The Valuation of American Exchange Options with Application to Real Options

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ABSTRACT

An American exchange option gives its owner the right to exchange one asset for another at any time prior to expiration. A model for valuing these options is developed using the Geske-Johnson approach for valuing American put options. The formula is shown to generalize much previous work in option pricing. Application of the general valuation formula to the timing option in capital investment theory and other real options is presented.

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I. INTRODUCTION

An American exchange option gives its owner the right to exchange one asset for another at any time up to and including expiration. Margrabe (1978) values a European exchange option which gives its owner the right to such an exchange only at expiration. Margrabe also proves that exercise of an American exchange option will only occur at expiration when neither underlying asset pays dividends. However, when the asset to be received in the exchange pays sufficiently large dividends, there is a positive probability that an American exchange option will be exercised strictly prior to expiration. This positive probability induces additional value for an American exchange option over its European counterpart.

The purpose of this paper is to develop a general formula for valuing American exchange options. The formula generalizes the Geske-Johnson (1984) solution for the value of an American put option. The generalization essentially involves redefining the exercise price to be the price of a traded asset. If either asset involved in the exchange has constant value over time, then an exchange option reduces to an ordinary call or put option. Consequently, this general formula for American exchange options may be used to value standard call or put options as special cases. Furthermore, the timing option inherent in a capital investment decision can also be valued.

The paper values American exchange options when both underlying assets pay dividends continuously. Any asset whose payoffs accrue over time may be considered to yield a continuous payout (e.g., a coupon bond). Furthermore, an asset may behave as if it pays dividends if, for example, it furnishes a convenience yield or earns a below equilibrium expected rate of return. Non-traded real assets may offer a below equilibrium return and may involve flexibilities to switch operating modes or exchange one asset for another. As a result, the general valuation formula may be used to value real options. For analytical tractability, the dividends from the underlying assets are presumed to provide a constant yield. When the dividend yield on the asset to be received in the exchange is strictly positive, American exchange options may be exercised early.

The Geske-Johnson approach is used here to value an American exchange option because it possesses two advantages over other methods. First, the solution may be differentiated to afford comparative statics results. Second, a polynomial approximation to the exact formula is computationally more efficient than either finite differences or the binomial method (see Geske and Shastri, 1985).

The paper is organized as follows. The next section reviews some of the relevant option pricing literature. The valuation formula for an American exchange option is derived in section III. The following section then incorporates some previous results as special cases of the general solution. Application of the general valuation model to the timing option in investment theory and other real options is discussed in section V. The final section concludes the paper.

II. LITERATURE REVIEW

This paper is concerned with valuing exchange options on dividend-paying assets which may rationally be exercised early. As an introduction, this section reviews previous work on valuing European exchange options and American puts. To focus the discussion, consider the European option to exchange asset D for asset V at time T. Asset D is referred to as the *delivery asset*, and asset V the *optioned asset*. The payoff to this European option at T is $\max(0, V_T - D_T)$ where V_T and D_T are the underlying assets' terminal prices. Suppose that the underlying asset prices V_t and

D_t prior to expiration follow a geometric Brownian motion of the form:

$$\begin{aligned} \frac{dV_t}{V_t} &= (\alpha_v - \delta_v)dt + \sigma_v dZ_t^v \\ \frac{dD_t}{D_t} &= (\alpha_d - \delta_d)dt + \sigma_d dZ_t^d \\ \operatorname{cov}\left(\frac{dV_t}{V_t}, \frac{dD_t}{D_t}\right) &= \sigma_{vd} dt, \quad t \in [0, T], \end{aligned} \tag{1}$$

where α_v and α_d are the expected rates of return on the two assets, δ_v and δ_d are the corresponding dividend yields, σ_v^2 and σ_d^2 are the respective variance rates, and dZ_t^v and dZ_t^d are increments of standard Wiener processes at time t. The rates of price changes, $\frac{dV_t}{V_t}$ and $\frac{dD_t}{D_t}$, can be correlated, with the covariance rate given by σ_{vd}. The parameters δ_v, δ_d, σ_v, σ_d, and σ_{vd} are assumed to be nonnegative constants, although they can be allowed to be deterministic functions of time.

