A FUNCTIONAL ANALYSIS APPROACH TO STATIC REPLICATION OF EUROPEAN OPTIONS

SÉBASTIEN BOSSU*, PETER CARR†, AND ANDREW PAPANICOLAOU†

Abstract. The replication of any European contingent claim by a static portfolio of calls and puts with strikes forming a continuum, formally proven by Carr and Madan (1998), is part of the more general theory of integral equations. We apply spectral decomposition techniques to show that replication may also be achieved with a discrete portfolio of special options. We propose a numerical application for fast pricing of vanilla options that may be suitable for large option books or high frequency option trading, and we use a reflected Brownian motion model to show how pricing formulas for the special options may be obtained.

1. Introduction

We consider the general problem of replicating a target European option[1] with a static portfolio of cash, underlying asset and a selection of "replicant" European options. Replication problems arise in many areas of finance, such as asset pricing theory where an asset is replicated with a finite number of other assets (e.g., Černý, 2016, ch. 1, 2) using the techniques of finite-dimensional linear algebra, or option pricing theory, where Carr and Madan (1998) formally proved that any European option may be replicated with a portfolio of cash, forward contracts, and European call and/or put options with a continuum of strike prices. A key consequence of payoff replication is that if the prices of the replicant options are known, then the price of the target European option is also known and enforced by no-arbitrage considerations.

Specifically, given a target European option's payoff $F(x)$ to be replicated, where $x \in \mathcal{X} \subseteq \mathbb{R}_+$ is the terminal price of the option's underlying asset, and a family of replicant European options' payoffs $G(x, y)$ indexed by $y \in \mathcal{Y} \subseteq \mathbb{R}$, we are looking for portfolio quantities or weights such that, for all $x \in \mathcal{X}$,

$$F(x) = c + q\,x + \int_{y\in\mathcal{Y}} G(x, y)\phi(y)\mathrm{d}\mu(y), \tag{1}$$

where c, q and $\phi(y)$ are the respective quantities of cash, underlying asset and replicant option with index y, and μ is an appropriate measure. In particular, if $\mathcal{Y}$ is discrete and μ is the counting measure, the above equation becomes $F(x) = c + q\,x + \sum_{y\in\mathcal{Y}} G(x, y)\phi(y)$ or, with the more habitual subscript notations for discrete sums,

$$F(x) = c + q\,x + \sum_{n\in\mathcal{Y}} \phi_n G_n(x).$$

Date: This version: June 29, 2019.

* NYU Courant. The first author thanks Stéphane Crépey for useful comments and suggestions. All remaining errors are his.

† NYU Tandon.

[1]We use the term "option" to designate any derivative contract, also known as "contingent claim", on a single underlying asset

We will especially focus on the case where both variables x, y belong to a continuous interval such as $[a, b]$ or (a, b) where $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ may be infinite, and μ is the Lebesgue measure, so that we may write equation (1) as

$$f(x) = \int_a^b G(x, y)\phi(y)\mathrm{d}y, \tag{2}$$

where $f(x) \coloneqq F(x) - c - q\,x$ is the target payoff function $F(x)$ up to linear terms. Observe how the second and higher derivatives of f and F coincide.

The origin of the Carr and Madan (1998) replication formula may be traced back to the work of Breeden and Litzenberger (1978) which showed that the terminal distribution of the underlying asset implicit in option prices, also known as the *implied distribution*, could be recovered by differentiating call prices twice against the strike price. This elegant theoretical result allowed to price any other European option payoff consistently with existing vanilla options. However, it was not until the 1990s that practitioners and researchers became particularly interested in replication and hedging strategies for non-vanilla option payoffs, on the back of the expansion of option markets and the search for option contract innovation. Evidence of such interest can be found in the works of Derman, Ergener, and Kani (1994), discussing static replication of especially barrier options, and Dupire (1993).

Much research (e.g., Demeterfi, Derman, Kamal, and Zou, 1999) has been devoted to the static replication of the log-contract first introduced by Neuberger (1990), leading to the development of volatility and variance swap markets. In this context, Carr and Madan (1998) offered a general replication result that did not solely apply to the log-contract and was also probability- and model-free. To this day, option practitioners refer to the idea that any European option payoff can be replicated with a continuous portfolio of vanilla calls and puts as the "Carr-Madan result". Its most visible impact may be seen in the new calculation methodology of the VIX (see *The CBOE volatility index—VIX* 2009), which was adopted in 2002 by the Chicago Board Options Exchange.

In other related literature, Carr and Wu (2013) consider the static hedging of a longer-dated vanilla option using a continuum of shorter-term options. Balder and Mahayni (2006) expand on this work and explore various discretization strategies when the strikes are pre-specified and the underlying price dynamics are unknown, and recently Wu and Zhu (2017) propose a model-free strategy of statically hedging options with nearby options in strike and maturity dimensions. Madan and Milne (1994) price options under Gaussian measure using Hermite polynomials as a basis. Carton de Wiart and Dempster (2011) use wavelet theory for partial differential equations used in derivatives pricing. Papanicolaou (2018) expresses a consistency condition between SPX Stochastic Volatility and VIX Market Models as an integral equation and solves it using an eigenseries decomposition.

Our main contribution is to show that perfect replication can be achieved with a discrete portfolio of special options forming an orthogonal eigensystem, rather than a continuous portfolio of vanilla options with overlapping payoffs. In practice, a satisfactory approximation may be achieved with a smaller number of these special options compared to integral discretization schemes.

The remainder of our paper is organized as follows: In Section 2, we show that the Carr-Madan result is part of the general theory of integral equations. In Section 3, we present key results of the theory about the existence and uniqueness of solutions, with particular focus on spectral decomposition within Hilbert spaces. In Section 4, we proceed with the spectral decomposition of the "straddle kernel", and we interpret our results in terms of option replication in Section 5. In Section 6, we propose a numerical application for fast pricing of vanilla options. In Section 7, we propose a theoretical application to derive pricing formulas when the underlying asset follows a reflected Brownian motion. Finally, in Section 8 we consider the case of the "straddle kernel" and derive equations for its eigensystem that may be solved numerically.

2. Carr-Madan as part of the theory of integral equations

In functional analysis, equation (2) is known as a *Fredholm linear integral equation of the first kind*, and $G(x, y)$ is called the *integral kernel* or, with slight abuse of terminology, the *integral operator*. A shorthand notation for the equation is often $f = \langle G, \phi\rangle$ or simply $f = G\phi$. When $f(x)$ is identically zero the equation is called *homogeneous*; otherwise it is called *inhomogeneous*. Many authors further categorize an integral equation as *singular* when it has a convergent improper integral, as in equation (2) when either bound a, b is infinite.

Many integral kernels that are relevant to finance vanish for $y \geq x$ or $y \leq x$, in which case equation (2) respectively simplifies to

$$f(x) = \int_a^x G(x, y)\phi(y)\mathrm{d}y, \qquad \text{or} \qquad f(x) = \int_x^b G(x, y)\phi(y)\mathrm{d}y.$$

These equations are known as a *Volterra integral equations of the first kind* and they have special properties and methods (e.g., Polyanin and Manzhirov, 2008, ch. 10, 11).

We will see in Section 3 that solving equation (2) is considerably easier when the integral kernel $G(x, y)$ is *symmetric* and *injective*, as defined later. Table 1 on p.4 lists several examples of kernels that are relevant to quantitative finance and indicates whether they are symmetric and/or injective. Note that, to a degree, log contracts and options trade on derivatives markets as options, futures and swaps on VIX and realized variance. Note also that, thanks to the development of electronic option markets, many option strategies combining vanilla options, such as straddles or butterfly spreads, quote and trade directly on dedicated platforms usually known as *complex order books*.

2.1. **Carr-Madan kernel.**

The kernel $G(x, y) \coloneqq (x - y)^+$ corresponds to the payoff replication problem with call options of various strike prices $y \in \mathcal{Y}$. When all strike prices form the continuum $\mathcal{Y} = \mathbb{R}_+$, the solution to equation (2) is then $\phi(y) = f''(y)$ as shown by Carr and Madan (1998) using standard calculus techniques. In fact, this solution can be viewed as a corollary of Taylor's theorem with remainder in integral form,

$$F(x) = F(0) + F'(0)x + \int_0^x (x - t)F''(t)\mathrm{d}t.$$

Substituting $(x - t)^+$ which is identically zero for $t > x$ yields the Carr-Madan formula at origin:

$$F(x) = F(0) + F'(0)x + \int_0^\infty (x - t)^+ F''(t)\mathrm{d}t.$$

The general Carr-Madan formula involves both call and put options whose strike prices are respectively above or below an arbitrary split level $x_0 \geq 0$:

$$F(x) = F(x_0) + F'(x_0)(x - x_0) + \int_0^{x_0} F''(y)(y - x)^+\mathrm{d}y + \int_{x_0}^\infty F''(y)(x - y)^+\mathrm{d}y. \tag{3}$$

Observe how the second term $F'(x_0)(x - x_0)$ corresponds to a long or short position in forward contracts with delivery price x_0. A convenient choice for practical applications is to set x_0 to the underlying asset's current forward price (respectively its current spot price), in which case all call and put options are out-of-the-money-forward (respectively out-of-the-money-spot).

Table 1. Examples of integral kernels†

European option payoff kernels			
Kernel	$G(x,y)$	Symmetric	Injective
Forward contracts	$x-y$	No	No
Calls and puts	$(x-y)^+, (y-x)^+$	No	Yes
Straddles	$\|x-y\|$	Yes	Yes
Powers of the above	$G(x,y)^c$		
Strangles	$(\|x-y\|-c)^+$	Yes	Yes ($\frac{1}{c} \in 2\mathbb{N}$)
Butterfly spreads	$(c-\|x-y\|)^+$	Yes	Yes
Binary options	$H(x-y), H(y-x)$	No	Yes
Risk reversals	$(x-y-c)^+ - (y-x-c)^+$	No	Yes ($\frac{1}{c} \in 2\mathbb{N}$)
Log contracts	$\ln x/y$	No	No
Log calls and puts	$(\ln x/y)^+, (\ln y/x)^+$	No	Yes
Mathematical kernels			
Kernel	$G(x,y)$	Symmetric	Injective
Power	x^y	No	Yes
Gaussian	$\frac{1}{\sqrt{2\pi}} e^{-(x-y)^2/2}$	Yes	Yes
Laplace transform	e^{-xy}	Yes	Yes
Fourier transform	$e^{-2i\pi xy}$	Yes	Yes

† $c > 0$ is a constant parameter, $H(\cdot)$ is Heaviside's step function, and i is the imaginary number.

The Carr-Madan formula (3) may be viewed as the solution $\phi(y) = f''(y)$ to the integral equation (2) with target function $f(x) \coloneqq F(x) - F(x_0) - F'(x_0)(x - x_0)$ and *Carr-Madan kernel*

$$G(x, y; x_0) \coloneqq (x-y)^+ H(y - x_0) + (y-x)^+ H(x_0 - y), \tag{4}$$

where $H(\cdot)$ is Heaviside's step function. An alternative proof to Taylor's theorem is to carefully differentiate both sides of equation (2) twice, either with the help of Dirac's delta functions or by invoking Leibniz's integral rule.

