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ALTERNATIVE CHARACTERIZATIONS OF AMERICAN PUT OPTIONS

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We derive alternative representations of the McKean equation for the value of the American put option. Our main result decomposes the value of an American put option into the corresponding European put price and the early exercise premium. We then represent the European put price in a new manner. This representation allows us to alternatively decompose the price of an American put option into its intrinsic value and time value, and to demonstrate the equivalence of our results to the McKean equation.

KEY WORDS: American put options, European put options, local time, free boundary-problem, optimal stopping problem.

The problem of valuing American options continues to intrigue finance theorists. For example, in the New Palgrave Dictionary of Economics, Ross (1987) writes:

> *This does not mean, however, that there are no important gaps in the (option pricing) theory. Perhaps of most importance, beyond numerical results . . . , very little is known about most American options which expire in finite time. . . . Despite such gaps, when judged by its ability to explain the empirical data, option pricing theory is the most successful theory not only in finance, but in all of economics.*

The history of the American option valuation problem spans over a quarter of a century (for a survey of this theory see Myneni (1992)). In the framework of Samuelson's equilibrium pricing model, McKean (1965) showed that the *optimal stopping problem* for determining an American option's price could be transformed into a *free-boundary problem.* This insight allowed him to derive rigorous valuation formulas for finite-lived and perpetual American options. Although the McKean equation explicitly represents the value of the finite-lived American option in terms of the *exercise boundary,* the solution reveals little about the underlying sources of value for an American option and does not lend itself to analysis or implementation.

Somewhat later, Black and Scholes (1973) and Merton (1973) developed a more satisfactory theory of option pricing using arbitrage-based arguments. Merton showed that while the Black-Scholes European option pricing methodology applied to American call

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options on non-dividend-paying stocks, it did not apply to American put options. He also observed that McKean's solutions could be adapted to valuing American put options by replacing the expected rate of return on the put and its underlying stock with the riskless rate. This insight foreshadowed the later development of *risk-neutral pricing* of Cox and Ross (1976) and the *equivalent martingale measure* technique of Harrison and Kreps (1979) and Harrison and Pliska (1981). The application of this technology to the optimal stopping problem for the American put option was studied by Bensoussan (1984) and Karatzas (1988). While the optimal stopping approach is both general and intuitive, it does not lead to tractable valuation results due to the difficulty involved in finding density functions for first passage times.

The intractability of the optimal stopping approach lead Brennan and Schwartz (1977) to investigate *numerical* solutions to the corresponding free boundary problem. Jaillet, Lamberton, and Lapeyre (1990) rigorously justify the Brennan-Schwartz algorithm for pricing American put options, using the theory of variational inequalities. Other numerical solutions were advanced by Parkinson (1977) and Cox, Ross, and Rubinstein (1979). Geske and Shastri (1985) compared the efficiency of these approaches and explained why an analytic solution may be more efficient. Furthermore, Geske and Johnson (1984) argued that numerical solutions do not provide the intuition which the comparative statics of an analytic solution afford.

Analytic *approximations* have been developed by Johnson (1983), MacMillan (1986), Omberg (1987), and Barone-Adesi and Whaley (1987). Blomeyer (1986) and Barone-Adesi and Whaley (1988) extend these approximations to account for discrete dividends. However, these approximations cannot be made arbitrarily accurate. In contrast, the Geske and Johnson (1984) formula is arbitrarily accurate, although difficult to evaluate unless extrapolation techniques are employed.

The purpose of this paper is to explore alternative characterizations of the American put's value. These characterizations enhance our intuition about the sources of value of an American put. They also provide computational advantages, new analytic bounds, and new analytic approximations for this value. Our first characterization decomposes the American put value into the corresponding European put price and the *early exercise premium*. In contrast to approximations by MacMillan (1987) and Barone-Adesi and Whaley (1988), we provide an exact determination of the early exercise premium. This decomposition was also derived independently in Jacka (1991) and Kim (1990), using different means.[1] We provide another proof of the result and offer intuition on the nature of the early exercise premium. In particular, we show that the early exercise premium is the value of an annuity that pays interest at a certain rate whenever the stock price is low enough so that early exercise is optimal.

As in McKean (1965), the formula for the American put value is a function of the exogenous variables and the exercise boundary. While the function relating the boundary to the exogenous variables remains an unsolved problem, the boundary can be determined numerically. Having priced American put options in terms of a boundary, we also value *European* put options in terms of a boundary. We prove that our result is equivalent to the Black-Scholes (1973) formula for the price of a European put. This work generalizes earlier papers by Siedenverg (1988) and Carr and Jarrow (1990), and should be of interest in its own right.

[1] Jacka (1991) obtains the result using probability theory applied to the optimal stopping problem, while Kim (1990) obtains it as a limit of the Geske-Johnson (1984) formula.

From our main valuation results and a particular choice of a boundary for our European put formula, we are able to decompose the American put value into its intrinsic value and its time value (or *delayed exercise value*). Just as the early exercise premium capitalizes the additional benefit of allowing exercise prior to maturity, the delayed exercise value yields the additional value of permitting exercise after the valuation date. A second boundary choice for our European put formula then recovers the McKean equation. In contrast to Geske and Johnson (1984), all of our characterizations for the value of an American put involve only one-dimensional normal distribution functions.

The outline for this paper is as follows. In Section 1, we decompose the American put value into the corresponding European put price and the early exercise premium. Section 2 represents the corresponding European put price in terms of an arbitrary boundary. In Section 3, we select boundaries in order to decompose the American put value into its intrinsic and time value, and to show the equivalence of our results to McKean's equation. Section 4 summarizes the paper and indicates some extensions and avenues for future research. An appendix contains proofs of our main results.

1. THE EARLY EXERCISE PREMIUM

Throughout the paper, we assume the standard model of perfect capital markets, continuous trading, no-arbitrage opportunities, a constant interest rate $r > 0$, and a stock price S_t following a geometric Brownian motion with no payouts; i.e.,

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t, \qquad \text{for all } t \in [0, T], \tag{1.1}$$

where the expected rate of return per unit time μ and the instantaneous volatility per unit time $\sigma > 0$ are constants. The term dW_t denotes increments of a standard Wiener process defined on the time set $[0, T]$ and on a complete probability space $(\Omega, \mathcal{F}, Q)$.