Under certain assumptions, McDonald and Siegel (1985) show that the value of a European exchange option on such dividend-paying assets is given by:

$$e(V, D, \tau) \; = Ve^{-\delta_v \tau} N_1(d_1(Pe^{-\delta\tau}, \sigma^2\tau)) - De^{-\delta_d \tau} N_1(d_2(Pe^{-\delta\tau}, \sigma^2\tau)), \tag{2}$$

where:

$N_1(d) \equiv \int_0^d \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz$ is the standard univariate normal distribution function,

$d_1(Pe^{-\delta\tau}, \sigma^2\tau) \equiv \frac{\ln(Pe^{-\delta\tau}) + \sigma^2\tau/2}{\sqrt{\sigma^2\tau}}$,

$P \equiv \frac{V}{D}$ is the price ratio of V to D,

$\delta \equiv \delta_v - \delta_d$ is the difference in the dividend yields,

$\sigma^2 \equiv \sigma_v^2 + \sigma_d^2 - 2\sigma_{vd}$ is the variance rate of $\frac{dP}{P}$, and

$d_2(Pe^{-\delta\tau}, \sigma^2\tau) \equiv d_1(Pe^{-\delta\tau}, \sigma^2\tau) - \sigma\sqrt{\tau}$.

(To simplify the notation, the second argument of d_1 and d_2 will be dropped whenever it can be inferred from the first argument.)

The underlying assets in the McDonald and Siegel model are not necessarily traded. Consequently, they develop their valuation formula using an equilibrium argument. When the underlying assets are traded, an arbitrage argument also leads to (2). Black and Scholes (1973) also showed that their valuation formula can be alternatively derived using an equilibrium model or an arbitrage argument. In the Black Scholes model, the expected rate of return on the underlying asset is irrelevant given the current asset price. Similarly, equation (2) indicates that the expected rates of return, α_v and α_d, are irrelevant given the current asset values V and D.

In contrast to the Black-Scholes formula however, the riskfree rate of interest, r, is also absent from the formula[1]. The reason for this is that the exchange option value is linearly homogeneous[2]in the asset prices V and D under the stochastic process (1). Consequently, the weights which eliminate risk in the hedge portfolio also make it costless. A no arbitrage equilibrium then implies that the hedge portfolio earns zero return rather than the interest rate, r. Since the expected rates of return and the interest rate are irrelevant given the current asset prices, investors need not agree on the dynamics of these rates. However, agreement is presumed on the constant variance rate, σ^2, and on the constant dividend yields, δ_v and δ_d.

If these dividend yields are set equal to zero, then Margrabe's (1978) formula for a European exchange option results. Under further parameter restrictions and the additional assumption of a constant (positive) riskless rate, r, formulas for European call and put options are obtained. To value a call option, suppose that we "zero out" the variance rate of the delivery asset ($\sigma_d^2 = 0$) so that its expected rate of return must be the riskless rate ($\alpha_d = r$) to avoid arbitrage. Further, suppose that the delivery asset pays dividends at the riskless rate ($\delta_d = r$) so that its value is constant over time ($\frac{dD}{D} = 0$). A call option is thus a special type of an exchange option, where the delivery asset, D, has a constant value over time. Under the asssumed parameter restrictions $\delta_d = r$ and $\sigma_d^2 = 0$, equation (2) reduces to Merton's (1973) formula for a

European call option on a dividend-paying stock:

$$c(V, D, \tau) \quad = V e^{-\delta_v \tau} N_1(d_1(P e^{-\delta \tau})) - D e^{-r\tau} N_1(d_2(P e^{-\delta \tau})), \tag{3}$$

where:

V is the current price of the underlying asset,

D is the exercise price of the call option,

$\delta = \delta_v - r$, and

$\sigma = \sigma_v$.

If the underlying asset pays no dividends ($\delta_v = 0$), then the standard Black-Scholes (1973) formula for a European call option emerges.

As is the case for a call, a put option is also a special kind of an exchange option. In contrast to a call, however, the delivery asset for a put option is risky, while the optioned asset, V, has a constant value over time. The value of asset V will similarly be constant ($\frac{dV}{V} = 0$) if its variance rate vanishes ($\sigma_v^2 = 0$) and if it yields dividends at the riskless rate r ($\delta_v = r$). Making these substitutions in (2) yields the formula for a European put option on an asset paying continuous dividends:

$$p(V, D, \tau) \quad = V e^{-r\tau} N_1(d_1(P e^{-\delta \tau})) - D e^{-\delta_d \tau} N_1(d_2(P e^{-\delta \tau})), \tag{4}$$

where:

V is the execise price of the put option,

D is the current price of the underlying asset,

$\delta = r - \delta_d$, and

$\sigma = \sigma_d$.

If the underlying asset pays no dividends ($\delta_d = 0$), the Black-Scholes European put option formula arises if we make use of the following identities:

$$d_1(P e^{-r\tau}) \quad = \quad d_1\left(\frac{V e^{-r\tau}}{D}\right) = -d_2\left(\frac{D}{V e^{-r\tau}}\right)$$

$$d_2(Pe^{-r\tau}) \quad = \quad d_2\left(\frac{Ve^{-r\tau}}{D}\right) = -d_1\left(\frac{D}{Ve^{-r\tau}}\right). \tag{5}$$

Up to this point, the focus has been exclusively on European options. Unfortunately, general equation (2) does not hold for American exchange options. If an American exchange option is sufficiently in the money, it will pay to exercise early when asset V has a positive dividend yield. For an American *put*, since this asset yields dividends at the riskless rate r, there is always a positive probability of premature exercise.