2.2. **Alternative expression.** It is worth noting that the Carr-Madan kernel (4) may be rewritten as

$$G(x, y; x_0) = (x-y)\left[H(x-y) - H(x_0 - y)\right],$$

by substituting $H(y - x_0) = 1 - H(x_0 - y)$ and then $(x-y)^+ - (y-x)^+ = x - y$ into equation (4). Substituting the above into (2), we obtain the Volterra equation of the first kind,

$$f(x) = \int_{x_0}^{x} (x-y)\phi(y)\mathrm{d}y,$$

which is forward for $x > x_0$ and backward for $x < x_0$.

2.3. **Limitations.** The Carr-Madan formula has two major limitations:

(1) In practice, only a finite number of vanilla option strikes are available and the formula must be discretized accordingly. Hedging is imperfect and approximation errors get in the way.

(2) In the theory of integral equations, the Carr-Madan kernel $G(x, y; x_0)$ (equation 4) is not symmetric and therefore it does not have an orthonormal decomposition.

In this paper we address the above limitations by substituting the "better" *straddle kernel* $G(x, y) := |x - y|$ which is symmetric and therefore admits an orthonormal decomposition. This kernel remains tractable in terms of practical applications as it corresponds to the family of all straddles with a continuum of strikes $y \in \mathbb{R}_+$. Moreover, the following identity shows that the straddle kernel has a one-to-one correspondence with the Carr-Madan kernel:

$$G(x, y; x_0) = \frac{|x-y|}{2} + \frac{x-y}{2}\left[H(y-x_0) - H(x_0-y)\right].$$

This identity is straightforwardly established by substituting $(\pm u)^+ = \frac{|u| \pm u}{2}$ into equation (4).

3. Existence and uniqueness of solutions

3.1. **Solving first-kind Fredholm equations.** Early theory for integral equations was developed by Volterra (1896), Fredholm (1903), Hilbert (1905), Schmidt (1907), and Riesz (1916). It turns out that first-kind Fredholm equations are very much related to second-kind equations,

$$f(x) = \lambda\phi(x) - \int_a^b G(x, y)\phi(y)\mathrm{d}y,$$

where λ is a nonzero complex parameter[2].

Much of the literature about integral equations is dedicated to the theoretical and numerical resolution of second-kind equations with a continuous kernel operating on continuous or square-integrable functions. Famously, the *Fredholm alternative* states that, for any $\lambda \neq 0$, either the homogeneous Fredholm integral equation of the second kind,

$$0 = \lambda\phi(x) - \int_a^b G(x, y)\phi(y)\mathrm{d}y,$$

has a nontrivial solution $\phi \not\equiv 0$ and λ is called an eigenvalue, or the inhomogeneous equation,

$$f(x) = \lambda\phi(x) - \int_a^b G(x, y)\phi(y)\mathrm{d}y,$$

always has a unique solution for any $f(x)$ and λ is called a regular value. Note that when λ is an eigenvalue, the second-kind inhomogeneous equation has either no solution or infinitely many solutions.

First-kind equations can be significantly more challenging to solve. It is worth emphasizing that there may be no solution at all, and that the theory about the existence and uniqueness of solutions is very limited compared to the Fredholm alternative available for second-kind equations. Fundamentally, the difficulty for finding a solution results from the smoothing property of integration. To illustrate this point, consider a well-behaved continuous kernel $G(x, y)$ and an input function $\phi(y)$ that is only piecewise continuous. The resulting output $\int_a^b G(x, y)\phi(y)\mathrm{d}y$ will be smoother than $\phi(y)$. Therefore, if $f(x)$ is a continuous target function, it is very possible that solutions $\phi(y)$ are all discontinuous, and that no solution exists within the class of continuous functions (e.g., Section 8 and Figure 4 p.18).

This observation is relevant to our payoff replication problem wherein a continuous solution is neither required nor expected; in fact, we will be mostly interested in square-integrable solutions.

[2] Observe that when $\lambda = 0$ we have a first-kind equation.

3.2. **Formal framework.** Let E denote the infinite-dimensional vector space of the payoff functions under consideration, such as $C([a,b])$ or $L^2([a,b])$. Define the linear operator:

$$\begin{array}{rccc} \mathcal{G}: & E & \to & E \\ & \phi & \mapsto & \mathcal{G}\phi : x \mapsto \int_a^b G(x,y)\phi(y)\mathrm{d}y. \end{array}$$

With these notations, the first-kind linear integral equation (2) may be written as $\mathcal{G}\phi = f$. The existence of solutions for all $f \in E$ then corresponds to $\mathcal{G}$ being a surjective operator, i.e., $\mathcal{G}(E) = E$, while the uniqueness of any solution corresponds to $\mathcal{G}$ being an injective operator, i.e. $\mathcal{G}^{-1}(0_E) = \{0_E\}$ where 0_E is the null function of E.

A standard theoretical requirement is for $\mathcal{G}$ to be a *compact operator* (see Kress, 2014, pp. 25—6, for a formal definition). It turns out that compact operators are never surjective (Kress, 2014, pp. 297—8), and thus there always are infinitely many target functions $f \in E$ for which the first-kind equation has no solution at all. In contrast, the identity operator $I : E \to E, \phi \mapsto \phi$ is trivially surjective and thus never compact (Kress, 2014, p. 27), and it can be shown that the second-kind operator $\lambda I - \mathcal{G}, \lambda \neq 0$ is surjective if and only if it is injective (Kress, 2014, p. 38). Within this framework, the Fredholm alternative translates into a discussion whether $\lambda I - \mathcal{G}$ is injective.

On the topic of eigenvalues, it is worth noting that three classic important properties from finite-dimensional linear algebra extend to infinite-dimensional Hilbert spaces E:

(1) For a large class of integral operators, the series of eigenvalues $\sum \lambda_n$ converges to the operators's trace $\int_a^b G(x,x)\mathrm{d}x$ (Lax, 2002, p. 329).
(2) *Perron-Frobenius theorem*: if the integral operator $\mathcal{G}$ is positive[3], it has a positive eigenvalue which is the largest in absolute value among all eigenvalues, and its eigenfunction is positive (Lax, 2002, p. 253).
(3) *Mercer's condition*: if the integral operator $\mathcal{G}$ is symmetric and satisfies $\int_a^b \int_a^b \phi(x)G(x,y)\phi(y)\mathrm{d}x\mathrm{d}y \geq 0$ then it is a positive-semidefinite operator and all its eigenvalues are nonnegative (Lax, 2002, p. 343).

3.3. **Spectral decomposition of continuous symmetric kernels.** When the vector space of payoff functions is the Hilbert space of square-integrable functions on a finite segment $E = L^2([a,b])$, the linear map $\mathcal{G}$ corresponding to the square-integrable kernel $G \in L^2([a,b] \times [a,b])$ is called a *Hilbert-Schmidt integral operator*. If the kernel $G(x,y)$ is continuous, the operator $\mathcal{G}$ is always compact and therefore never surjective, i.e. there always are target functions $f \in L^2([a,b])$ for which the first-kind integral equation $\mathcal{G}\phi = f$ has no solution at all.

By Hilbert-Schmidt theory, when the kernel $G(x,y)$ is continuous and symmetric, all eigenvalues of $\mathcal{G}$ are real and form a finite or countable subset of $\mathbb{R}$ and there is an orthonormal system of eigenfunctions (ϕ_n). In practical applications, we can find all nonzero eigenvalues λ_n of $\mathcal{G}$ and their associated eigenfunctions ϕ_n by solving the homogeneous integral equation of the second kind $(\lambda_n I - \mathcal{G})\,\phi_n \equiv 0$, for which numerous methods exist. Moreover, we have the spectral decomposition (Eidelman, Milman, and Tsolomitis, 2004, p. 94),

$$G(x,y) = \sum_n \lambda_n \phi_n(x)\phi_n(y), \tag{5}$$

where the convergence of the series is understood in the sense of $L^2([a,b] \times [a,b])$. As a corollary, $\sum_n \lambda_n^2 = \int_a^b \int_a^b G(x,y)^2 \mathrm{d}x\mathrm{d}y$. Substituting the above spectral decomposition identity (5) into equation (2) we obtain

[3] Here, an operator is positive when the function $\mathcal{G}\phi$ is positive for any nonnegative and nonnull function ϕ.

that, when a solution ϕ exists, the target function f is attained by a linear combination of all eigenfunctions ϕ_n,

$$f(x) = \sum_n \lambda_n \phi_n(x) \int_a^b \phi_n(y)\phi(y)\mathrm{d}y.$$

The financial interpretation of the above equation is that the target option payoff $F(x)$ discussed in Section 1 is perfectly replicated by a combination of cash and underlying asset together with a *discrete* portfolio of independent "spectroreplicant" options, i.e.,

$$F(x) = c + q\,x + \sum_n w_n \phi_n(x), \tag{6}$$

where c, q are the quantities of cash and underlying asset, and $w_n := \lambda_n \int_a^b \phi_n(y)\phi(y)\mathrm{d}y$ is the weight or quantity of the n^{th} spectroreplicant option paying off $\phi_n(x)$.

3.4. **Unique square-integrable solution for continuous, symmetric and injective kernels.** In some cases an explicit solution $\phi(y)$ to a first-kind equation with symmetric kernel may be obtained using non-spectral techniques, such as the convolution method for difference kernels (e.g., Srivastava and Buschman, 2013, ch. 3). However, many equations do not solve in this manner. Fortunately, theory provides for a criterion about the existence of a unique solution when the continuous and symmetric kernel $G(x, y)$ induces an injective integral operator $\mathcal{G}$ on the Hilbert space of square-integrable functions $E = L^2([a, b])$ or $E = L^2((a, b))$.

Indeed, when $\mathcal{G}$ is symmetric and injective the orthonormal eigensystem (ϕ_n) is complete and therefore a basis of E, and all eigenvalues are real. Denoting $f_n := \int_a^b f(x)\phi_n(x)\mathrm{d}x$ the coordinates of any target function $f \in E$ in the basis, it is then easy to see that the function

$$\phi(y) := \sum_n \frac{f_n}{\lambda_n} \phi_n(y)$$

is a well-defined element of E if and only if the series $\sum f_n^2/\lambda_n^2$ converges, in which case it is the unique solution to the first-kind integral equation $f = \mathcal{G}\phi$.

Note that if $\mathcal{G}$ is symmetric but *not injective*, its nullspace is necessarily of finite dimension n_0 and solutions exist if and only if the series $\sum_{n>n_0} f_n^2/\lambda_n^2$ converges. The solution set is then the affine space $\hat{\phi} + \mathcal{G}^{-1}(0_E)$ where $\hat{\phi} := \sum_{n>n_0} f_n\phi_n/\lambda_n$. In the context of payoff replication it is worth emphasizing that the nullspace portfolios $(\phi_n)_{0\leq n\leq n_0}$ replicate the null payoff and thus always have zero cost. As such, they do not change the economics of replicating the target payoff and may be ignored. For ease of exposition we only consider injective kernels.