Consider an American put option on the stock with strike price K and maturity date T. Let P_t denote the value of the American put at time $t \in [0, T]$. For each time $t \in [0, T]$, there exists a *critical stock price* B_t below which the American put should be exercised early; i.e.,

$$\text{if } S_t \leq B_t, \text{ then } P_t = \max[0, K - S_t], \tag{1.2}$$

$$\text{and if } S_t > B_t, \text{ then } P_t > \max[0, K - S_t]. \tag{1.3}$$

The exercise boundary is the time path of critical stock prices B_t, $t \in [0, T]$. This boundary is independent of the current stock price S_0 and is a smooth, nondecreasing function of time t terminating in the strike price; i.e., $B_T = K$. The put value is also a function, denoted $P(S, t)$, mapping its domain $\mathcal{D} \equiv (S, t) \in [0, \infty) \times [0, T]$ into the nonnegative real line. The exercise boundary B_t, $t \in [0, T]$, divides this domain $\mathcal{D}$ into a *stopping region* $\mathcal{S} \equiv [0, B_t] \times [0, T]$ and a *continuation region* $\mathcal{C} \equiv (B_t, \infty) \times [0, T]$. Equation (1.2) indicates that in the stopping region, the put value function $P(S, t)$ equals its exercise value, $\max[0, K - S]$. In contrast, the inequality expressed in (1.3) shows that in the continuation region, the put is worth more "alive" than "dead." Since the

American put value is given by (1.2) if the stock price starts in the stopping region, we henceforth assume that the put is alive at the valuation date 0; i.e., $S_0 > B_0$.

The partial derivatives, $\partial P/\partial t$, $\partial P/\partial S$, and $\partial^2 P/\partial S^2$ exist[2] and satisfy the *Black-Scholes partial differential equation*[3] in the continuation region $\mathcal{C}$; i.e.,

$$\frac{\sigma^2 S^2}{2}\frac{\partial^2 P(S, t)}{\partial S^2} + rS\frac{\partial P(S, t)}{\partial S} - rP(S, t) + \frac{\partial P(S, t)}{\partial t} = 0 \quad \text{for } (S, t) \in \mathcal{C}. \tag{1.4}$$

McKean's analysis implies that the American put value function $P(S, t)$ and the exercise boundary B_t jointly solve a free-boundary problem, consisting of (1.4) subject to the following boundary conditions:

$$P(S, T) = \max[0, K - S], \tag{1.5}$$

$$\lim_{S \uparrow \infty} P(S, t) = 0, \tag{1.6}$$

$$\lim_{S \downarrow B_t} P(S, t) = K - B_t \tag{1.7}$$

$$\lim_{S \downarrow B_t} \frac{\partial P(S, t)}{\partial S} = -1. \tag{1.8}$$

Equation (1.5) states that the American put is European at expiration. Expression (1.6) shows that the American put's value tends to zero as the stock price approaches infinity. The *value-matching conditions* (1.7) and (1.2) imply that the put price is continuous across the exercise boundary. Furthermore, the *high contact conditions* (1.8) and (1.2) further imply that the *slope* is continuous. This condition was postulated by Samuelson (1965) and proved by McKean (1965). Equations (1.7) and (1.8) are jointly referred to as the *smooth fit conditions*.

Working within Samuelson's equilibrium framework, McKean (1965) solved the free-boundary problem for the American *call* option. By applying his analysis to the American *put* option, and by replacing the expected rate of return on the option and stock by the riskless rate, one obtains an analytic valuation formula for the put value and an integral equation for the exercise boundary B_t. Numerical evaluation of this integral equation is complicated by the fact that the integrand depends on the slope of the exercise boundary, which becomes infinite at maturity ($\lim_{t \uparrow T} \partial B_t/\partial t = \infty$). To avoid this difficulty, we seek an alternative characterization for the American put's value which does not involve the slope of the exercise boundary. Our first theorem obtains such a characterization.[4]

THEOREM 1.1 (Main Decomposition of the American Put). *On the continuation region $\mathcal{C}$, the American put value P_0 can be decomposed into the corresponding European put price p_0 and the early exercise premium e_0:*

$$P_0 = p_0 + e_0, \tag{1.9}$$

[2]See Jaillet, Lamberton, and Lapeyre (1990, Theorem 3.6) or Van Moerbeke (1976, p. 116, Theorem 1).
[3]See McKean (1965, p. 38), and Merton (1973, p. 173).
[4]Independently, Jacka (1991) and Kim (1990) derive the same result by different means.

where

$$e_0 = rK \int_0^T e^{-rt} N\left(\frac{\ln(B_t/S_0) - \rho_2 t}{\sigma\sqrt{t}}\right) dt,$$

$$e_2 = r - \frac{r^2}{2}, \text{ and,}$$

$$N(x) \equiv \int_0^x \frac{\exp(-z^2/2)}{\sqrt{2\pi}}\, dz$$

is the standard normal distribution function.

To understand this decomposition, consider the following trading strategy which converts an American put option into a European one. Suppose that an investor holds one American put[5] whenever the stock price is above the exercise boundary. When the stock price is at or below the boundary, the investor duplicates the put's exercise value by keeping K dollars in bonds and staying short one stock. Since the American put is worth more alive than dead above the boundary, the value of this portfolio at any time $t \in [0, T]$ is the larger of the put's holding and exercise values, i.e., $\max[P_t, K - S_t]$.

The strategy's opening cost is the initial American put price P_0, since the stock price starts above the boundary by assumption (i.e., $S_0 > B_0$). If and when the stock price crosses the exercise boundary from above, the investor exercises his put by shorting one share of stock to the writer and by investing the exercise price received in bonds. The "smooth fit" conditions (1.7) and (1.8) guarantee that these transitions at the exercise boundary are self-financing. However, when the stock price is below the boundary, interest earned on the K dollars in bonds must be siphoned off to maintain a level bond position. If and when the stock price crosses the exercise boundary from below, the investor liquidates this bond position, using the K dollars to buy one put for $K - S$ dollars and to close his short stock position for S dollars. The "smooth fit" conditions again guarantee self-financing at the exercise boundary. At expiration, the strategy's liquidation value matches the payoff of a European put, $\max[0, K - S_T]$, since the alive American put is worthless above the boundary.

The present value of this terminal payoff is the initial European put price p_0. The initial early exercise premium e_0, as defined in (1.9), equals the present value of interest accumulated while the stock price is below the boundary. The decomposition (1.9) then states that the initial investment in the trading strategy, P_0, equates to the present value of the terminal payoff, p_0, and the present value of these intermediate interest withdrawals, e_0.

The price of a *European* put at the valuation date 0 is given by the Black-Scholes formula:

$$p_0 = Ke^{-rT}N(k_{2T}) - S_0N(k_{1T}), \tag{1.10}$$

where $k_{2T} \equiv [\ln(K/S_0) - \rho_2 T]/\sigma\sqrt{T}$, $k_{1T} \equiv k_{2T} - \sigma\sqrt{T} = [\ln(K/S_0) - \rho_1 T]/\sigma\sqrt{T}$, and $\rho_1 \equiv \rho_2 + \sigma^2 = r + \sigma^2/2$. Consequently, in the continuation region $\mathcal{C}$, the initial American put value may be expressed as a function of the exogenous variables (S_0, K, T, r, σ) and the exercise boundary (B_t, $t \in [0, T]$):

[5] Alternatively, if the put is mispriced, the investor can manage the self-financing portfolio of stocks and bonds which replicates the put's payoff. We determine this portfolio shortly.

$$P_0 = Ke^{-rT}N(k_{2T}) - S_0N(k_{1T}) + rK\int_0^T e^{-rt}N(b_{2t})\,dt, \tag{1.11}$$

where $b_{2t} \equiv [\ln(B_t/S_0) - \rho_2 t]/\sigma\sqrt{t}$, $\rho_2 \equiv r - \sigma^2/2$.

