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Pricing and hedging in incomplete markets[☆]

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Abstract

We present a new approach for positioning, pricing, and hedging in incomplete markets that bridges standard arbitrage pricing and expected utility maximization. Our approach for determining whether an investor should undertake a particular position involves specifying a set of probability measures and associated floors which expected payoffs must exceed in order for the investor to consider the hedged and financed investment to be acceptable. By assuming that the liquid assets are priced so that each portfolio of assets has negative expected return under at least one measure, we derive a counterpart to the first fundamental theorem of asset pricing. We also derive a counterpart to the second fundamental theorem, which leads to unique derivative security pricing and hedging even though markets are incomplete. For products that are not spanned by the liquid assets of the economy, we show how our methodology provides more realistic bid–ask spreads. © 2001 Published by Elsevier Science S.A.

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1. Introduction

Opportunities to transform initial certain wealth into random future wealth abound in a thriving economy. Thus, a fundamental problem in investment theory concerns the issue of whether or not to undertake such an opportunity. To simplify the problem, one often assumes that the opportunity vanishes forever if it is initially rejected. For further simplification, the scale of the opportunity is usually taken to be fixed, at least at the outset.

The purpose of this paper is to propose a new approach for deciding whether or not to accept opportunities of this type. Our approach is intermediate between expected utility theory and arbitrage pricing theory in terms of both the initial information required and the power of the conclusions derived. Hence, to place our approach in the proper context, we now review these two fundamental paradigms.

Expected utility maximization is a powerful tool for deciding whether or not to accept a project at a given time and scale. So long as an investor's behavior is consistent with the von Neumann Morgenstern axioms (see Varian, 1984), an investor accepts an opportunity if and only if it increases her expected utility. While expected utility maximization has a long history and a strong theoretical appeal, it has had limited acceptance in practice. While this negative result could be due to real-world violation of the behavioral axioms, a more compelling reason involves the difficulty inherent in specifying the required inputs to the optimization. These inputs include the current endowment, the joint stochastic process over all assets, and the utility function over all certain wealth levels. In practice, corporations typically fail to fully specify these three fundamental constructs in making capital budgeting decisions, and our experience is that even professional investors are generally unwilling to explicitly specify these three inputs when making investment decisions. Even if the constructs are inferred through past decisions, the revealed constructs are often inconsistent over time and across assets.¹ This inconsistency would be benign if the recommended action were robust to the particular specification of endowments, beliefs, and preferences. Unfortunately, the maximization is notoriously sensitive to these inputs, whose formulation is suspect at the outset. This shortcoming renders the methodology potentially useless, primarily because the decisions consistent with the inputs used in the optimization may be seriously disputed by other perspectives.

To draw some inferences on these other perspectives, it is widely acknowledged that market prices of related instruments provide useful informational inputs for the decision process. So long as these prices are liquid, their levels reflect a panoply of endowments, beliefs, and preferences. If an opportunity is undertaken, the relevance of related liquid market prices is enhanced by the

¹See the substantial literature on the Allais (1987) paradox.

observation that unforeseen contingencies can often induce the liquidation of a position prior to the anticipated exit time. The possibility of early exit underscores the advantages of developing a marking model, which links future market prices of these instruments to the associated liquidation value of the proposed opportunity. This pricing function imbues initial market prices with additional significance, in that these prices are closely related to their expected future values. Furthermore, the historical variability of these prices are often used in conjunction with the pricing model in order to assess the future variability of liquidation levels. When this variability is deemed to be too large, opportunities which increase a decision-maker's expected utility over a particular horizon are sometimes rejected in practice. Since the decision maker's endowments, preferences, and subjective probabilities are not market-determined constructs, reliance on them may be subsequently regretted if the position is reversed prematurely. As a consequence, methods that place greater emphasis on market observables provide greater protection against early reversals.

Arbitrage pricing is an example of a theory that relies heavily on the existence of related market prices. The decision of whether or not to undertake a specific opportunity is easily solved in this framework when the payoffs from the opportunity are spanned by the payoffs from traded assets. When the opportunity's payoff is spanned, an associated present value can be determined and compared to the initial cost of the opportunity. An investor preferring more to less takes those and only those spanned opportunities that result in a positive net present value (NPV). By undertaking the opportunity and assuming an opposite position in a replicating portfolio, the investor synthesizes an arbitrage opportunity. Note that the identification of the opportunity as an arbitrage does not require the investor to specify personal constructs such as positions, probabilities, and preferences. Instead, the investor specifies certain market constructs such as price processes and trading opportunities. Part of the appeal of this approach is that the policy of accepting all arbitrage opportunities is known to be consistent with expected utility maximization for any combination of endowments, beliefs, and increasing preferences. Thus, the difficulties inherent in specifying these constructs are neatly bypassed whenever the payoffs from the opportunity are spanned by the payoffs from the traded assets.

One is assured that the payoffs to any opportunity are in fact spanned whenever the market is taken to be complete. This feature likely explains the prevalence in practice of models which presume market completeness. Unfortunately, the conditions required for models to produce market completeness are generally quite stringent. For example, in the models of Black and Scholes (1973), Merton (1973), and Heath et al. (1992), the state space is reduced significantly by requiring continuous price processes, while the asset space is inflated substantially by allowing continuous trading. Unfortu-

nately, empirical work seems to suggest that realistic price processes and trading opportunities do not typically permit market completeness. For example, Figlewski and Green (1999) recently document the substantial risks involved in hedging the sale of options using the Black-Scholes model, even when optimal forecasts of volatility are used in pricing and hedging. Furthermore, it is likely that the artifices inherent in complete market models cannot be alleviated by generalizations that expand the state space to include stochastic volatility and jumps, since these models must also expand the asset space to accommodate continuous trading in options. As the latter markets are not liquid enough at present to support even an approximation of continuous trading, it is widely appreciated both in academia and in practice that market incompleteness is a pervasive phenomenon that must be addressed in order to actualize any proposed theory of asset allocation.

Unfortunately, when the payoffs from an opportunity do not lie in the span of the traded assets, the investment decision becomes more complicated. Although the present value operator is not unique, it can nonetheless happen that the investment opportunity can be combined with the traded assets so as to become an arbitrage. When this incidence occurs, an investor should definitely accept the opportunity. However, when an arbitrage cannot be formed, then it is not clear that the opportunity under consideration should be rejected. Thus, the complications that arise in incomplete markets essentially surround the rejection criterion. An investor who rejects all opportunities that cannot be converted to arbitrages is likely to be foregoing opportunities that would increase expected utility for a wide range of plausible beliefs and preferences. Empirical work suggests that the bid–ask spreads observed in practice reflect the willingness of market-makers to accept controlled risks (see Figlewski, 1989). Essentially, market-makers who can measure their risks can set their spreads to levels at which the potential for losses by competitors is inevitable.

The asset pricing literature has attempted to identify the nature of opportunities that should be accepted, despite their potential for losses. For example, Cochrane and Saá Requejo (2000), Ledoit (1995), and Bernardo and Ledoit (2000) all use the Sharpe ratio as a criterion for acceptability. As an alternative Bernardo and Ledoit (1999) consider investments with high ratios of gains to losses. These approaches require the investor to specify subjective probabilities, and minimize the role of related market prices.

To avoid these drawbacks, alternative work has focussed on selecting a present value operator out of the arbitrage-free set, and then using this operator to operationalize the standard NPV rule. For example, Rubinstein (1994) chooses the risk-neutral density that minimizes the distance to a prior lognormal density among those consistent with observed option prices. Similarly, Buchen and Kelly (1996), Stutzer (1996), and Avellaneda et al. (1997) minimize cross-entropy relative to a prior density. Alternatively, Hull

and White (1990), Heston (1993), Bates (1996), and Madan et al. (1998) all specify a tractable and flexible family of risk-neutral processes and then estimate parameters from option prices. Aït-Sahalia and Lo (1998) take a non-parametric approach to the same problem. In all of these papers, the relevance of the selected density remains an important issue. Empirical work is needed to assure that alternative criteria are not being used by the market in selecting a risk-neutral density from the possible set.

The purpose of this paper is to propose a new criterion for deciding when the rewards received from accepting an opportunity outweigh the accompanying risks. Our starting point is the observation that an arbitrage is an opportunity that absolutely everyone would accept. It follows from the continuity of preferences that opportunities exist, with mild risks, which would be considered acceptable to all but the most risk-averse. For example, in a simple economy with a riskless asset and a single risky asset with a positive risk premium, Huang and Litzenberger (1988) show, every investor invests a positive amount in the risky asset, unless they have infinite risk aversion. Thus, so long as there are a finite number of individuals each with finite risk aversion, then there always exists a risky investment acceptable to every individual. Hence, we propose to generalize the concept of an arbitrage opportunity to include such risky opportunities. We term an opportunity that is agreeable to a wide variety of reasonable individuals to be an *acceptable opportunity*. The precise definition of acceptable opportunity controls for the reasonableness of the individuals involved by endogenizing this requirement into the definition. We note that this concept contains the class of arbitrage opportunities as an important special case.

The central idea in our definition of acceptability is that every reasonable person would take the view that the benefits engendered by the gains adequately compensate for the costs imposed by the losses. One can regard these persons as counterparties willing to take the other side should one decide to exit after entering. By requiring that each person in a specified set finds the trade agreeable, one can enter the trade assured that there are multiple avenues for exit. Since an expected gain of negative infinity under any probability measure will clearly obviate the required unanimity, a necessary condition on acceptability by the group is that the expected gain under each measure be bounded from below by a finite constant. This constant should furthermore not be allowed to be positive, since requiring that expected gains exceed a positive constant would rule out the acceptability of certain arbitrages. Formally, then, our concept of acceptability requires the specification of a set of probability measures called test measures, and an associated set of non-positive constants called floors. An investment is accepted if and only if the expected gain under each measure at least weakly exceeds its associated floor, for each measure in the specified set. We recognize that many investments require financing. Further, we recognize that investments should be judged

from a portfolio context rather than in isolation. As a result, we define an investment to be acceptable if it can be financed and hedged so that the expected gain under each measure weakly exceeds its associated floor, for each probability measure in the specified set.

