ADOL: Markovian approximation of a rough lognormal model

Peter Carr and Andrey Itkin apply a Markovian approximation of the fractional Brownian motion, known as the Dobrić-Ojeda process, to the fractional stochastic volatility model, where the instantaneous variance is modelled using a lognormal process with drift and fractional diffusion

Gatheral *et al* (2014) discovered that historical volatility time series exhibit behaviour that is much rougher than that of a Brownian motion for a wide range of assets. These authors also showed the dynamics of log volatility can be well modelled by a fractional Brownian motion with a Hurst parameter of order 0.1. In the literature, there exist various opinions as to whether the Hurst index should be less than 1/2 (short memory) or above 1/2 (long memory), depending on the particular asset class. As mentioned in Funahashi & Kijima (2017), it is well known that (i) the decrease in market volatility smile amplitude is much slower than predicted by the standard stochastic volatility models and (ii) the term structure of the at-the-money volatility skew is well approximated by a power-law function with an exponent close to zero. These stylised facts cannot be captured by standard models, and while (i) has been explained using a fractional volatility model with the Hurst index $H > 1/2$, (ii) is proven to be satisfied by a rough volatility model with $H < 1/2$ under a risk-neutral measure. For more detail, see Funahashi & Kijima (2017) and the references therein. In Livieri *et al* (2018), the value of the Hurst exponent obtained via high-frequency volatility estimations from historical price data is revisited by studying implied volatility-based approximations of the spot volatility. Using at-the-money options on the Standard & Poor's 500 index with short maturity, the authors confirm the volatility is rough and the Hurst parameter is of order 0.3, ie, slightly larger than that usually obtained from historical data.

Despite the fact that rough volatility models have already been elaborated on and there exists a rich literature on the subject, due to the non-Markovian nature of the fractional Brownian motion, one can face some technical problems when it comes to derivatives pricing. Pricing variance swaps is even more complicated. Therefore, in this article, we attempt to attack the rough volatility problem using some approximation to the fractional Brownian motion, which, however, is a semi-martingale.

The Dobrić-Ojeda process

As mentioned, this article aims to construct a rough lognormal model by replacing the fractional Brownian motion, which drives the instantaneous volatility, using a similar process to that introduced in Dobrić & Ojeda (2009). Below, we provide a short description of this process, which, for the sake of brevity, we call the DO process, following Dobrić & Ojeda (2009) and Conus & Wildman (2016).

The DO process is a Gaussian Markov process with similar properties to those of a fractional Brownian motion; namely, its increments are dependent in time. The DO process is defined by first considering the fractional Gaussian field $Z = Z_H(t)$, $(t, H) \in [0, \infty) \times (0, 1)$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined by covariance (compare this with a standard fractional Brownian motion, where $\alpha_{H,H'} = 1$ and $H = H'$):

$$\mathbb{E}[Z_H(t) Z_{H'}(s)] = \frac{\alpha_{H,H'}}{2}[|t|^{H+H'} + |s|^{H+H'} - |t-s|^{H+H'}]$$

$$\alpha_{H,H'} = \begin{cases} -\dfrac{2\eta}{\pi}\xi(H)\xi(H')\cos\left[\dfrac{\pi}{2}(H'-H)\right]\cos\left[\dfrac{\pi}{2}(H'+H)\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad H + H' \neq 1 \\ \xi \sin^2(\pi H) \equiv \alpha_H \equiv \alpha_{H'}, \quad H + H' = 1 \end{cases}$$

$$\xi(H) = [\Gamma(2H+1)\sin(\pi H)]^{1/2}, \qquad \eta = \Gamma(-(H+H'))$$

$$\xi = [\Gamma(2H+1)\Gamma(3-2H)]^{1/2} \tag{1}$$

Here, $\Gamma(x)$ is the Gamma function. Obviously, if $H = H'$, Z_H is a fractional Brownian motion; so, if $H = H' = 1/2$, it is a standard Brownian motion. It is established in Dobrić & Ojeda (2009) that Z_H exists.

Further, Dobrić & Ojeda (2009) seek a process of the form $\psi_H(t) M_H(t)$ that in some sense approximates a fractional Brownian motion; they assume $\psi_H(t)$ is a deterministic function of time and $M_H(t)$ is a stochastic process. They construct $M_H(t)$ as follows. On the Gaussian field Z, define $M_H(t)$, $t \in [0, \infty)$, as:

$$M_H(t) = \mathbb{E}[Z_{H'}(t) \mid \mathcal{F}_t^H] \tag{2}$$

where $\mathcal{F}_t^H$ is a filtration generated by a sigma-algebra $Z_H(r)$, $0 \leqslant r \leqslant t$. It is shown that $M_H(t)$ is a martingale with respect to $(\mathcal{F}_t^H)_{t \geqslant 0}$ and that $M_H(t)$ is a Gaussian centred process with independent increments and covariance:

$$\mathbb{E}[M_H(t) M_H(s)] = c_H \alpha_H \bar{B}(3/2 - H)(s \wedge t)^{2-2H}$$

$$c_H = \frac{\alpha_H}{2H\Gamma(3/2 - H)\Gamma(H + 1/2)} \tag{3}$$

where $\bar{B}(x) = B(x, x)$, $B(x, y)$ is the Beta function.

The coefficient $\psi_H(t)$ could be determined by minimising the difference $\mathbb{E}[(Z_H(t) - \psi_H(t) M_H(t))^2]$ to provide:

$$\psi_H(t) = \frac{\mathbb{E}[Z_H(t) M_H(t)]}{\mathbb{E}[M_H^2(t)]} = \frac{\Gamma(3-2H)}{c_H \Gamma^2(3/2 - H)} t^{2H-1} \tag{4}$$

In summary, this construction introduces the DO process $V_H(t)$, $t \in [0, \infty]$, defined as $V_H(t) = \psi_H(t) M_H(t)$, where $\psi_H(t)$ is given in (4) and $M_H(t)$ is given in (2) with $H + H' = 1$.