The initial boundary value B_0 is the initial stock price S_0 which implicitly solves the value-matching condition (1.7):

$$Ke^{-rT}N(k_{2T}) - S_0N(k_{1T}) + rK\int_0^T e^{-rt}N(b_{2t})\,dt = K - S_0. \tag{1.12}$$

Since the critical stock price B_0 depends on future boundary values, B_t, $t \in (0, T]$, it must be determined by setting the terminal boundary value to the strike price ($B_T = K$) and working backwards through time.

Our equations (1.11) and (1.12) do not involve the slope of the exercise boundary, as in McKean's equation. In addition, we have localized the effect of the exercise boundary B_t, $t \in [0, T]$, on the American put value to the last term in (1.11). Unfortunately, the boundary satisfies the nonlinear integral equation (1.12), which has no known analytic solution. However, solving (1.12) numerically for the exercise boundary should prove easier than in McKean's formulation.

The early exercise premium is increasing in the boundary. This observation allows us to bound the American put value analytically. Suppose that an estimate for the boundary is known to be always greater (lesser) than the true boundary B_t. This estimate along with (1.11) then generates an upper (lower) bound on the put option. For example, the true boundary B_t always lies between the strike price K and the exercise boundary for the perpetual put, B_∞ (i.e., $K \geq B_t \geq B_\infty$ for all $t \in [0, T]$). McKean (1965) and Merton (1973) calculate the perpetual boundary to be $B_\infty = rK/\rho_1$. Consequently, we can bound the American put value P_0 analytically:

$$p_0 + rK\int_0^T e^{-rt}N\left(\frac{\ln(K/S_0) - \rho_2 t}{\sigma\sqrt{t}}\right)dt \geq P_0 \geq p_0 + rK\int_0^T e^{-rt}N\left(\frac{\ln(B_\infty/S_0) - \rho_2 t}{\sigma\sqrt{t}}\right)dt. \tag{1.13}$$

Section 3 shows that for at or out-of-the-money puts ($S_0 \geq K$), the upper bound in (1.13) is tighter than the bound given by the price of the corresponding European put with strike price growing at the riskless rate. As far as we know, there are no explicit tighter bounds to the exercise boundary. However, another upper bound on the American put value can be generated by using that initial stock price S_0, which equates the right side of (1.13) to the exercise value $\max[0, K - S_0]$. Since this quantity lies between B_t and K, inserting it in (1.11) yields an even tighter upper bound than the left side of (1.13). This procedure can also be used to generate lower bounds and can be applied iteratively.

Our characterization also allows us to *approximate* the American put value by replacing the exercise boundary B_t in (1.11) with an estimate for it, $\hat{B}_t$:

$$P_0 \approx p_0 + rK\int_0^T e^{-rt}N\left(\frac{\ln(\hat{B}_t/S_0) - \rho_2 t}{\sigma\sqrt{t}}\right)dt.$$

At a minimum, an estimator should satisfy the characteristics of the exercise boundary described at the start of this section. An example of such an estimator which leads to an analytic approximation is the discounted strike price $\hat{B}_t = Ke^{-\theta(T-t)}$, $\theta \geq 0$. For small times to maturity, Van Moerbeke (1976) shows that the exercise boundary B_t is approximately $Ke^{-\alpha\sigma\sqrt{T-t}}$, where α is a unitless constant.[6] Conversely, for very long times to maturity, B_t converges at an exponential rate to B_∞. An estimator which also accounts for both of these characteristics is an exponentially weighted average[7] of the strike price and the perpetual boundary, $\hat{B}_t = Ke^{-\theta\sqrt{T-t}} + B_\infty(1 - e^{-\theta\sqrt{T-t}})$, $\theta \geq 0$.

Besides the above benefits, our characterization also facilitates the analysis of limiting values and comparative statics. For example, as the initial stock price approaches infinity, (1.11) indicates that the premiums for the European put and early exercise both tend to zero. Consequently, the American put value also vanishes, verifying the boundary condition (1.6) and confirming intuition. Differentiating (1.11) with respect to the initial stock price yields the "delta" for the American put:

$$\frac{\partial P_0}{\partial S_0} = -N(k_{1T}) - rK\int_0^T \frac{e^{-rt}N'(b_{2t})}{S_0\sigma\sqrt{t}}\,dt \leq 0, \tag{1.14}$$

$$\text{where } N'(x) \equiv \frac{\exp(-x^2/2)}{\sqrt{2\pi}}$$

is the standard normal density function. Thus, as the initial stock price falls, the premiums for the European put and early exercise both rise. The early exercise premium rises because of the increased probability of stock price trajectories entering into the stopping region. As the initial stock price falls below the critical stock price, the American put is valued by (1.2). Consequently, as the stock price approaches zero, the American put value approaches the strike price, which acts as an upper bound. Although the observed American put value remains constant at the strike price over time, this does not represent an arbitrage opportunity, since all puts written with positive strike prices are immediately exercised.

Beyond indicating the sensitivity of the American put value to stock price changes, the delta of an alive American put also represents the number[8] of shares to hold when replicating it in a self-financing strategy. Consequently, since the early exercise premium's delta is negative (before expiration), more stock is shorted than for a European put because of the possibility of early exercise.

The delta can also be used to determine a simpler integral equation for the exercise boundary B_t, $t \in [0, T]$. If the high-contact condition (1.8) is used, the critical stock price B_0 is the initial stock price S_0, which solves the integral equation

$$rK\int_0^T \frac{e^{-rt}N'(b_{2t})}{S_0\sigma\sqrt{t}}\,dt = N(-k_{1T}). \tag{1.15}$$

Once again, the entire exercise boundary is generated numerically by working backwards through time.

Calculation of the other derivatives verifies that (1.11) satisfies the free-boundary prob-

[6] See Van Moerbeke (1976), p. 144.

[7] See Barone-Adesi and Whaley (1987) for a similar approximation.

[8] The amount of dollars invested in bonds when replicating an alive American put in a self-financing strategy is given by $P_0 - (\partial P_0/\partial S_0)S_0$.

lem (1.4)–(1.8). Kim (1990) also verifies that the limiting value of (1.11) yields the perpetual put formulae given in McKean (1965) and Merton (1973).