In terms of input requirements, our acceptability criterion is intermediate between expected utility maximization and arbitrage pricing theory. In contrast to expected utility maximization and much of the previous literature, we do not require the investor to specify endowments, beliefs, or preferences in the standard manner. We do require a specification of test measures and floors, which can be interpreted as a mechanism for expressing preferences and beliefs. However, our experience suggests that this mechanism is more natural than specifying a single probability measure and utility function. Nonetheless, our input requirements are stronger than those required by arbitrage pricing theory, which merely requires a specification of the state space and the asset space.

Our acceptability criterion is intermediate between arbitrage pricing theory and expected utility maximization in terms of the power of its implications as well. In contrast to arbitrage pricing theory, our approach can decide which risky investments are worth pursuing. However, when compared to expected utility maximization, our approach is comparatively silent with respect to many important economic questions. For example, we do not attempt to solve the difficult problem of determining optimal positions to be taken by one or more market participants. Thus, we do not solve individual decision problems, nor do we offer a general equilibrium theory of market price determination. Instead, our limited focus is on the development of criteria which can be used to support a particular investment decision.

We note that the decisions obtained by applying either traditional arbitrage pricing theory or expected utility maximization to the investment decision can be obtained as special limiting cases of our theory. The policy of accepting only arbitrage opportunities is enacted by having one zero floor measure for each state, such that each measure places unit mass on its associated state and zero mass on all others. Furthermore, the policy of accepting only opportunities which increase expected utility can always be obtained by equating the convex set generated by our test measures with the set of investments which increase the expected utility of the decision-maker. Thus, one can regard our approach as endogenizing the determination of an indifference curve at the decision maker's present endowment level. The approach differs from expected utility theory in that the specification of indifference curves at other wealth levels is not required.

We use our definition of an acceptable opportunity to refine the notion of market efficiency. We define a market to be efficient if there is no portfolio of the liquid assets that would be viewed as beneficial under all of our test measures. Since the restrictions imposed by measures with strictly negative

floors can be nullified by scaling down the portfolio, only the measures with zero floors play any role in determining whether or not a market is efficient by our definition. Measures with a strictly negative floor are termed stress test measures, since their only role is to prevent the excessive scaling up of opportunities which are acceptable on the margin. Measures with a floor of zero are termed valuation test measures since we will show that they play an important role in determining the values to be assigned to opportunities on the margin.

Treating our definition of an acceptable opportunity as a generalization of an arbitrage opportunity, we revisit the two fundamental theorems of Harrison and Kreps (1979). Their first fundamental theorem showed that if the liquid assets are priced so that there are no free lunches, then there exists a positive state pricing function. In the single-period context, a state pricing function is a vector of positive weights summing to one, whose inner product with each asset's payoffs across states yields the asset's forward market price. In analogy to the first fundamental theorem presented in Harrison and Kreps (1979), we examine the implications of the liquid assets being priced so that there are no strictly acceptable opportunities among them. We show that this condition is equivalent to the existence of a vector of positive weights summing to one, whose inner product with each asset's expected payoff across valuation measures yields the asset's forward market price. Thus, just as the first fundamental theorem from Harrison and Kreps (1979) reduces the valuation problem down to the selection of a vector of positive weights to attach to the states, our version of this theorem reduces the valuation problem down to the determination of the vector of positive weights to attach to the valuation measures. The weight on each measure may be viewed as reflecting the importance of that measure in determining the market prices of the liquid assets. In fact, the weight on each measure is the forward price of a portfolio that has an expected payoff of one dollar under that measure and no expected payoff under any other valuation measure. We can thus interpret this portfolio as an Arrow Debreu security whose payoff is defined over the space of valuation measure outcomes, instead of over the original state space.

The importance of the first fundamental theorem stems from its restriction on the range of valuations for new securities that are consistent with the prices of liquid assets. In general, the fewer are the number of states relative to the number of independent liquid asset payoffs, the smaller will be the range of such market consistent values. In our analog of the first fundamental theorem, the payoff in each state is replaced with the expected payoff under each valuation test measure. Thus, when the number of valuation test measures is significantly smaller than the number of states, one would expect a tighter range for the values of new securities. Since prohibiting acceptable opportunities is stronger than banning arbitrage opportunities, this outcome is to be expected.

The second fundamental theorem of Harrison and Kreps (1979) examines the implications for the state pricing function when the number of states equals the number of non-redundant assets. In the single-period context, the result of this market completeness is that the state pricing function is unique. In our context, markets are said to be acceptably complete when the number of non-redundant assets weakly exceeds the number of independent valuation measures. We show that, when markets are acceptably complete, the convex combination of valuation test measures which is consistent with market prices, is uniquely determined. As a result, we are able to uniquely value and hedge assets. When the number of non-redundant assets weakly exceeds the number of independent valuation measures, but is less than the number of states, then one can uniquely value and hedge claims on cash flows, even though the market is classically incomplete. This powerful result arises because our hedges do not eliminate all risk, and instead require that the remaining risk form an acceptable opportunity.

We go on to consider the question of the pricing of claims for non-marginal trades. In incomplete markets, arbitrage pricing theory can be used to determine the bid–ask spread for such trades. The minimum asking price for a derivative security is obtained by determining the smallest initial cost of forming a portfolio of the liquid assets which super-replicates the claim's payoff. Similarly, the maximum bid for a derivative security can be obtained by determining the largest initial inflow received from shorting a portfolio of the liquid assets which sub-replicates the claim's payoff. We generalize these ideas by replacing the non-negativity of the difference between the claim's payoff and its hedge by the acceptability of this difference. The resulting theory is shown to deliver spreads that are substantially smaller than those generated by arbitrage pricing theory, and are thus closer to the spreads observed in practice. We also find that the size of the spread increases with the proposed size of the trade, which is consistent with market practice.

The plan of the paper is as follows. Section 2 presents a simple example illustrating our main ideas. Section 3 sets out the economic model. Section 4 is devoted to the first fundamental theorem, while the second fundamental theorem is considered in Section 5. Section 6 presents our ideas in a continuous state lognormal setting. Section 7 considers the market-maker's problem of determining the bid and ask prices of derivative securities. Section 8 summarizes and concludes.

2. Two example economies

In this section, we present two examples illustrating the operation of our methodology. The first example shows how derivative securities may be uniquely priced in our approach, even when markets are incomplete. Consider

Table 1
Net payoff and weighting matrices for the single-period economy

Panel A describes the payoffs to two assets in the three states and Panel B describes the weighting by two measures on three states.

	States		
	State 1	State 2	State 3
<i>Panel A: Payoffs by asset choice</i>			
Bond	0	0	0
Stock	2	0	-1
<i>Panel B: Weighting by measures</i>			
Measure 1	1/3	1/3	1/3
Measure 2	0	0	-1

a simple single-period economy, with dates 0 and 1. For simplicity, we consider a model with three states, ω_1 , ω_2 , ω_3 and two assets, a unit bond and a stock with payoffs $[3, 1, 0]$ across states. Suppose that each asset costs one dollar initially, and that both assets are financed by borrowing.

Table 1, Panel A displays the net payoff matrix. With 3 states and only 2 opportunities, the market is clearly incomplete. However, one can create any payoff proportional to $[2, 0, -1]$. Additionally, the economy has no arbitrage opportunities.

To determine whether or not there are any acceptable opportunities, we consider two valuation measures, Measure 1 and 2 (see Table 1, Panel B). Recall that a valuation measure has a zero floor by definition. An opportunity has zero cost by definition. By definition, then, an opportunity is acceptable when its expected payoff is non-negative under both measures. An opportunity is strictly acceptable if the expected payoff is also positive under at least one measure. We observe that the second measure, Measure 2, forces acceptable payoffs to have non-negative payoffs in the third state. We further note that all arbitrage opportunities are strictly acceptable. The expected payoff from the financed bond position is zero under each measure, and so it is not strictly acceptable. The expected payoff from the financed stock position is $1/3$ under Measure 1 and -1 under Measure 2, so it is not strictly acceptable. Any portfolio of λ stocks and κ bonds has expected value under Measure 1 of $\lambda/3$ and expected value under Measure 2 of $-\lambda$. Hence, as one of these must be negative, there are no acceptable opportunities in this economy.

Just as the absence of arbitrage implies that no zero-cost portfolio has a payoff in the positive orthant of the space generated by the 3 states, the absence of acceptable opportunities implies that no zero-cost portfolio has expected payoffs in the positive orthant of the space generated by the two valuation measures. Our generalization of the fundamental theorems implies the existence of a pair of weights, which sum to one, and which re-price the

financed stock, as follows:

$$w \frac{1}{3} + (1 - w)(-1) = 0. \quad (1)$$

The solution to Eq. (1) is $w = 3/4$, which implies that the weight vector is $[3/4, 1/4]$. To value a call struck at two, note that the payoff is $[1, 0, 0]$. The expected value of this payoff under Measure 1 is $1/3$ and under Measure 2 is 0, so the call value is $(3/4)(1/3) + (1/4)(0) = 1/4$.

Our second example considers a more general setting in which assets are also uniquely priced by our approach. This example illustrates one technique which can be used to generate the test measures and associated floors that are needed to determine whether an opportunity is acceptable. Although our technique uses the same constructs as an expected utility maximization, one should not interpret the use of these constructs as necessary. Consider a simple single-period economy with dates 0 and 1. To delineate the issues involved in defining acceptable opportunities with minimal complexity, we consider a model with five states and three assets. Uncertainty is completely resolved at Time 1 in one of five states labeled $j = 1, \dots, 5$. The three assets are a unit bond, a stock, and an at-the-money straddle written on the stock. The Time 1 payoff matrix of the three assets over the five states is displayed in Panel A of Table 2. Let this 3×5 payoff matrix be denoted by A . With five states and only three assets, the market is clearly incomplete. The Time 0 prices of the three assets are given by the 3×1 column vector $\pi = [\pi_1, \pi_2, \pi_3]' = [0.9091, 88.1899, 12.3173]'$.

