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The Valuation of Sequential Exchange Opportunities

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ABSTRACT

Sequential exchange opportunities are valued using the techniques of modern option-pricing theory. The vehicle for analysis is the concept of a compound exchange option. This security is shown to exist implicitly in several contractual settings. A valuation formula for this option is derived. The formula is shown to generalize much previous work in option pricing. Several applications of the formula are presented.

FINANCIAL CONTRACTS FREQUENTLY ASSIGN one party the right to exchange one asset for another. For example, bondholders can often convert debt into equity. Alternatively, in an exchange offer, the target firm's shareholders can exchange their shares for those of the acquiring firm. In each case, the party having the option to exchange is said to own an exchange option. Margrabe [6] defines an exchange option as the right to exchange one asset for another within a specified period of time. If either asset has constant value over time, then an exchange option degenerates into an ordinary call or put.

A complication that arises in valuing these opportunities occurs when one exchange leads to another. In the above examples, a bondholder converting into stock may find himself or herself later exchanging the shares received for those of an acquiring firm. This paper also identifies several other situations involving multiple exchanges, most of which are more likely but less obvious.

Sequential exchange opportunities exist whenever an exchange of assets creates the potential for further exchange. This paper is concerned with the valuation of a sequence of potential exchanges. For simplicity, the timing and terms of each possible transaction are assumed to be known in advance. The paper integrates work on compound option pricing[1] by Geske [4] with work on exchange option pricing by Fischer [2] and Margrabe [6]. A valuation formula for a security called a compound exchange option is developed. Exercise of this instrument involves delivering one asset in return for an exchange option. The option received upon delivery may then be used to make another exchange at a later date. The paper

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[1] A compound call option is a call option written on another call option.

demonstrates that any finite number of exchanges can be valued. However, for simplicity only the last two transactions in any series will usually be examined. To keep the analysis tractable, the asset delivered in each exchange is assumed to be the same. Despite this limitation, the formula can be used in a wide variety of contexts.

The remainder of this section briefly describes some of these applications. These illustrations need not literally involve the physical exchange of assets if cash settlement is equivalent. For example, a firm usually retires its expiring debt with cash. However, no harm is done in imagining that the firm delivers simultaneously maturing bonds instead.

In their seminal paper, Black and Scholes [1] discuss how the shareholders of a levered firm are regarded as owning a call option written on the firm's assets. They also illustrate how coupon payments compound this option's character. With only two payments remaining, the shareholders have the option either to pay the coupon or to default. By making the penultimate payment, the shareholders receive the option to default at the bond's maturity. Modeling the equity in this manner, Geske [3] explicitly values it as a compound call option. The equity can also be viewed as a compound exchange option if it is imagined that maturing bonds are delivered instead of cash paid. Viewed this way, it is also possible to value the equity if the debt is denominated in a different currency from the assets. In this case, the equity is a compound exchange option, where the asset delivered in each exchange is a foreign bond.

Compound exchange options may similarly be used to value sinking-fund debt or floating-rate debt subject to default risk. As in the above example, consider the shareholder's position when there are only two payments outstanding. In both cases, the equity of the firm has the option either to make the penultimate outlay or to default. By making the required payment, the shareholders receive the option to default at the bond's maturity. Consequently, the equity may be valued as a compound option. Furthermore, the equity is also an exchange option because, in both cases, the size of the cash payment is tied to the market value of a traded asset. In the sinking-fund case, the payments are tied to the corporate bond's market value, while, under the floating-rate scenario, the payments are linked to the value of some reference instrument. Note, however, that pricing the equity as a compound exchange option is tricky because the underlying simple option does not trade. Once the equity can be valued, the Modigliani-Miller Theorem can be invoked to price the debt.

Turning to the asset side of the balance sheet, this paper demonstrates how compound exchange option pricing may be used in capital budgeting. The opportunity to invest in a project may be characterized as a simple call option on the value of the project's cash inflows, with the exercise price equal to the required investment. If investing in a project unveils further opportunities, then compound option pricing is appropriate. Furthermore, the analogy best captures the essence of the opportunity when the series of investments required are random. If the randomness in the investments can be hedged using traded assets, then the valuation technique developed in this paper can be implemented.

The pricing of compound exchange options also has implications outside of corporate finance. Margrabe [6] discusses how performance-incentive fees may be valued as exchange options. Suppose that a portfolio manager's fee will consist of a portion of the difference in value, if positive, between his or her managed portfolio and some benchmark portfolio. Then the manager's fee is proportional to the profit obtained from exchanging the benchmark portfolio for the managed one. Furthermore, if the contract is guaranteed in a manner described later in the paper, then the manager's fee can be valued using the formula developed in this paper.

The plan of this work is as follows. Section I presents assumptions and some distribution-free results on the valuation of compound exchange options. Section II then uses these results along with a distributional assumption to derive an exact pricing formula for this security. It is shown that the valuation formulas of Black and Scholes [1], Margrabe [6], and Geske [4] are all special cases of the general result. The following two sections then elaborate on the applications described above. The final section summarizes the paper and contains suggestions for future research. A comparative-statics analysis in Appendix A suggests that the formula aaccords well with intuition and with distribution-free results. Appendix B contains two extensions to the valuation formula.

I. Assumptions and Distribution-Free Results

This section develops distribution-free results for exchange options analogous to those given in Merton [9]. The first major result is a generalization of the put-call parity relation to compound exchange options. A second result develops conditions under which an American exchange option may be valued as European.[2] Finally, pricing bounds are developed for compound exchange options.

The following terminology is employed in this paper. An asset is either primary or derivative. All of the derivative assets examined in this paper are exchange options. Such an option gives its owner the right to exchange one asset for another. The asset given up is termed the delivery asset. The asset received is called the optioned asset. These assets are referred to collectively as the underlying assets. Exchange options may be simple or compound. If either underlying asset is another option, then the original option is considered compound. Otherwise, the exchange option is simple. Exchange options may be further categorized as American or European. The valuation formula developed in this paper will be for European compound exchange options. However, the formula is shown to apply to American options under certain assumptions.

These assumptions are as follows.

(A1) Frictionless markets: There are no transaction costs, indivisibilities, or differential taxes. Short selling is allowed without restriction.

This assumption ensures that a portfolio of assets with payoffs that weakly exceed those of a second portfolio must sell for at least as much. If short-selling restrictions are present, a no-dominance assumption must be added. In the next section, the market structure is strengthened to allow for continuous trading.

[2] An American option gives its owner the right to exercise at any date prior to expiration. In contrast, a European option can only be exercised at expiration.

(A2) Identical delivery assets: All options in the series have the same delivery asset.

This assumption is not required in this section but is used subsequently. It is only made here to reduce the notational burden.

(A3) Known terms of exchange: The times and terms of each possible transaction are known on the valuation date.

While the *identity* of the delivery asset must be the same for each option, the *quantity* required can vary across options. However, these quantities must be known in advance. The above assumptions are made throughout the paper. In contrast, the next two assumptions will be relaxed later.