2. REPRESENTING EUROPEAN PUTS IN TERMS OF A BOUNDARY

The previous section priced American puts in terms of the exercise boundary. This section represents the value of a *European* put in terms of an arbitrary boundary. The appendix proves that this representation is mathematically equivalent to the Black-Scholes formula (1.10). We then select alternative boundaries to generate various valuation formulas for a European put. These formulas enhance intuition on the sources of value of a European put and are employed in the next section to generate additional characterizations of the American put's value.

As in the Black-Scholes dynamic hedge, we consider a trading strategy in stocks and bonds whose liquidation value at the expiration date T is the put's terminal payoff, $\max[0, K - S_T]$. Consider a strategy with the amount m_t dollars held in bonds earning interest continuously at constant rate r and with the number of shares of stock equal to n_t. The value of this strategy at any time t is

$$V_t \equiv m_t + n_t S_t. \tag{2.1}$$

Suppose transitions in stock holdings occur only at a positive, smooth, but otherwise arbitrary boundary A_t, which terminates at the strike price

$$A_T = K. \tag{2.2}$$

Examples of such a boundary include the strike price itself, K, the exercise boundary for an American put, B_t, or an estimator for this boundary, $\hat{B}_t$, as given in Section 1. We study an example of this type of strategy, termed the *stop-loss start-gain* strategy, defined by

$$m_t = 1_{\{S_t \le A_t\}} A_t, \quad n_t = -1_{\{S_t \le A_t\}} \qquad \text{for all } t \in [0, T], \tag{2.3}$$

where $1_{\{B\}}$ is the indicator function of the set $\{B\}$. This strategy involves keeping A_t dollars in bonds whenever the stock price S_t is at or below the boundary A_t. Funds are injected and withdrawn as required after accounting for the interest earned. The strategy also requires that one share of stock be held short when the stock price is at or below the boundary. No bonds or stocks are held above the boundary. The stop-loss start-gain strategy for specified boundaries has been previously studied by Hull and White (1987), Ingersoll (1987), and Siedenverg (1988), among others.

Substituting (2.3) in (2.1) implies that the value of the stop-loss start-gain strategy at any time $t \in [0, T]$ is

$$V_t = 1_{\{S_t \le A_t\}} A_t - 1_{\{S_t \le A_t\}} S_t = \max[0, A_t - S_t]. \tag{2.4}$$

Consequently, from (2.2), this strategy replicates the payoff of a European put; i.e., $V_T = \max[0, K - S_T]$. From (2.4), the initial investment in the strategy is $V_0 = \max[0, A_0 - S_0]$. Since the strategy replicates the European put's payoff, the put's value is given by this initial investment plus the present value of the external financing required

to implement this strategy. The appendix determines this present value, yielding the following theorem.

THEOREM 2.1 (Main Decomposition of the European Put). *The European put price* p_0 *is given by*

$$(2.5)\quad p_0 = \max[0, A_0 - S_0] + \frac{\sigma^2 S_0}{2}\int_0^T \frac{N'(a_{1t})}{\sigma\sqrt{t}}\,dt + \int_0^T N(a_{2t})d(A_t e^{-rt}),$$

where $a_{1t} \equiv [\ln(A_t/S_0) - \rho_1 t]/\sigma\sqrt{t}$, $\rho_1 \equiv r + \sigma^2/2$ *and* $a_{2t} \equiv [\ln(A_t/S_0) - \rho_2 t]/\sigma\sqrt{t}$, $\rho_2 \equiv r - \sigma^2/2$.

In general, the European put value decomposes into three terms. The first term in (2.5) is the initial investment in the stop-loss start-gain strategy (2.3). The next term represents the present value of the external financing required because of adverse movements of the stock price at the arbitrary boundary A_t. The final term represents the present value of funds injected and withdrawn in order to keep A_t dollars in bonds whenever the stock price is at or below this boundary. The appendix proves that our representation (2.5) is equivalent to the Black-Scholes formula.

In addition, our representation (2.5) is a generalization of the formula given in Carr and Jarrow (1990). To get this formula, we use the exponential boundary

$$(2.6)\qquad A_t = Ke^{-r(T-t)}.$$

Substituting (2.6) into Theorem 2.1 yields a decomposition[9] of the initial European put price into its intrinsic and time value:

$$(2.7)\quad p_0 = \max[0, Ke^{-rT} - S_0] + \frac{\sigma^2 S_0}{2}\int_0^T \frac{1}{\sigma\sqrt{t}} N'\left(\frac{\ln(K/F_0) - \sigma^2 t/2}{\sigma\sqrt{t}}\right) dt,$$

where $F_0 \equiv S_0 e^{rT}$ is the initial forward price of the stock. The corresponding result for the binomial model is developed in Siedenverg (1988).

Equation (2.7) indicates that the payoff of a European put is replicated by holding a pure discount bond paying K dollars at T and being short one stock whenever the stock price is at or below the present value of the strike price, $Ke^{-r(T-t)}$. While this strategy is self-financing below this boundary, external financing is required at the boundary. The first term in (2.7) is the initial investment in the strategy, while the second is the present value of this external financing.

Suppose that on the valuation date 0, we wish to price a European put with current strike price K growing at the riskless rate r. By the expiration date T, the exercise price will be Ke^{rT}. Let the current price of this put be g_0. Replacing K in (2.7) with Ke^{rT} yields

$$(2.8)\quad g_0 = \max[0, K - S_0] + \frac{\sigma^2 S_0}{2}\int_0^T \frac{1}{\sigma\sqrt{t}} N'\left(\frac{\ln(K/S_0) - \sigma^2 t/2}{\sigma\sqrt{t}}\right) dt.$$

[9]To directly prove the equivalence of (2.7) to the Black-Scholes formula (1.10), use (1.10) to express the forward price of the European put $\hat{p}(T)$ in terms of the forward price of the stock $\hat{S}$, and differentiate it with respect to time to maturity T, holding the forward price constant. Then integrate back over T, using the boundary condition $\hat{p}(0) = \max[0, K - \hat{S}]$ to determine the constant of integration.

Margrabe (1978) shows that this put is an upper bound for an American put with strike price K. In Section 3, we prove that the upper bound generated in the last section is tighter than (2.8), if the American put is at or out-of-the-money ($S_0 \geq K$).

A second boundary choice in our representation of the European put option leads to another important decomposition. Consider a constant boundary equal to the strike price:

$$A_t = K. \tag{2.9}$$

Substituting (2.9) into Theorem 2.1 leads to the following decomposition[10] of the European put value:

$$p_0 = \max[0, K - S_0] + \int_0^T \left[\frac{\sigma^2 S_0}{2} \frac{N'(k_{1t})}{\sigma\sqrt{t}} - rKe^{-rt}N(k_{2t}) \right] dt, \tag{2.10}$$

where recall $k_{1t} \equiv [\ln(K/S_0) - \rho_1 t]/\sigma\sqrt{t}$, $\rho_1 \equiv r + \sigma^2/2$ and $k_{2t} \equiv [\ln(K/S_0) - \rho_2 t]/\sigma\sqrt{t}$, $\rho_2 \equiv r - \sigma^2/2$. Equation (2.10) indicates that the payoff of a European put can be replicated by keeping K dollars in bonds and being short one share whenever the put is at or in-the-money. No bonds or stocks are held when the put is out-of-the-money. The first term represents the initial investment in the strategy, while the other term gives the present value of the external financing needed to implement the strategy.