An arbitrage opportunity is a column vector $\alpha = [\alpha_1, \alpha_2, \alpha_3]'$ representing the number of units of the assets held over the period, such that $\pi'\alpha = 0$, $\alpha'A \geq 0$, and $\alpha'A \neq 0$. One can verify that this economy has no arbitrage opportunities.

Table 2

Cash flows for three assets and wealth levels of three individuals in five states of a single-period economy

Panel A describes the cash flows in five states and Panel B describes the wealth for three individuals in five states.

	States				
	State 1	State 2	State 3	State 4	State 5
<i>Panel A: Cash flows</i>					
Bond	1	1	1	1	1
Stock	80	90	100	110	120
Straddle	20	10	0	10	20
<i>Panel B: Wealth</i>					
Individual 1	100	100	100	100	100
Individual 2	80	90	100	110	120
Individual 3	120	110	100	90	80

However, our interest is in defining a larger class of opportunities, which we term acceptable opportunities.

For this purpose, we consider three valuation test measures on our five dimensional space of cash flows. To generate these valuation measures, we consider a set of three investors who have reasonable positions, reasonable preferences, and reasonable prior probability beliefs. The first holds the bond, the second holds the stock, while the third holds the bond and is short the stock. The time one wealth levels of these three positions are given by the matrix shown in Table 2, Panel B.

For simplicity, we assume that all three investors regard all five states as equally likely. We also assume that all three investors have the same utility function, given by $-(W/100)^{-4}$, where W is the level of their wealth at Time 1 and the coefficient of relative risk aversion is 5. Under this assumption, the respective state contingent marginal utilities are proportional to the three column vectors, shown in Table 3.

Let B denote the above 5×3 matrix, shown in Table 3, Panel A, of state contingent marginal utilities for our three individuals. We note that the elements of each column of B are all positive and thus can be made to sum to one upon normalizing. Up to a constant, the columns of B represent the valuation test measures in our economy. Since the valuation test measures are being employed to define sets using non-negativity conditions, it is not necessary to perform the renormalization. If B were to correspond to spot prices, then we would normalize the vector such that the entries would sum to the price of a bond,

Table 3
Marginal utility based and stress motivated measures for five states

Panel A describes the B matrix for three utility based measures and Panel B describes the augmented B matrix with two stress measures.

	Marginal utility			Stress	
	Indiv. 1	Indiv. 2	Indiv. 3	1	2
<i>Panel A: Marginal utility</i>					
First state	1	3.05	0.40		
Second state	1	1.69	0.62		
Third state	1	1	1		
Fourth state	1	0.62	1.69		
Fifth state	1	0.40	3.05		
<i>Panel B: The augmented B</i>					
First state	1	3.05	0.40	1	0
Second state	1	1.69	0.62	0	0
Third state	1	1	1	0	0
Fourth state	1	0.62	1.69	0	0
Fifth state	1	0.40	3.05	0	1

while for futures prices the entries in B would sum to unity. From the viewpoint of defining acceptability, neither transformation is necessary.

Our example should not be construed as the only possible way to generate valuation test measures. The example uses heterogeneity in initial positions as a way to generate heterogeneous valuation measures. Clearly, heterogeneous beliefs or preferences, or both, could also have been used. For example, the three columns of B could have employed differing risk aversion coefficients in each column, as well as differing subjective state probabilities across columns in place of our illustrative use of a uniform relative risk aversion coefficient of 5 and state probabilities of 0.2.

The valuation measures are not risk-neutral measures, nor do they necessarily represent the historical frequency with which states occur. Each measure merely represents a plausible map from payoffs to values which an individual might use to assess the desirability of an investment. For example, in robustly approximating an arbitrage, an investor may wish to require that expected values are positive for each of a set of possible regimes deemed relevant from historical experience, without adopting a single measure that further averages across these regimes.

An opportunity is acceptable when all three individuals view the benefits from potential gains in a marginal position as outweighing the costs of potential losses. Hence, for our individuals, a zero cost portfolio with payoff vector $x = [x_1, x_2, \dots, x_5]'$ satisfies this property if

$$x'B \geq 0. \quad (2)$$

Our hypothetical investors may also be concerned that losses not exceed 50 dollars in the extreme states ω_1 and ω_5 . Letting e_1 and e_5 denote the first and last columns of the 5×5 identity matrix shown in Table 2, Panel B, we thus add the further restrictions:

$$x'e_1 \geq -50 \quad \text{and} \quad x'e_5 \geq -50. \quad (3)$$

Any payoffs meeting conditions shown in Eqs. (2) and (3) are regarded as acceptable, since benefits outweigh costs at the margin and losses are controlled. This restriction leads to an augmented matrix B that is 5×5 with the vectors e_1 and e_2 being the additional two columns. The augmented 5×5 matrix is shown in Table 3, Panel B. Acceptability requires that $x'B \geq f$ where $f = (0, 0, 0, -50, -50)'$.

We note that all non-negative cash flows $x \geq 0$ are acceptable. Hence arbitrage opportunities are always acceptable. However, there are acceptable opportunities with possible losses, such as the payoff $x = [-1, 2, 2, 2, -1]$. On the other hand, the payoff $[1, -2, -2, -2, 1]$ is not acceptable.

All acceptable opportunities should be executed in our economy, as all individuals regard them as worthwhile. Once the prices of liquid assets have responded to these trades, we anticipate that no acceptable opportunities will

remain among the traded assets. This expectation means that if a portfolio has zero cost, $\pi'\alpha = 0$, then it is not the case that it has a positive expected payoff under some specific valuation measure, and a non-negative payoff under all other valuation measures, such that

$$\alpha'AB \geq 0, \quad \alpha'AB \neq 0. \quad (4)$$

These are the only conditions that are necessary, since by scaling down the position, one can also ensure that the conditions expressed in Eq. (3) are also met.

Just as the absence of arbitrage implies that no zero cost portfolio has a payoff in the positive orthant of the space generated by the 5 states, the absence of acceptable opportunities implies that no zero cost portfolio has expected payoffs in the positive orthant of the space generated by the three valuation measures. In our context, the valuation operator for a portfolio is given by the 3×3 matrix AB , and so classical results imply the existence of a non-negative vector of weights w , such that

$$\pi = ABw. \quad (5)$$

The vector $q = Bw$ is, by construction, a state price vector assembled from the columns of B , which are the valuation test measures. Since this state price vector is determined by averaging over the valuation test measures, we term it a representative state pricing vector. When our results are specialized so that the only excluded opportunities are arbitrages, then the matrix B becomes the identity matrix. The vector w is then classically interpreted as yielding the prices of the Arrow Debreu securities, which pay a dollar in a particular state and zero otherwise. Under the stronger hypothesis of no acceptable opportunities, the columns of B , shown in Table 3, Panel A, represent valuation test measures. The vector w now yields the prices of securities with an expected payoff of one under a particular test measure, and no expected payoff otherwise. We note that the normalization of B is appropriately adjusted for by the valuation prices w , for if we were to double the entries in B , then the prices of the valuation measures in w would appropriately be halved. If the columns of B are normalized to sum to unity, then the vector w includes time discounting and represents spot prices of securities with unit expected payoffs under a particular measure, and zero under others. However, if B is normalized to sum to the price of a bond, then the entries of w sum to unity and represent forward prices in period 1 for these securities.

Since the matrix AB is 3×3 and invertible, we must also have that w is uniquely given by

$$w = (AB)^{-1}\pi. \quad (6)$$

The invertibility of AB is not essential. In general for the uniqueness of w . We present such a case for simplicity. All that is required is that AB have full

column rank. More generally, we would then have w given by $(B'A'AB)^{-1}B'A'\pi$.

For our sample economy, one may explicitly evaluate that the non-negative weight vector is

$$w = [0.0085, 0.085, 0.0427]. \quad (7)$$

The representative state pricing function is given by Bw , which in our example equals

$$q = [0.2861, 0.1796, 0.1366, 0.1338, 0.1731]. \quad (8)$$

We note that this representative state pricing function is negatively skewed, as reflected by the relatively greater weight given to lower states relative to the corresponding higher states. The U-shaped structure of the function follows from the positive weighting received by the short stock position. The relatively greater weighting of the long stock position accounts for the asymmetry.

The intuition behind our reformulation of the two fundamental theorems arises from recognizing that the objects of choice are not dollars in particular states, but rather the expected values under particular valuation measures. Just as the absence of arbitrage forces all non-trivial zero cost portfolios to be exposed to states in which payoffs are negative, so it is that the absence of acceptable opportunities forces all non-trivial zero cost portfolios to be exposed to a probability measure in which the expected gain is negative. Furthermore, just as a complete market has the property that the hedging error is zero in every state, so it is that our acceptably complete market has the property that the hedging residual is just acceptable for every valuation measure.

Acceptable completeness does differ from completeness, in that the residuals are non-zero whenever a claim's payoff is not in the span of the liquid assets. Mirroring this property is the fact that our acceptably complete markets renders uniqueness to only w , the market prices of the valuation test measures, and not to the full set of state prices. Our uniqueness and completeness results reside only in the reduced space of valuation test measures. Portfolios with the same outcomes under the valuation test measures are regarded as equivalent, and hence their difference vanishes in this space, even if it does not vanish in the original state space. If the decision-maker is not indifferent between two such portfolios, then additional measures should be introduced to differentiate them. While acceptable completeness may be lost as a result, the ranges of values consistent with no acceptable opportunities is still likely to be much tighter than the range of values consistent with the weaker hypothesis that excludes arbitrage opportunities.

