrier case was also considered.

In our test, although we assumed the barrier H to be constant in time, the whole framework is developed for the general case where the barrier is some arbitrary function of time.

From the computational point of view, the proposed solution is very efficient, as we have shown. Using theoretical analysis justified by a test example, we conclude that our method is at least of the same complexity, and can be even faster than the forward FD method. On the other hand, our approach provides high accuracy in computing the options prices, as this is regulated by the order of a quadrature rule used to discretize the kernel. Therefore, the accuracy of the method in z space can be easily increased by using high order quadratures. However, doing the same for the FD method is not easy, i.e., it significantly increases the complexity of the method, see Itkin (2017), for example.

Another advantage of the approach advocated in this article is the computation of option Greeks. Since the option prices in both the BP and GIT methods are represented in closed form via integrals, the explicit dependence of prices on the model parameters is available and transparent. Therefore, explicit representations of the option Greeks can be obtained by a simple differentiation under the integrals. This means that the values of the Greeks can be calculated simultaneously with the prices, with almost no increase in time. This is because differentiation under the integrals changes the integrands slightly, and these changes could be represented as changes in weights of the quadrature scheme used to compute the integrals numerically. Since the majority of the computational time has to be spent on the computation of densities that contain special functions, they can be saved during the calculation of the prices, and then reused for computation of the Greeks.

Note that the FD method also provides the values of delta, gamma, and theta on the FD grid, while, for instance, for vega one needs to bump the model volatility and rerun the whole scheme. However, for the BP and GIT methods, the computation of delta or vega are done uniformly. Also, the ability to complete a fast computation of the Greeks is important for model calibration. Therefore, one can efficiently calibrate the CIR and CEV models to market data by using the BP and GIT methods, since the semi-explicit nature of the final expressions allows quasi-analytical formulae for the gradient of the loss function.

APPENDIX

GENERAL CONSTRUCTION OF THE POTENTIAL METHOD

In this section, we generalize the construction of the potential method originally proposed for the heat equation. For convenience, we follow the notation of Tikhonov and Samarskii (1963).

Consider a PDE

$$\frac{\partial V(t, x)}{\partial t} = \mathcal{L}(V(t, x)) \quad (\text{A1})$$

where the operator \mathcal{L} is a linear differential operator with time-independent coefficients. An example of such an equation is the heat equation and the Bessel equation in Eq. (5). Suppose that the fundamental solution (or the Green's function) of Eq. (A1) $G(x, t | \xi, \tau)$ is known in closed form.

Suppose we need to solve Eq. (A1) subject to the homogeneous initial condition

$$V(0, x) = 0. \quad (\text{A2})$$

Note that this doesn't restrict our consideration because through the change of variables in Eq. (54), the problem with inhomogeneous initial condition can be transformed to the problem with $V(0, x) = 0$.

We are interested in solving the first boundary-value problem, assuming that the boundaries are some functions of time. Consider, for simplicity just a semi-bounded region with $x \geq y(t)$, $x < \infty$, $t > 0$, with the following boundary conditions

$$V(t, x) \Big|_{x \rightarrow \infty} = 0, \quad V(t, y(t)) = \phi(t). \quad (\text{A3})$$

The assumption that the problem has just one time-dependent boundary can easily be relaxed, as this is demonstrated in the article, and is not a restriction of the method.

Let us introduce the single layer potential, (Tikhonov and Samarskii 1963)

$$\Pi(x, t) = \int_0^t \Psi(\tau) \frac{\partial G}{\partial \xi}(x, t | \xi, \tau) \Big|_{\xi=y(\tau)} d\tau, \quad (\text{A4})$$

where $\Psi(t)$ the potential density. The single layer potential is a continuous and twice differentiable function in x , and continuous and differentiable in t . Then the below proposition follows

Proposition A1. *The potential function in Eq. (A4) solves Eq. (A1). The potential density is determined by the boundary condition in Eq. (A3) and solves the Volterra equation of second kind*

$$\phi(t) = b\Psi(t) + \int_0^t \Psi(\tau) \frac{\partial G}{\partial \xi}(y(t), t | y(\tau), \tau) \Big|_{\xi=y(\tau)} d\tau, \quad (\text{A5})$$

where b is a constant.