3. VARIOUS AMERICAN PUT REPRESENTATIONS

This section uses our main decomposition of the American put value in Theorem 1.1 and our representation of the European put price in Theorem 2.1 to derive two alternative characterizations of the American put's value.

Substituting the European put formula (2.10) arising from the constant boundary $A_t = K$ into Theorem 1.1 yields a decomposition of the American put value into its intrinsic value, $\max[0, K - S_0]$, and its time value, also called the *delayed exercise value* d_0:

$$P_0 = \max[0, K - S_0] + d_0, \tag{3.1}$$

where

$$d_0 = \frac{\sigma^2 S_0}{2} \int_0^T \frac{N'(k_{1t})}{\sigma\sqrt{t}}\, dt - rK \int_0^T e^{-rt}[N(k_{2t}) - N(b_{2t})]\, dt.$$

This representation can be given a financial interpretation by rearranging it as

$$\begin{aligned} P_0 - \max[0, K - S_0] &= d_0 \\ &= \int_0^T \left\{ \frac{\sigma^2 S_0}{2} \frac{N'(k_{1t})}{\sigma\sqrt{t}} - rKe^{-rt}[N(k_{2t}) - N(b_{2t})] \right\} dt. \end{aligned} \tag{3.2}$$

[10]Note that if the interest rate vanishes ($r = 0$), then the boundary in (2.6) simplifies to that in (2.9) and the European put value in (2.10) simplifies to that in (2.7) (with $r = 0$).

The following strategy duplicates the time value of the American put in the continuation region. Suppose an investor holds an American put when it is at or out-of-the-money. When the stock price is strictly between the strike price and the exercise boundary, the investor continues to hold the American put, and, in addition, holds one share of the stock while keeping K dollars in borrowings. When the stock price enters the stopping region, the investor exercises his put by delivering the share held and using the strike price received to pay off his borrowing. Consequently, the investor holds nothing in this region. The value of this strategy at any time $t \in [0, T]$ is $V_t = 1_{\{S_t > B_t\}}(P_t - \max[0, K - S_t])$, which is the time value in the continuation region. Since the stock price starts in the continuation region ($S_0 > B_0$), the strategy has an initial investment of $P_0 - \max[0, K - S_0]$. Since the exercise boundary terminates at the strike price ($B_T = K$), this strategy has zero terminal value. The delayed exercise value, as determined in (3.1), equals the present value of the intervening cash flows. Since there is no terminal payoff, (3.2) then states that the initial investment in this strategy equates to the present value of these cash flows. The American put has the same value as a claim which pays the exercise value immediately, and a flow equal to the sum of the stock price movement "around" the strike price less interest on the strike price paid while the put is in-the-money but optimally held alive (i.e., $K > S_t > B_t$).

The exercise boundary B_t, $t \in [0, T]$, can be determined implicitly from the condition (1.7) that there is no value in delaying exercise at this boundary ($d_t|_{S_t = B_t} = 0$). Consequently, the critical stock price B_0 is the initial stock price S_0 that solves

$$\frac{\sigma^2}{2}\int_0^T \frac{e^{-rt}N'(k_{2t})}{\sigma\sqrt{t}}\,dt = r\int_0^T e^{-rt}[N(k_{2t}) - N(b_{2t})]\,dt. \tag{3.3}$$

Thus, the put is exercised as soon as the present value of the flow arising from movement of the stock price at the strike price equates to the present value of interest paid, while the stock price is between the exercise boundary and the strike price.

The decomposition (3.1) into intrinsic and time values can also be used to bound the American put's value. Since the difference in cumulative normals in (3.2) is nonnegative, setting the difference to zero yields the following upper bound:

$$P_0 \le \max[0, K - S_0] + \frac{\sigma^2 S_0}{2}\int_0^T \frac{N'(k_{1t})}{\sigma\sqrt{t}}\,dt, \tag{3.4}$$

where $k_{1t} \equiv [\ln(K/S_0) - \rho_1 t]/\sigma\sqrt{t}$. From (2.10), this is the same upper bound as in (1.13). Comparing (3.4) with (2.8), we see that our upper bound given by (3.4) is tighter if the American put is at[11] or out-of-the-money ($S_0 \ge K$). Using a result from Hadley and Whitin (1963, Appendix 4, Property 9), our upper bound can be rewritten in terms of standard normal distribution functions:

$$P_0 \le \max[0, K - S_0] + \frac{S_0}{\lambda}\left\{[1_{\{S_0 \le K\}} - N(k_{1T})] - \left(\frac{K}{S_0}\right)^{\lambda}[1_{\{S_0 \le K\}} - N(k_{1T} + \lambda\sigma\sqrt{T})]\right\},$$

[11] A slightly weaker condition is $S_0 \ge Ke^{-\rho_1 T}$.

where $\lambda \equiv 2\rho_1/\sigma^2$.

Finally, to generate McKean's characterization of the American put value, let the boundary used to price the European put be the exercise boundary for the American put:

$$A_t = B_t. \tag{3.5}$$

Then substituting (3.5) into Theorem 2.1 yields the following representation for the European put value:

$$\begin{aligned} p_0 &= \max[0, B_0 - S_0] + \frac{\sigma^2 S_0}{2}\int_0^T \frac{N'(b_{1t})}{\sigma\sqrt{t}}\,dt + \int_0^T N(b_{2t})\,d(B_t e^{-rt}) \\ &= \frac{\sigma^2 S_0}{2}\int_0^T \frac{N'(b_{1t})}{\sigma\sqrt{t}}\,dt + \int_0^T N(b_{2t})\,d(B_t e^{-rt}), \qquad \text{since } S_0 > B_0. \end{aligned} \tag{3.6}$$

Equation (3.6) indicates that the payoff to a European put can be achieved by keeping B_t dollars in bonds and staying short one share whenever the stock price is in the stopping region. No bonds or stocks are held when the stock price is in the continuation region. Since the stock price starts in this region, no investment is initially required. However, transitions at the exercise boundary and the bond position below it are not self-financing. The first term in (3.6) gives the present value of the external financing required at the boundary, while the second term gives this present value below it.