It is clear from the example that the uniqueness of the representative state pricing vector arises from the number of non-redundant assets weakly exceeding the number of valuation test measures. What is at issue here are

the dimensions of the matrix AB . The number of states is irrelevant. In fact, one may have a continuum of states, and obtain the elements of the analog to the product AB by integrating over this continuous state space. In the interests of focusing on the main ideas, we restrict attention here primarily to the case of a finite state model, but we will outline the extension to continuous states in passing.

3. The economic model

We first discuss the two classes of measures central to defining acceptable opportunities. The first class of measures collectively constitute a rational view responsible for determining when marginal benefits exceed marginal costs. It is useful to keep in mind the special case of no arbitrage, which arises when the valuation measures place unit mass on one state and zero on all others. We note that accepting an opportunity with a positive expected value under one such measure is by itself neither rational nor particularly reasonable. But when such a result is available for the entire collective of measures, then the case for action is strong indeed. Analogously, it is the collective of the valuation measures which makes the case for the rationality of the proposed action.

To capture the notion of risk aversion to larger trades, this collective is enhanced by adding in the second class of measures, called stress test measures. These measures seek to limit expected losses for non-marginal trades. This second class of measures are irrelevant for pricing, as is illustrated in our example. The reason is that the constraints these measures impose can always be avoided by scaling down the position, so long as the number of stress test measures is finite. The role of these limits on expected losses will become important in Section 7, when we consider the problem of determining reservation prices for derivative securities whose payoffs cannot be perfectly replicated. The size of the proposed position is then a critical determinant of the level of the bid and ask prices derived.

3.1. Probability measures and floors

We initially consider the task of deciding whether to accept or reject a derivative security at a given price and at a marginal scale. Such a decision is strongly supported in the affirmative if, from a variety of perspectives, one evaluates that the benefits outweigh the costs at the margin. One is essentially inquiring into the size of the community of individuals who would approve the deal in small quantities, and on a financed and hedged basis. In incomplete markets, personal valuations will vary across market participants, reflecting the risk aversion, probability beliefs, initial endowments, transactions costs, or

other constraints of the individuals considering the trade. As a result, the collection of valuation test measures could be chosen to vary broadly over the relevant dimensions of preferences, beliefs, and endowments. As in our example, endowments that are short to the market may evaluate potential opportunities differently from those that are long. Similarly, delta-neutral endowments that display long gamma properties will evaluate the opportunity differently from those that display short gamma properties. If the evaluation is positive from all of these perspectives, then the opportunity may be safely executed at the margin. In aiming at robustness, one may combine these elements across each other, as well as across the historical experience on these entities. After defining the valuation test measures, one may consider whether the size involved may be safely increased.

Size considerations introduce the second class of measures, which we term stress test measures. The stress test measures may, by design, be concentrated on situations in which one only has expected losses, and it is therefore unreasonable to demand that these valuations be non-negative. Instead, the concern shifts to limiting the size of potential losses. Thus, outcomes with disastrous consequences in certain states can explicitly be avoided by placing a negative lower bound on the expected gain. Given the focus on limiting expected losses, these stress tests are specified in terms of probability measures, reflecting conditional probabilities that are influenced by preferences and endowments. For example, losses which arise in situations where one is otherwise well covered are not as important as losses which arise when one already has a considerable exposure. Alternatively, one may have a greater concern for one of two equivalent loss outcomes, as a matter of personal preference. For example, one may be more concerned about a loss which would be borne in isolation, rather than an equally sized loss which is incurred by many parties.

For the purpose of deciding whether to accept an opportunity, we assume the existence of a set \mathcal{S} , consisting of indexes for $M \geq 1$ probability measures that we refer to as test measures. Note that these measures correspond to the columns of the B matrix, shown in Table 3, Panel B, in our example. However, to invoke the idea of probability measures the measures in B have been normalized to a unit column sum. As already noted, this transformation occurs without loss of generality. With this convention, discounting to reflect time value of money is not required for the measures. Thus, the prices we determine via these measures are forward prices. Associated with each measure P_m , $m \in \mathcal{S}$, is a real number, $f_m \leq 0$, that we refer to as the associated floor which we will use in defining acceptability. Measures paired with floors of zero are termed valuation test measures, while measures paired with negative floors are termed stress test measures. Let \mathcal{S}^v denote the set of valuation test measures and let \mathcal{S}^s denotes the complementary set of stress test measures. We assume there is at least one valuation test measure in \mathcal{S}^v , so that no anti-arbitrage cash flows can

become acceptable. We also let $l \leq M - 1$ be the number of stress test measures in \mathcal{S}^s .

It is important to require that the stress test measures evaluate outcomes believed possible by some valuation test measure. Otherwise, one would accept opportunities with no payoff in states supporting the valuation test measures and which also generate acceptable losses under the stress test, even though such opportunities have non-positive cash flows over all states. To prevent the acceptability of such anti-arbitrage cash flows, we require that every state believed possible by a stress test measure is also believed possible by some valuation test measure. Thus, for $m = 1, \dots, l$, let

$$\Omega_m^s = \{\omega_{1m}, \omega_{2m}, \dots, \omega_{K_m}\} \tag{9}$$

be the set of states charged with positive probability mass by the m th stress test measure. Let $\Omega^s = \cup_{m=1}^l \Omega_m^s$ be the aggregate support of the stress test measures. Similarly, for $m = 1, \dots, M - l$, define Ω_m^v as the set of states charged with positive probability mass by the m th valuation test measure, and let $\Omega = \cup_{m=1}^{M-l} \Omega_m^v$ be the aggregate support of the valuation test measures. We require that $\Omega^s \subseteq \Omega$, and we let K be the total number of states in Ω .

3.2. Payoffs and traded assets

A payoff is a real-valued function defined on Ω that describes the state contingent cash flows received. This payoff is said to be risky if it is not constant on Ω . In our finite state economy, a generic payoff is denoted by a K dimensional vector x . We denote the set of all possible payoffs by X .

We assume the economy has a riskfree bond, with payoff $x_0(\omega) = 1$ and initial price $\pi_0 > 0$. The economy also has $N \geq 1$ traded risky assets, indexed by $n = 1, \dots, N$, all of which payoff $x_n(\omega)$ at time $t = 1$. Further, all market prices π_n are determined at time $t = 0$. An opportunity is a payoff $x \in X$ which is being offered to the decision-maker for zero-cost. To apply our results to potential investments with non-zero initial cost, simply combine the investment with a position having an offsetting initial cost, and adjust the payoffs accordingly. Given the ability to borrow or lend, our focus on zero-cost opportunities is without loss of generality, and is merely used to equate gains with payoffs in time 1. The method of financing an investment can also include trading in other risky assets, which would, in general, change the risk profile of the investment. Thus, an opportunity is defined not only by the payoffs offered by the derivative security in question, but also by the specific financing and hedging mechanisms available through investing in the traded assets. The problem of choosing a financing and hedging strategy is considered in greater detail in Section 7.

3.3. Defining acceptable opportunities

For an opportunity to be acceptable, it must have a non-negative expected value under each valuation test measure. To limit the scale of the positions taken in acceptable opportunities, we also require consistency with the limits on expected losses imposed by the set of stress test measures.

We define an opportunity with payoff $x \in X$ as being *acceptable* if, for all $m \in \mathcal{S}$

$$\sum_{\omega \in \Omega} P_m(\omega)x(\omega) \geq f_m, \quad f_m \leq 0. \quad (10)$$

Let \mathcal{A} denote the set of acceptable opportunities. For an opportunity to be acceptable, the expectation of the payoff ($x(\omega), \omega \in \Omega$) with respect to each set of valuation test measures ($P_m(\omega), \omega \in \Omega$) must exceed the associated value for the floor f_m . Equivalently, one may define acceptability by writing

$$E^{P_m}[x] \geq f_m \quad \text{for all } m \in \mathcal{S}. \quad (11)$$

It is important to note that it is the universal domination of the associated floors across all the test measures that gives an opportunity its acceptability. It is the collective that defines acceptability, and not the measures taken individually, no matter how reasonable or unreasonable they may be when considered on an individual basis. We observe that all classical arbitrages meet these requirements for acceptability.

3.3.1. Further discussion of acceptable opportunities

The set of acceptable opportunities \mathcal{A} , is a convex set and hence, a blend of two acceptable opportunities is itself acceptable. Acceptable opportunities also contain the set of all non-negative payoffs. This feature is built into the definition by formulating acceptable opportunities as a generalization of arbitrage opportunities.

Our notion of an acceptable opportunity is inspired by earlier work on measuring risk by Artzner et al. (1998). In their paper, these authors develop four criteria that every risk measure should satisfy. The four criteria are sub-additivity, positive homogeneity, monotonicity, and a risk-free condition. Measures meeting these criteria are termed coherent. In a significant advance, they characterize all coherent risk measures as the maximum expected loss evaluated under a convex set of probability measures, termed generalized scenario measures and these authors further go on to define acceptable positions as those for which this maximum expected loss is non-positive, or stated another way, for which the minimum expected worth is non-negative. Our definition of an acceptable opportunity reflects this structure and adds one consideration as well. Under the coherency axioms, the set of acceptable opportunities is a cone. Our imposition of limits on expected losses through the

stress test measures reduces this cone to a convex set. If the acceptable opportunity is not also an arbitrage opportunity, then the imposition of stress test measures limits the ability to scale the opportunity upwards and retain acceptability. However, arbitrage opportunities retain their acceptability as they are scaled upwards without limit. We note that in a later work Artzner et al. (1999) also extend their definition of acceptability by introducing constants, that we term floors, associated with their scenario measures to form general convex sets of acceptable opportunities. In their context, however, these constants may be positive.