(A4) No payouts: No asset makes any payouts over the lives of the options.

This assumption ensures that American options are never rationally exercised early.

(A5) Last two exchanges: An investor can purchase a compound exchange option (CEO) written on a simple exchange option (SEO).

This final assumption serves to simplify notation by focusing attention on the last two exchanges in any series. The units of the primary assets are assumed to be normalized so that the simple option involves a one-for-one exchange. In contrast, the compound exchange option is allowed to involve the exchange of an arbitrary quantity of the delivery asset in return for one SEO. This arbitrary quantity is termed the exchange ratio.

The notation used throughout the paper is given below. Let

q be the exchange ratio of the CEO,[3]
t be the valuation date,
T_c be the expiration date of the CEO,
$\tau_c \equiv T_c - t$ be the time to expiration of the CEO,
T_s be the maturity date of the SEO where $T_s > T_c$,
$\tau_s \equiv T_s - t$ be the time to maturity of the SEO,
$\tau \equiv \tau_s - \tau_c$ be the time between the expirations of the two options,
C denote the value of an American CEO,
c denote the value of a European CEO,
S denote the value of an American SEO,
s denote the value of a European SEO,
V denote the value of the optioned asset of the SEO, and
D denote the value of the common delivery asset.

Using this notation, the functional relationship between the value of the simple exchange option and the relevant state variables may be expressed as $s(V, D, \tau_s)$. At the option's maturity date, this functional relationship is known since the option must sell for its exercise value to avoid arbitrage:

$$s(V, D, 0) = \max(0, V - D).$$

[3] The exchange ratio q is taken to be constant or, at most, a deterministic function of time.

Similarly, the functional governing the compound exchange option's value $c(s, qD, \tau_c)$ is known at expiration to be $\max(0, s - qD)$.

Another distribution-free result is Margrabe's generalization of put-call parity to European exchange options. Consider a portfolio consisting of a simple exchange option on V and its delivery asset D. At the maturity of the option, this portfolio is worth the more valuable underlying asset since $\max(0, V - D) + D = \max(D, V)$. A second portfolio that pays off this maximum consists of a simple exchange option on D and its delivery asset V. Since the two portfolios have identical terminal value, arbitrage is avoided only if their current values are equal:[4]

PARITY THEOREM: *European exchange options satisfy*

$$s(V, D, \tau_s) = V - D + s(D, V, \tau_s).$$

The limited liability of an exchange option implies that its value can never be negative. Consequently, the Parity Theorem implies that a lower bound for the European option price is the difference in the underlying asset prices:

$$s(V, D, \tau_s) \geq \max(0, V - D) \geq V - D. \tag{1}$$

Transitivity implies the same lower bound for an American option since it must be worth at least as much as its European counterpart:

$$S(V, D, \tau_s) \geq s(V, D, \tau_s) \geq V - D. \tag{2}$$

Since an American option is never priced below its exercise value, its early-exercise privilege is redundant. As this feature is the only distinction between an American option and its European cousin, a second theorem holds:[5]

EQUIVALENCE THEOREM: *An American exchange option will be priced as if it were European*:

$$S(V, D, \tau_s) = s(V, D, \tau_s). \tag{3}$$

This powerful theorem allows the parity equation to be restated in terms of American options:

$$S(V, D, \tau_s) = V - D + S(D, V, \tau_s). \tag{4}$$

Furthermore, since CEOs are themselves exchange options, the Equivalence and Parity Theorems apply to them, i.e.,

$$C(S, qD, \tau_c) = c(s, qD, \tau_c) \tag{5}$$

and

$$C(S, qD, \tau_c) = S - qD + C(qD, S, \tau_c). \tag{6}$$

[4] All of the theorems in this section depend on assumptions (A1) to (A5). The no-payout assumption is particularly critical for the Equivalence Theorem.

[5] When both assets pay dividends at constant rates, the Equivalence Theorem holds when the dividend rate of the optioned asset is less than that of the delivery asset. This result explains why American calls are not exercised early while American puts may be.

Substituting (4) into (6) leads to a parity result unique to compound options:

$$C(S(V, D, \tau_s), qD, \tau_c) = V - (1 + q)D + S(D, V, \tau_s) + C(qD, S(V, D, \tau_s), \tau_c). \qquad (7)$$

Since option values are never negative, the CEO value is bounded below by the difference in values between the optioned asset and the sum of the delivery assets:

$$C(S(V, D, \tau_s), qD, \tau_c) \geq \max(0, V - (1 + q)D) \geq V - (1 + q)D. \qquad (8)$$

The right-hand side is the present value of the profit obtained if both options must be exercised. Since neither option need be exercised, the option value exceeds this bound. Limited liability also implies non-negativity.

Turning from lower to upper bounds, it should be clear that an option is always *less* valuable than its optioned asset:

$$C(S, qD, \tau_c) \leq S(V, D, \tau_s) \leq V. \qquad (9)$$

The reason is that the payoffs from ownership of the optioned asset can always be achieved through exercise of the option, but this requires delivery of an asset of positive value.

To summarize, this section has shown how certain assumptions led to a parity theorem for compound exchange options. This parity theorem was used to show that an American option may be valued as European under these assumptions. The parity theorem also bounded the value of a compound exchange option. The next section makes a distributional assumption that leads to an exact valuation formula for this option.

II. Distributional Assumption and Formula

In this section, an assumption concerning the underlying assets' return dynamics is added in order to derive a closed-form solution for the value of a compound exchange option. The formula is shown to contain the solutions of Black and Scholes [1], Margrabe [6], and Geske [4] as special cases.

Assume that investors agree on the following stochastic process for asset returns.

(A6) Underlying-asset-return dynamics: The rates of return on the underlying assets V and D are described by the stochastic differential equations:

$$\frac{d\tilde{V}}{V} = \alpha_v dt + \sigma_v d\tilde{Z}_v \qquad (10)$$

and

$$\frac{d\tilde{D}}{V} = \alpha_d dt + \sigma_d d\tilde{Z}_d, \qquad (11)$$

where α_v and α_d are the expected rates of return on the two assets per unit time, σ_v^2 and σ_d^2 are the corresponding variance rates, and dZ_v and dZ_d are the increments to standard Wiener processes with correlation coefficient ρ. While the expected rates of return α_v and α_d can depend on the state variables, the return variances σ_v^2 and σ_d^2 and correlation coefficient ρ are assumed constant although they can be allowed to be deterministic functions of time. Similarly, while investors may differ on the expected rates of return, α_v and α_d, agreement is required on the constants in the covariance[6] matrix σ_v^2, σ_d^2, and ρ.

These dynamics have the important property that the return distribution over any interval is independent of the initial price level. For processes with this characteristic, the following theorem can be proven:

HOMOGENEITY THEOREM: *The CEO pricing function C is linearly homogeneous in the underlying asset prices V and D*:

$$\forall \lambda \geq 0, \quad C(S(\lambda V, \lambda D, \tau_s), q\lambda D, \tau_c) = \lambda C(S(V, D, \tau_s), qD, \tau_c).$$

Intuitively, doubling the underlying asset prices V and D doubles the price of the simple exchange option, which then doubles the price of a compound option.