The most useful property of the DO process is that it is a semi-martingale and can be represented as an Itô diffusion. This means (see Dobrić & Ojeda 2009) there exists a Brownian motion process W_t, $t \in [0, \infty)$, adapted to the filtration $\mathcal{F}_t^H$ such that:

$$\begin{aligned} \mathrm{d}V_H(t) &= \frac{2H-1}{t} V_H(t)\,\mathrm{d}t + B_H t^{H-1/2}\,\mathrm{d}W_t \\ B_H &= \frac{2^{3-4H}\csc^4(\pi H)\Gamma(2-H)}{\Gamma(3/2-H)^2\Gamma(H)} \end{aligned} \tag{5}$$

In contrast to Conus & Wildman (2016), where the DO process was used as noise in the Black-Scholes framework, we apply it to model dynamics of the instantaneous variance. The main advantage of this model compared with the rough volatility models is that the semi-martingale property of the DO process allows utilisation of Itô's calculus.

In Harms (2019), it is shown that a fractional Brownian motion can be represented as an integral over the family of Ornstein-Uhlenbeck (OU) processes. Thus, the DO process can be considered a particular case of that construction. However, as we show below, the DO approximation of the fractional Brownian motion provides some additional tractability but is less accurate.

An adjusted DO process

An adjusted DO (ADO) process is constructed in Carr & Itkin (2019) (which is an extended version of this article) and is defined as:

$$\begin{aligned} \mathcal{V}_H(t) &= \psi_H(t)M_H(t) + \mathrm{i}d_H t^H = V_H(t) + \mathrm{i}d_H t^H \\ d_H^2 &= 1 - 2H\frac{\Gamma(3-2H)\Gamma(H+1/2)}{\Gamma(3/2-H)} \end{aligned} \tag{6}$$

where i is the imaginary unit. The ADO process inherits a semi-martingale property from $V_H(t)$. Also, $\psi_H(t)$ as it is defined in (4) minimises the difference $\mathbb{E}[\mathcal{Y}_H^2(t)] = \mathbb{E}[(Z_H(t) - \mathcal{V}_H(t))^2]$, and the minimum value of this difference is:

$$\mathbb{E}[\mathcal{Y}_H^2(t)] = \mathbb{E}[\{Z_H(t) - (\psi_H(t)M_H(t) + \mathrm{i}d_H t^H)\}^2] = 0 \tag{7}$$

This is in contrast to the DO process V_H, which approximates Z_H with a relative L^2 error of at most 12% (see Dobrić & Ojeda 2009, figure 1). At a lower H, the discrepancy is bigger and can reach 80–100% at very small H. However, using the ADO process requires an extension of the traditional measure theory into the complex domain (see, for example, Carr & Wu 2004).

From the definition $V_H(t) = \mathcal{V}_H(t) - \mathrm{i}d_H t^H$, (5) can be transformed into:

$$\begin{aligned} \mathrm{d}\mathcal{V}_H(t) = \bigg[\mathrm{i}(1-H)d_H t^{H-1} + \frac{2H-1}{t}\mathcal{V}_H(t)\bigg]\mathrm{d}t \\ + B_H t^{H-1/2}\,\mathrm{d}W_t \end{aligned} \tag{8}$$

with the same Brownian motion as in (5). In other words, the ADO process can also be represented as an Itô diffusion. If $H < 1/2$, it exhibits mean reversion.

It is worth underlining that the ADO process is no longer a martingale under $\mathcal{F}_t^H$ due to the adjustment made. However, as we use this process for modelling the instantaneous variance, it should not be a martingale. Hence, the only property we need is that the ADO process is a semi-martingale; this can be represented as an Itô diffusion in (8).

Conus & Wildman (2016) emphasise that the term $1/t$ in the drift of $\mathcal{V}_H(t)$ causes an explosion of the DO process at $t = 0$. To remedy this issue, Conus & Wildman define a modified process in which the drift is 0 until time $t = \varepsilon > 0$. Here, we exploit this idea for the ADO process as well.

The ADOL model

As mentioned in the introduction, an analysis of the market data has revealed the rough nature of implied volatility. Therefore, there has been a trend in the literature of proposing fractional alternatives of well-known stochastic volatility (SV) models such as the Heston model. One of the SV models recommended (eg, in Sepp 2016) is the mean-reverting lognormal model. However, it is not affine. Therefore, a closed-form solution for the characteristic function (CF) of $\log S_T$ is not yet known, although some approximations have been reported in the literature (Sepp 2016). Therefore, the main idea of this article is to propose a tractable version of the rough lognormal model. In doing so, we take the following steps:

- For the instantaneous volatility process, we use the ADO process instead of the fractional Brownian motion.
- We assume the mean-reversion level $\theta = 0$ (this, however, can be relaxed; see the discussion at the end of this article).