We further comment on the relationship between our acceptable opportunity set and preference orderings in general. If the origin is identified with an individual's current position, then the set of opportunities preferred to the status-quo by an investor with increasing concave utility is a convex set containing the positive orthant. These are the investor's "better than status-quo" sets. The intersection of several such "better than status-quo" sets for a number of investors is again such a convex set. Our acceptable opportunities may be regarded as an approximation of the convex set arising from requiring unanimity across several investors in preferring opportunities over the status-quo.

In general, acceptable opportunities do allow arbitrarily large losses in some states, provided they are compensated for by sufficiently large profits in others. However, the use of stress test measures does allow one to limit losses in particular states. For example, one may place a lower bound for a measure P_m such that $P_m(\omega) = 1$ for a particular ω , while it is zero for all other ω 's.

By removing the stress test conditions, one obtains a cone containing our acceptable opportunities \mathcal{A} . Fig. 1 shows the geometry when there are just two states. The horizontal axis is measured in State 1 dollars ($\$_1$), while the vertical axis is measured in State 2 dollars ($\$_2$). The cone is the set between the outer two arrows, while the convex set it contains is formed by adding two more constraints. One of these constraints prevents very negative losses in State 1. The convex set contains the positive orthant and excludes the negative orthant. The perimeter of the convex set plays the same role as an indifference curve, in that opportunities strictly inside the convex set are preferred to those outside the set. However, we do not require that preferences be specified over all levels of all goods. In particular, we do not rank two opportunities inside the convex set of acceptable opportunities. In this regard, our analysis is consistent with arbitrage pricing theory, which does not rank arbitrages.

We note from Eq. (10) that if an opportunity x is acceptable, then there is no guarantee that the opportunity λx is also acceptable for λ sufficiently greater than one.² Conversely, we note that if an opportunity in the cone is not

²To ensure that opportunities are never scaled up arbitrarily high, floors must be imposed on states like the one imposed on State 1 in Fig. 1.

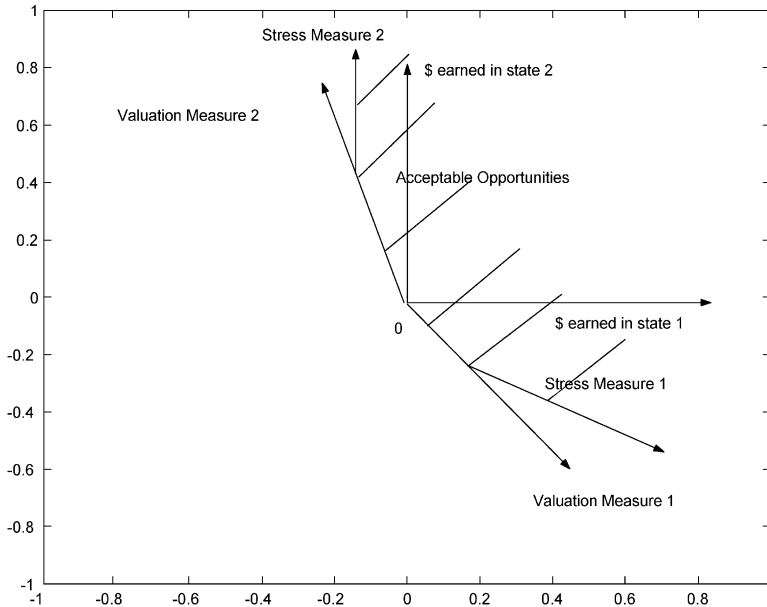


Fig. 1. Graph of acceptable opportunity set created by two valuation measures defining a cone containing the positive orthant and two further stress test measures reducing the acceptable opportunity set to a convex set still containing the positive orthant.

acceptable because a floor constraint is violated, then it will be rendered acceptable by scaling the position downwards, since for a sufficiently small λ

$$E^{P_m}[\lambda x] \geq f_m, \quad \text{all } m \in \mathcal{S}. \tag{12}$$

Critical to this conclusion is the finiteness of the set of stress test measures.

The next section investigates the implications for derivatives pricing when the traded assets are priced so that there are no acceptable opportunities among them.

4. The first fundamental theorem

By definition, acceptable opportunities are widely viewed as sufficiently meritorious so as to be undertaken at the margin by many market participants. Hence, just as arbitrage opportunities vanish quickly, one would expect that these more general opportunities will also disappear in efficient economies. In this section, we investigate the implications for asset pricing when economies satisfy the condition of no acceptable opportunities, thereby generalizing the corresponding result regarding the implications of no arbitrage.

We first note that, if there are no acceptable opportunities, then there are also no arbitrage opportunities. Hence, the classical result of Harrison and Kreps (1979) implies that a state pricing function exists:

Definition 1. A state pricing function is a strictly positive vector q of dimension K satisfying

$$\pi_n = \pi_0 \sum_{\omega \in \Omega} q(\omega)x_n(\omega), \quad n = 0, 1, \dots, N. \tag{13}$$

State pricing functions determine asset prices π_n as the present value given by discounting the forward market price by π_0 . The forward market price is obtained as the value of an equivalent portfolio that holds $x_n(\omega)$ units of the Arrow Debreu security indexed by ω , that pays a dollar in state ω and zero elsewhere. The forward market price of the Arrow Debreu security with index ω is $q(\omega)$. In incomplete markets, this state pricing function is not unique, and many authors have proposed a variety of methods for choosing an economically relevant state pricing function. We will show that the absence of acceptable opportunities provides us with additional structural restrictions on the state pricing function, which are contained in the following definition:

Definition 2. A state pricing function q is a representative state pricing function (RSPF) if there exists a set of strictly positive weights $w_m, m \in S^v$ such that

$$q(\omega) = \sum_{m \in S^v} w_m P_m(\omega) \quad \text{for all } \omega \in \Omega. \tag{14}$$

The RSPF determines forward Arrow Debreu prices $q(\omega)$ in the cone generated by the valuation test measures, $P_m(\omega)$, using the forward prices of valuation test measures w_m for the formation of the linear combination. If it exists, an RSPF derives its name from the observation that every valuation test measure with positive mass on a given state contributes to the price assigned to that state. Obversely, we note that stress test measures do not contribute to the price. This outcome is natural, as prices apply only to marginal units of trade and, as noted earlier, such trades never violate stress test limits.

The necessity of positive weighting for only the valuation test measures may be further appreciated on noting that re-arranging Eq. (13) implies:

$$\sum_{\omega \in \Omega} q(\omega)x_n(\omega) - \frac{\pi_n}{\pi_0} = 0, \quad n = 0, 1, \dots, N. \tag{15}$$

Thus, by the very definition of a state pricing function, the risk-neutral expected payoff from any zero cost portfolio of the liquid assets just meets a floor of zero. Placing a positive weight on a stress test measure would be inconsistent with this property, as such measures by design meet negative floor values.

The formulation of the condition of the absence of acceptable opportunities proceeds in analogy with that of the absence of arbitrage opportunities. Just as the latter recognizes that the creation of a zero payoff at zero cost is not an arbitrage, we need to define a strictly acceptable opportunity.

Definition 3. An acceptable opportunity $x \in \mathcal{A}$ is *strictly acceptable* if in addition for some $m \in \mathcal{S}^v$

$$\sum_{\omega \in \Omega} P_m(\omega)x(\omega) > 0. \quad (16)$$

Let \mathcal{A}^+ denote the set of strictly acceptable opportunities.

For strict acceptability, we only require that the acceptable opportunity have a strictly positive valuation under some valuation test measure. In other words, an opportunity is strictly acceptable if and only if it is acceptable and some valuation test measure assigns it a positive value. Our interest is then in the asset pricing implications for economies that meet the condition of eliminating all strictly acceptable opportunities.

Condition 1. The economy satisfies the condition of *no strictly acceptable opportunities* (NSAO), if there is no portfolio of the traded assets holding α_n units of asset n such that

$$\begin{aligned} (i) \quad & \sum_{n=0}^N \alpha_n \pi_n = 0, \\ (ii) \quad & \sum_{n=0}^N \alpha_n x_n \in \mathcal{A}^+. \end{aligned} \quad (17)$$

Under this condition, the liquid assets are priced so that no zero-cost portfolio exists that is strictly acceptable. Note that when strictly acceptable opportunities have payoffs restricted to the positive orthant, then the condition of No Acceptable Opportunities reduces to the classical condition of No Arbitrage Opportunities.

4.1. Equivalence of existence of RSPF and NSAO

This subsection demonstrates and discusses the implications of the following fundamental theorem:

Theorem 1. The economy has no strictly acceptable opportunities (NSAO) if and only if there exists a representative state price density (RSPF)

The proof of the theorem is contained in Appendix A. Theorem 1 tells us that when the liquid assets are priced so that there are no strictly acceptable opportunities among them, then one can take a convex combination of the

valuation test measures to be a state pricing function. Thus, the NSAO restriction reduces the problem of identifying a state pricing function down to that of identifying the strictly positive set of forward prices for the valuation test measures $w_m, m \in \mathcal{S}^v$ that define the representative state pricing function. We note in this regard that Artzner et al. (1998) argue in their Condition 4.3 that it would be desirable if some convex combination of their scenario measures was a pricing measure. Theorem 1 shows that this property is implied by our NSAO condition.

If the fair value of a derivative security is assessed using all of the different permitted weight assignments, then in general a range of fair values will result. In general, this range will be narrower than the one generated by requiring no arbitrage as the relevant state price densities are now restricted to being representative. If the market price for buying the derivative is below the smallest fair value, then the derivative should be purchased. Furthermore, a hedging strategy should be undertaken so that the hedged derivative position is strictly acceptable. Conversely, if the market price for selling the derivative is above the largest fair value, a sale and a hedge should ensue. In either case, the investor should also investigate both whether the scale of the transaction can be increased, and if the opportunity remains acceptable at the new price and in the new scale.