Exact pricing of a compound exchange option begins by postulating that the current CEO value C is a twice-differentiable function of the state variables V, D, and τ_c. Thus, by Euler's theorem, linear homogeneity implies that

$$C(S, qD, \tau_c) = \frac{\partial C}{\partial V} V + \frac{\partial C}{\partial D} D. \tag{12}$$

Consequently, a portfolio H consisting of long one compound exchange option C, short $\frac{\partial C}{\partial V}$ units of V and short[7] $\frac{\partial C}{\partial D}$ units of D, is costless:

$$H \equiv C - \frac{\partial C}{\partial V} V - \frac{\partial C}{\partial D} D = 0.$$

For a self-financing portfolio, the change in value over any infinitesimal increment of time is given by

$$d\tilde{H} = d\tilde{C} - \frac{\partial C}{\partial V} d\tilde{V} - \frac{\partial C}{\partial D} d\tilde{D}. \tag{13}$$

As the compound option value C is ultimately a function of the state variables V, D, and τ_c, Itô's Lemma implies that its stochastic differential may be written as

$$d\tilde{C} = \left[\frac{1}{2}\sigma_v^2 V^2 \frac{\partial^2 C}{\partial V^2} + \rho\sigma_v\sigma_d VD \frac{\partial^2 C}{\partial V \partial D} + \frac{1}{2}\sigma_d^2 D^2 \frac{\partial^2 C}{\partial D^2} - \frac{\partial C}{\partial \tau_c}\right] dt + \frac{\partial C}{\partial V} d\tilde{V} + \frac{\partial C}{\partial D} d\tilde{D}. \tag{14}$$

[6] Actually, investors need only agree on $\sigma^2 \equiv \sigma_v^2 + \sigma_d^2 - 2\rho\sigma_v\sigma_d$, which is the variance of percentage changes in the price ratio $P \equiv V/D$.

[7] $\frac{\partial C}{\partial D}$ will turn out to be negative, so the portfolio actually involves a long position in the delivery asset D.

Substituting (14) into (13) leads to the result that the change in the portfolio value over the time increment is riskless:

$$dH = \left[\frac{1}{2}\sigma_v^2 V^2 \frac{\partial^2 C}{\partial V^2} + \rho\sigma_v\sigma_d\, VD \frac{\partial^2 C}{\partial V \partial D} + \frac{1}{2}\sigma_d^2 D^2 \frac{\partial^2 C}{\partial D^2} - \frac{\partial C}{\partial \tau_c}\right] dt. \tag{15}$$

As the hedge portfolio is both costless and riskless, the no-arbitrage condition implies that it must have zero return:

$$\frac{1}{2}\sigma_v^2 V^2 \frac{\partial^2 C}{\partial V^2} + \rho\sigma_v\sigma_d\, VD \frac{\partial^2 C}{\partial V \partial D} + \frac{1}{2}\sigma_d^2 D^2 \frac{\partial^2 C}{\partial D^2} - \frac{\partial C}{\partial \tau_c} = 0. \tag{16}$$

This fundamental equation partially describes the behavior of any contingent-claim value that is linearly homogeneous in both of its underlying asset prices. Appending appropriate boundary and terminal conditions completely describes this behavior. For a compound exchange option, these conditions were developed in Section I:

$$S \geq C \geq 0 \tag{17}$$

and

$$C(S, qD, 0) = \max(0, S - qD). \tag{18}$$

Equations (16) to (18) completely summarize the behavior of the CEO's value. However, the boundary and terminal conditions are expressed in terms of the possibly unobservable value of the underlying option S. As the objective is to express the compound-option value in terms of the observable asset prices V and D, these conditions must be rewritten in terms of these state variables.

Since the value of a *simple* exchange option is linearly homogeneous in its underlying asset prices, it must also be governed by the fundamental equation (16); i.e.,

$$\frac{1}{2}\sigma_v^2 V^2 \frac{\partial^2 S}{\partial V^2} + \rho\sigma_v\sigma_d\, VD \frac{\partial^2 S}{\partial V \partial D} + \frac{1}{2}\sigma_d^2 D^2 \frac{\partial^2 S}{\partial D^2} - \frac{\partial S}{\partial \tau_s} = 0. \tag{19}$$

Margrabe [6] shows that the unique solution of (19), subject to the boundary conditions

$$V \geq S \geq 0 \tag{20}$$

and the terminal condition

$$S(V, D, 0) = \max(0, V - D), \tag{21}$$

is

$$S(V, D, \tau_s) = VN_1(d_1(P, \tau_s)) - DN_1(d_2(P, \tau_s)), \tag{22}$$

where

$N_1(d)$ is the value of the standard univariate normal distribution function evaluated at d,

$d_1(y, \tau) \equiv \dfrac{\ln y + \sigma^2\tau}{\sigma\sqrt{\tau}}$,

$d_2(y, \tau) \equiv d_1(y, \tau) - \sigma\sqrt{\tau}$,

$P \equiv \dfrac{V}{D}$ is the price ratio of V to D, and

$\sigma^2 \equiv \sigma_v^2 + \sigma_d^2 - 2\rho\sigma_v\sigma_d$ is the instantaneous variance of the percentage change in this price ratio $\left(\text{var}\left(\dfrac{dP}{P}\right) = \sigma^2 dt\right)$.

To simplify notation, the second argument of d_1 and d_2 will be dropped when it can be understood from the context. Using Margrabe's solution, the boundary conditions (17) become

$$VN_1(d_1(P)) - DN_1(d_2(P)) \geq C \geq 0, \tag{23}$$

and the terminal condition (18) is

$$C(V, qD, 0) = \max[0,\ VN_1(d_1(P)) - DN_1(d_2(P)) - qD], \tag{24}$$

where the second argument of d_1 and d_2 is understood to be $\tau \equiv \tau_s - \tau_c$.

The compound exchange option will be exercised at expiration if the simple option value exceeds the cost of exercise:

$$VN_1(d_1(P)) - DN_1(d_2(P)) \geq qD.$$

The simple option value depends on the random prices of the two underlying assets. The dimensionality of the problem can be halved by taking the delivery asset to be the numeraire. Dividing by the delivery-asset value D makes the exercise condition depend on only a single random variable, namely the price ratio $P \equiv \dfrac{V}{D}$:

$$PN_1(d_1(P)) - N_1(d_2(P)) \geq q. \tag{25}$$

The left-hand side of (25) is the Black-Scholes formula for the value of a simple call option on the price ratio P with an exercise price of one.[8] Since this option price is increasing in P, there is a unique value of P where (25) holds with equality. This critical price ratio P^* is defined by

$$P^*N_1(d_1(P^*)) - N_1(d_2(P^*)) = q. \tag{26}$$

Note that the critical ratio P^* can be calculated from (26) at any time. Consequently, P^* is a parameter that may legitimately be included in the pricing equation for a compound exchange option.

[8] The risk-free rate in the Black-Scholes formula is zero.