Geske and Johnson (1984) account for this possibility of early exercise when they derive a valuation formula for American put options. Their approach is to view an American put option as the limit to a sequence of pseudo-American puts. A pseudo-American option can only be exercised at a finite number of discrete exercise points. As the number of possible exercise points grows, the value of a pseudo-American option approaches that of a true American one. Unfortunately, for a large number of exercise points, the valuation formula becomes cumbersome. The authors circumvent this problem by extrapolating from the values of puts with a small number of exercise points. The valuation formulae for these lower order puts can be easily implemented.

The next section generalizes the Geske-Johnson approach to American exchange options on dividend-paying assets. The resulting solution incorporates many of the option pricing formulae which have appeared in the earlier literature. In particular, the formulae discussed in this section arise as special cases.

III. VALUATION OF THE AMERICAN EXCHANGE OPTION

This section derives the valuation formula for an American exchange option on dividend-paying assets. Let t be the valuation date, and T the option expiration date. The first step involves dividing the option's time to maturity, $\tau \equiv T - t$, into n equal intervals. Let $E_n(\tau)$ be the value of a pseudo-American exchange option with time to maturity τ. The subscript n indicates that the option can be exercised at any of the n end points of each interval. Then $E_1(\tau)$ is just the value of a European exchange option as given by (2).

$E_2(\tau)$ is the value of an exchange option which may be exercised at $\frac{T}{2}$ or at T. This option will not be exercised at mid-life if the opportunity cost of exercise, i.e., the value of the option from (2), exceeds the cash proceeds of exercise, i.e., if:

$$Ve^{-\delta_v \triangle t} N_1(d_1(Pe^{-\delta \triangle t})) - De^{-\delta_d \triangle t} N_1(d_2(Pe^{-\delta \triangle t})) > V - D, \text{ where } \triangle t = \frac{\tau}{2}. \quad (6)$$

Both V and D are random prices as of the valuation date, t. However, the exercise condition can be re-expressed in terms of just one random variable by taking the delivery asset as numeraire. Dividing by the delivery asset price, D, and substituting the price ratio P for $\frac{V}{D}$ yields:

$$e^{-\delta_v \triangle t} N_1(d_1(Pe^{-\delta \triangle t})) - e^{-\delta_d \triangle t} N_1(d_2(Pe^{-\delta \triangle t})) > P - 1. \quad (7)$$

Let P^* be the unique value of the price ratio, P, which makes the above an equality. That is, the critical price ratio, P^*, is defined by:

$$P^* e^{-\delta_v \triangle t} N_1(d_1(P^* e^{-\delta \triangle t})) - e^{-\delta_d \triangle t} N_1(d_2(P^* e^{-\delta \triangle t})) = P^* - 1. \quad (8)$$

For values of the price ratio P greater than the critical price ratio P^*, the option is exercised to yield proceeds of $V - D$ at the intermediate exercise date $\frac{T}{2}$. Otherwise, the option is held and would pay off $\max(0, V - D)$ at the expiration date T. The risk-neutral valuation relationship of Cox and Ross (1976) may be used to value these contingent payoffs as:

$$E_2(\tau) \;=\; V\left[e^{-\delta_v \triangle t} N_1\left(d_1\left(\frac{Pe^{-\delta \triangle t}}{P^*}\right)\right) + e^{-\delta_v T} N_2\left(-d_1\left(\frac{Pe^{-\delta \triangle t}}{P^*}\right), d_1(Pe^{-\delta T}); -\sqrt{\frac{1}{2}}\right)\right]$$

$$- \quad D\left[e^{-\delta_d \triangle t} N_1\left(d_2\left(\frac{Pe^{-\delta\triangle t}}{P^*}\right)\right) + e^{-\delta_d T} N_2\left(-d_2\left(\frac{Pe^{-\delta\triangle t}}{P^*}\right), d_2(Pe^{-\delta T}); -\sqrt{\frac{1}{2}}\right)\right], \tag{9}$$

where $N_2(x_1, x_2; \rho)$ is the standard bivariate normal distribution function evaluated at x_1 and x_2 with correlation coefficient ρ, given by:

$$N_2(x_1, x_2; \rho) \equiv \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} \frac{\exp\left\{-\frac{1}{2(1-\rho^2)}[z_1^2 - 2\rho z_1 z_2 + z_2^2]\right\}}{2\pi\sqrt{1-\rho^2}} dz_2 dz_1.$$