When the number of liquid assets equals or exceeds the number of valuation test measures in \mathcal{S}^v , then the representative state pricing function can, potentially, be uniquely identified. The NSAO condition therefore offers a significant advance in solving the problem of choosing a state pricing function in incomplete markets. Recall that under the weaker no arbitrage condition, the number of non-redundant assets must equal the number of states for unique identification of the state pricing function. The next section derives the implications of uniqueness of the RSPF. Before proceeding on to this issue, the next subsection comments briefly on the extension of Theorem 1 to the important case of a one-period model with a continuous state space.

4.1.1. Continuous state extension

For simplicity, we have restricted attention so far to finite state spaces. However, as the example of Section 2 indicates, the states are being integrated out, and thus Theorem 1 easily generalizes to infinite state spaces for the one-period model with finitely many assets and valuation test measures. We sketch an outline of the proof in this subsection. First, we fix a reference probability measure R with respect to which all of the valuation test measures are presumed absolutely continuous on some infinite state space. Let this probability space be $(\Omega, \mathfrak{F}, R)$. Let the set of valuation measures be given by \mathfrak{F} measurable R integrable densities $q_m(\omega)$ for $m \in \mathcal{S}^v$, and suppose that all assets have payoffs $x_n(\omega)$ that are \mathfrak{F} measurable random variables integrable with respect to all of the valuation test measures. We may then define the

matrix C of asset valuation test measure outcomes by

$$C_{nm} = \int x_n(\omega)q_m(\omega)R(d\omega) \quad (18)$$

and the NSAO condition would state that $\alpha'\pi = 0$ implies that it is not the case that $\alpha'C \geq 0$ and $\alpha'C \neq 0$. Classical results allow us to deduce that

$$\pi = Cw \quad (19)$$

for some positive vector w , or equivalently that

$$\pi_n = \int x_n(\omega)q(\omega)R(d\omega), \quad (20)$$

$$\text{where } q(\omega) = \sum_m w_m q_m(\omega). \quad (21)$$

The converse also holds for if an RSPF exists, then we have a strictly positive solution to $\pi = Cw$. Hence it is not possible that $\alpha'\pi = 0$, $\alpha'C \geq 0$, and $\alpha'C \neq 0$, so NSAO fails.

The extension to infinite asset spaces or to dynamic trading in finitely many assets is more involved, and we leave these matters to future research.

5. The second fundamental theorem

The second fundamental theorem of asset pricing, which is due to Harrison and Kreps (1979), shows that completeness is equivalent to the uniqueness of the state pricing function, which exists under the no arbitrage condition as a consequence of their first fundamental theorem. In a complete market with no arbitrage opportunities, the residual risk after hedging is zero. Acceptable opportunities are designed to allow investors to undertake opportunities which result in a non-zero residual after hedging. This idea suggests a modification of the idea of completeness that we shall term acceptable completeness. We have already seen that a representative state pricing function exists whenever there are no strictly acceptable opportunities. The question then arises as to whether the uniqueness of a representative state pricing function is associated with some modified concept of market completeness. This section derives an affirmative answer to this question. We begin by defining acceptable completeness.

5.1. Acceptable completeness

With the concept of an acceptable opportunity well-defined, a liability can reasonably be regarded as hedged if the residual is regarded as acceptable. By definition, the expected payoff of the hedged and financed investment is then at or above the floor associated with each test measure. In an economy with a

sufficiently rich array of assets, it may be possible that the floor on the expected payoff from the hedged opportunity is just reached for all test measures. An acceptably complete market is defined to have exactly this property:

Definition 4. The market is *acceptably complete* if for all $x \in X$, there exists a hedge portfolio $\alpha_n, n = 0, \dots, N$ such that for all $m \in S$

$$\sum_{\omega \in \Omega} P_m(\omega) \left[-x(\omega) + \sum_{n=0}^N \alpha_n x_n(\omega) \right] = f_m. \quad (22)$$

In acceptably complete markets, liabilities may be hedged by strategies that make the hedging residual just acceptable. The residual will typically be non-zero, but it will just meet the requirements to be acceptable.

We view x as a liability that has been issued and has been hedged with positions in the traded assets given by α_n . An economy which is acceptably complete has a sufficient number of assets relative to the number of test measures, so that the hedged investment is just acceptable for each test measure. The hedging error being zero in classically complete markets is weakened to the requirement that the excess of the expected payoff over the floor be zero for each test probability.

When concerned with the hedging of claims at the margin in the sense of the size of ones position the definition of acceptability may be restricted to just the valuation measures with the zero floors. Such a reduced acceptable completeness is of course easier to attain than having acceptable completeness with respect to all measures including the stress test measures. The wider acceptability is relevant for hedging positions in size. The arguments for equivalence of the reduced acceptable completeness to the uniqueness of a representative state pricing function, that is in the span of the valuation measures, are identical to those for the equivalence of the expanded acceptable completeness to the uniqueness of a state pricing function in the larger space spanned by all the measures including the stress test measures. The existence by the first fundamental theorem of a representative state pricing function in the smaller space spanned by the valuation measures coupled with the uniqueness in the larger space spanned by all the measures also yields uniqueness in the smaller space of the valuation measure span. In some practical circumstances it may well be the case that one only has the reduced form of acceptable completeness and this condition is sufficient for trading at the margin. In the interests of some generality we study the concept in the expanded form.

5.2. Uniqueness of the representative state pricing function

This subsection defines what is meant by uniqueness of the representative state pricing function.

Definition 5. The representative state pricing function is *unique in the wide sense* if there exists at most one set of positive weights w_m , $m \in \mathcal{S}$ such that

$$q(\omega) = \sum_{m \in \mathcal{S}} w_m P_m(\omega) \quad (23)$$

is a state pricing function.

The choice of a state pricing function is a crucial component of risk management as it is necessary for assessing profitability and evaluating exposures. In an incomplete market, the many choices for the state pricing function can lead to radically different derivative security prices. As noted in the introduction, this has led many authors to propose a variety of criteria for selecting a state pricing function from among the class of solutions.

5.3. Equivalence of acceptable completeness and uniqueness of RSPF

This subsection establishes the equivalence of acceptable completeness and the uniqueness of an RSPF, assuming that the asset space is richer than the space of test measures. Specifically what is needed is that the asset space span the space of test results. What we do not want is the situation of over-testing, which occurs for example, if the asset space has 3 dimensions and is tested using 10 measures. In such a case the space of achievable test results is at most a three dimensional subspace of the space of test results. In this case acceptable completeness is not sufficient to derive uniqueness of the representative state pricing function. For most practical situations, we expect the asset space to be much richer and capable of generating all possible test results. We term this condition under testing the asset space.

Condition 2. The tests measures satisfy the condition of under testing (UT), if for any set of potential test results $c = (c_m, m \in \mathcal{S})$, there exist a cash flow x such that

$$x' B = c. \quad (24)$$

Condition 2 leads us to the statement of Theorem 2, as follows.

Theorem 2. Under UT, markets are acceptably complete if and only if the representative state pricing function is unique in the wide sense.

The proof of Theorem 2 can be found in Appendix B. For a price-taker contemplating whether or not to accept a derivative security, the existence of a unique representative state price vector means that the derivative's fair price can be determined and compared to the market price. If the market price paid for entering a long position is below the fair price, then the derivative should be

accepted and a hedging strategy enacted so that the resultant opportunity is strictly acceptable. Conversely, if the market price received for entering a short position is above this fair price, then the short position should be accepted and a hedging strategy should again be enacted, so as to lock in the strictly acceptable opportunity.

5.3.1. Continuous state space considerations

For the case of the single-period model with a continuous state space, the UT condition is much easier to satisfy. This model only requires that, for all test results $c = (c_1, \dots, c_M)$, there exists an \mathfrak{J} measurable random variable such that

$$\int x(\omega)q_m(\omega)R(d\omega) = c_m. \tag{25}$$

Uniqueness of an RSPF is equivalent to the null space of C being $\{0\}$, where C is the asset valuation test measure outcome matrix defined in Eq. (18). Acceptable completeness is equivalent to the existence of α such that

$$\alpha' C = c + f',$$

where c is defined by (25) and x is the claim to be hedged. Under UT, acceptable completeness is equivalent to the the range of C' being \mathfrak{R}^M . This requirement in turn, is equivalent from linear algebra to the null space of C being $\{0\}$, or equivalently to the uniqueness of the RSPF.

6. Example in a single-period lognormal economy

This section applies our pricing theory to a single-period continuous state economy with lognormal valuation test measures. Thus, consider an economy open for trading only at date $t = 0$ and at date $t = T$. The economy consists of a non-dividend paying stock with stock price realizations at time T on the positive half-line, and a bond paying one unit at time T . The current prices of these two assets are S_0 for the stock and e^{-rT} for the bond. For simplicity, we have just two valuation test measures. These measures are given by a lognormal distribution for the stock with a mean continuously compounded rate of return of $\mu_d < r$ and volatility $\sigma_d > 0$, and another lognormal distribution for the stock with a continuous compounded mean rate of return $\mu_u > r$ and volatility $\sigma_u > \sigma_d$. These measures are consistent with two prior lognormal distributions for the stock price with means ζ_u, ζ_d and volatilities σ_u, σ_d , respectively. Suppose further that the two perspectives are those of two individuals. The first investor evaluates wealth in dollars, while the other uses the stock as the numeraire asset and measures wealth in the number of shares. With coefficients of relative risk aversion of γ_u, γ_d , respectively, one may

determine the valuation measures as described in Brennan (1979) or Rubinstein (1976) to get $\mu_u = \zeta_u - \gamma_u \sigma_u^2$, while $\mu_d = \zeta_d - \gamma_d \sigma_d^2$. For further simplicity, we have no stress test measures in this example.