This pricing equation is the solution to the fundamental equation (16) subject to the boundary conditions (23) and the terminal condition (24):

$$C(S(V, D, \tau_s), qD, \tau_c) = VN_2\left(d_1\left(\frac{P}{P^*}, \tau_c\right), d_1(P, \tau_s)\right) - DN_2\left(d_2\left(\frac{P}{P^*}, \tau_c\right), d_2(P, \tau_s)\right) - qDN_1\left(d_2\left(\frac{P}{P^*}, \tau_c\right)\right), \tag{27}$$

where $N_2(a, b)$ is the standard bivariate normal distribution function evaluated at a and b with correlation coefficient $\sqrt{\frac{\tau_c}{\tau_s}}$.

This formula is the solution to a previously unsolved problem and will be applied repeatedly in this paper.[9] The solution shares all of the appealing features of its predecessors. Like the Black-Scholes formula, there is no direct dependence on the unobservable expected rates of return α_v and α_d. Like the Margrabe formula, there is no presumption that the term structure of interest rates be flat or even known. Like Geske's formula, the result does not depend directly on the value of the simple option S, which may be unobservable in certain applications.

Under certain parameter restrictions, the solution (27) yields the valuation formulas of Margrabe [6], Geske [4], and Black and Scholes [1] as special cases. If the exchange ratio vanishes ($q = 0$) so that the compound exchange option can be exercised freely, the formula collapses to Margrabe's result for a simple exchange option with maturity τ_s:

$$C(S(V, D, \tau_s), 0 \times D, \tau_c) = VN_1(d_1(P)) - DN_1(d_2(P)) = S(V, D, \tau_s). \tag{28}$$

On the other hand, if the exchange ratio q remains positive but the delivery assets are nonrandom ($\sigma_d = 0$), then the formula reduces to Geske's solution for a compound call option, under the additional assumption of a constant interest rate r:

$$C(S(V, F, \tau_c), K, \tau_s) = VN_2\left(d_1\left(\frac{V}{V^* e^{-r\tau_c}}\right), d_1\left(\frac{V}{Fe^{-r\tau_s}}\right)\right) - Fe^{-r\tau_s}N_2\left(d_2\left(\frac{V}{V^* e^{-r\tau_c}}\right), d_2\left(\frac{V}{Fe^{-r\tau_s}}\right)\right) - Ke^{-r\tau_c}N_1\left(d_2\left(\frac{V}{V^* e^{-r\tau_c}}\right)\right), \tag{29}$$

where K and F are the exercise prices of the simple and compound calls, respectively, and where the critical value V^* satisfies

$$V^*N_1\left(d_1\left(\frac{V^*}{Fe^{-r\tau}}\right)\right) - Fe^{-r\tau}N_1\left(d_2\left(\frac{V^*}{Fe^{-r\tau}}\right)\right) = K.$$

[9] Again, notation is simplified by dropping the second argument of d_1 and d_2 whenever possible. This argument will always be τ_c for the first argument of N and τ_s for its second argument.

If the exchange ratio and the variance of the delivery assets both vanish, the Black-Scholes formula for a call option results:

$$C(S(V,\, F,\, \tau_s),\, 0 \times K,\, \tau_c) \;=\; VN_1\!\left(d_1\!\left(\frac{V}{Fe^{-r\tau_s}}\right)\right) - Fe^{-r\tau_s}N_1\!\left(d_2\!\left(\frac{V}{Fe^{-r\tau_s}}\right)\right).$$

III. Applications

In the next two sections, the valuation formula is applied to several different sequential exchange opportunities. Two applications presented by Geske involving compound call options are extended by considering the effect of stochastic exercise prices. Similarly, an application presented by Margrabe involving simple exchange options is extended to more than one exchange date. In the next section, some new applications for compound exchange option pricing are presented. In both sections, the emphasis is on displaying the versatility of CEO pricing, rather than attempting to analyze thoroughly each application.

Compound call options will have stochastic exercise prices if their strike prices are denominated in a foreign currency. Foreign discount bonds can be used to hedge the compound call in this case. The valuation formula (27) for a compound exchange option can be applied if the foreign interest rate is assumed constant at rate r and if the exchange rate follows geometric Brownian motion.[10]

A. Foreign Call Option on a Stock with Foreign Debt

Black and Scholes [1] model the stock of a levered firm as a call option on the value of the firm's assets. If the debt is denominated in a currency different from that of the assets, then the exercise price of this call option is random. Furthermore, an exchange-traded call on this stock is compound.[11] If the call's exercise price is denominated in the same currency as the debt, then the call is priced by the CEO valuation formula developed in this paper.

To be concrete, consider a call option trading in Canada on all of the shares of a firm operating in the U.S.[12] Suppose that the firm's sole liability is denominated in Canadian dollars. Let F be the face value of this debt and K the exercise price of the call, where both amounts are fixed in Canadian dollars. Then the U.S. dollar value of this call is given by equation (27) for $C(S(V, D, \tau_s), qD, \tau_c)$, where

S is the stock value of the firm,
V is the value of the firm's assets,
$D \equiv EFe^{-r\tau_s}$ is the domestic value of a discount bond paying F Canadian dollars at the expiration of the firm's debt at T_s, and
$q \equiv \dfrac{K}{F}e^{r\tau}$ so that, at the call's expiration ($\tau = \tau_s$), the product of q and D (i.e., the quantity delivered when exercising a CEO) will equal $E \times K$, the domestic value of the foreign option's exercise price.

[10] The assumptions concerning the behavior of interest rates and exchange rates are not necessary. For stochastic interest rates, alternatively assume that the conversion factor relating future foreign currency to current dollars follows a lognormal diffusion.

[11] The call must expire before the debt matures.

[12] The price of a call on a hundred shares of stock should be scaled down appropriately.

The stock value is given by Margrabe's formula (22) for a simple exchange option $S(V, D, \tau_s)$. By the Parity Theorem of Section I, this value can also be written as $V - D + S(D, V, \tau_s)$. The first two terms represent the equity value if the debt were riskless. The third term is the value of the shareholder's default option. The debt value is given by the firm value less this equity value, i.e., $V - S(V, D, \tau_s)$. Thus, by the same Parity Theorem, the value of the debt can alternatively be written as $D - S(D, V, \tau_s)$, i.e., an equivalent default-free bond, less the default option written to the shareholders.

B. Stock of a Firm with a Coupon Bond

Suppose a firm's sole liability is a bond with one coupon payment and one final payment remaining. If the shareholders skip the coupon, the firm is bankrupt and the shareholders get nothing. On the other hand, by paying the coupon, the shareholders are in a position characterized by a simple call on the value of the firm's assets. Thus, the coupon payment may be regarded as the exercise price of a call on this call, i.e., a compound call. Geske's compound-call formula (29) may be employed to value the equity under the standard assumptions.[13]

If the coupon bond is denominated in a foreign currency, then both the coupon payment K and the final payment F will be random in dollar terms. Thus, the equity of a firm with foreign denominated debt is a compound exchange option. The equity may be valued by formula (27) for $C(S(V, D, \tau_s), qD, \tau_c)$, where

S is the stock value if there are no coupon payments,[14]
V is the value of the firm's assets,
$D \equiv EFe^{-r\tau_s}$ is the dollar value of the final payment F at time T_s, and
$q \equiv \frac{K}{F} e^{r\tau}$ so that, at the coupon-payment date ($\tau = \tau_s$), the product of q and D will equal $E \times K$, the domestic value of the coupon payment.