Also, for simplicity, we use the symbol $\mathcal{V}_t$ instead of $\mathcal{V}_H(t)$ and $\nu(t) = B_H t^{H-1/2}$. Then, assuming real-world dynamics (ie, under the $\mathbb{P}$ measure), our model can be represented as:

$$\left.\begin{aligned} \mathrm{d}S_t &= S_t\mu\,\mathrm{d}t + S_t\sigma_t\,\mathrm{d}W_t^{(1)} \\ \mathrm{d}\sigma_t &= \sigma_t[-\kappa + \xi D_v]\,\mathrm{d}t + \sigma_t\xi\nu(t)\,\mathrm{d}W_t^{(2)} \\ \mathrm{d}\mathcal{V}_t &= D_v\,\mathrm{d}t + \nu(t)\,\mathrm{d}W_t^{(2)} \\ D_v &= \bigg[\mathrm{i}(1-H)d_H t^{H-1} + \frac{2H-1}{t}\mathcal{V}_t\bigg]\mathbf{1}_{t>\varepsilon} \\ S_t|_{t=0} &= S_0, \quad \sigma_t|_{t=0} = \sigma_0, \quad \mathcal{V}_t|_{t=0} = \mathcal{V}_0 \end{aligned}\right\} \tag{9}$$

where μ is the drift. This model is a two-factor model (actually, we introduce three stochastic variables, S_t, σ_t and $\mathcal{V}_t$, but two of them, σ_t and $\mathcal{V}_t$, are fully correlated).

The model in (9) is a stochastic volatility model, where the speed of mean reversion of the instantaneous volatility σ_t is stochastic but fully correlated with σ_t. In the literature, there have already been some attempts to consider an extension of the Heston model by assuming the mean-reversion level θ is stochastic (see Bi *et al* 2016; Gatheral 2008). In particular, it is shown in Gatheral (2008) that such a model is able to replicate a term structure of volatility index options. However, to the best of our knowledge, stochastic mean-reversion speed has not yet been considered. In what follows, for the sake of brevity, we call this the ADO lognormal (ADOL) model. In our model, the volatility-of-volatilities (vol-of-vol) is time dependent.

From the definition in (9), the drift D_v vanishes at $t = 0$ and the mean-reversion speed of σ_t at the origin becomes $-k$, ie, it is well defined for all $H \in [0, 1]$.

It is worth mentioning the drift in the last equation of (9) is complex. However, the only requirement of the model is that the market 'observables' produced by the model are real. In our case, these are the stock price S_t and the volatility σ_t. All other parameters of the model not explicitly observable may be complex and could be found by calibrating the model to the market data. As will be seen below, under the risk-neutral measure, σ_t is real when all the parameters of the model are real.

The ADOL partial differential equation (PDE)

In order to price options written on the underlying stock price S_t, which follows the ADOL model, a standard approach can be utilised (Rouah 2013).

Denote by $V = V(S, \sigma, \mathcal{V}, t)$ the price of this option. Then, as shown in Carr & Itkin (2019), this solves the following PDE:

$$0 = \frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \tfrac{1}{2}\xi^2\sigma^2\nu^2(t)\frac{\partial^2 V}{\partial \sigma^2} + \tfrac{1}{2}\nu^2(t)\frac{\partial^2 V}{\partial \mathcal{V}^2}$$
$$+ \rho S\xi\sigma^2\nu(t)\frac{\partial^2 V}{\partial S\partial\sigma} + \rho S\sigma\nu(t)\frac{\partial^2 V}{\partial S\partial\mathcal{V}} + \xi\sigma\nu^2(t)\frac{\partial^2 V}{\partial\mathcal{V}\partial\sigma}$$
$$+ (r-q)S\frac{\partial V}{\partial S} + (\bar{D}_v - f)\frac{\partial V}{\partial \mathcal{V}} + \sigma[-\kappa + \xi(\bar{D}_v - f)]\frac{\partial V}{\partial\sigma} - rV \tag{10}$$

To proceed, we choose an explicit form of $f(S, v, \mathcal{V}, t)$ to be $f = \bar{D}_v + \lambda + m(t)\mathcal{V}$, with $m(t)$ some function of time t. Since in (9) the drift of σ_t is already a linear function of $\mathcal{V}_t$ under a physical measure, the proposed construction keeps it linear under the risk-neutral measure as well. With this definition, (10) takes the form:

$$0 = \frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \tfrac{1}{2}\xi^2\sigma^2\nu^2(t)\frac{\partial^2 V}{\partial \sigma^2}$$
$$+ \tfrac{1}{2}\nu^2(t)\frac{\partial^2 V}{\partial \mathcal{V}^2} + \rho S\xi\sigma^2\nu(t)\frac{\partial^2 V}{\partial S\partial\sigma} + \rho S\sigma\nu(t)\frac{\partial^2 V}{\partial S\partial\mathcal{V}}$$
$$+ \xi\sigma\nu^2(t)\frac{\partial^2 V}{\partial\mathcal{V}\partial\sigma} + (r-q)S\frac{\partial V}{\partial S} - [\lambda + m(t)\mathcal{V}]\frac{\partial V}{\partial\mathcal{V}}$$
$$- [\kappa + \xi(\lambda + m(t)\mathcal{V})]\sigma\frac{\partial V}{\partial\sigma} - rV. \tag{11}$$

As shown in Carr & Itkin (2019), the PDE in (11) corresponds to the following model under the risk-neutral measure $\mathbb{Q}$:

$$\left.\begin{aligned} dS_t &= S_t(r-q)\,dt + S_t\sigma_t\,dW^Q_{1,t} \\ d\sigma_t &= -[\kappa + \xi(\lambda + m(t)\mathcal{V}_t)]\sigma_t\,dt + \sigma_t\xi\nu(t)\,dW^Q_{2,t} \\ d\mathcal{V}_t &= -[\lambda + m(t)\mathcal{V}_t]\,dt + \nu(t),dW^Q_{2,t} \end{aligned}\right\} \tag{12}$$

When this model is used for option pricing, with parameters obtained by calibrating the model to market options prices, one is already in the risk-neutral setting (see Carr & Itkin (2019) for more detail). This allows us to set the market price of volatility risk λ equal to zero.

The model for σ_t in (12) is similar to that introduced in Benth & Khedher (2016) (see Carr & Itkin (2019) for more detail).