This economy is grossly incomplete, as there are only two assets and an infinite number of terminal states. Suppose that an investor wishes to value a European call of maturity T written on the stock. The matrix of asset valuation test measure outcomes C in this case is given by taking the appropriate expectations:

$$C = \begin{pmatrix} S_0 e^{(\mu_u - r)T} & S_0 e^{(\mu_d - r)T} \\ e^{-rT} & e^{-rT} \end{pmatrix}.$$

For any zero cost trading strategy $\alpha = (\alpha_0, \alpha_1)$, we must have

$$\alpha_0 S_0 + \alpha_1 e^{-rT} = 0$$

or

$$\alpha_1 = -\alpha_0 S_0 e^{rT}.$$

Hence, we have that

$$\alpha' C = \alpha_0 S_0 [(e^{(\mu_u - r)T} - 1), (e^{(\mu_d - r)T} - 1)].$$

Since $\mu_u > r > \mu_d$, no strictly acceptable opportunities exist. By the continuous state extension of Theorem 1, discussed in Section 4.1.1, the representative state pricing density exists, and has the form

$$q(S, T) = w_d \frac{\exp\{-(1/2\sigma_d^2 T)[\ln(S/S_0) - (\mu_d - \sigma_d^2/2)T]^2\}}{\sqrt{2\pi T \sigma_d S}} + w_u \frac{\exp\{-(1/2\sigma_u^2 T)[\ln(S/S_0) - (\mu_u - \sigma_u^2/2)T]^2\}}{\sqrt{2\pi T \sigma_u S}}, \quad (26)$$

where w_d and w_u are both positive.

Since the number of assets equals the number of valuation measures in this economy, the representative state pricing density could potentially be uniquely determined. It is simple to see that UT holds. Furthermore, as C is invertible, we have uniqueness of the RSPF. To determine it, one deduces from the fact that bonds are correctly priced that w_u and w_d sum to one. To simplify notation, let $w_u = w$ and $w_d = 1 - w$. The condition that the stock be priced by the density expressed in Eq. (26) leads to the solution for w of

$$w = \frac{e^{rT} - e^{\mu_d T}}{e^{\mu_u T} - e^{\mu_d T}}. \quad (27)$$

European calls are then uniquely priced in this economy by

$$C(K) = w BS_u + (1 - w)BS_d, \quad (28)$$

where

$$BS_u \equiv BS(S_0, K, r, r - \mu_u, \sigma_u, T), \quad BS_d \equiv BS(S_0, K, r, r - \mu_d, \sigma_d, T). \quad (29)$$

In Eq. (29) BS represents the Black-Scholes formula for a spot price of S_0 , a strike of K , an interest rate of r , a dividend yield of $r - \mu$, a volatility of σ , and an expiration of T . Thus, the acceptability-free price of the call is obtained by averaging the Black-Scholes value in a high expected growth rate, high volatility environment with the Black-Scholes value in a low expected growth rate, low volatility environment. This pricing model has 4 free parameters, namely μ_u, μ_d, σ_u , and σ_d .

To calculate the number of shares held, α , and the amount invested in the riskless asset, β , we require that the hedged and financed portfolio be just acceptable as defined by Eq. (22) of Theorem 2. These conditions yield one solution:

$$\alpha = \frac{BS_u - BS_d}{S_0(e^{(\mu_u-r)T} - e^{(\mu_d-r)T})} \quad \text{and} \quad \beta = \frac{-e^{(\mu_d-r)T} BS_u + e^{(\mu_u-r)T} BS_d}{e^{(\mu_u-r)T} - e^{(\mu_d-r)T}}. \quad (30)$$

It is interesting to observe that the hedge position in stock is a proper delta type calculation, except that the changes in the stock price and the call price result from changes in expected value across measures rather than across states. If we set $\sigma_u = \sigma_d = 0$, then the number of states reduces to two and both valuation measures degenerate into indicator functions. The representative state pricing function given by Eq. (27) turns into the state pricing function implied by the absence of arbitrage in the single-period binomial model, with the upward jump $u \equiv e^{\mu_u T}$ and the downward jump $d \equiv e^{\mu_d T}$. Similarly, the call value in Eq. (28) and the hedges in Eq. (30) turn into those obtained by excluding arbitrage in the single-period binomial model.

7. Pricing derivative securities

The focus thus far has been on deciding whether to accept or reject a proposed opportunity at a given market price and the associated implications of market efficiency. We now address the question of how this market price might be determined. We assume that market prices are being determined by a competitive set of market-makers, who must set the bid price and the ask price at each scale of the investment. These prices represent the demand schedule and the supply schedule, respectively. The decision to accept or reject the investment is then reserved for the potential clientele attracted to the market-makers. This section examines the implications of our framework for a market-maker who must determine a pricing schedule and a hedging strategy for a derivative security.

7.1. Definition of bid and ask

Whether or not markets are acceptably complete, a market-maker can usually charge enough for issuing a liability so that the hedged liability is acceptable. This result is merely the consequence of a finite set of test measures, even if the state space is infinite. In a competitive market, it is useful for a market-maker to know the minimum amount which can be charged to be consistent with acceptability. If the market-maker sets his ask price to this reservation price, then the market-maker can be assured that all liabilities issued can be hedged so that the residual is acceptable. Thus, although the market maker does not directly accept or reject the opportunity, all opportunities taken up by the market-maker are nonetheless accepted by him.

Consider some derivative security with non-negative payoff vector $x \geq 0$ at some fixed scale. The competitive market-maker wishes to find the lowest amount which must be charged for selling the derivative, so that the hedged liability is acceptable. The problem of determining the asking price for this derivative can be formulated as the following linear program:

$$\min_{\alpha} \alpha' \pi \text{ subject to } -x'B + \alpha'AB \geq f', \quad (31)$$

where all quantities are defined in the proof to Theorem 1 (see Appendix A). If a solution α^* exists to this problem, then the ask price $a(x) = \alpha^{*'}\pi$. The objective in this linear program, shown in Eq. (31) is to minimize the cost of setting up a hedge portfolio of riskless and risky assets. The constraint in this linear program is that the hedged liability is acceptable. Equivalent formulations for continuous state spaces require working directly with the matrix of asset valuation test measure outcomes, C , as defined by Eq. (18) and replacing $x'B$ by the vector c as given by Eq. (25).

One can also find the bid price for buying a derivative security. Here the market-maker must find the largest amount of cash which can be generated immediately by issuing liabilities, so that the payoff from the offset derivative asset is still acceptable. The problem of determining the bid price for a derivative can be formulated as the following linear program:

$$\max_{\alpha} \alpha' \pi \text{ subject to } x'B - \alpha'AB \geq f'. \quad (32)$$

If a solution α^* exists to this problem, then the bid $b(x) = \alpha^{*'}\pi$. The objective in this linear program is to maximize the cash received from issuing liabilities which potentially includes riskfree borrowing. The constraint in this linear program is that the portfolio of the derivative and these liabilities is acceptable.

7.2. Properties of the bid–ask spread

Let $s(x) \equiv a(x) - b(x)$ be the bid–ask spread for a derivative security with non-negative payoff x . We first prove the following sensible result:

Theorem 3. The bid–ask spread is non-negative:

$$s(x) \geq 0.$$

The proof of Theorem 3 is contained in Appendix C. If a derivative security is valued using an RSPF, then the resulting value will lie between the bid and the ask. The reason for this condition is that the dual solution Eq. (C.1) from Appendix C, to the ask problem makes it clear that the ask is determined by maximizing over RSPFs. Similarly, the dual solution from Eq. (C.2) of Appendix C, to the bid problem shows that the bid is being determined by minimizing over RSPFs. By setting his bid–ask spread in the manner indicated in the proof of Theorem 3, the market-maker is ensuring that any transactions he is forced into are acceptable.

In an acceptably complete market, Theorem 2 implies there is only one RSPF and thus the bid and the ask are equal. In an acceptably complete market with no strictly acceptable opportunities among the liquid assets, the requirement that derivatives be priced so that no strictly acceptable opportunities are introduced induces a unique price. The hedging strategy is also identified and amounts to ensuring that the hedged position is just acceptable.

When markets are not acceptably complete, a derivative security may still have the property that the hedged investment is just acceptable. The following theorem shows that in this case, the spread is still zero.

Theorem 4. For a given opportunity x , if there exists a vector α such that

$$(-x' + \alpha' A)B = f', \quad (33)$$

then $s(x) = 0$.

The proof of Theorem 4 is contained in Appendix D. When markets are not acceptably complete and the derivative security cannot be hedged so as to be just acceptable, then the bid must be below the ask for the derivative security to be acceptable. In this case, a market-maker may find that his spread is too wide. The following theorem shows that the bid–ask spread decreases if the scale is decreased and the market-maker could consider reducing the scale of the trade.

Theorem 5. The bid–ask spread is an increasing function of the scale

$$s(\lambda x) \leq s(x) \quad \text{for } \lambda \in (0, 1).$$

The proof to Theorem 5, contained in Appendix E, shows that the role of the stress test measures is to cause the bid–ask spread to widen as the scale increases. We are thus able to generate imperfect liquidity in derivatives as a consequence of our theory.

8. Summary and conclusions

We considered the problem of pricing and hedging in incomplete markets by expanding the role played by arbitrage opportunities to acceptable opportunities. An acceptable opportunity was defined to be a zero cost investment, whose expected payoff under each test measure weakly exceeds a non-positive floor associated with that measure. We further defined a representative state pricing function as a strict convex combination of the zero floor test measures. We demonstrate that there exists a representative state pricing function if and only if the economy has no strictly acceptable opportunities. The weights attached to the valuation test measures in defining the representative state pricing function are interpretable as forward prices of portfolios delivering unit expected payoff under one test measure and zero expected payoff under all others.

The concept of hedging is reformulated to one of attaining acceptable residual risks. Markets are defined to be acceptably complete if all claims can be hedged in such a way that the floor on expected payoffs is just met under each test measure in the scenario set. We show that markets are acceptably complete if and only if the representative state pricing function is unique. When markets are acceptably complete, prices and hedges are uniquely determined. When markets are not acceptably complete, a bid–ask spread for derivatives can still be determined.