From the Parity Theorem for compound exchange options, the value of the equity may also be written as

$$V - (1 + q)D + S(D, V, \tau_s) + C(qD, S(V, D, \tau_s), \tau_c).$$

The first two terms reflect the value of the equity if the debt is riskless. The other two terms are default options, one for each payment date. Again the debt value is given by the firm value less the above equity value. From the parity result, this debt value is also given by

$$D + qD - S(D, V, \tau_s) - C(qD, S(V, D, \tau_s), \tau_c).$$

The first two terms are the value of an equivalent riskless bond with two payments remaining. This value is reduced by the two default options issued to the shareholders.

[13] Assuming that the coupon payments are financed through the sale of assets violates the presumption that the value of the firm's assets V follows a diffusion. One can alternatively assume that the coupon payments are financed through a rights offering.

[14] Thus, S is a nontraded asset. This result shows why formula (27) for a compound exchange option is expressed in terms of the underlying asset price V and not S.

If the foreign bond has more than two payments outstanding, then a formula for a nested series of exchange options is required to value the equity. This formula is presented in Appendix B as equation (33). This equation may also be used to value a call[15] option on the stock if its exercise price is denominated in the same foreign currency as the debt. Given this call value, the corresponding put value[16] may be derived through the parity relationship.

C. Performance-Incentive Fee

Margrabe [6] shows how a performance-incentive fee can be valued using his formula for a simple exchange option. A portfolio manager may be paid a fee of the form

$$\tilde{F} = m \times \max(0, \tilde{R}_v - \tilde{R}_b), \tag{30}$$

where

$\tilde{R}_v$ is the return on his or her managed portfolio V,
$\tilde{R}_b$ is the return on the benchmark portfolio B (e.g., the S&P 500), and
m is a multiplier converting returns to dollars.

The property that the fee cannot be negative arises either contractually or through the possibility of default by the portfolio manager. Substituting return definitions into (30) implies that the fee is proportional to the payoffs from a simple exchange option:

$$\begin{aligned} \tilde{F} &= m \times \max\left(0, \frac{\tilde{V}}{V_0} - \frac{\tilde{B}}{B_0}\right) \\ &= \frac{m}{V_0} \times \max\left(0, \tilde{V} - \frac{V_0}{B_0}\tilde{B}\right) \\ &= \alpha \max(0, \tilde{V} - \tilde{D}), \end{aligned}$$

where $\alpha \equiv \dfrac{m}{V_0}$ and $\tilde{D} \equiv \dfrac{V_0}{B_0} \times \tilde{B}$. Thus, if the contract terminates at time T_s, then the current value of the manager's contract is given by α times Margrabe's formula for a simple exchange option $S(V, D, \tau_s)$.

Now suppose instead that the portfolio manager guarantees a certain performance level by an intermediate date T_c. Specifically, the managed portfolio return must exceed the benchmark portfolio return by a specified proportion to avoid termination at time T_c. When translated into price space, the guarantee is equivalent to requiring that the ratio of portfolio values $P \equiv \dfrac{V}{D}$ exceed some specified critical value P^* at the intermediate date T_c. The current value of this fee is given by equation (27) for a compound exchange option, where the third term is ignored. The formula for a nested series of exchange options can similarly

[15] The call must expire before the next coupon payment.

[16] Since the stock is assumed not to pay dividends, the call option valued may be American. However, the corresponding put value is European.

be modified to value a contract that requires a set of hurdles to be cleared before payment is finally received.

IV. New Applications

In this section, applications of CEO pricing are presented that were not previously developed as illustrations of compound call options or simple exchange options. In particular, the focus is on variable-rate corporate debt and the investment decision. Both of these applications involve features that do not permit them to be valued using the pricing of compound calls or simple exchange options alone.

A. Variable-Rate Corporate Debt

Suppose a corporation has a coupon bond outstanding that has one coupon payment due at time T_c and one final payment due at time T_s. As in Section III*B*, the equity can be valued as a compound option since the shareholders can default at either date. Further, suppose that the coupon payment is calculated as a specified fraction q of the market value of another debt instrument D. With stochastic interest rates, the coupon payment is random since the market value of the reference bond D is also random at the coupon date. Consequently, the equity is a compound exchange option, the delivery asset of which is the reference bond. If the reference bond is the default-free equivalent of the corporate bond, then equation (27) can be used, where S and V are defined as in Section III*B*. The nested formula (33) may be used for a series of coupon payments, all calculated as some fraction of the contemporaneous market value of the reference bond. As in the last section, the debt can be valued by subtracting this equity value from the firm value.

B. The Investment Decision

The traditional approach to capital budgeting has been to accept a project if its net present value (NPV) is positive. Some authors have recognized that the mutually exclusive alternative of delaying the project may have a greater NPV than accepting a positive NPV project immediately. McDonald and Siegel [8] have valued this timing option as an American exchange option, which is infinitely lived. In this setting, investment in a project corresponds to the exercise of an option. The value of a *finite-lived* American exchange option can be determined using CEO pricing theory.

Let $\tilde{V}$ be the value of the revenues from the project and $\tilde{D}$ the value of the costs. Since the values of the revenues and costs are not prices of traded assets, their expected rates of change need not equal the expected rate of return required for their risk in equilibrium. This subsection follows McDonald and Siegel in supposing that the difference between these two expected rates is constant over time. Thus, the dynamics of the underlying assets are amended to

$$\frac{d\tilde{V}}{V} = (\mu_v - \delta_v)dt + \sigma_v d\tilde{Z}_v$$

and

$$\frac{d\tilde{D}}{D} = (\mu_d - \delta_d)dt + \sigma_d d\tilde{Z}_d,$$

where δ_v and δ_d are the assumed constant difference between the equilibrium expected rates of return, μ_v and μ_d, and the expected percentage rates of change. These dynamics also correspond to those of traded assets paying dividend yields δ_v and δ_d, respectively. The formal development in this section allows for either interpretation. McDonald and Siegel [7] show that, in either case, the current value of a *European* exchange option is now given by

$$s(V, D, \tau_s; \delta_v, \delta_d) = Ve^{-\delta_v \tau_s} N_1(d_1(Pe^{-\delta\tau_s})) - De^{-\delta_d \tau_s} N_1(d_2(Pe^{-\delta\tau_s})), \tag{31}$$

where $\delta \equiv \delta_v - \delta_d$.