4. Spectral decomposition of the straddle kernel

The straddle kernel $G(x, y) := |x - y|$ is continuous and symmetric and thus admits a spectral decomposition over any finite segment $[a, b]$. Moreover, there must be at least one negative and one positive eigenvalue since the kernel trace vanishes: $\int_a^b |x - x|\mathrm{d}x = 0$. In fact, since the straddle kernel induces a positive operator, it must have a positive eigenvalue which is the largest among all eigenvalues.

For ease of exposure, and without loss of generality, we first derive the spectral decomposition of the straddle kernel on the unit interval $[a, b] = [0, 1]$ with corresponding integral equation

$$f(x) = \int_0^1 |x-y|\phi(y)\mathrm{d}y, \quad 0 \le x \le 1.$$

The decomposition for an arbitrary interval $[a, b]$ is then straightforwardly obtained through the affine map $x \mapsto a + (b-a)x$ and similarly for y. Note that differentiating the above integral equation twice against x yields the solution $\phi(x) = \frac{1}{2}f''(x)$ which is unique[4]. In particular, the homogeneous equation only has the trivial solution and thus the kernel is injective. Furthermore, we can see that when $f(x) \equiv 0$, i.e. the target payoff function $F(x)$ is purely linear, the integral equation only has the trivial solution.

To find the eigensystem we must solve the homogeneous second-kind equation

$$\lambda\phi(x) = \int_0^1 |x-y|\phi(y)\mathrm{d}y, \tag{7}$$

for $\lambda \neq 0$. Again, differentiating twice against x yields that eigenfunctions must satisfy the homogeneous second-order linear differential equation

$$\lambda\phi''(x) = 2\phi(x), \quad 0 \le x \le 1,$$

whose general solution is of the form

$$\phi(x) = \begin{cases} \alpha e^{2\omega x} + \beta e^{-2\omega x} & \text{if } \lambda > 0, \quad (8a) \\ \alpha \cos 2\omega x + \beta \sin 2\omega x & \text{if } \lambda < 0, \quad (8b) \end{cases}$$

where α, β are real coefficients and $\omega \coloneqq 1/\sqrt{2|\lambda|}$ is the semi-angular frequency associated with λ.

Following the notations of Section 3.3 we index eigenelements by nonnegative integers $n \in \mathbb{N}$ from largest to smallest absolute eigenvalue $|\lambda_n|$. In the next section 4.1 we will see that there is only one positive eigenvalue λ_0 which is the largest among all absolute eigenvalues.

4.1. **Eigenfunction associated with the positive eigenvalue.** Substituting (8a) into equation (7) and simplifying, the straddle integral operator maps an eigenfunction ϕ_0 with positive eigenvalue $\lambda_0 > 0$ to

$$\int_0^1 |x-y|\phi_0(y)\mathrm{d}y = \lambda_0 \Big[\phi_0(x) + \left(\beta - \alpha e^{2\omega}\right)\left(1 + e^{-2\omega}\right)\omega x \\ - \frac{\alpha}{2}e^{2\omega}\left(1 - 2\omega + e^{-2\omega}\right) - \frac{\beta}{2}e^{-2\omega}\left(1 + 2\omega + e^{2\omega}\right)\Big]. \tag{9}$$

For the remainder terms which are linear in x to vanish we must have $\beta = \alpha e^{2\omega}$. After substitution into equation (9) and simplifications, we obtain that ω must be the only fixed point of the hyperbolic cotangent $\omega_0 \approx 1.19968$; equivalently, the only positive eigenvalue of the straddle kernel is:

$$\lambda_0 = \frac{1}{2\omega_0^2} \approx 0.34741.$$

Finally, solving $\int_0^1 \phi_0^2(y)\mathrm{d}y = 1$ for α we obtain the normalized eigenfunction

$$\phi_0(x) = \frac{\sqrt{2}}{\cosh \omega_0} \cosh \omega_0(1-2x) \approx 0.78126 \times \cosh[1.19968 \times (1-2x)], \tag{10}$$

[4]For a formal proof of uniqueness, suppose that $\tilde{\phi}$ is another solution; then $\int_0^1 |x-y| \left(\phi(y) - \tilde{\phi}(y)\right) \mathrm{d}y = 0$ and differentiating twice against x yields $\tilde{\phi} \equiv \phi$.

which is a positive function as expected from the Perron-Frobenius theorem.

4.2. **Eigenfunctions associated with negative eigenvalues.** Substituting (8b) into equation (7) and simplifying through trigonometric identities, the straddle integral operator maps an eigenfunction $\phi_n, n \geq 1$ with negative eigenvalue $\lambda_n < 0$ to

$$\int_0^1 |x-y|\phi_n(y)\mathrm{d}y = \lambda_n \Big[\phi_n(x) + 2\omega\cos\omega\,(\alpha\sin\omega - \beta\cos\omega)\,x + \left(\beta\omega - \frac{\alpha}{2}\right)\cos 2\omega - \left(\alpha\omega + \frac{\beta}{2}\right)\sin 2\omega - \frac{\alpha}{2}\Big]. \quad (11)$$

The remainder terms linear in x vanish when either

(a) $\beta = 0$ and $\omega = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$; or
(b) $\beta = \alpha\tan\omega$, where $\omega \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$ satisfies $\cos\omega + \omega\sin\omega = 0$, i.e. it is an opposite fixed point of the cotangent function.

Solving $\int_0^1 \phi_n^2(y)\mathrm{d}y = 1$ for α and simplifying through trigonometric identities, we obtain the alternating system of normalized eigenfunctions

$$\phi_n(x) = \begin{cases} \sqrt{2}\cos n\pi x & \text{if } n \geq 1 \text{ is odd,} \\ \dfrac{\sqrt{2}}{\cos\omega_n}\cos\omega_n(1-2x) & \text{if } n \geq 2 \text{ is even,} \end{cases} \quad (12)$$

where ω_n is the only opposite fixed point of the cotangent function in the interval $\left(\frac{(n-1)\pi}{2}, \frac{(n+1)\pi}{2}\right)$ when $n \geq 2$ is even. With the convention $\omega_n := \frac{n\pi}{2}$ when $n \geq 1$ is odd, the negative eigenvalues λ_n are indexed from largest to smallest in absolute value:

$$\lambda_n = -\frac{1}{2\omega_n^2}, \quad n \geq 1.$$

Asymptotically, when $n \geq 2$ is even, we have $\omega_n \sim \frac{n\pi}{2}$ as $n \to \infty$. Indeed, inverting the opposite fixed point equation $\cot\omega_n = -\omega_n$ produces $\omega_n = -\operatorname{arccot}\omega_n + \frac{n\pi}{2}$, and the inverse cotangent function is bounded. Therefore, for large n, we have $\omega_n = \frac{n\pi}{2}$ if n is odd and $\omega_n \sim \frac{n\pi}{2}$ if n is even.

4.3. **Remarks about the straddle eigensystem.** Note the following with regard to the eigensystem derived in sections 4.1 and 4.2:

(a) The eigenfunctions $\phi_n, n \geq 1$ take positive and negative values. This may have a numerical benefit when replicating a target payoff $f(x)$ which is small in absolute value.
(b) The eigensystem is consistent with the spectral decomposition of linear and symmetric Toeplitz matrices (Bünger, 2014) which are a discrete version of the straddle kernel.
(c) All eigenfunctions satisfy $\phi_n(0) = \sqrt{2}$ and $\phi_n(1) = (-1)^n\sqrt{2}$.
(d) Since the kernel trace vanishes, we have:

$$\sum_{k=0}^{\infty} \lambda_{2k+2} = \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} - \lambda_0 = \frac{2}{\pi^2}\frac{\pi^2}{8} - \lambda_0 \approx -0.09741.$$

(e) By definition, for each eigenfunction we have $\phi_n'' = \frac{2}{\lambda_n}\phi_n = -4\omega_n^2\phi_n$ for $n \geq 1$ and $\phi_0'' = 4\omega_0^2\phi_0$

4.4. **Spectral decomposition on the unit interval.** Substituting the normalized eigenfunction expressions of equations (10) and (12) into the spectral decomposition equation (5), and then simplifying, the spectral decomposition of the straddle kernel on the unit interval $[0,1]$ is

$$\begin{aligned} |x-y| &= c_0 \cosh \omega_0(1-2x)\cdot \cosh \omega_0(1-2y) \\ &\quad + \sum_{k=0}^{\infty} c_{2k+1} \cos[(2k+1)\pi x]\cdot \cos[(2k+1)\pi y] \\ &\quad + \sum_{k=0}^{\infty} c_{2k+2} \cos \omega_{2k+2}(1-2x)\cdot \cos \omega_{2k+2}(1-2x), \end{aligned} \tag{13}$$

where c_n are the scaling coefficients:

$$c_n := \begin{cases} 1/(\omega_0 \cosh \omega_0)^2 & \text{if } n=0, \\ -4/(n\pi)^2 & \text{if } n \geq 1 \text{ is odd,} \\ -1/(\omega_n \cos \omega_n)^2 & \text{if } n \geq 2 \text{ is even.} \end{cases}$$