Proof. First, it can be checked that substituting the definition in Eq. (A4) into Eq. (A1) we get

$$\begin{aligned} \frac{\partial \Pi(x, t)}{\partial t} &= \int_0^t \Psi(\tau) \frac{\partial}{\partial \xi} \frac{\partial G}{\partial t} \Big|_{\xi=y(\tau)} d\tau + \Psi(t) \frac{\partial G}{\partial y(\tau)}(x, y(t) | t, t) \\ &= \int_0^t \Psi(\tau) \frac{\partial}{\partial \xi} \frac{\partial G}{\partial t} \Big|_{\xi=y(\tau)} d\tau, \\ \mathcal{L}(\Pi(t, x)) &= \int_0^t \Psi(\tau) \frac{\partial}{\partial \xi} \mathcal{L}(G) \Big|_{\xi=y(\tau)} d\tau. \end{aligned} \quad (\text{A6})$$

Combining these two expressions yields

$$\int_0^t \Psi(\tau) \frac{\partial}{\partial \xi} \left[\frac{\partial G}{\partial t} - \mathcal{L}(G) \right] \Big|_{\xi=y(\tau)} d\tau = 0. \quad (\text{A7})$$

The first line follows from the fact that Green's function solves Eq. (A1), and $G(x, y(\tau) | t, \tau) \Big|_{\tau=t} = G(x, y(t) | t - \tau) \Big|_{\tau=t} = \delta(x - y(t)) = 0$ as $x \neq y(t)$. The second line is a consequence of time-independence of the operator coefficients.

To summarize what we obtained: the potential function satisfies the PDE for $x \geq y(t)$, is bounded at infinity, and has a zero initial value for any choice of $\Psi(t)$. Thus, it is the solution of Eq. (A1), i.e.,

$$V(t, x) = \int_0^t \Psi(\tau) \frac{\partial G}{\partial \xi}(x, t | \xi, \tau) \Big|_{\xi=y(\tau)} d\tau. \quad (\text{A8})$$

Now using the boundary condition at $x = y(t)$, we obtain from Eq. (A8)

$$\phi(t) = \int_0^t \Psi(\tau) \frac{\partial G}{\partial \xi}(\gamma(t), t | \xi, \tau) \Big|_{\xi=\gamma(\tau)} d\tau. \quad (A9)$$

However, as shown in Tikhonov and Samarskii (1963), for $x = \gamma(t)$, the RHS is discontinuous, but with the finite limiting value at $x = \gamma(t) + 0$. The limiting value could be represented as

$$b\Psi(t) + \int_0^t \Psi(\tau) \frac{\partial G}{\partial \xi}(\gamma(t), t | \gamma(\tau), \tau) \Big|_{\xi=\gamma(\tau)} d\tau.$$

The constant b depends on the particular form of the operator \mathcal{L} . In particular, if \mathcal{L} is a second order parabolic operator with the diffusion coefficient a , then $b = 1/(2a)$, (Tikhonov and Samarskii 1963). Thus, for instance, for $a = 1/2$, we have $b = 1$.

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ADDITIONAL READING

Physics and Derivatives: On Three Important Problems in Mathematical Finance

ALEXANDER LIPTON AND VADIM KAUSHANSKY

The Journal of Derivatives

<https://jod.pm-research.com/content/early/2020/02/21/jod.2020.1.098>

ABSTRACT: In this article, we use recently developed extension of the classical heat potential method in order to solve three important but seemingly unrelated problems of financial engineering: (A) American put pricing; (B) default boundary determination for the structural default problem; and (C) evaluation of the hitting time probability distribution for the general time-dependent Ornstein–Uhlenbeck process. We show that all three problems boil down to analyzing behavior of a standard Wiener process in a semi-infinite domain with a quasi-square-root boundary.

Quantum Option Pricing and Quantum Finance

SERGIO FOCARDI, FRANK J. FABOZZI,

AND DAVIDE MAZZA

The Journal of Derivatives

<https://jod.pm-research.com/content/early/2020/05/28/jod.2020.1.111>

ABSTRACT: In this article, the authors discuss the use of quantum probability, that is, the probability theory of quantum mechanics, for option pricing and for finance in general. The authors discuss the motivations for applying quantum probability to finance. The critical issues are replacing random variables with operators, self-reflexivity of markets, and the existence of incompatible observations. The authors outline quantum probability theory, quantum stochastic processes, and the pricing of options in a quantum context.