Unfortunately, for a strictly positive "dividend yield" ($\delta_v > 0$), premature exercise of American options is rational for large enough values of the optioned asset V. Consequently, the above formula need not apply to American options unless $\delta_v = 0$. To value an American exchange option, suppose initially that it can be exercised only at times T_c or T_s. Thus, if a project is rejected now, then it may be accepted only at time T_c or at time T_s. If the project is also rejected at the intermediate date T_c, then the firm's pseudo-American option will revert into a simple exchange option with value given by (31). On the other hand, if the project is accepted at time T_c, then the firm receives the NPV of the project, $\tilde{V} - \tilde{D}$. The firm will further delay the project at time T_c if and only if the benefit of delaying exceeds the opportunity cost:

$$Ve^{-\delta_v \tau} N_1(d_1(Pe^{-\delta\tau})) - De^{-\delta_d \tau} N_1(d_2(Pe^{-\delta\tau})) \geq V - D.$$

The problem is again simplified by treating the delivery asset as numeraire. Dividing by D and recalling that the price ratio P is $\frac{V}{D}$ gives

$$Pe^{-\delta_v \tau} N_1(d_1(Pe^{-\delta\tau})) - e^{-\delta_d \tau} N_1(d_2(Pe^{-\delta\tau})) > P - 1.$$

Let Q^* be the unique value of P that makes the above an equation. For values of P below Q^*, the option won't be exercised at T_c and reverts to a European exchange option with value $S(V, D, \tau)$. Otherwise, the option is exercised to yield $V - D$. These contingent payoffs are duplicated by a portfolio of three European exchange options as indicated by Table I.

The table generalizes one developed by Roll [10] for pricing American call options on stocks paying discrete dividends. The first option is simply the

Table I

At Valuation Date	At time T_c: If $P \leq Q^*$	At time T_c: If $P > Q^*$
Long $s(V, D, \tau_s)$	$s(V, D, \tau)$	$s(V, D, \tau)$
Long $s(V, Q^* \times D, \tau_c)$	0	$V - Q^* D$
Short $c(s(V, D, \tau_s), (Q^* - 1) \times D, \tau_c)$	0	$Q^* D - D - s(V, D, \tau)$
	$s(V, D, \tau)$	$V - D$

European counterpart to the American option to be replicated. The second option involves exchanging Q^* units of D for one unit of V. The third option is a compound exchange option, which involves exchanging $Q^* - 1$ units of asset D for the first European option. The last two options in the table both expire at T_c.

The reasoning leading to this replicating portfolio is straightforward. The pseudo-American exchange option is similar to a portfolio of the first two European exchange options, with each expiring at a possible exercise date. However, if an American option is exercised early, then it cannot be exercised at expiration. This feature is accounted for by writing a compound option on the long-maturity option. The terms of exchange on the compound and short-maturity options are set so that they are both exercised whenever early exercise is rational. In this case, simultaneous exercise of the two European options leads to the receipt of the exercise value and the surrender of the long-maturity option. If early exercise is unwarranted, then these two options expire worthless and a European option is retained.

The first two options' value may be calculated using (31). The third option's value can be calculated using the following formula for a *European* CEO on assets paying dividends at a constant yield:

$$\begin{aligned} c(s(V, D, \tau_s), qD, \tau_c; \delta_v, \delta_d) = {} & Ve^{-\delta_v \tau_s} N_2\left(d_1\left(\frac{Pe^{-\delta\tau_c}}{P^*}\right), d_1(Pe^{-\delta\tau_s})\right) \\ & - De^{-\delta_d \tau_s} N_2\left(d_2\left(\frac{Pe^{-\delta\tau_c}}{P^*}\right), d_2(Pe^{-\delta\tau_s})\right) \\ & - qD^{-\delta_d \tau_c} N_1\left(d_2\left(\frac{Pe^{-\delta\tau_c}}{P^*}\right)\right), \end{aligned}$$

where P^* satisfies

$$P^* e^{-\delta_v \tau} N_1(d_1(P^* e^{-\delta\tau})) - e^{-\delta_d \tau} N_1(d_2(P^* e^{-\delta\tau})) = q.$$

Combining the three formulas leads to the value of the pseudo-American timing option:[17]

$$\begin{aligned} \text{t.o.} = {} & Ve^{-\delta_v \tau_s} N_2\left(-d_1\left(\frac{Pe^{-\delta\tau_c}}{Q^*}\right), d_1(Pe^{-\delta\tau_s}); -\sqrt{\frac{\tau_c}{\tau_s}}\right) \\ & - De^{\delta_d \tau_s} N_2\left(-d_2\left(\frac{Pe^{-\delta\tau_c}}{Q^*}\right), d_2(Pe^{-\delta\tau_s}); -\sqrt{\frac{\tau_c}{\tau_s}}\right) \\ & + Ve^{-\delta_v \tau_c} N_1\left(d_1\left(\frac{Pe^{-\delta\tau_c}}{Q^*}\right)\right) \\ & - De^{-\delta_d \tau_c} N_1\left(d_2\left(\frac{Pe^{-\delta\tau_c}}{Q^*}\right)\right), \end{aligned} \tag{32}$$

where the last argument in the bivariate normals is the correlation coefficient.

[17] The identity $N_2(-a, b; -\rho) = N_1(b) - N_2(a, b; \rho)$ has been used to simplify the formula.

If the timing-option value exceeds the current NPV of the project, $V - D$, it will be worthwhile to delay accepting the project. Note that, as the investment decision is isomorphic to the abandonment decision and the replacement decision, the approach taken here can also be applied to these problems.

Appendix B presents the valuation formula for an exchange option with an arbitrary number of possible exercise dates. As the number of these dates increases, the formula value becomes arbitrarily close to the value of a true American exchange option, i.e., one with a continuum of possible exercise dates. By appropriately restricting dividend yields and variances, the formula specializes to those for American or European call or put options. In particular, the American put-option formula was previously advanced by Geske and Johnson [5].

Unfortunately, formula evaluation requires calculating distribution functions with order equal to the number of possible exercise dates. As in Geske and Johnson, the computational burden can be eased by extrapolating from the values of options with a small number of possible exercise dates. For example, the two-point Richardson extrapolation for the value of an American exchange option E is

$$E \approx E_1 + \tfrac{1}{3}(E_2 - E_1),$$

where

E_1 is the value of the corresponding European option and
E_2 is a pseudo-American exchange option with two possible exercise dates, with the first occurring when the option has half its current time to maturity.

All options have the same underlying assets and time to expiration. E_1 and E_2 can be valued using (31) and (32), respectively.

V. Summary

This paper values sequential exchange opportunities using modern option-pricing theory. In particular, the notion of a compound exchange option is developed. Some distribution-free valuation results for the option are presented. A pricing formula is derived for a particular stochastic process. The formula may be extended and applied in a variety of ways as discussed. The author has also applied it to valuing sinking-fund bonds, tax-timing options, bankruptcy options, pension-fund obligations, two-tier exchange offers, and convertible debt.

One of the most important avenues for future research in this area would relax the requirement that the delivery assets of the compound and simple options be the same. The principal difficulty in developing this formula lies in the exercise condition. This condition generates a complex functional relationship between critical values of the underlying assets. Numerical integration techniques may be employed to overcome this difficulty.