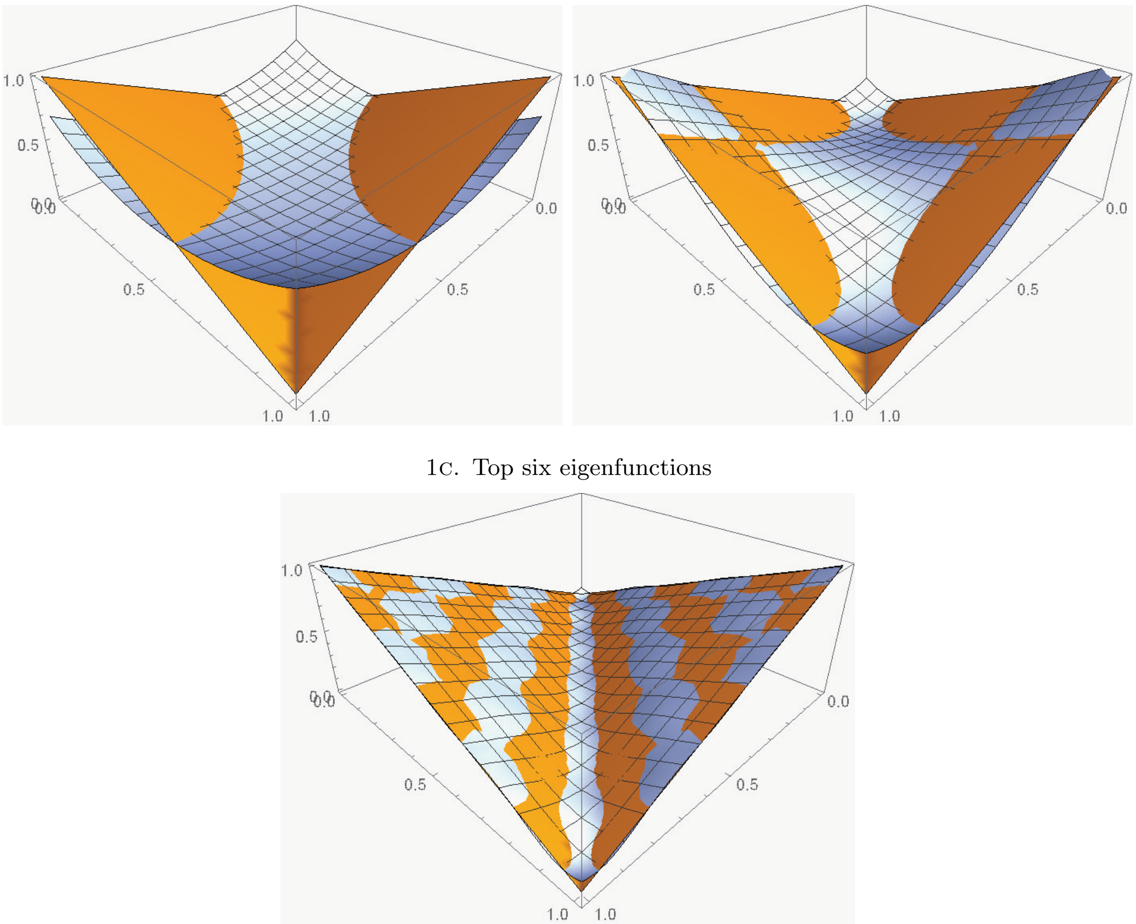
In Table 2 p. 10, we report numerical estimates of λ_n, ω_n, c_n together with the L^2 norm of the running spectral decomposition error[5] $|x-y| - \sum_{i=0}^{n} \lambda_i \phi_i(x)\phi_i(y)$. Figure 1 p. 11 illustrates the goodness of fit using 1, 2 and 6 eigenfunctions associated with top eigenvalues. As predicted by the rapidly decaying error norm, we can see that few eigenfunctions are needed to obtain a visually excellent fit.

TABLE 2. Top 20 eigenvalues and related coefficients of the straddle kernel

n	λ_n ($\times 10^{-3}$)	ω_n	c_n	Err. Norm
0	347.408 269 0	1.199 678 640	0.212 046 516	0.214 416
1	−202.642 367 3	1.570 796 327	−0.405 284 735	0.070 073
2	−63.849 095 79	2.798 386 046	−0.144 005 020	0.028 871
3	−22.515 818 59	4.712 388 980	−0.045 031 637	0.018 071
4	−13.344 112 79	6.121 250 467	−0.027 400 487	0.012 186
5	−8.105 694 691	7.853 981 634	−0.016 211 389	0.009 099
6	−5.758 866 886	9.317 866 462	−0.011 650 392	0.007 045
7	−4.135 558 516	10.995 574 29	−0.008 271 117	0.005 703
8	−3.206 946 639	12.486 454 40	−0.006 455 031	0.004 716
9	−2.501 757 621	14.137 166 94	−0.005 003 515	0.003 998
10	−2.042 994 806	15.644 128 37	−0.004 102 685	0.003 437
11	−1.674 730 308	17.278 759 59	−0.003 349 461	0.003 001
12	−1.415 208 556	18.796 404 37	−0.002 838 428	0.002 646
13	−1.199 067 262	20.420 352 25	−0.002 398 135	0.002 359
14	−1.038 184 585	21.945 612 88	−0.002 080 680	0.002 118
15	−0.900 632 744	23.561 944 90	−0.001 801 265	0.001 917
16	−0.794 086 718	25.092 910 41	−0.001 590 696	0.001 745
17	−0.701 184 662	26.703 537 56	−0.001 402 369	0.001 598
18	−0.627 008 356	28.238 936 58	−0.001 255 589	0.001 470
19	−0.561 336 198	29.845 130 21	−0.001 122 672	0.001 359

[5]In the orthonormal eigensystem the error norm is $\| \sum_{i=n+1}^{\infty} \lambda_i \phi_i \| = \sqrt{\sum_{i=n+1}^{\infty} \lambda_i{}^2}$

Figure 1. Straddle kernel fit with top eigenfunctions



1a. Top eigenfunction

1b. Top two eigenfunctions

1c. Top six eigenfunctions

4.5. **Spectral decomposition on a finite segment** $[a,b]$**.** Using affine transformations, it is easy to show that an orthonormal eigensystem for the straddle kernel defined over an arbitrary finite segment $[a,b]$ is simply

$$\left(\frac{1}{\sqrt{b-a}}\,\phi_n\left(\frac{x-a}{b-a}\right)\right), a \leq x \leq b, n \geq 0,$$

with associated eigenvalues $(b-a)^2\lambda_n$, where ϕ_n, λ_n are defined in sections 4.1 and 4.2. The corresponding spectral decomposition is then given as

$$|x-y| = (b-a)c_0 \cosh\omega_0\left(1-2\frac{x-a}{b-a}\right)\cdot\cosh\omega_0\left(1-2\frac{y-a}{b-a}\right) \tag{14}$$

$$+(b-a)\sum_{k=0}^{\infty} c_{2k+1}\cos\left[(2k+1)\pi\frac{x-a}{b-a}\right]\cdot\cos\left[(2k+1)\pi\frac{y-a}{b-a}\right] \tag{15}$$

$$+(b-a)\sum_{k=0}^{\infty} c_{2k+2}\cos\omega_{2k+2}\left(1-2\frac{x-a}{b-a}\right)\cdot\cos\omega_{2k+2}\left(1-2\frac{y-a}{b-a}\right), \tag{16}$$

where the coefficients c_n are given in equation (13).

5. Consequences for option replication and pricing

Because equation (2) with straddle kernel has the unique solution $\phi(y) = \frac{1}{2}f''(y) = \frac{1}{2}F''(y)$ when it exists, the weights of the spectroreplicant options in equation (6) may be further specified as

$$w_n = \frac{b-a}{2}\lambda_n\int_a^b \phi_n\left(\frac{x-a}{b-a}\right)F''(x)\mathrm{d}x. \tag{17}$$

To determine the cash and underlying asset quantities c, q we need two independent conditions. For instance, integrating the right-hand side of equation (1) by parts and evaluating at the boundaries $x = a, b$ with straddle kernel $G(x,y) := |x-y|$, we obtain

$$\begin{cases} F(a) = c+q\,a+\frac{1}{2}\int_a^b (y-a)F''(y)\mathrm{d}y = c+q\,a+\frac{1}{2}\left[(b-a)F'(b)-(F(b)-F(a))\right], \\ F(b) = c+q\,b+\frac{1}{2}\int_a^b (b-y)F''(y)\mathrm{d}y = c+q\,b-\frac{1}{2}\left[(b-a)F'(a)-(F(b)-F(a))\right]. \end{cases}$$

Solving for c, q yields

$$\begin{cases} c = \frac{1}{2}\left[F(a)+F(b)-aF'(a)-bF'(b)\right], \\ q = \frac{1}{2}\left[F'(a)+F'(b)\right]. \end{cases}$$

When equation (6) holds and all relevant quantities converge in L^2, the price of the target option is simply given as

$$F = c+qX+\sum_{n=0}^{\infty} w_n\Phi_n, \tag{18}$$

where F, X are the respective prices of the target option and underlying asset, Φ_n is the price of the n^{th} spectroreplicant option, and all prices are forward (i.e. paid on the common maturity date.)

The above pricing equation can be established using classical arbitrage arguments under the assumptions that short-selling and the instant trading of infinitely many securities are both feasible. In practice, just as with the Carr-Madan formula, the latter assumption is not realistic and must be mitigated by selecting a finite number of replicant options. For example, a proxy of order $n+1$ based on the largest absolute eigenvalues would be

$$\hat{F}_{n+1} := c+qX+\sum_{i=0}^{n} w_i\Phi_i. \tag{19}$$

Since practical implementations of the Carr-Madan formula and the spectral decomposition method are both approximate, and since the spectroreplicant options induced by the straddle kernel clearly do not trade, one may wonder what benefit there is in choosing the latter method over the former. The key benefit here is that spectroreplicant options are orthogonal in the sense of the standard scalar product of functions $\langle f, g\rangle = \int f(x)g(x)\mathrm{d}x$. In contrast, the continuum of call and put replicants in Carr-Madan are very codependent due to their overlapping payoff functions. This suggests that, for non-pathological target payoff $F(x)$, a limited number of spectroreplicant options should be enough to achieve satisfactory replication and pricing accuracy.

An obvious practical disadvantage of equation (19) is that the fair prices $(\Phi_i)_{0\leq i\leq n}$ of spectroreplicant options must be discovered by another method. One such method could simply be the Carr-Madan formula for option prices, either discretized along listed option strikes or using a numerical integration scheme together with the Black and Scholes (1973) formula and a model of the implied volatility smile.

Here, it is worth emphasizing that the weights w_i and fair prices Φ_i only need to be precomputed once for a limited selection of spectroreplicant options with largest absolute eigenvalues, based on the desired level of accuracy. Once this preliminary step is done, computing the proxy $\hat{F}_{n+1}$ for the target option price is immediate. The corresponding gain of speed is likely to be very relevant for electronic market-making, risk management of large portfolios of options or high frequency option trading. Moreover, the computational cost of refreshing the prices Φ_i throughout the trading day can be mitigated using Greek sensitivities.

6. Numerical application: Fast vanilla option pricing

6.1. **Proxy formula for vanilla option prices.** For the vanilla call target payoff $F(x) \coloneqq (x-K)^+$ where K is the strike price, the second-order derivative is Dirac's delta function $F''(x) = \delta(x-K)$; substituting into equation (17) we obtain the proxy formula for the call price

$$\hat{c}_{n+1}(K) = \frac{X-K}{2} + \sum_{i=0}^{n} w_n(K)\Phi_n, \tag{20}$$

with weights

$$w_n(K) = \frac{b-a}{2}\lambda_n \phi_n\left(\frac{K-a}{b-a}\right).$$

Similarly, the put proxy formula is given as

$$\hat{p}_{n+1}(K) = \frac{K-X}{2} + \sum_{i=0}^{n} w_n(K)\Phi_n$$

with the same weights $w_n(K)$.

6.2. **Numerical results.** We repriced 30-day out-of-the-money options on the S&P 500 index using the top 20 spectroreplicant options, based on sample bid and offer data as of 20 November 2018. We report the spectroreplicant option prices Φ_n in Table 3 obtained with a VIX-style discretization of the Carr-Madan formula (3). Then, we compute the proxy option prices for strike prices ranging from $a = 1225$ to $b = 3075$ using the formulas above.

In Figure 2a p.15 we plot our results for listed strikes between 1225 and 3075 on a scale from 0 to 1, where 0 corresponds to the market bid and 1 corresponds to the market offer price. Remarkably, all but two proxy option prices lie within the bid-offer range.