CF of log S_T under the ADOL model

One of the main reasons the Heston model is so popular is the CF of log S_T in this model is known in closed form. Then, any fast Fourier transform-based method can be used to price European (and even American) options written on the underlying stock S_t.

Let us denote by T the option maturity. We use:

$$\mathbb{E}[e^{iu\log S_T} \mid S, v, \mathcal{V}] = e^{iu\log S}\psi(u; x, \sigma, \mathcal{V}, t)$$

to represent the CF, where:

$$\psi(u; x, \sigma, \mathcal{V}, \tau) = \mathbb{E}[e^{iux}] \quad \text{and} \quad x = \log S_T/S$$

As per the Feynman-Kac theorem, $\psi(u; x, \sigma, \mathcal{V}, \tau)$ solves a PDE similar to (11) but with no discounting term rV:

$$0 = \frac{\partial\psi}{\partial t} + \tfrac{1}{2}\sigma^2\frac{\partial^2\psi}{\partial x^2} + \tfrac{1}{2}\xi^2\sigma^2\nu^2(t)\frac{\partial^2\psi}{\partial\sigma^2} + \tfrac{1}{2}\nu^2(t)\frac{\partial^2\psi}{\partial\mathcal{V}^2}$$
$$+ \rho\xi\sigma^2\nu(t)\frac{\partial^2\psi}{\partial x\partial\sigma} + \rho\sigma\nu(t)\frac{\partial^2\psi}{\partial x\partial\mathcal{V}} + \xi\sigma\nu^2(t)\frac{\partial^2\psi}{\partial\mathcal{V}\partial\sigma}$$
$$+ (r - q - \tfrac{1}{2}\sigma^2)\frac{\partial\psi}{\partial x} - m(t)\mathcal{V}\frac{\partial\psi}{\partial\mathcal{V}} - (\kappa + \xi m(t)\mathcal{V})\sigma\frac{\partial\psi}{\partial\sigma} \tag{13}$$

subject to the initial condition $\psi(u; x, \sigma, \mathcal{V}, T) = 1$.

We will search for a solution of this PDE in the form:

$$\psi(u; x, \sigma, \mathcal{V}, t) = e^{iux}z(u; t, \sigma, \mathcal{V}) \tag{14}$$

Substituting (14) into (13) yields:

$$0 = \frac{\partial z}{\partial t} + \tfrac{1}{2}\xi^2\nu^2(t)\sigma^2\frac{\partial^2 z}{\partial\sigma^2} + \tfrac{1}{2}\nu^2(t)\frac{\partial^2 z}{\partial\mathcal{V}^2} + \xi\nu^2(t)\sigma\frac{\partial^2 z}{\partial\mathcal{V}\partial\sigma}$$
$$+ [-(\kappa + \xi m(t)\mathcal{V}) + iu\rho\xi\nu(t)\sigma]\sigma\frac{\partial z}{\partial\sigma}$$
$$+ [iu\rho\nu(t)\sigma - m(t)\mathcal{V}]\frac{\partial z}{\partial\mathcal{V}} + [-\tfrac{1}{2}u(i+u)\sigma^2 + iu(r-q)]z \tag{15}$$

which should be solved subject to the initial condition $z(u; T, \sigma, \mathcal{V}) = 1$.

To the best of our knowledge, this PDE does not have a closed-form solution. However, an approximate solution can be constructed. In particular, in what follows, we assume the vol-of-vol parameter ξ is small. More rigorously, we observe that in the second line of (12) the term $\Psi = \xi\nu(t)\,dW^Q_{2,t}$ is dimensionless. As $dW^Q_{2,t} \propto 1/(2\sqrt{t})$ and $\nu(t) = B_H t^{H-1/2}$, we have:

$$\Psi \propto \xi B_H t^H/2 = (\xi B_H T^H/2)(t/T)^H$$

Suppose we consider only time intervals $0 \leqslant t \leqslant T$; hence, $0 \leqslant (t/T)^H \leqslant 1$. Then, our assumption on ξ being small means:

$$\frac{\xi B_H T^H}{2} \ll 1 \tag{16}$$

With allowance for this assumption, we construct the solution of (14) as follows. Let us represent the solution of (14) as a series:

$$z(u; t, \sigma, \mathcal{V}) = \sum_{i=0}^{\infty}\xi^i z_i(u; t, \sigma, \mathcal{V}) \tag{17}$$

Substituting this representation into (14) yields:

$$0 = \sum_{i=0}^{\infty}\xi^i\frac{\partial z_i}{\partial t} + \sum_{i=0}^{\infty}\xi^i\mathcal{L}z_i + \tfrac{1}{2}\sum_{i=0}^{\infty}\xi^{i+2}\nu^2(t)\sigma^2\frac{\partial^2 z_i}{\partial\sigma^2}$$
$$+ \sum_{i=0}^{\infty}\xi^{i+1}\left[\nu^2(t)\sigma\frac{\partial^2 z_i}{\partial\mathcal{V}\partial\sigma} + (m(t)\mathcal{V} + iu\rho\nu(t)\sigma)\sigma\frac{\partial z_i}{\partial\sigma}\right] \tag{18}$$

$$\mathcal{L} = \tfrac{1}{2}\nu^2(t)\frac{\partial^2}{\partial\mathcal{V}^2} - \kappa\sigma\frac{\partial}{\partial\sigma} + [iu\rho\nu(t)\sigma - m(t)\mathcal{V}]\frac{\partial}{\partial\mathcal{V}}$$
$$+ [-\tfrac{1}{2}u(i+u)\sigma^2 + iu(r-q)]$$

It is clear that the terms in the second line of (18) have a higher order in ξ and as such do not contribute, for example, to the zero-order solution. For higher-order approximations, however, they appear as source terms. In other words, the terms in the second line have no influence on Green's function of (18). This makes finding the solution to (18) much easier.