Appendix A

Comparative-Statics Analysis

The comparative-statics results for compound exchange options accord well with our intuition and distribution-free results. Differentiating (27) with respect to the underlying asset prices V and D yields

$$\frac{\partial C}{\partial V} = N_2\left(d_1\left(\frac{P}{P^*}\right),\ d_1(P)\right) > 0,$$

$$\frac{\partial C}{\partial D} = -\left[N_2\left(d_2\left(\frac{P}{P^*}\right),\ d_1(P)\right) + qN_1\left(d_2\left(\frac{P}{P^*}\right)\right)\right] < 0.$$

As expected, the value of a compound exchange option is increasing in the value of the optioned asset and decreasing in the value of the common delivery asset. The magnitude of these partials determines the number of underlying assets that one must short to hedge a compound exchange option.

Inspection of the CEO-pricing formula (27) indicates that the relevant variance for this option $\sigma^2 \equiv \sigma_v^2 + \sigma_d^2 - 2\rho\sigma_v\sigma_d$, which is the variance of the price ratio. When the underlying assets have equal variance ($\sigma_v^2 = \sigma_d^2$) and perfect negative correlation ($\rho = -1$), then this variance vanishes. Under this restriction, the CEO-pricing formula (27) reduces to

$$C(S(V, D, \tau_s), qD, \tau_c) = V - (1 + q)D.$$

The right-hand side was shown to be a lower bound for C in Section I. Thus, this bound is sharp under the distributional assumption in Section II. As the price-ratio variance rises from zero, the CEO value will rise above this lower bound since the option value is increasing in it:

$$\frac{\partial C}{\partial \sigma^2} = \frac{\partial C}{\partial S}\frac{\partial S}{\partial \sigma^2} = \frac{N_2(\cdot)}{N_1(\cdot)}\frac{Dn_1(d_2(P, \tau_s))\sqrt{\tau_s}}{2\sigma} > 0,$$

where $n_1(z) \equiv \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is the standard-normal probability-density function. The partial derivative can be used in conjunction with standard numerical techniques to imply out the variance used by the market if the market price is observable.

The CEO value is *not* monotonically increasing in the individual asset variances σ_v^2 and σ_d^2 when, inter alia, the correlation coefficient is held constant:

$$\frac{\partial C}{\partial \sigma_v^2} = \frac{\partial C}{\partial \sigma^2}\frac{\partial \sigma^2}{\partial \sigma_v^2} = \frac{N_2(\cdot)}{N_1(\cdot)}\frac{Dn_1(d_2(P, \tau_s))\sqrt{\tau_s}}{2\sigma}(1 - B_{dv}) \gtreqless 0,$$

where $B_{dv} \equiv \rho\frac{\sigma_d}{\sigma_v}$ and

$$\frac{\partial C}{\partial \sigma_d^2} = \frac{\partial C}{\partial \sigma^2}\frac{\partial \sigma^2}{\partial \sigma_d^2} = \frac{N_2(\cdot)}{N_1(\cdot)}\frac{Dn_1(d_2(P, \tau_s)\sqrt{\tau_s}}{2\sigma}(1 - B_{vd}) \gtreqless 0,$$

where $B_{vd} \equiv \rho\frac{\sigma_v}{\sigma_d}$.

The sign of the partials depends on the stochastic relationship between the rates of return of the underlying assets as measured by the beta coefficient of a regression between them. If the rates of return on the underlying assets V and D

are positively correlated, a rise in either's volatility σ_v^2 or σ_d^2 can reduce the relevant variance and thus the CEO value.

The greater the correlation between the rates of return of the two assets, the less likely it is that the price of the optioned asset V will exceed that of the delivery asset D at maturity and the lower should be the CEO value. This intuition is confirmed by the sign of the partial derivative:

$$\frac{\partial C}{\partial \rho} = \frac{\partial C}{\partial \sigma^2}\frac{\partial \sigma^2}{\partial \rho} = -\frac{N_2(\cdot)}{N_1(\cdot)}\frac{Dn_1(d_2(P,\,\tau_s))}{\sigma}\sigma_V\sigma_D < 0.$$

As one might suspect, the compound-option value is increasing in both its own time to maturity and that of the underlying simple option, i.e., $\dfrac{\partial C}{\partial \tau_c} > 0$ and $\dfrac{\partial C}{\partial \tau_s} > 0$.

Finally, the partial derivative of a compound option with respect to its exchange ratio is negative as expected:

$$\frac{\partial C}{\partial q} = -DN_1\left(d_2\left(\frac{P}{P^*}\right)\right) < 0.$$

The CEO value is also declining in q when the exchange ratio of the simple option is $1 - q$ instead of 1. Consequently, if the holder of an option can deliver any fraction of the delivery asset early, the position value is maximized by choosing this fraction to be zero. In other words, prepayment never occurs voluntarily, and, consequently, a simple exchange option is held. In a similar manner, any individual who has the option of exchanging uncertain quantities will find it beneficial to procrastinate for as long as possible. This result can break down when the underlying assets pay dividends or provide other benefits of ownership.

Appendix B

Extensions

This appendix presents two extensions to the valuation formula (27). The first gives the value of a nested series of exchange options. The second values a pseudo-American exchange option with an arbitrary number of possible exercise dates. To value a nested series, let

E^0 be the current value of the optioned asset V,
$E^1(E^0,\, q_nD,\, \tau_n)$ be the current value of the simple exchange option to receive E^0 for q_nD at time T_n, and
$E^2(E^1,\, q_{n-1}D,\, \tau_{n-1})$ be the current value of the compound exchange option to receive E^1 for $q_{n-1}D$ at time T_{n-1}.

The formula required is for $E^n(E^{n-1},\, q_1D,\, \tau_1)$, i.e., the current value of the compound exchange option, to receive E^{n-1} for q_1D at time T_1.[18] By induction,

[18] Note that the superscript n on E is not an exponent but refers to the degree of compoundness of the exchange option.

one can show that

$$E^n(E^{n-1}, q_1 D, \tau_1)$$

$$= VN_n\left(d_{1n}; \left\{\sqrt{\frac{\tau_i}{\tau_j}}\right\}\right) - D \textstyle\sum_{t=1}^{n} q_t N_t\left(d_{2t}; \left\{\sqrt{\frac{\tau_i}{\tau_j}}\right\}\right), \quad (33)$$

where

$N_n\left(\cdot; \left\{\sqrt{\frac{\tau_i}{\tau_j}}\right\}\right)$ is the standard n-variate normal-distribution function with correlation matrix $\left\{\sqrt{\frac{\tau_i}{\tau_j}}\right\}$,

d_{1n} is an $n \times 1$ column vector with elements $\left(d_1\left(\frac{P}{P_1^*}, \tau_1\right), \cdots, d_1\left(\frac{P}{P_n^*}, \tau_n\right)\right)'$,

d_{2t} is a $t \times 1$ column vector with elements $\left(d_2\left(\frac{P}{P_1^*}, \tau_1\right), \cdots, d_2\left(\frac{P}{P_t^*}, \tau_t\right)\right)'$,

$\left\{\sqrt{\frac{\tau_i}{\tau_j}}\right\}$ is an $n \times n$ symmetric matrix whose i, jth element is

$$\sqrt{\frac{\tau_i}{\tau_j}}, \qquad i = 1, \cdots, j, \quad j = 1, \cdots, n,$$

q_n is normalized to be 1,

P_t^* is the unique value of $P \equiv \frac{V}{D}$ that solves

$$\frac{E^{\tau-t}(E^{\tau-(t+1)}, q_{t+1}(D, \tau_{t+1})}{D} = q_t, \quad t = 1, \cdots, n-1,$$

and $P_n^* = 1$.