TABLE 3. Spectroreplicant option prices for the S&P 500 option market as of 20 November 2018

n	Φ_n	n	Φ_n
0	0.963 625 844 3	10	−0.215 640 803 7
1	−1.010 953 323	11	0.230 884 669 6
2	−0.086 058 597 69	12	−0.177 179 030 5
3	0.562 828 561	13	0.099 073 697 14
4	−0.873 351 989 5	14	−0.025 792 482 18
5	0.682 335 075	15	−0.031 588 463 23
6	−0.353 171 368 4	16	0.057 404 496 08
7	0.083 213 913 68	17	−0.065 055 145 40
8	0.138 417 330 6	18	0.052 725 231 43
9	−0.215 640 803 7	19	−0.031 390 284 03

A valuable additional benefit of the spectral decomposition method is to provide a natural "fit" of the implied volatility smile for arbitrary strikes $a \leq K \leq b$. In Figure 2b, we show our results in the slightly extended range $[1000, 3300]$. We can see that the fit is visually pleasing and the extrapolated values on the left and right regions of the chart look plausible.

6.3. **Arbitrage considerations.** It is worth emphasizing that the proxy formula of equation (20) is not theoretically free of arbitrage due to the oscillatory nature of the spectroreplicant options. Indeed, the tails of the corresponding implied distribution $h_{n+1}(K) := \hat{c}''_{n+1}(K)$ can become negative as shown in Figure 3 p.18, indicating the theoretical existence of butterfly arbitrages. However, our empirical results shown in Figure 2a suggest that such arbitrages are unlikely to have any practical relevance once bid-offer spreads are taken into account.

As expected, in the limit as $n \to \infty$, the proxy formula is arbitrage-free as long as all spectroreplicant prices Φ_n are known and priced off a valid implied distribution $h(K)$. This can be verified by substituting $\Phi_n = \int_a^b \phi_n\left(\frac{x-a}{b-a}\right) h(x)\mathrm{d}x$ into (20) to get

$$c(K) := \frac{X-K}{2} + \sum_{n=0}^{\infty} w_n(K)\Phi_n = \frac{X-K}{2} + \int_a^b h(x)dx \sum_{n=0}^{\infty} w_n(K)\phi_n\left(\frac{x-a}{b-a}\right).$$

Substituting into the above the expression for $w_n(K)$, recognizing the spectral decomposition (5) of the straddle kernel $|x-K|$, and differentiating both sides twice against K we recover the implied distribution $c''(K) = h(K)$. In addition, note that the truncated implied distribution $h_{n+1}(K)$ has the simple expression:

$$h_{n+1}(K) := \hat{c}''_{n+1}(K) = \sum_{i=0}^{n} \Phi_i \phi_i\left(\frac{K-a}{b-a}\right),$$

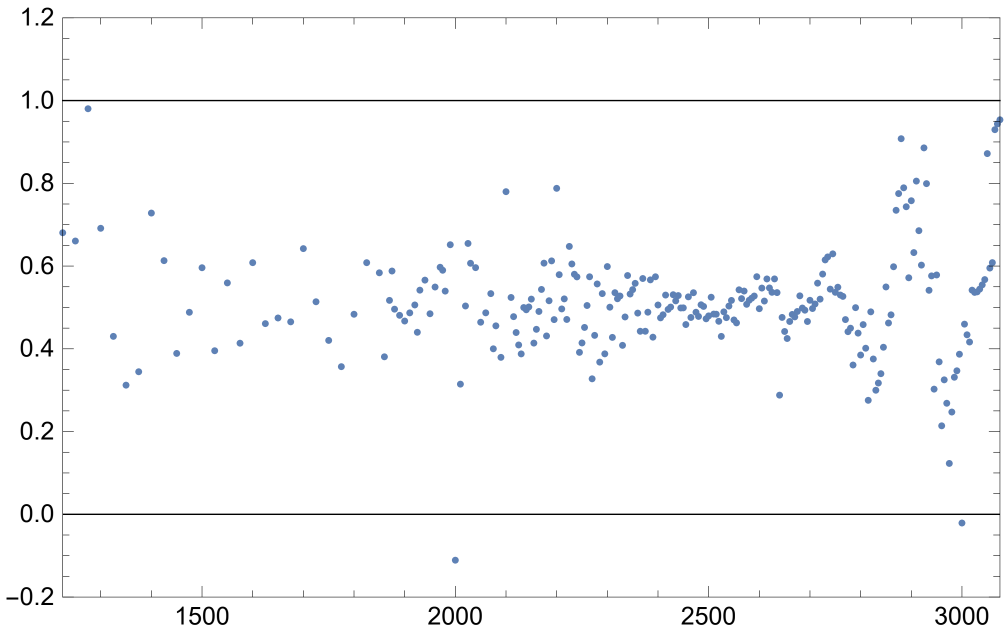
which is obtained by differentiating (20) twice against K using $\phi''_n(y) = \frac{2}{\lambda_n}\phi_n(y)$.

7. Theoretical application: formulas for spectroreplicant option prices when the characteristic function is known

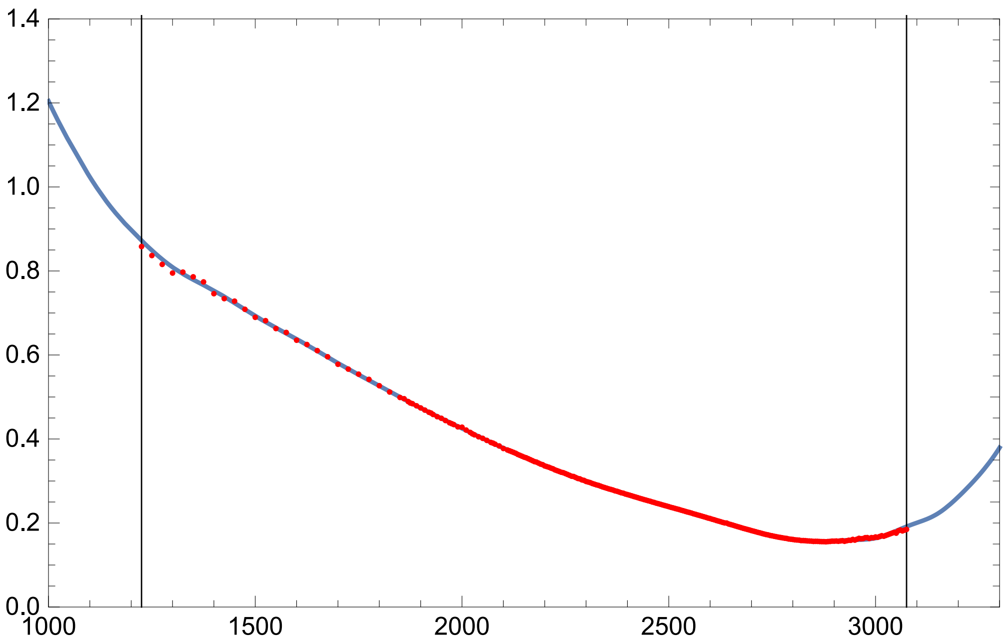
Consider an option pricing model where the characteristic function $\varphi(z) := \mathbb{E}(e^{izX_T}), z \in \mathbb{C}$ of the terminal underlying value X_T at time T is known in closed form. The characteristic function for $\tilde{X}_T := \frac{X_T - a}{b-a}$

FIGURE 2. Proxy OTM option prices and corresponding proxy implied volatility smile

2A. Proxy prices on a market bid-offer 0-1 scale



2B. Proxy implied volatility smile



is then

$$\tilde{\varphi}(z) = e^{-i\frac{a}{b-a}z}\varphi\left(\frac{z}{b-a}\right),$$

and we may recover pricing formulas for spectroreplicant options through the identities

$$\mathbb{E}\cosh\omega(1-2\tilde{X}_T) = \frac{1}{2}e^{\omega}[\tilde{\varphi}(2i\omega)+\tilde{\varphi}(-2i\omega)], \tag{21a}$$

$$\mathbb{E}\cos\omega\tilde{X}_T = \frac{1}{2}[\tilde{\varphi}(\omega)+\tilde{\varphi}(-\omega)], \tag{21b}$$

$$\mathbb{E}\cos\omega(1-2\tilde{X}_T) = \frac{1}{2}e^{i\omega}\tilde{\varphi}(-2\omega)+\frac{1}{2}e^{-i\omega}\tilde{\varphi}(2\omega)]. \tag{21c}$$

While the characteristic function of the *log-price* is known for many classical models such as Black and Scholes (1973), Heston (1993), and Merton (1976), we cannot use this knowledge here because the spectral decomposition (14) assumes a finite segment (a, b) for the range of underlying values. Instead, we turn to a model where the stochastic process for the underlying level $(X_t)_{t\geq 0}$ is a reflected Brownian motion initiated at x_0 with constant drift μ, volatility σ and reflecting bounds a, b. This type of model would be suitable for mean-reverting underlyings such as interest rates, energy, or the VIX. Note that, for option pricing, the model parameters x_0, μ, σ must be chosen such that $\mathbb{E}X_T = X$, i.e. the expected terminal underlying level must match the futures price.

The spectral expansion of the characteristic function of the terminal level $\tilde{X}_T \in (0, 1)$ can be found in Appendix A with the substitutions $r = 1, T \mapsto \sigma^2 T, \mu \mapsto \frac{\mu}{\sigma^2}$. Substituting the expansion into the identities (21) and simplifying, we can obtain formulas for spectroreplicant option prices as desired; for example, the formula for odd-indexed eigenfunctions $\phi_{2k+1}, k \in \mathbb{N}$ simplifies to:

$$\begin{aligned}
\mathbb{E}\phi_{2k+1}(\tilde{X}_T) &= \sqrt{2}\,\mathbb{E}\cos{(2k+1)\pi\tilde{X}_T} \\
&= \frac{\sqrt{2}}{2}[\tilde{\varphi}((2k+1)\pi)+\tilde{\varphi}(-(2k+1)\pi)] \\
&= -\frac{m^2\sqrt{2}\coth m}{m^2+(2k+1)^2\pi^2/4} - 4m\sqrt{2}\,(2k+1)^2\pi^2 e^{-mx_0}\times \\
&\quad \sum_{n=1}^{\infty}\frac{n\pi\,[1+(-1)^n e^m]\,\psi_n(x_0)\,e^{-(m^2+n^2\pi^2)\sigma^2 T/2}}{(m^2+n^2\pi^2)\,(m^2+(n-2k-1)^2\pi^2)\,(m^2+(n+2k+1)^2\pi^2)}
\end{aligned}$$

where $m := \frac{\mu}{\sigma^2}$ must be nonzero.