Future research in this area would involve extending our results to multiple periods and to continuous time. Research should also go into methodologies for determining the set of test measures and associated floors. It is worth noting that the primitive setting for this unified view on pricing and hedging in classically incomplete markets is the use that notions similar to acceptable opportunities are fundamental to the operation of financial markets and an inherent part of the formulation of any financial markets model. We therefore encourage further research on the use of this approach in financial decision-making, and on its resulting welfare and social policy implications.

Appendix A

Theorem 1. The economy has no strictly acceptable opportunities (NSAO) if and only if there exists a representative state price density (RSPF).

Proof of Theorem 1. Let A be the $(N + 1)$ by K matrix of payoffs to the $N + 1$ traded assets indexed $n = 0, \dots, N$ in each of the K states indexed by the set Ω . Let B be the K by M matrix with columns B_m given by the M probability mass

functions for $m \in \mathcal{S}$. Let $f \leq 0$ be the M dimensional vector of floors on expected payoffs with $f_m = 0$ for $m \in \mathcal{S}^v$ and $f_m < 0$ for $m \in \mathcal{S}^s$. Let π denote the $N + 1$ dimensional vector of initial asset prices. Under NSAO there does not exist an $N + 1$ dimensional vector α such that $\alpha' \pi = 0$ and $\alpha' AB \geq f'$, and $\alpha' AB_m > 0$ for some $m \in \mathcal{S}^v$. The existence of an RSPF asserts the existence of an M dimensional vector $w \geq 0$, with $w_m > 0$ for $m \in \mathcal{S}^v$ and $w_m = 0$ for $m \in \mathcal{S}^s$ such that $\pi = ABw$. Furthermore, by the construction of the vectors f and w , we have $f'w = 0$.

Suppose that we have an RSPF given by the weights w . It then follows that if $\alpha' \pi = 0$ then $\alpha' ABw = 0$. Now suppose NSAO fails and $\alpha' A$ is strictly acceptable. We must then have that $\alpha' AB_{m_0} > 0$ for some $m_0 \in \mathcal{S}^v$, while $\alpha' AB_m \geq 0$ for all $m \in \mathcal{S}^v$. This contradicts $\alpha' ABw = 0$ on noting that $w_m = 0$ for $m \in \mathcal{S}^s$.

For the converse, suppose now that NSAO holds. Let $L = \{\alpha' A | \alpha' \pi = 0\}$ be the linear space of payoffs resulting from zero cost portfolios. Under NSAO, L does not intersect the convex set \mathcal{A}^+ . Define the matrix B^v as the submatrix of B that contains only the columns B_m for $m \in \mathcal{S}^v$. Let $C_1 \equiv \{\alpha' AB^v | \alpha' \pi = 0\}$ be the expected payoffs from zero cost portfolios generated by the valuation measures. Furthermore, let $k = M - l$ be the column dimension of B^v and let $C_2 \equiv \{y | y \geq 0, \mathbf{1}'_k y = 1\}$ denote the unit simplex of k dimensional space. Under NSAO, the convex sets C_1 and C_2 do not intersect. It follows from the separating hyperplane theorem (see for example Duffie, 1992) that there exists a k dimensional vector $w^v \geq 0$ such that $\pi = AB^v w^v$. Thus, the representative state pricing function exists and is given by $B^v w^v$ or Bw defining w to be zero for the columns corresponding to the stress test measures. \square

Appendix B

Theorem 2. Under UT, markets are acceptably complete if and only if the representative state pricing function is unique in the wide sense.

Proof of Theorem 2. The representative state pricing function is unique in the wide sense if and only if there is at most one solution in w to the equation defining the representative state pricing function, i.e.

$$\pi = ABw.$$

This setting is equivalent to the matrix AB introduced having a null space equal to $\{0\}$. Results of linear algebra imply that this property holds if and only if the range of transpose, $B'A'$ is all of \mathcal{R}^M . Now markets are acceptably complete if for all x , there exists an $N + 1$ dimensional vector α such that

$$\alpha' AB = x'B + f'.$$

Since under UT, the range of B' is \mathfrak{R}^M , it follows that under acceptable completeness the range of $B'A'$ is all of \mathfrak{R}^M . Hence, markets are acceptably complete if and only if the RSPF is unique in the wide sense. \square

Appendix C

Theorem 3. The bid–ask spread is non-negative:

$$s(x) \geq 0.$$

Proof of Theorem 3. The dual to the ask problem (31) is

$$\max_w x'Bw + f'w \text{ subject to } ABw = \pi \text{ and } w \geq 0. \quad (\text{C.1})$$

The maximand in this dual is reduced if we impose the constraint that $f'w = 0$:

$$\max_w x'Bw + f'w = 0 \text{ subject to } ABw = \pi \text{ and } w \geq 0 \text{ and } f'w = 0.$$

The maximand is further reduced if we minimize instead:

$$\min_w x'Bw + f'w \text{ subject to } ABw = \pi \text{ and } w \geq 0 \text{ and } f'w = 0.$$

Since $f'w = 0$, the objective is unchanged if $f'w$ is subtracted rather than added:

$$\min_w x'Bw - f'w \text{ subject to } ABw = \pi \text{ and } w \geq 0 \text{ and } f'w = 0.$$

Finally, the minimand is decreased if the constraint $f'w = 0$ is removed:

$$\min_w x'Bw - f'w \text{ subject to } ABw = \pi \text{ and } w \geq 0. \quad (\text{C.2})$$

This problem is the dual solution to the bid problem of Eq. (32). \square

Appendix D

Theorem 4. For a given opportunity x , if there exists a vector α such that

$$(-x' + \alpha'A)B = f',$$

then $s(x) = 0$.

Proof of Theorem 4. Let α^a solve Eq. (33). Since the ask linear program describing the ask problem, shown in Eq. (31) is a minimization

$$\alpha^a \pi \geq a(x). \quad (\text{D.1})$$

As α^a solves Eq. (33) and $f' \leq 0$, α^a also solves the constraints of the linear program establishing the bid problem in (32). Since Eq. (32) is a maximization

$$\alpha^a \pi \leq b(x). \quad (\text{D.2})$$

The inequalities in Eqs. (D.1) and (D.2) imply

$$b(x) \geq a(x)$$

or equivalently, $s(x) \leq 0$. Combining this result with Theorem 3 gives the desired result. \square

Appendix E

Theorem 5. The bid–ask spread is an increasing function of the scale

$$s(\lambda x) \leq s(x) \quad \text{for } \lambda \in (0, 1).$$

Proof of Theorem 5. Let α_λ^a solve the following ask linear program for λx :

$$\min_{\alpha} \alpha' \pi \quad \text{subject to } -\lambda x' B + \alpha' AB \geq f' \tag{E.1}$$

and let $a^*(\lambda x) \equiv \alpha_\lambda^a \pi / \lambda$ be the asking price per unit of x when the scale is λ . Now α_1^a solves the LP in Eq. (31) and consequently satisfies the constraint

$$\alpha_1^a AB \geq x' B + f'.$$

Multiplying by $\lambda \in (0, 1)$ implies that

$$\lambda \alpha_1^a AB \geq \lambda x' B + \lambda f' \geq \lambda x' B + f',$$

since $f \leq 0$. Thus $\lambda \alpha_1^a$ solves the constraints in Eq. (E.1) and so $\lambda \alpha_1^a \pi \equiv \lambda a^*(x)$ is greater than the minimand of Eq. (E.1):

$$\alpha_\lambda^a \pi \leq \lambda a^*(x).$$

Dividing by λ implies

$$a^*(\lambda x) \leq a^*(x)$$

so that the asking price per unit of x increases with the scale λ .

Similarly, let α_λ^b solve the following bid LP for λx :

$$\min_{\alpha} \alpha' \pi \quad \text{subject to } \lambda x' B - \alpha' AB \geq f' \tag{E.2}$$

and let $b^*(\lambda x) \equiv \alpha_\lambda^b \pi / \lambda$ be the bid per unit of x when the scale of the transaction is λ . Now α_1^b solves the LP in Eq. (32) and consequently satisfies the constraint

$$\alpha_1^b AB \leq x' B - f'.$$

Multiplying by $\lambda \in (0, 1)$ implies that

$$\lambda \alpha_1^b AB \leq \lambda x' B - \lambda f' \leq \lambda x' B - f',$$

since $f \leq 0$. Thus $\lambda\alpha_1^b$ solves the constraints in Eq. (E.2) and so $\lambda\alpha_1^b \equiv \lambda b^*(x)$ is less than the maximand of Eq. (E.2):

$$\alpha_\lambda^{b'} \pi \geq \lambda b^*(x).$$

Dividing by λ implies

$$b^*(\lambda x) \geq b^*(x)$$

so that the bid for one unit of x decreases as the scale λ increases.

As $\lambda \downarrow 0$, we know that the dual shown in Eq. (C.1) of Appendix C to the ask LP reduces to the dual solution without the affine constraints

$$\max_{w \geq 0} x' Bw \text{ subject to } ABw = \pi, \quad (\text{E.3})$$

while the dual solution shown in Eq. (C.2) of Appendix C to the bid LP similarly reduces to

$$\min_{w \geq 0} x' Bw \text{ subject to } ABw = \pi. \quad (\text{E.4})$$

The solution to the ask dual of Eq. (E.3) is the total asking price for λ units, while the solution to the bid dual of Eq. (E.4) is the total bid price for λ units. The solution to the maximization is clearly greater than the solution to the minimization and both solutions scale proportionally with x . Thus, the ask and bid per unit of x is constant with respect to scale for small λ . Once λ becomes sufficiently large, an affine constraint becomes binding. Hence, the ask rises and the bid falls, so that the spread widens. \square

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