Zero-order solution of (18). In the zero-order approximation of ξ, (18) transforms into:

$$0 = \frac{\partial z_0}{\partial t} + \mathcal{L}z_0 \tag{19}$$

This equation could be solved in a few steps. First, we make a change to the dependent variable:

$$z_0(u; t, \sigma, \mathcal{V}) \mapsto y_0(u; t, \sigma, \mathcal{V})\exp[\alpha(t) + \gamma(t)\sigma^2 + \beta(t)\sigma\mathcal{V}] \tag{20}$$
$$\alpha(t) = -iu(r-q)(t-T), \quad \beta(t) = -i\rho u\frac{1}{\nu(t)}$$
$$\gamma(t) = -\frac{u[1 + u(1-\rho)^2]}{4\kappa}(1 - e^{2\kappa(t-T)})$$

With the new variable $y_0(u;t,\sigma,\mathcal{V})$, (19) transforms into:

$$0 = \frac{\partial y_0}{\partial t} + \tfrac{1}{2}\nu^2(t)\frac{\partial^2 y_0}{\partial \mathcal{V}^2} - m(t)\mathcal{V}\frac{\partial y_0}{\partial \mathcal{V}} - \kappa\sigma\frac{\partial y_0}{\partial \sigma} + \mathrm{i}\rho\sigma\mathcal{V}\frac{\nu'(t) + \nu(t)[\kappa + m(t)]}{\nu(t)^2} y_0 \quad (21)$$

and should be solved subject to the initial (terminal) condition:

$$y_0(u;T,\sigma,\mathcal{V}) = \mathrm{e}^{-\beta(T)\sigma\mathcal{V}}$$

Second, we introduce a new independent variable $\sigma \mapsto g = \sigma\mathcal{V}$ and search for a solution to the dependent variables $y_0(u;t,g,\mathcal{V})$ in the form:

$$y_0(u;t,g,\mathcal{V}) = Y_1(u;t,g)Y_2(u;t,\mathcal{V}) \quad (22)$$

It turns out that, after some algebra on (21), the new variables can be represented in the form:

$$\frac{1}{Y_1}\frac{\partial Y_1}{\partial t} - \frac{\kappa g}{Y_1}\frac{\partial Y_1}{\partial g} + \mathrm{i}\rho g\frac{\nu'(t) + \nu(t)[\kappa + m(t)]}{\nu(t)^2} = -\frac{1}{Y_2}\frac{\partial Y_2}{\partial t} + m(t)\mathcal{V}\frac{1}{Y_2}\frac{\partial Y_2}{\partial \mathcal{V}} - \frac{n(t)^2}{2Y_2}\frac{\partial^2 Y_2}{\partial \mathcal{V}^2} \quad (23)$$

This equation has to be solved subject to the terminal condition:

$$Y_1(t,g)Y_2(t,\mathcal{V}) = \mathrm{e}^{-\beta(T)g}$$

Hence, we may set the independent terminal conditions Y_1 and Y_2 as:

$$Y_1(u;T,g) = \mathrm{e}^{-\beta(T)g}, \qquad Y_2(u;T,\mathcal{V}) = 1 \quad (24)$$

A standard approach tells us that since the left-hand side of (23) is a function of (t,g) only and the right-hand side of (23) is a function of $(t,\mathcal{V})$ only, both parts must be a function of t only. In our case, we can choose this function to be zero. This splits (23) into two independent equations:

$$\left.\begin{aligned} \frac{\partial Y_1}{\partial t} &= \kappa g\frac{\partial Y_1}{\partial g} - \mathrm{i}\rho g\frac{\nu'(t) + \nu(t)[\kappa + m(t)]}{\nu(t)^2}Y_1 \\ \frac{\partial Y_2}{\partial t} &= m(t)\mathcal{V}\frac{\partial Y_2}{\partial \mathcal{V}} - \tfrac{1}{2}\nu(t)^2\frac{\partial^2 Y_2}{\partial \mathcal{V}^2} \end{aligned}\right\} \quad (25)$$

The first equation in (25) is a first-order PDE (of the hyperbolic type), which can easily be solved in closed form to get:

$$Y_1(u;t,g) = \exp\left[-\beta(T)g\mathrm{e}^{k(t-T)} - \mathrm{i}g\rho u\int_T^t \frac{\nu(t)[k + m(t)] + \nu'(t)}{\nu(t)^2}\,\mathrm{d}t\right] \quad (26)$$

The second equation is a convection-diffusion PDE, which can be reduced to the heat equation. For instance, this can be achieved by performing a change of independent variables:

$$Y_2(u;t,\mathcal{V}) = \mathrm{e}^{\alpha_1(t)\mathcal{V}+\tau(t)}w(\tau,\varsigma), \quad (27)$$

$$\alpha_1(t) = \mathrm{e}^{\int_T^t m(t)\,\mathrm{d}t}, \qquad \tau(t) = -\frac{1}{2}\int_T^t \nu^2(s)\alpha_1^2(s)\,\mathrm{d}s$$

$$\varsigma = \alpha_1(t)\mathcal{V} + 2\tau(t)$$

In particular, by this transformation, the terminal point $t = T$ is mapped to $\tau = 0$. However, due to the terminal condition $Y_2(u;T,\mathcal{V}) = 1$, the solution is just a constant $Y_2(u;t,\mathcal{V}) = 1$ for all $t \in [0,T]$.