8. Spectral decomposition of the butterfly kernel

To further underscore the generality of our approach, in this final section we consider the butterfly kernel

$$G(x, y; c) := (c-|x-y|)^+ = (x-y+c)^+ - 2(x-y)^+ + (x-y-c)^+$$

for finite domain $[a, b]$ and fixed call spread parameter $0 < c \leq \frac{1}{3}(b-a)$. As stated in Table 1 p.4 this kernel is symmetric and injective. Indeed, we can write $G(x, y; c) = cK(x-y)$ where $K(z) := (1-|z|/c)^+$, and G is a positive-definite kernel if and only if K is a positive-definite *function*. By Bochner's theorem (Lax, 2002, p. 144) a function is positive-definite if and only if it is the Fourier transform of a probability density, and it is easy to verify that

$$\hat{K}(u) := \frac{1}{c\pi}\cdot\frac{1-\cos cu}{u^2}$$

is such a density. Indeed, by Fubini and then the property that the real number $\phi(y)$ is equal to its conjugate, we may write

$$\begin{aligned}\int_a^b \int_a^b \phi(x)K(x-y)\phi(y)\mathrm{d}x\,\mathrm{d}y &= \int_a^b \int_a^b \phi(x)\phi(y) \int_{-\infty}^{\infty} e^{i(x-y)u}\hat{K}(u)\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty} \hat{K}(u)\mathrm{d}u \int_a^b \phi(x)e^{iux}\mathrm{d}x \int_a^b \overline{\phi(y)e^{iuy}}\mathrm{d}y \geq 0,\end{aligned}$$

and equality implies $\phi \equiv 0$. Therefore, the butterfly kernel only has strictly positive eigenvalues and it is injective.

For ease of exposure, and without loss of generality, we assume $[a,b] = [0,1], 0 < c \leq \frac{1}{3}$ as we did in Section 4. Differentiating the integral equation (2) with butterfly kernel $G(x,y;c)$ twice against x we obtain the linear recurrence equation for ϕ

$$f''(x) = \phi(x-c) - 2\phi(x) + \phi(x+c),$$

with the convention $\phi(x) \equiv 0$ for $x < 0$ or $x > 1$. When $c = \frac{1}{N}, N \in \mathbb{N} \setminus \{0,1,2\}$, the solution is

$$\begin{cases} \phi(x) &= -\dfrac{N-n}{N+1}\displaystyle\sum_{k=0}^{n}(n+1-k)f''(x-kc) \\ & \quad -\dfrac{n+1}{N+1}\displaystyle\sum_{k=1}^{N-n-1}(N-n-k)f''(x+kc), \\ x &= nc+r, \quad 0 \leq r < c, \quad n \in \{0,1,\cdots,N-1\}, \end{cases} \tag{22}$$

wherein n is the Euclidean quotient of x by c with remainder r (i.e., x modulo c). In particular, the homogeneous equation with $f(x) \equiv 0$ only has the trivial solution $\phi(x) \equiv 0$, thereby confirming that the butterfly kernel is injective when $c = 1/N$.

It is worth noting that the solution (22) is typically discontinuous at every step c, and that the integral $\int_0^1 G(x,y;c)\phi(y)\mathrm{d}y$ matches $f(x)$ up to linear terms. Figure 4 p.18 shows the solution obtained for $F(x) = e^x$ and $c = 1/6$.

In the fashion of Section 4, it is possible to identify the general form of eigenfunctions $\phi(x)$ satisfying

$$\lambda\phi''(x) = \begin{cases} -2\phi(x) + \phi(x+c) & \text{for } 0 \leq x < c, \\ \phi(x-c) - 2\phi(x) + \phi(x+c) & \text{for } c \leq x \leq 1-c, \\ \phi(x-c) - 2\phi(x) & \text{for } 1-c < x \leq 1, \end{cases}$$

for an eigenvalue $\lambda > 0$. This may be done by splitting the domain $[0,1]$ at every step c and solving the system of second-order ordinary linear differential equations

$$\lambda\mathbf{u}''(x) = -\mathbf{A}.\mathbf{u}(x), \quad 0 \leq x < c,$$

where $\mathbf{u}(x) := (\phi(x), \phi(x+c), \cdots, \phi(x+1-c))^T$ is a vector of length N and $\mathbf{A}$ is the familiar $N \times N$ tridiagonal matrix

$$\mathbf{A} := \begin{bmatrix} 2 & -1 & & (0) \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \\ (0) & & -1 & 2 \end{bmatrix},$$

FIGURE 3. Implied distribution tails corresponding to proxy call prices

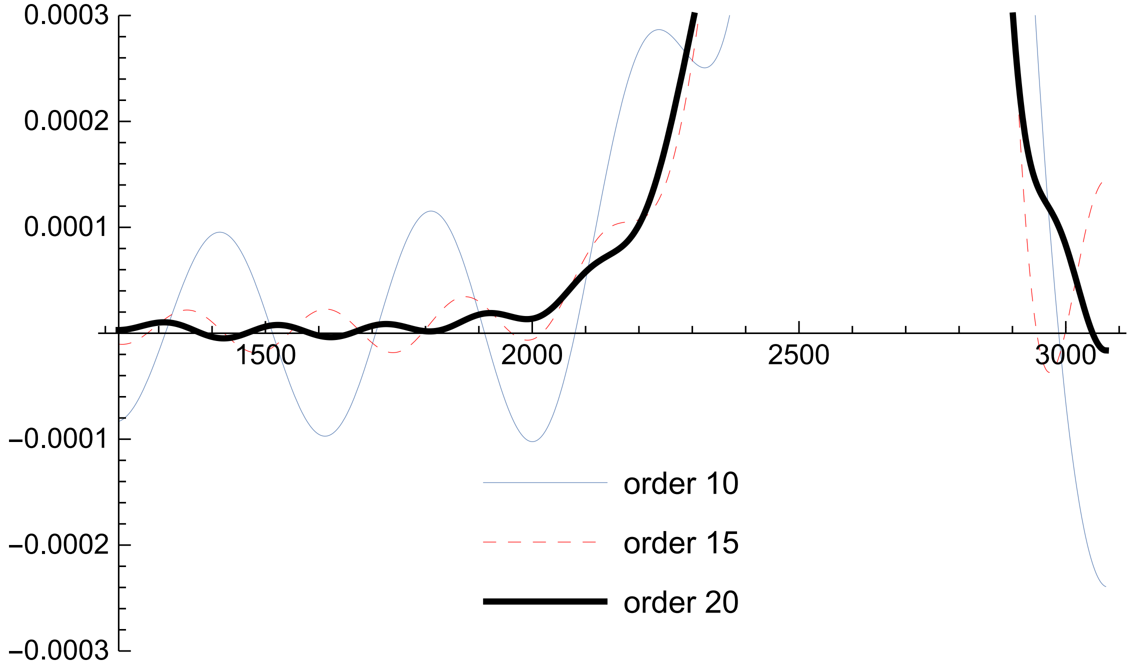
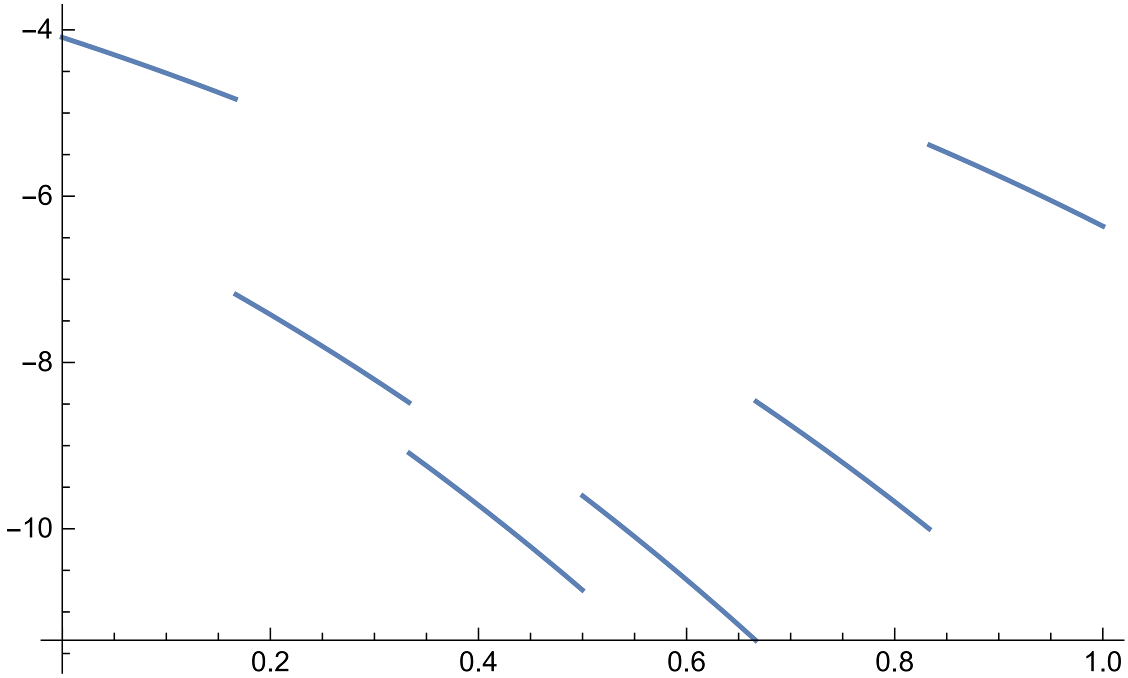


FIGURE 4. Solution to integral equation with butterfly kernel $G(x, y; c), c = \frac{1}{6}$, target function $F(x) = e^x$, domain $[a, b] = [0, 1]$



which is positive-definite with principal square root $\mathbf{A}^{\frac{1}{2}}$; formulas for the spectral elements of $\mathbf{A}$ can be found in e.g., Smith (1985), pp. 55, 154–156. The homogeneous second-kind integral equation with butterfly kernel may then be written in terms of $\mathbf{u}$ as

$$\lambda \mathbf{u}(x) = \int_0^c \mathbf{G}(x,y).\mathbf{u}(y)\mathrm{d}y \tag{23}$$

where $\mathbf{G}(x,y) \coloneqq ((c - |x-y+(n-p)c|)^+)_{0\le n,p\le N-1}$ is a $N\times N$ matrix defined for $0\le x,y<c$.