Substituting formula (3.6) for the European put price into Theorem 1.1 yields the following formula[12] for the value of an American put:

$$\begin{aligned} P_0 &= \frac{\sigma^2 S_0}{2}\int_0^T \frac{N'(b_{1t})}{\sigma\sqrt{t}}\,dt + \int_0^T N(b_{2t})\,d(B_t e^{-rt}) + rK\int_0^T e^{-rt}N(b_{2t})\,dt \\ &= \frac{\sigma^2 S_0}{2}\int_0^T \frac{N'(b_{1t})}{\sigma\sqrt{t}}\,dt - \int_0^T N(b_{2t})\,d((K - B_t)e^{-rt}). \end{aligned} \tag{3.7}$$

To understand this decomposition, consider a strategy of holding one American put whenever the stock price is above the exercise boundary. When the stock price is in the stopping region, keep $K - B_t$ dollars in bonds, but hold no puts or stocks. Since the stock price starts in the continuation region, the startup cost of the strategy is the initial American put price P_0. Since the American put is worthless in the continuation region at expiration and $B_T = K$, the strategy has no terminal payoff. In contrast to the strategy underlying Theorem 1.1, this strategy is not self-financing at the exercise boundary. The first term in (3.7) is the present value of the external financing at this boundary, while the second is this present value below it. Since there is no terminal payoff to this strategy, (3.7) indicates that its startup cost equates to the present value of its external financing requirement.

[12] Equation (3.7) is in fact equivalent to the McKean equation for the American put value. To see this, replace $S_0N'(b_{1t})$ with the equivalent value $B_t e^{-rt}N'(b_{2t})$ in the first term in (3.7) and integrate the second term by parts.

4. SUMMARY AND EXTENSIONS

This paper makes several contributions. We decompose the American put's value into the corresponding European put price and the early exercise premium. This formulation leads to increased understanding, more efficient numerical evaluation, tighter analytic bounds, and new analytic approximations. In addition, we employ a stop-loss start-gain strategy at an arbitrary smooth boundary to obtain a new European put valuation formula. This allows us to alternatively decompose the American put price into intrinsic and time value and to prove the equivalence of our results to the McKean equation.

Although we considered the case of no dividends, our results easily extend to the case where the underlying asset pays continuous proportional dividends.[13] This extension to continuous dividends can be used to price American options on commodities, foreign currencies, or futures prices. By letting interest rates be the underlying state variable, American bond options can also be priced (see Jamshidian (1989) and El Karoui, Myneni, and Viswanthan (1991)). By generalizing the payoff function, other American claims can be valued by this approach, such as compound options, prepayment options, or callable bonds. It is also possible to relax the assumption that the stock price follows a geometric Brownian motion. The results in the paper easily generalize to the case where the stock price follows an arbitrary diffusion process.

There are at least three important avenues for future research. First, as the integral equations determining the exercise boundary remain unsolved, it would be useful to investigate the nature of the solution and its approximations based on a study of those equations. A second significant avenue for future research involves valuing American puts when the underlying asset has discrete payouts. A third avenue involves multiple state variables, for example, combining stochastic stock prices with stochastic interest rates and/or dividends.

5. APPENDIX

Proof of Theorem 1.1

We wish to prove[14] that

$$P_0 = p_0 + rK \int_0^T e^{-rt} N\left(\frac{\ln(B_t/S_0) - \rho_2 t}{\sigma\sqrt{t}}\right) dt.$$

Let $Z_t \equiv e^{-rt}P_t$ be the discounted put price, defined in the region $\mathcal{D} \equiv \{(S, t) : S \in [0, \infty), t \in [0, T]\}$. In this region, the pricing function $P(S, t)$ is convex in S for all t, continuously differentiable in t for all S, and a.e. twice continuously differentiable in S for all t. Consequently, the discounted pricing function

$$Z(S, t) \equiv e^{-rt}P(S, t) \tag{5.1}$$

[13]The only nontrivial change is the determination of the exercise boundary at expiration. For American put options, if the dividend rate $\delta \geq 0$ is less than or equal to the riskless rate r, then the exercise boundary B_t converges to the strike price K at expiration as before (i.e., $B_T = K$). However, if the dividend rate δ exceeds the riskless rate, then $B_T = (r/\delta)K < K$, reflecting the reduced incentive to exercise early; Van Moenbeke (1976) p. 142.

[14]D. Lamberton communicated this proof to us. It was motivated by our earlier proof based on Fourier transforms.

inherits these properties. Although the partial derivative $\partial^2 P/\partial S^2$ is discontinuous at the boundary B_t, Itô's lemma extends[15] to $Z(S, t)$, so that

$$Z_T = Z_0 + \int_0^T \frac{\partial Z(S_t, t)}{\partial S}\, dS_t + \int_0^T \left[\frac{\partial^2 Z(S_t, t)}{\partial S^2} \frac{\sigma^2 S_t^2}{2} + \frac{\partial Z(S_t, t)}{\partial t} \right] dt.$$

Therefore from (5.1)

$$e^{-rT} P_T = P_0 + \int_0^T e^{-rt} \frac{\partial P(S_t, t)}{\partial S}\, dS_t + \int_0^T \left[e^{-rt} \frac{\partial^2 P(S_t, t)}{\partial S^2} \frac{\sigma^2 S_t^2}{2} - r e^{-rt} P(S_t, t) + e^{-rt} \frac{\partial P(S_t, t)}{\partial t} \right] dt.$$

Now $P_T = \max[0, K - S_T]$, and there exists[16] a probability measure $\tilde{Q}$, equivalent to Q, such that

$$dS_t = rS_t\, dt + \sigma S_t\, d\tilde{W}_t, \tag{5.2}$$

where $\tilde{W}_t \equiv W_t - [(\mu - r)/\sigma]t$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \tilde{Q})$. Separating the put value into the two regions, $P(S_t, t) = 1_{\{S_t > B_t\}} P(S_t, t) + 1_{\{S_t \le B_t\}}(K - S_t)$, we have