Thus, combining all of the above expressions into (20), we obtain:

$$\begin{aligned} z_0(u;t,\sigma,\mathcal{V}) &= \exp[\alpha(t) + \gamma(t)\sigma^2 + \bar\beta(t)\sigma\mathcal{V}] \\ \bar\beta(t) &= \beta(t) - \beta(T)\mathrm{e}^{k(t-T)} - \mathrm{i}\rho u\int_T^t \frac{\nu(t)[k + m(t)] + \nu'(t)}{\nu(t)^2}\,\mathrm{d}t \\ &= \mathrm{i}\rho u\left[\frac{\mathrm{e}^{k(t-T)}}{\nu(T)} - \frac{1}{\nu(t)} - \int_T^t \frac{k + m(t)}{\nu(t)}\,\mathrm{d}t\right] \end{aligned} \quad (28)$$

With this expression, the final representation of the CF in (14) reads:

$$\begin{aligned} \psi(u;x,\sigma,\mathcal{V},t)|_{t=0} &= \exp[\mathrm{i}ux + \alpha(0) + \gamma(0)\sigma^2 + \bar\beta(0)\sigma\mathcal{V}] \\ \alpha(0) = \mathrm{i}u(r-q)T, \quad & \gamma(0) = -\frac{u[1 + u(1-\rho)^2]}{4\kappa}(1 - \mathrm{e}^{-2\kappa T}) \\ \bar\beta(0) &= \mathrm{i}\rho u\left[\frac{\mathrm{e}^{-kT}}{\nu(T)} - \frac{1}{\nu(0)} + \int_0^T \frac{k + m(t)}{\nu(t)}\,\mathrm{d}t\right] \end{aligned} \quad (29)$$

As $\nu(t) = B_H t^{H-1/2}$, the expression for $\bar\beta(0)$ is well defined only for $H < 1/2$. Then, $1/\nu(0) = 0$.

■ **First- and second-order solution of (18).** For the construction of the first- and second-order approximations, we refer the reader to Carr & Itkin (2019). In brief, in the first-order approximation, we obtain $z(u;t,\sigma,\mathcal{V}) = z_0(u;t,\sigma,\mathcal{V}) + \xi z_1(u;t,\sigma,\mathcal{V})$. It can be shown that:

$$z_1(u;t,\sigma,\mathcal{V}) = \Phi_1 I \quad (30)$$

$$I = \int_t^T\int_{-\infty}^{\infty}\int_0^{\infty} z_0(u;k,\sigma',\mathcal{V}')\mathcal{G}_1(\sigma,\sigma',\mathcal{V},\mathcal{V}',k-t)\,\mathrm{d}\sigma'\,\mathrm{d}\mathcal{V}'\,\mathrm{d}k$$

$$\Phi_1 = \nu^2(t)\sigma\frac{\partial^2}{\partial\mathcal{V}\partial\sigma} + [m(t)\mathcal{V} + \mathrm{i}u\rho\nu(t)\sigma]\sigma\frac{\partial}{\partial\sigma}$$

where $\mathcal{G}_1(\sigma,\sigma',\mathcal{V},\mathcal{V}',t)$ is Green's function, which can be expressed via Green's function for the zero-order approximation and thus is known. As shown in Carr & Itkin (2019):

$$I = \int_t^T \frac{1 - \Theta(t-\chi) + \Theta(0)}{2\sqrt{\pi(\chi - t)}}\mathrm{e}^{\alpha(\chi)-\chi+f_1(\chi)\omega}\mathcal{J}(t,\varsigma,\omega;\chi)\,\mathrm{d}\chi \quad (31)$$

$$\mathcal{J}(t,\varsigma,\omega;\chi) = \int_{-\infty}^{\infty}\exp\left[\omega^2\mathrm{e}^{-2\kappa\chi}\frac{\gamma(\chi)\alpha_1^2(\chi)}{(\varsigma' - 2\chi)^2} + \varsigma' - \frac{(\varsigma' - \varsigma)^2}{4(\chi - t)}\right]\mathrm{d}\varsigma'$$

$$f_1(\chi) = \bar\beta(\chi)\mathrm{e}^{-\kappa\chi} + \int_0^{\chi - t} a_1(k)\,\mathrm{d}k$$

$$a_1(t) = 2\mathrm{i}\rho u\frac{\nu(t)(\kappa + m(t)) + \nu'(t)}{\nu(t)^4}\mathrm{e}^{-\kappa t - 2\int_T^t m(t)\,\mathrm{d}t}$$

Here, $\Theta(\tau)$ is the Heaviside theta function, $\omega = \mathrm{e}^{\kappa t}\sigma\mathcal{V}$. From the definition of $\gamma(\chi)$ in (20), $\gamma(\chi) \leqslant 0$. Therefore, the second integral in (31) is well defined. Finally, to obtain $z_1(u;t,\sigma,\mathcal{V})$, in (31) we set $t = 0$.

■ **An example.** Looking closely at the integrand of $\mathcal{J}(0,\varsigma,\omega;\chi)$ in (31), one can observe that it behaves as follows:

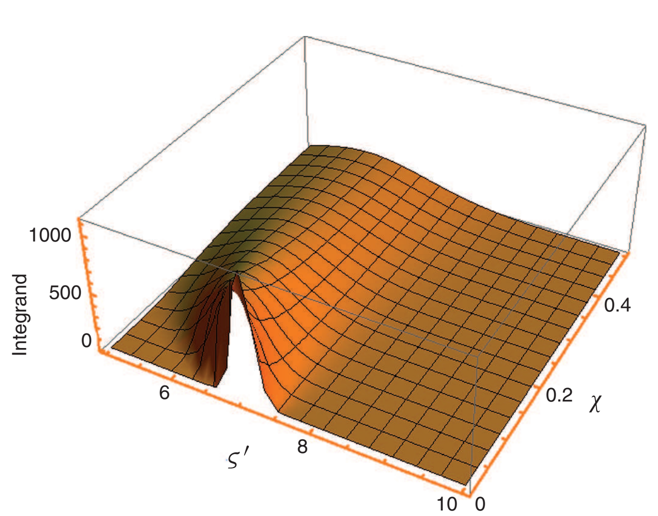
- Suppose we consider options with maturities $T < 1$ year. Since $0 \leqslant \chi \leqslant T$, at ς' far away from ς the term $(\varsigma' - \varsigma)^2/4(\chi - t)$ is large. Therefore, for these regions of ς' the integrand almost vanishes.
- Also, at large $|\varsigma'|$ where $(\varsigma' - 2x)^2$ is also large, the term:

$$\omega^2\mathrm{e}^{-2\kappa\chi}\frac{\gamma(\chi)\alpha_1^2(\chi)}{(\varsigma' - 2\chi)^2} = -\sigma^2\mathcal{V}^2\frac{u[1 + u(1-\rho)^2]}{4\kappa}(1 - \mathrm{e}^{2\kappa(\chi - T)})\frac{\alpha_1^2(\chi)}{(\varsigma' - 2\chi)2}$$

is small for fixed $\sigma, \mathcal{V}, u$.

A. Parameters of the test								
κ	H	T	σ_0	$\mathcal{V}_0$	ρ	u	ϱ	π
2.0	0.3	0.5	0.3	5.0	−0.5	1.0	1.0	0.5

1 Integrand of $\mathcal{J}(0,\varsigma,\omega;\chi)$ as a function of (ς',χ) at $0 \leqslant \chi \leqslant T$



Thus, the integrand of $\mathcal{J}(0,\varsigma,\omega;\chi)$ has a bell shape with a maximum close to the point $\varsigma' = \varsigma_*$; this solves the equation:

$$\partial_{\varsigma'}\left[\omega^2 \mathrm{e}^{-2\kappa\chi}\frac{\gamma(\chi)\alpha_1^2(\chi)}{(\varsigma'-2\chi)^2}+\varsigma'-\frac{(\varsigma'-\varsigma)^2}{4(\chi-t)}\right]=0 \tag{32}$$

Indeed, consider an example with the explicit form of the function $m(t) = \varrho t^{\pi}$, $\varrho, \pi \in \mathbb{R}$, $\pi \geqslant 0$, so the stochastic differential equation (SDE) for $\mathcal{V}_t$ in (12) is mean reverting. With this $m(t)$, one can find:

$$\gamma(t) = \frac{B_H^2}{2(1+\pi)}\mathrm{e}^{-(\varrho/(1+\pi))T^{1+\pi}}\left[t^{2H} E\left(1-\frac{2H}{1+\pi}, -\frac{\varrho t^{1+\pi}}{1+\pi}\right) - T^{2H} E\left(1-\frac{2H}{1+\pi}, -\frac{\varrho T^{1+\pi}}{1+\pi}\right)\right] \tag{33}$$

where $E(k,z)$ is the exponential integral function. Let us also use the values of our model parameters given in table A.

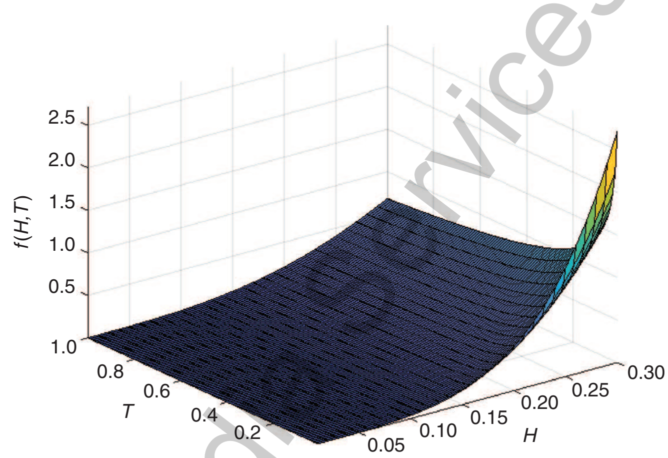
Now the integrand of $\mathcal{J}(0,\varsigma,\omega;\chi)$ can be computed explicitly, and the result is presented in figure 1. The bell shape of this function can be clearly seen at small T.

The bell shape of the integrand implies the integral $\mathcal{J}(0,\varsigma,\omega;\chi)$ can be computed approximately in closed form to yield (Carr & Itkin 2019):

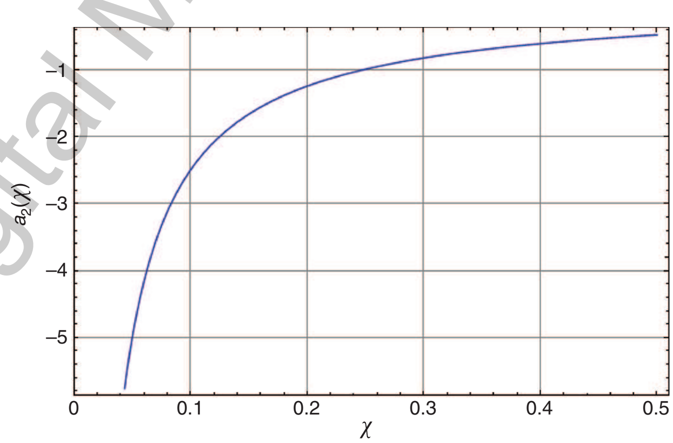
$$\mathcal{J}(0,\varsigma,\omega;\chi) = \sqrt{\frac{\pi}{-a_2}}\mathrm{e}^{a_0-(a_1^2/4a_2)}$$
$$a_0 = \varsigma + \frac{k(\chi)}{(\varsigma-2\chi)^2}, \qquad a_1 = 1 - \frac{2k(\chi)}{(\varsigma-2\chi)^3}$$
$$a_2 = -\frac{1}{4\chi} + \frac{3k(\chi)}{(\varsigma-2\chi)^4}, \qquad k(\chi) = \gamma(\chi)\alpha_1^2(\chi)\sigma\mathcal{V} \tag{34}$$

Figure 3 demonstrates function $a_2(\chi)$ computed in this experiment, which turns out to be negative for all values of $0 \leqslant \chi \leqslant T$. Then, figure 4 presents the percentage difference between the value of $\mathcal{J}(0,\varsigma,\omega;\chi)$ computed numerically and using (34). The difference is about 5 basis points, so in our test this approximation works pretty well.