The general solution to the system of second-order ordinary linear differential equations $\lambda\mathbf{u}'' = -\mathbf{A}.\mathbf{u}$ is known to be

$$\mathbf{u}(x) = \cos\left(\frac{x}{\sqrt{\lambda}}\mathbf{A}^{\frac{1}{2}}\right).\mathbf{k}_1 + \sin\left(\frac{x}{\sqrt{\lambda}}\mathbf{A}^{\frac{1}{2}}\right).\mathbf{k}_2,$$

where $\mathbf{k}_1, \mathbf{k}_2$ are two column vectors of N constant coefficients. Substituting into (23) and integrating by parts twice we obtain[6]

$$\begin{aligned}\lambda\mathbf{u}(x) = &-\lambda\left[\mathbf{G}(x,y)\mathbf{A}^{-1}\mathbf{u}'(y) - \mathbf{G}_y(x,y)\mathbf{A}^{-1}\mathbf{u}(y)\right]_{y=0}^{y\to c^-}\\ &-\lambda\int_0^c \mathbf{G}_{yy}(x,y)\mathbf{A}^{-1}\mathbf{u}(y)\mathrm{d}y,\end{aligned}$$

where $\mathbf{G}_y, \mathbf{G}_{yy}$ are the first- and second-order partial derivatives of $\mathbf{G}(x,y)$ against y. Substituting the identities $\mathbf{G}(x,c) = x\mathbf{I} + (c-x)\mathbf{L}, \mathbf{G}(x,0) = (c-x)\mathbf{I} + x\mathbf{L}^T, \mathbf{G}_y(x,c) = -\mathbf{I} + \mathbf{L}, \mathbf{G}_y(x,0) = \mathbf{I} - \mathbf{L}^T$, where $\mathbf{L}$ is the lower shift matrix with ones on the subdiagonal, we may rewrite the bracket in the above equation as the block matrix expression

$$\begin{aligned}\mathbf{b}_\lambda(x) \coloneqq &\begin{bmatrix}\mathbf{I}-\mathbf{L} & x(\mathbf{I}-\mathbf{L}) + c\mathbf{L}\end{bmatrix}\begin{bmatrix}\mathbf{C}_\lambda & \mathbf{S}_\lambda\\ -\frac{1}{\sqrt{\lambda}}\mathbf{A}^{\frac{1}{2}}\mathbf{S}_\lambda & \frac{1}{\sqrt{\lambda}}\mathbf{A}^{\frac{1}{2}}\mathbf{C}_\lambda\end{bmatrix}\begin{bmatrix}\mathbf{A}^{-1}\mathbf{k}_1\\ \mathbf{A}^{-1}\mathbf{k}_2\end{bmatrix}\\ &+\begin{bmatrix}\mathbf{I}-\mathbf{L}^T & \frac{x}{\sqrt{\lambda}}(\mathbf{I}-\mathbf{L}^T)\mathbf{A}^{\frac{1}{2}} - \frac{c}{\sqrt{\lambda}}\mathbf{A}^{\frac{1}{2}}\end{bmatrix}\begin{bmatrix}\mathbf{A}^{-1}\mathbf{k}_1\\ \mathbf{A}^{-1}\mathbf{k}_2\end{bmatrix}\end{aligned}$$

where $\mathbf{C}_\lambda \coloneqq \cos\left(\frac{c}{\sqrt{\lambda}}\mathbf{A}^{\frac{1}{2}}\right)$ and $\mathbf{S}_\lambda \coloneqq \sin\left(\frac{c}{\sqrt{\lambda}}\mathbf{A}^{\frac{1}{2}}\right)$. The vector $\mathbf{b}_\lambda(x)$ is linear in x and will vanish if and only if the intercept and slope vectors $\mathbf{b}_\lambda(0), \mathbf{b}'_\lambda$ are zero, leading to the homogeneous block matrix equation in $\mathbf{k}_1, \mathbf{k}_2$

$$\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix} = \left(\begin{bmatrix}\mathbf{I}-\mathbf{L} & c\mathbf{L}\\ \mathbf{O} & \mathbf{I}-\mathbf{L}\end{bmatrix}\begin{bmatrix}\mathbf{C}_\lambda & \mathbf{S}_\lambda\\ -\frac{1}{\sqrt{\lambda}}\mathbf{A}^{\frac{1}{2}}\mathbf{S}_\lambda & \frac{1}{\sqrt{\lambda}}\mathbf{A}^{\frac{1}{2}}\mathbf{C}_\lambda\end{bmatrix} + \begin{bmatrix}\mathbf{I}-\mathbf{L}^T & -\frac{c}{\sqrt{\lambda}}\mathbf{A}^{\frac{1}{2}}\\ \mathbf{O} & (\mathbf{I}-\mathbf{L}^T)\frac{1}{\sqrt{\lambda}}\mathbf{A}^{\frac{1}{2}}\end{bmatrix}\right)\begin{bmatrix}\mathbf{A}^{-1}\mathbf{k}_1\\ \mathbf{A}^{-1}\mathbf{k}_2\end{bmatrix},$$

where $\mathbf{0}$ is the null column vector of $\mathbb{R}^N$ and $\mathbf{O}$ is the null matrix of $\mathbf{R}^{N\times N}$. It is worth noting that solving the above equation is difficult: we need to find λ such that the $2N\times 2N$ block matrix between parentheses is singular, and then find the corresponding nullspace to identify non-trivial solutions $\mathbf{k}_1, \mathbf{k}_2$. However, with some algebra we can simplify this problem for some eigenvalues λ, as detailed below.

Left-multiplying both sides of the previous equation by $\begin{bmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \sqrt{\lambda}\mathbf{A}^{-\frac{1}{2}}\end{bmatrix}\begin{bmatrix}\mathbf{I}-\mathbf{L} & c\mathbf{L}\\ \mathbf{O} & \mathbf{I}-\mathbf{L}\end{bmatrix}^{-1}$, we obtain

$$\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix} = \left(\begin{bmatrix}\mathbf{C}_\lambda & \mathbf{S}_\lambda\\ -\mathbf{S}_\lambda & \mathbf{C}_\lambda\end{bmatrix} + \begin{bmatrix}\mathbf{I} & \mathbf{O}\\ \mathbf{O} & \sqrt{\lambda}\mathbf{A}^{-\frac{1}{2}}\end{bmatrix}\begin{bmatrix}\mathbf{I}-\mathbf{L} & c\mathbf{L}\\ \mathbf{O} & \mathbf{I}-\mathbf{L}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{I}-\mathbf{L}^T & -\frac{c}{\sqrt{\lambda}}\mathbf{A}^{\frac{1}{2}}\\ \mathbf{O} & (\mathbf{I}-\mathbf{L}^T)\frac{1}{\sqrt{\lambda}}\mathbf{A}^{\frac{1}{2}}\end{bmatrix}\right)\begin{bmatrix}\mathbf{A}^{-1}\mathbf{k}_1\\ \mathbf{A}^{-1}\mathbf{k}_2\end{bmatrix}. \tag{24}$$

[6]Observe that the matrix versions of $\cos, \sin$ commute with any power of the argument matrix, and that $-\lambda\mathbf{A}^{-1}\mathbf{u}'(x), -\lambda\mathbf{A}^{-1}\mathbf{u}(x)$ are respectively first- and second-order antiderivatives of $\mathbf{u}(x)$.

It is easy to show that the second term above between parentheses simplifies to

$$\begin{bmatrix} \mathbf{e}\mathbf{v}^T - \mathbf{L}^T & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^{-\frac{1}{2}}(\mathbf{e}\mathbf{v}^T - \mathbf{L}^T)\mathbf{A}^{\frac{1}{2}} \end{bmatrix} - \frac{c}{\sqrt{\lambda}} \begin{bmatrix} \mathbf{w} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{A}^{\frac{1}{2}}\mathbf{v} \end{bmatrix}^T,$$

where $\mathbf{e} := (1, \cdots, 1)^T$ is the first diagonal vector of $\mathbb{R}^N$, $\mathbf{v} := (1, 0, \cdots, 0)^T$ is the first coordinate vector, and $\mathbf{w} := (1, 2, \cdots, N)^T$. Equation (24) may thus be rewritten as

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} = \left(\mathbf{M}_\lambda - \frac{c}{\sqrt{\lambda}} \begin{bmatrix} \mathbf{w} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{A}^{\frac{1}{2}}\mathbf{v} \end{bmatrix}^T \right) \begin{bmatrix} \mathbf{A}^{-1}\mathbf{k}_1 \\ \mathbf{A}^{-1}\mathbf{k}_2 \end{bmatrix},$$

where

$$\mathbf{M}_\lambda := \begin{bmatrix} \mathbf{C}_\lambda & \mathbf{S}_\lambda \\ -\mathbf{S}_\lambda & \mathbf{C}_\lambda \end{bmatrix} + \begin{bmatrix} \mathbf{e}\mathbf{v}^T - \mathbf{L}^T & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^{-\frac{1}{2}}(\mathbf{e}\mathbf{v}^T - \mathbf{L}^T)\mathbf{A}^{\frac{1}{2}} \end{bmatrix}.$$

When $\mathbf{M}_\lambda$ is invertible, the Sherman-Morrison formula (e.g., Golub and Loan, 1996, p. 51) states that $\left(\mathbf{M}_\lambda - \frac{c}{\sqrt{\lambda}} \begin{bmatrix} \mathbf{w} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{A}^{\frac{1}{2}}\mathbf{v} \end{bmatrix}^T \right)$ is singular if and only if λ satisfies the scalar equation

$$1 - \frac{c}{\sqrt{\lambda}} \begin{bmatrix} \mathbf{0} \\ \mathbf{A}^{\frac{1}{2}}\mathbf{v} \end{bmatrix}^T \mathbf{M}_\lambda^{-1} \begin{bmatrix} \mathbf{w} \\ \mathbf{0} \end{bmatrix} = 0, \tag{25}$$

and in this case $\mathbf{M}_\lambda^{-1} \begin{bmatrix} \mathbf{w} \\ \mathbf{0} \end{bmatrix}$ is in the nullspace, giving a nontrivial solution

$$\begin{bmatrix} \mathbf{k}_1 \\ \mathbf{k}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} \end{bmatrix} \mathbf{M}_\lambda^{-1} \begin{bmatrix} \mathbf{w} \\ \mathbf{0} \end{bmatrix}.$$

In Figure 5 p. 21 we plot the top eigenfunction that we obtained by numerically solving equation (25) for $N = 3$ and $N = 10$ and then computing $\mathbf{k}_1, \mathbf{k}_2$ as written above. As expected, the eigenfunctions are continuous and smooth. Note that there may be eigenvalues λ for which $\mathbf{M}_\lambda$ is not invertible, in which case equation (25) cannot be relied upon.