[15] A sketch of the proof for the extension can be obtained as follows. Unless specified otherwise, all theorem and equation references are to Karatzas and Shreve (1988). The proof of (7.4), p. 219, can be modified and extended to apply to $f : [0, \infty) \times \mathfrak{R} \to \mathfrak{R}$, denoted $f(x, t)$, where f is convex in x for all t, continuously differentiable in t for all x, and a.e. twice continuously differentiable in x for all t. Itô's lemma (see Theorem 3.6, p. 153) is used to get a slight modification of (7.5), p. 219:

$$f_n(X_t, t) = f_n(X_0, 0) + \int_0^t \frac{\partial f_n(X_s, s)}{\partial s}\, ds + \int_0^t f_n'(X_s, s)\, dM_s + \int_0^t f_n'(X_s, s)\, dV_s + \frac{1}{2}\int_0^t f_n''(X_s, s)\, d\langle M\rangle_s,$$

where the prime(s) on f denote partial differentiation with respect to the first argument of f. The identical argument gives

$$f_n(X_t, t) \to f(X_t, t), \qquad \int_0^t \frac{\partial f_n(X_s, s)}{\partial s}\, ds \to \int_0^t \frac{\partial f(X_s, s)}{\partial s}\, ds,$$

$$\int_0^t f_n'(X_s, s)\, dV_s \to \int_0^t f'(X_s, s)\, dV_s \text{ a.e.},$$

$$\int_0^t f_n'(X_s, s)\, dM_s \to \int_0^t f'(X_s, s)\, dM_s$$

in probability for every fixed t, and $\frac{1}{2}\int_0^t f_n''(X_s, s)\, d\langle M\rangle_s$ converges to a limit in probability. Since $f(x, t)$ is a.e. twice continuously differentiable in x, this limit is determined as

$$\frac{1}{2}\int_0^t f_n''(X_s, s)\, d\langle M\rangle_s = \frac{1}{2}\int_{x\in\mathfrak{R}} \int_0^t f_n''(x, s)\, d_s l^x\, dx \quad \text{from Rogers and Williams (1987, p. 104 (45.4))}$$

$$\to \frac{1}{2}\int_{x\in\mathfrak{R}} \int_0^t f''(x, s)\, d_s l^x\, dx \quad \text{from Karatzas and Shreve (1988, top of p. 215)}$$

$$= \frac{1}{2}\int_0^t f''(X_s, s)\, d\langle M\rangle_s \quad \text{from Rogers and Williams (1987, p. 104 (45.4)).}$$

[16] Define $\tilde{Q}$ by its Radon-Nikodym derivative

$$\frac{d\tilde{Q}}{dQ} = \exp\left[\frac{\mu - r}{\sigma} W_T - \left(\frac{\mu - r}{\sigma}\right)^2 \left(\frac{T}{2}\right)\right].$$

$$
\begin{aligned}
e^{-rT}&\max[0, K - S_T] \\
&= P_0 + \int_0^T e^{-rt}\left[1_{\{S_t>B_t\}}\frac{\partial P(S_t, t)}{\partial S} - 1_{\{S_t\le B_t\}}\right][rS_t\,dt + \sigma S_t\,d\tilde{W}_t] \\
&\quad + \int_0^T e^{-rt}\left\{1_{\{S_t>B_t\}}\left[\frac{\partial^2 P(S_t, t)}{\partial S^2}\frac{\sigma^2 S_t^2}{2} - rP(S_t, t) + \frac{\partial P(S_t, t)}{\partial t}\right]\right. \\
&\quad \left. + 1_{\{S_t\le B_t\}}[-r(K - S_t)]\right\} dt.
\end{aligned}
$$

On the continuation region, the pricing function $P(S, t)$ satisfies the Black-Scholes partial differential equation (1.4). Consequently, the terms multiplying $1_{\{S_t>B_t\}}$ sum to zero, leaving

$$
e^{-rT}\max[0, K - S_T] = P_0 - rK\int_0^T e^{-rt}1_{\{S_t\le B_t\}}\,dt + \int_0^T e^{-rt}\sigma S_t\frac{\partial P}{\partial S}\,d\tilde{W}.
$$

Taking expectations with respect to the martingale measure $\tilde{Q}$ establishes the result

$$
p_0 \equiv \tilde{E}\{e^{-rT}\max[0, K - S_T]\} = P_0 - rK\int_0^T e^{-rt}N\left(\frac{\ln(B_t/S_0) - \rho_2 t}{\sigma\sqrt{t}}\right)dt. \quad \square
$$

Proof of Theorem 2.1

To determine the discounted external financing of the strategy (2.3), define the process

$$
\begin{aligned}
D_t &\equiv e^{-rt}\max[0, A_t - S_t] \qquad \text{for all } t \in [0, T] \\
&= e^{-rt}A_t\max[0, 1 - S_t/A_t] \\
&= e^{-rt}A_t\max[0, 1 - Y_t],
\end{aligned}
$$

where $Y_t \equiv S_t/A_t$. Using integration by parts, we find

$$
\begin{aligned}
A_T e^{-rT}\max[0, 1 - Y_T] &= A_0 e^{-r0}\max[0, 1 - Y_0] \\
&\quad + \int_0^T A_t e^{-rt}\,d(\max[0, 1 - Y_t]) \\
&\quad + \int_0^T \max[0, 1 - Y_t]\,d(A_t e^{-rt}).
\end{aligned} \tag{5.3}
$$

Now, from the Tanaka-Meyer formula:[17]

$$
\max[0, 1 - Y_t] = \max[0, 1 - Y_0] - \int_0^t 1_{\{Y_t<1\}}\,dY_t + \Lambda_Y(1, t),
$$

where $\Lambda_Y(1, t)$ is the *local time* of the process Y at 1 by time t. Consequently, (5.3) becomes

$$
A_T e^{-rT}\max[0, 1 - Y_T] = A_0\max[0, 1 - Y_0] \tag{5.4}
$$

[17] See Karatzas and Shreve (1988, p. 220 (7.7)).

$$+ \int_0^T A_t e^{-rt}[-1_{\{Y_t<1\}}\, dY_t + d\Lambda_Y(1, t)]$$

$$+ \int_0^T 1_{\{Y_t<1\}}\, d(A_t e^{-rt}) - \int_0^T 1_{\{Y_t<1\}} Y_t\, d(A_t e^{-rt}).$$

Since $Y_t \equiv S_t/A_t$ and $A_T = K$ by (2.2), the left side of (5.4) becomes $e^{-rT}\max[0, K - S_T]$, while the right side is

$$\max[0, A_0 - S_0] - \int_0^T A_t e^{-rt} 1_{\{S_t/A_t<1\}}\left[\frac{dS_t}{A_t} - \left(\frac{S_t}{A_t^2}\right) dA_t\right] + \int_0^T A_t e^{-rt}\, d\Lambda_Y(1, t)$$

$$+ \int_0^T 1_{\{S_t/A_t<1\}}\, d(A_t e^{-rt}) - \int_0^T 1_{\{S_t/A_t<1\}}\left(\frac{S_t}{A_t}\right) e^{-rt}\, dA_t$$

$$+ \int_0^T 1_{\{S_t/A_t<1\}} S_t r e^{-rt}\, dt$$

$$= \max[0, A_0 - S_0] - \int_0^T 1_{\{S_t<A_t\}}[e^{-rt}\, dS_t - S_t r e^{-rt}\, dt]$$

$$+ \int_0^T \frac{e^{-rt}}{A_t} S_t 1_{\{S_t<A_t\}}\, dA_t + \int_0^T e^{-rt} A_t\, d\Lambda_Y(1, t)$$

$$+ \int_0^T 1_{\{S_t<A_t\}}\, d(A_t e^{-rt}) - \int_0^T \frac{e^{-rt}}{A_t} S_t 1_{\{S_t<A_t\}}\, dA_t$$

$$= \max[0, A_0 - S_0] - \int_0^T 1_{\{S_t<A_t\}}\, d(S_t e^{-rt})$$

$$+ \int_0^T e^{-rt} A_t\, d\Lambda_Y(1, t) + \int_0^T 1_{\{S_t<A_t\}}\, d(A_t e^{-rt}).$$