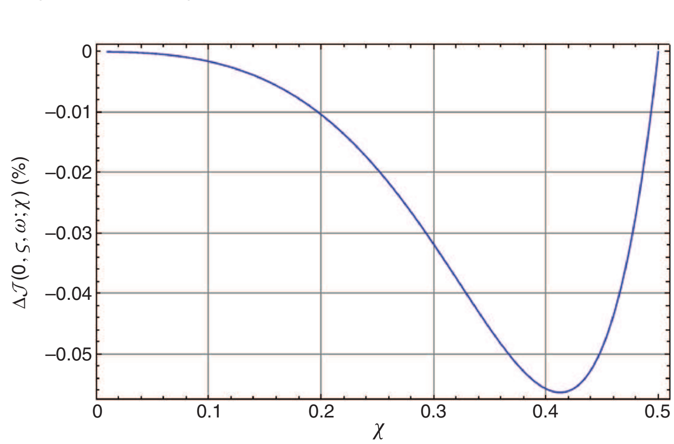
2 Function $f(H,T)$ for $H \in [0, 0.3]$ and $T \in [0, 1]$ years



3 Function $a_2(\chi)$ computed for our experiment



4 Difference in percent between $\mathcal{J}(0,\varsigma,\omega;\chi)$ obtained using numerical integration and using (34)



Discussion

As the CF of $\log S_T$ is known in closed form (in our case, this is an approximation of the exact solution constructed using power series in ξ), pricing options can be conducted in a standard way using fast Fourier transforms. In turn, pricing variance swaps can be conducted using a forward CF, similar to in Itkin & Carr (2010). Using the forward time t, the forward CF is defined as:

$$\phi_{t,T} = \mathbb{E}_{\mathbb{Q}}[\exp(\mathrm{i}u\mathcal{X}_{t,T}) \mid S_0, \sigma_0]$$

where $\mathcal{X}_{t,T} = \mathcal{X}_T - \mathcal{X}_t$ and $\mathcal{X}_t = \log S_t$. Then, under a discrete set of observations of the stock price at times t_i, $i \in [1, N]$, the quadratic variation $\mathcal{Q}_N(x)$ of S_t is given by (Itkin & Carr 2010):

$$\begin{aligned}\mathcal{Q}_N(s) &= \frac{1}{T}\sum_{i=1}^{N}\mathbb{E}_{\mathbb{Q}}[(\mathcal{X}_{t_i} - \mathcal{X}_{t_{i-1}})^2] \\ &= \frac{1}{T}\sum_{i=1}^{N}\mathbb{E}_{\mathbb{Q}}[\mathcal{X}^2_{t_i,t_{i-1}}] = -\frac{1}{T}\sum_{i=1}^{N}\frac{\partial^2\phi_{t_i,t_{i-1}}(u)}{\partial u^2}\bigg|_{u=0} \end{aligned} \tag{35}$$

As:

$$\begin{aligned}\mathcal{X}_{t_i,t_{i-1}} &= \log S_{t_i} - \log S_{t_{i-1}} \\ &= x_{T-t_{i-1}} - x_{T-t_i}\end{aligned}$$

$\mathcal{Q}_N(s)$ in (35) can be computed in a similar way to in our section titled 'CF of $\log S_T$ under the ADOL model'.

Therefore, the proposed model could be useful, eg, for pricing options and swaps, as, on the one hand, it captures some properties of rough volatility while, on the other hand, it is more tractable. We underline that our approach allows the CF to be found as the solution to the PDE in (13). This PDE, in general, can be solved numerically. However, in this article we provide a closed-form series solution obtained by assuming the vol-of-vol ξ to be small. This condition is defined in (16), so, as can be seen, it is a function of the Hurst exponent H and the time to maturity T. The function $f(H,T) = 2/(B_H T^H)$ for various values of H and T is represented in figure 2 (as the range $H \in [0, 0.3]$ is reported in the literature to be important).

Overall, the values of $f(H,T)$ look reasonable compared with those reported in the literature, for instance, for the Heston model. In other words, the values of the vol-of-vol parameter ξ, found by calibration of the Heston model to market prices of European vanilla options, could be of the order of magnitude required to conform to (16) for $H > 0.1$ and $T > 0.1$. However, this should definitely be justified by independent calibration of the ADOL model to that market data. This calibration would require solving the PDE in (13) numerically so we are not relying on the assumption (16). These results will be reported elsewhere.

In addition, the proposed model should be calibrated to the option market to figure out whether it is capable of capturing the behaviour of options skew as rough volatility models do. We recognise the importance of this step, which is why this is the subject of another article we are currently working on.

Another assumption we made when deriving (9) is the mean-reversion level $\theta = 0$. As shown in Carr & Itkin (2019), this can be easily relaxed.

Several questions remain open concerning the proposed model (this was, in particular, mentioned by reviewers). One is whether the SDEs in (9) and (12) have solutions for all times $t > 0$. Another is whether the solution of the PDE in (10) is real analytic or at least smooth in the vol-of-vol parameter. ■

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