Appendix A. Characteristic function of the reflected Brownian motion

The density of a reflected Brownian motion over a finite interval $(0, r)$ with constant drift μ and unit diffusion coefficient is given as (Linetsky, 2005, equation (25) with our notations)[7]

$$p(x) = \frac{2\mu\, e^{2\mu x}}{e^{2\mu r} - 1} + \frac{2}{r} e^{\mu(x - x_0)} \sum_{n=1}^{\infty} \frac{e^{-\left(\mu^2 + n^2\pi^2/r^2\right)T/2}}{\mu^2 + n^2\pi^2/r^2} \psi_n(x_0)\psi_n(x),$$

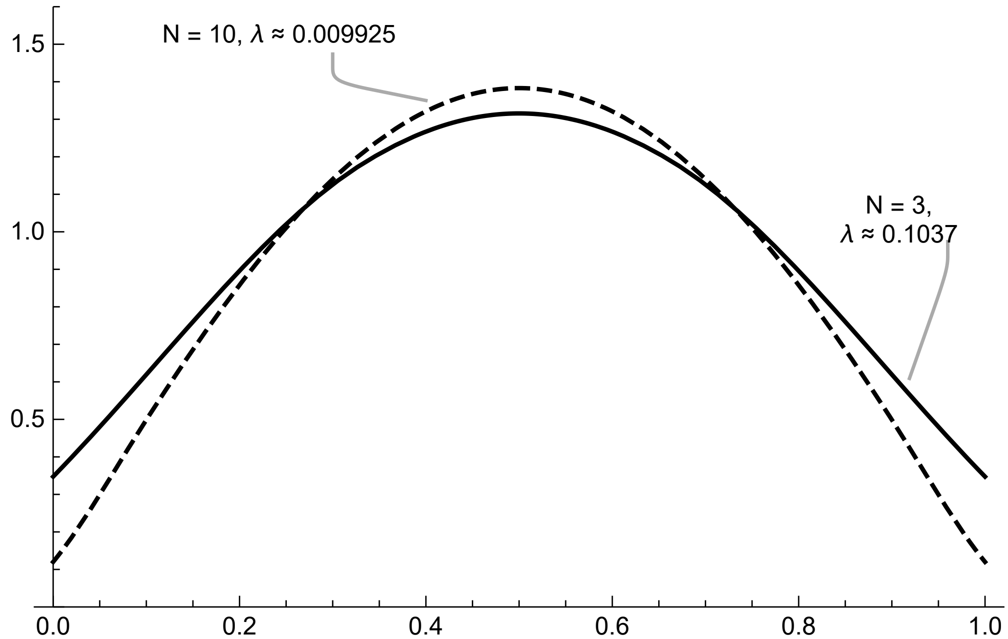
where $x_0, x \in (0, r)$ are respectively the initial and terminal levels, and

$$\psi_n(x) := \frac{n\pi}{r} \cos\frac{n\pi x}{r} + \mu \sin\frac{n\pi x}{r}.$$

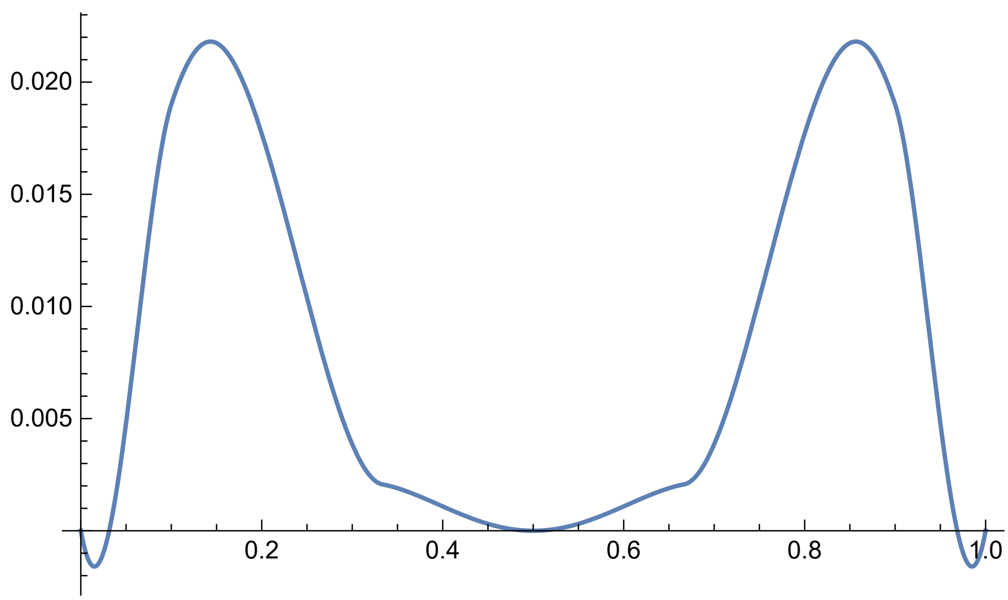
[7] Note that the first term in equation (25) of Linetsky (2005) contains a typo and should depend on y rather than x in the author's notations.

FIGURE 5. Comparison of top eigenfunctions of the butterfly kernel $G(x, y; c), c = \frac{1}{N}$ over the domain $[a, b] = [0, 1]$

5A. Top normalized eigenfunction for $N = 3$ and $N = 10$



5B. Difference after rescaling



The corresponding characteristic function may then be calculated term by term. Isolating the terms and factors that only depend on x we have, for any complex number z,

$$\begin{cases} \int_0^r e^{2\mu x+izx}\mathrm{d}x & = & 2\dfrac{e^{2(\mu+iz)r}-1}{2\mu+iz}, \\ \int_0^r e^{\mu x+izx}\psi_n(x)\mathrm{d}x & = & i\dfrac{n\pi z}{r}\dfrac{(-1)^n e^{(\mu+iz)r}-1}{(\mu+iz)^2+n^2\pi^2/r^2}. \end{cases}$$

Substituting into $\hat{p}(z) := \int_0^r p(x)e^{izx}\mathrm{d}x$ we obtain the spectral expansion of the characteristic function as

$$\begin{aligned} \hat{p}(z) = & \frac{2\mu}{2\mu+iz}\frac{e^{(2\mu+iz)r}-1}{e^{2\mu r}-1} \\ & + 2i\frac{\pi z}{r^2}e^{-\mu x_0}\sum_{n=1}^{\infty} n\cdot\frac{e^{-\left(\mu^2+n^2\pi^2/r^2\right)T/2}}{\mu^2+n^2\pi^2/r^2}\cdot\frac{(-1)^n e^{(\mu+iz)r}-1}{(\mu+iz)^2+n^2\pi^2/r^2}\cdot\psi_n(x_0). \end{aligned}$$

Corresponding formulas for the reflected Brownian motion with constant diffusion coefficient σ may then be obtained by scaling $T \mapsto \sigma^2 T$ and $\mu \mapsto \frac{\mu}{\sigma^2}$.

References

Balder, Sven and Antje Mahayni (2006). "Robust Hedging with Short-Term Options". In: *Wilmott Magazine* 9, pp. 72–78.

Black, Fischer and Myron Scholes (1973). "The pricing of options and corporate liabilities". In: *Journal of political economy* 81.3, p. 637.

Breeden, Douglas T. and Robert H. Litzenberger (1978). "Prices of state-contingent claims implicit in option prices". In: *Journal of business*, pp. 621–651.

Bünger, Florian (2014). "Inverses, determinants, eigenvalues, and eigenvectors of real symmetric Toeplitz matrices with linearly increasing entries". In: *Linear Algebra and its Applications* 459, pp. 595–619.

Carr, Peter and Dilip Madan (1998). "Towards a theory of volatility trading". In: *Volatility: New estimation techniques for pricing derivatives*. Ed. by Robert A. Jarrow. Vol. 29. Risk books, pp. 417–427.

Carr, Peter and Liuren Wu (2013). "Static hedging of standard options". In: *Journal of Financial Econometrics* 12.1, pp. 3–46.

Carton de Wiart, Benjamin and Michael A. H. Dempster (2011). "Wavelet Optimized Valuation of Financial Derivatives". In: *International Journal of Theoretical and Applied Finance* 14.07, pp. 1113–1137.

Černý, Aleš (2016). *Mathematical Techniques in Finance. Tools for Incomplete Markets*. 2nd ed. Princeton University Press.

Demeterfi, Kresimir, Emanuel Derman, Michael Kamal, and Joseph Zou (1999). *More than you ever wanted to know about volatility swaps*. Goldman Sachs quantitative strategies research notes. Goldman, Sachs & Co New York.

Derman, Emanuel, Deniz Ergener, and Iraj Kani (1994). *Static options replication*. Goldman Sachs quantitative strategies research notes. Goldman, Sachs & Co New York.

Dupire, Bruno (1993). *Arbitrage pricing with stochastic volatility*. Tech. rep. Banque Paribas.

Eidelman, Yuli, Vitali D. Milman, and Antonis Tsolomitis (2004). *Functional analysis: an introduction*. Vol. 66. American Mathematical Society.

Fredholm, Ivar (1903). "Sur une classe d'équations fonctionnelles". In: *Acta mathematica* 27.1, pp. 365–390.

Golub, Gene H. and Charles F. Van Loan (1996). *Matrix Computations*. 3rd ed. Johns Hopkins University Press.

Heston, Steven L. (1993). "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options". In: *The Review of Financial Studies* 6.2, pp. 327–343.

Hilbert, David (1905). "Über eine Anwendung der Integralgleichungen auf ein Problem der Funktionentheorie". In: *Verhandlungen des dritten internationalen Mathematiker Kongresses.* (Heidelberg, Aug. 13, 1904). Ed. by Adolf Krazer. Leipzig: Teubner, pp. 233–240.

Kress, Rainer (2014). *Linear integral equations.* 3rd ed. Vol. 82. Springer Verlag, New York.

Lax, Peter D. (2002). *Functional Analysis.* New York: Wiley-Interscience.

Linetsky, Vadim (2005). "On the transition densities for reflected diffusions". In: *Advances in Applied Probability* 37.2, pp. 435–460.

Madan, Dilip and Frank Milne (1994). "Contingent Claims Valued and Hedged by Pricing and Investing in a Basis". In: *Mathematical Finance* 4.3, pp. 223–245.

Merton, Robert C. (1976). "Option pricing when underlying stock returns are discontinuous". In: *Journal of Financial Economics* 3, pp. 125–144.

Neuberger, Anthony (1990). *Volatility trading.* Working paper. London Business School.

Papanicolaou, Andrew (2018). *Consistent Inter-Model Specification for Time-Homogeneous SPX Stochastic Volatility and VIX Market Models.* Working paper. arXiv.org.

Polyanin, Andrei D. and Alexander V. Manzhirov (2008). *Handbook of Integral Equations.* 2nd ed. Chapman and Hall/CRC.

Riesz, Friedrich (1916). "Über lineare funktionalgleichungen". In: *Acta Mathematica* 41.1, pp. 71–98.

Schmidt, Erhard (1907). "Zur Theorie der linearen und nichtlinearen Integralgleichungen". In: *Mathematische Annalen* 63.4, pp. 433–476.

Smith, Gordon D. (1985). *Numerical solution of partial differential equations: finite difference methods.* 3rd ed. Clarendon Press/Oxford University Press.

Srivastava, Hari M. and Robert G. Buschman (2013). *Theory and applications of convolution integral equations.* Ed. by Michiel Hazewinkel. Rev., enl., updated ed. of *Convolution integral equations, with special function kernels* c1977. Vol. 79. Springer Science & Business Media.

The CBOE volatility index—VIX (2009). Tech. rep. Chicago Board Options Exchange.

Volterra, Vito (1896). "Sulla inversione degli integrali definiti". In: *Rendiconti della Classe di Scienze Fisiche, Matematiche e Naturali.* Vol. 5. Accademia Nazionale dei Lincei, pp. 177–185, 289–300.

Wu, Liuren and Jingyi Zhu (2017). "Simple Robust Hedging with Nearby Contracts". In: *Journal of Financial Econometrics* 15.1, pp. 1–35.