Taking expectations using the equivalent martingale measure $\tilde{Q}$ implies

$$(5.5)\quad p_0 \equiv \tilde{E}\{e^{-rT}\max[0, K - S_T]\}$$

$$= \max[0, A_0 - S_0] + \int_0^T A_t e^{-rt}\tilde{E}\, d\Lambda_Y(1, t) + \int_0^T \tilde{E} 1_{\{S_t<A_t\}}\, d(A_t e^{-rt}),$$

from Fubini's theorem and from the fact that the expectation of $\int_0^T 1_{\{S_t<A_t\}}\, d(S_t e^{-rt})$, a martingale, vanishes. To evaluate the first integral, recall $Y_t \equiv S_t/A_t$. From (5.2) and Itô's lemma:

$$\frac{dY_t}{Y_t} = (r - \mu_a)\, dt + \sigma\, d\tilde{W}_t, \qquad \text{where } \mu_a \equiv \frac{dA_t}{dt}\frac{1}{A_t}.$$

Then from the properties of local time, the first integral can be rewritten as

$$\int_0^T A_t e^{-rt}\tilde{E}\, d\Lambda_Y(1, t) = \frac{\sigma^2}{2}\int_0^T A_t e^{-rt}\ell(1, t; Y_0, 0)\, dt,$$

where $\ell(Y_t, t; Y_0, 0)$ denotes the (lognormal) transition density function:

$$\ell(Y_t, t; Y_0, 0) \equiv \frac{1}{Y_t\sqrt{2\pi\sigma^2 t}} \exp\left\{ -\frac{1}{2}\left[\frac{\ln(Y_t/Y_0) - \rho_2 t + \int_0^t \mu_a\, du}{\sigma\sqrt{t}}\right]^2\right\}.$$

Consequently, in the standard normal notation of (1.14), the first integral in (5.5) has the form

(5.6)

$$\begin{aligned}\int_0^T A_t e^{-rt}\tilde{E}d\Lambda_Y(1, t) &= \frac{\sigma^2}{2}\int_0^T \frac{A_t e^{-rt}}{\sigma\sqrt{t}} N'\left(\frac{\ln(1/Y_0) - \rho_2 t + \int_0^t \mu_a\, du}{\sigma\sqrt{t}}\right) dt \\ &= \frac{\sigma^2}{2}\int_0^T \frac{A_t e^{-rt}}{\sigma\sqrt{t}} N'\left(\frac{\ln\left(A_0 \exp\left(\int_0^t \mu_a\, du\right)/S_0\right) - \rho_2 t}{\sigma\sqrt{t}}\right) dt \\ &= \frac{\sigma^2}{2}\int_0^T \frac{A_t e^{-rt}}{\sigma\sqrt{t}} N'\left(\frac{\ln(A_t/S_0) - \rho_2 t}{\sigma\sqrt{t}}\right) dt.\end{aligned}$$

Substituting (5.6) in (5.5) yields

$$\begin{aligned}p_0 = \max[0, A_0 - S_0] &+ \frac{\sigma^2}{2}\int_0^T A_t e^{-rt}\frac{N'(a_{2t})}{\sigma\sqrt{t}}\, dt \\ &+ \int_0^T \int_0^{A_t} \ell(S_t, t; S_0, 0)\, dS_t\, d(A_t e^{-rt}),\end{aligned}$$

where $a_{2t} \equiv [\ln(A_t/S_0) - \rho_2 t]/\sigma\sqrt{t}$. To simplify the first integral, use the identity

$$A_t e^{-rt} N'(a_{2t}) = S_0 N'(a_{1t}), \tag{5.7}$$

where $A_t > 0$ and

$$a_{1t} \equiv a_{2t} - \sigma\sqrt{t} = \frac{\ln(A_t/S_0) - \rho_1 t}{\sigma\sqrt{t}}. \tag{5.8}$$

To simplify the second integral, perform the change of variables $z = [\ln(S_t/S_0) - \rho_2 t]/\sigma\sqrt{t}$. Thus,

$$p_0 = \max[0, A_0 - S_0] + \frac{\sigma^2 S_0}{2}\int_0^T \frac{N'(a_{1t})\, dt}{\sigma\sqrt{t}} + \int_0^T N(a_{2t})\, d(A_t e^{-rt}). \tag{5.9}$$

□

Proof of Equation (5.9)

To prove that (5.9) is equivalent to the Black-Scholes formula (1.10), use integration by parts on the last integral in (5.9):

$$
\begin{aligned}
(5.10)\quad p_0 &= \max[0, A_0 - S_0] + \frac{\sigma^2 S_0}{2}\int_0^T \frac{N'(a_{1t})}{\sigma\sqrt{t}}\,dt \\
&\quad + A_t e^{-rt}N(a_{2t})\Big|_0^T - \int_0^T A_t e^{-rt}N'(a_{2t})\,\frac{da_{2t}}{dt}\,dt \\
&= 1_{\{S_0<A_0\}}(A_0 - S_0) + S_0\int_0^T N'(a_{1t})\,\frac{\sigma}{2\sqrt{t}}\,dt \\
&\quad + Ke^{-rT}N(k_{2T}) - 1_{\{S_0<A_0\}}A_0 - S_0\int_0^T N'(a_{1t})\,\frac{da_{2t}}{dt}\,dt \\
&= -1_{\{S_0<A_0\}}S_0 + Ke^{-rT}N(k_{2T}) - S_0\int_0^T N'(a_{1t})\left[\frac{da_{2t}}{dt} - \frac{\sigma}{2\sqrt{t}}\right]dt,
\end{aligned}
$$

where the second equality follows from (2.2) and (5.7). Differentiating (5.8) and substituting in (5.10) leads to

$$
\begin{aligned}
p_0 &= -1_{\{S_0<A_0\}}S_0 + Ke^{-rT}N(k_{2T}) - S_0\int_0^T N'(a_{1t})a_1'(t)\,dt \\
&= -1_{\{S_0<A_0\}}S_0 + Ke^{-rT}N(k_{2T}) - S_0N(a_{1t})\Big|_0^T \\
&= -1_{\{S_0<A_0\}}S_0 + Ke^{-rT}N(k_{2T}) - S_0N(k_{1T}) + 1_{\{S_0<A_0\}}S_0 \\
&= Ke^{-rT}N(k_{2T}) - S_0N(k_{1T}),
\end{aligned}
$$

which is the Black-Scholes formula (1.10). □

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