Next, the formula for the value of an option with n possible exercise dates is presented. To allow for the possibility of early exercise, constant dividend yields on the underlying assets are assumed. Let t be the valuation date and T the expiration date. The first step is to divide the option's time to maturity $\tau \equiv T - t$ into n equal intervals. Let E_n be the value of a quasi-American exchange option that may be exercised at any of the n endpoints of each interval. Then, E_1 is just the value of a European exchange option as given by (31). E_2 is the value given by (32) of an exchange option that may be exercised at $\frac{T}{2}$ or at T.

Once E_2 can be valued, it can be used to value the pseudo-American exchange option E_3. This option can be exercised at $\frac{T}{3}, \frac{2T}{3}$ or at T. Whether the option is exercised early depends on whether the price ratio reaches certain critical values at the intermediate dates $\frac{T}{3}$ and $\frac{2T}{3}$. The option will not be exercised at the first exercise point $\frac{T}{3}$ if the opportunity cost of exercise, E_2, exceeds the cash proceeds from exercise $V - D$. Dividing by the delivery-asset price D leads to the defining equation for the first critical value P_1^*. Similarly, the second critical value P_2^* can be determined by equating the holding value E_1 to the exercise value $V - D$ and dividing by the delivery-asset price D.

Proceeding in a similar fashion leads to the valuation formula for the pseudo-American exchange option E_n:

$$E_n = Vw_v - Dw_d, \tag{34}$$

where

$$\begin{aligned} w_v \equiv{} & e^{-\delta_v \Delta t} N_1\left(d_1\left(\frac{Pe^{-\delta\Delta t}}{P_1^*}\right)\right) \\ & + e^{-\delta_v 2\Delta t} N_2\left(-d_1\left(\frac{Pe^{-\delta\Delta t}}{P_1^*}\right), d_1\left(\frac{Pe^{-\delta 2\Delta t}}{P_2^*}\right); -\sqrt{\frac{1}{2}}\right) \\ & + e^{-\delta_v 3\Delta t} N_3\left(-d_1\left(\frac{Pe^{-\delta\Delta t}}{P_1^*}\right), -d_1\left(\frac{Pe^{-\delta 2\Delta t}}{P_2^*}\right), d_1\left(\frac{Pe^{-\delta 3\Delta t}}{P_3^*}\right); \left\{\sqrt{\frac{i}{j}}\right\}\right) \\ & + \cdots + e^{-\delta_v \tau} N_n\left(-d_1\left(\frac{Pe^{-\delta\Delta t}}{P_1^*}\right), \ldots, -d_1\left(\frac{Pe^{-\delta(n-1)\Delta t}}{P_{n-1}^*}\right), d_1(Pe^{-\delta\tau}); \left\{\sqrt{\frac{i}{j}}\right\}\right), \end{aligned}$$

and

$$\begin{aligned} w_d \equiv{} & e^{-\delta_d \Delta t} N_1\left(d_2\left(\frac{Pe^{-\delta\Delta t}}{P_1^*}\right)\right) \\ & + e^{-\delta_d 2\Delta t} N_2\left(-d_2\left(\frac{Pe^{-\delta\Delta t}}{P_1^*}\right), d_2\left(\frac{Pe^{-\delta 2\Delta t}}{P_2^*}\right); -\sqrt{\frac{1}{2}}\right) \\ & + e^{-\delta_d 3\Delta t} N_3\left(-d_2\left(\frac{Pe^{-\delta\Delta t}}{P_1^*}\right), -d_2\left(\frac{Pe^{-\delta 2\Delta t}}{P_2^*}\right), d_2\left(\frac{Pe^{-\delta 3\Delta t}}{P_3^*}\right); \left\{\sqrt{\frac{i}{j}}\right\}\right) \\ & + \cdots + e^{-\delta_d \tau} N_n\left(-d_2\left(\frac{Pe^{-\delta\Delta t}}{P_1^*}\right), \ldots, -d_2\left(\frac{Pe^{-\delta(n-1)\Delta t}}{P_{n-1}^*}\right), d_2(Pe^{-\delta\tau}); \left\{\sqrt{\frac{i}{j}}\right\}\right), \end{aligned}$$

where

$\Delta t = \frac{\tau}{n}$,

N_n is the standard n-variate normal distribution function with correlation matrix $\left\{\sqrt{\frac{i}{j}}\right\}$,

$\left\{\sqrt{\frac{i}{j}}\right\}$ is the $n \times n$-symmetric matrix whose i, jth element is

$$= \sqrt{\frac{i}{j}}, \qquad i = 1, \cdots, j, \quad j = 1, \cdots, n-1, \text{ and } i = j = n,$$

$$= -\sqrt{\frac{i}{j}}, \qquad i = 1, \cdots, n-1, \quad j = n,$$

and P_k^* is the critical value of P at $k\Delta t$, $k = 1, \cdots, n-1$.

Arbitrary accuracy can be achieved for sufficiently large values of n.

REFERENCES

1. Fischer Black and Myron Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81 (May/June 1973), 637–54.
2. Stanley Fischer. "Call Option Pricing When the Exercise Price is Uncertain, and the Valuation of Index Bonds." *Journal of Finance* 33 (March 1978), 169–76.
3. Robert Geske. "The Valuation of Corporate Liabilities as Compound Options." *Journal of Financial and Quantitative Analysis* 12 (November 1977), 541–52.
4. ———. "The Valuation of Compound Options." *Journal of Financial Economics* (March 1979), 63–81.
5. ——— and Herbert Johnson. "The American Put Option Valued Analytically." *Journal of Finance* 39 (December 1984), 1511–24.
6. William Margrabe. "The Value of an Option to Exchange One Asset for Another." *Journal of Finance* 33 (March 1978), 177–86.
7. Robert McDonald and Daniel Siegel. "Investment and the Valuation of Firms When There is an Option to Shut Down." *International Economic Review* 26 (June 1985), 331–49.
8. ———. "The Value of Waiting to Invest." *Quarterly Journal of Economics* (November 1986), 707–27.
9. Robert C. Merton. "Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science* 4 (Spring 1973), 141–83.
10. Richard Roll. "An Analytic Valuation Formula for Unprotected American Call Options on Stocks with Known Dividends." *Journal of Financial Economics* 5 (November 1977), 251–58.