The above functional form for $E_2(\cdot)$ in turn can be used to determine the pseudo-American exchange option value, $E_3(\tau)$. This option can be exercised at times $\frac{T}{3}$, $\frac{2T}{3}$, or at T. Whether the option is exercised early or not depends on whether the price ratio, P, reaches certain critical values at the intermediate dates $\frac{T}{3}$ and $\frac{2T}{3}$. The option will not be exercised at the first exercise point, $\frac{T}{3}$, if the opportunity cost of exercise, $E_2(\frac{2\tau}{3})$, exceeds the cash proceeds from exercise, $V - D$. Dividing again by the delivery asset price, D, leads to the defining equation for the first critical value, P_1^*:

$$\begin{aligned} &P_1^*\left[e^{-\delta_v \triangle t} N_1\left(d_1\left(\frac{P_1^* e^{-\delta\triangle t}}{P^*}\right)\right) + e^{-\delta_v 2\triangle t} N_2\left(-d_1\left(\frac{P_1^* e^{-\delta\triangle t}}{P^*}\right), d_1(P_1^* e^{-\delta 2\triangle t}); -\sqrt{\frac{1}{2}}\right)\right] \\ &- \left[e^{-\delta_d \triangle t} N_1\left(d_2\left(\frac{P_1^* e^{-\delta\triangle t}}{P^*}\right)\right) + e^{-\delta_d 2\triangle t} N_2\left(-d_2\left(\frac{P_1^* e^{-\delta\triangle t}}{P^*}\right), d_2(P_1^* e^{-\delta 2\triangle t}); -\sqrt{\frac{1}{2}}\right)\right] \\ &= P_1^* - 1, \text{ where } \triangle t = \frac{\tau}{3}. \end{aligned} \tag{10}$$

Assuming that the pseudo-American option survives its first exercise point, $\frac{T}{3}$, it will also not be exercised at the next exercise point, $\frac{2T}{3}$, if its value alive, $E_1(\frac{\tau}{3})$, exceeds its exercise value, $V - D$. Again, dividing by the delivery asset price, D, leads to the defining equation for the second critical value, P_2^*:

$$P_2^* e^{-\delta_v \triangle t} N_1(d_1(P_2^* e^{-\delta\triangle t})) - e^{-\delta_d \triangle t} N_1(d_2(P_2^* e^{-\delta\triangle t})) = P_2^* - 1.$$

Risk-neutral valuation can again be employed to write the valuation formula for the pseudo-American exchange option $E_3(\tau)$ as:

$$\begin{aligned}E_3(\tau) \;&= V[e^{-\delta_v \triangle t} N_1(d_1(\tfrac{Pe^{-\delta \triangle t}}{P_1^*})) \\ &+e^{-\delta_v 2\triangle t} N_2(-d_1(\tfrac{Pe^{-\delta \triangle t}}{P_1^*}), d_1(Pe^{-\delta 2\triangle t}); -\sqrt{\tfrac{1}{2}}) \\ &+e^{-\delta_v \tau} N_3(-d_1(\tfrac{Pe^{-\delta \triangle t}}{P_1^*}), -d_1(\tfrac{Pe^{-\delta 2\triangle t}}{P_2^*}), d_1(Pe^{-\delta\tau}); \Omega_3)] \\ &-D[e^{-\delta_d \triangle t} N_1(d_2(\tfrac{Pe^{-\delta \triangle t}}{P_1^*})) \\ &+e^{-\delta_d 2\triangle t} N_2(-d_2(\tfrac{Pe^{-\delta \triangle t}}{P_1^*}), d_2(Pe^{-\delta 2\triangle t}); -\sqrt{\tfrac{1}{2}}) \\ &+e^{-\delta_d \tau} N_3(-d_2(\tfrac{Pe^{-\delta \triangle t}}{P_1^*}), -d_2(\tfrac{Pe^{-\delta 2\triangle t}}{P_2^*}), d_2(Pe^{-\delta\tau}); \Omega_3)],\end{aligned} \tag{11}$$

where $\triangle t = \frac{\tau}{3}$, $N_3(x_1, x_2, x_3; \Omega_3)$ is the standard trivariate normal distribution function evaluated at x_1, x_2, and x_3 with correlation matrix Ω_3, given by:

$$N_3(x_1, x_2, x_3; \Omega_3) \equiv \int_{\infty}^{x_1} \int_{\infty}^{x_2} \int_{\infty}^{x_3} (2\pi)^{-3/2} |\Omega_3|^{-1/2} \exp\left\{-\frac{1}{2} z' \Omega_3^{-1} z\right\} dz_1 dz_2 dz_3,$$

with z as the 3×1 vector:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix},$$

and Ω_3 as the 3×3 symmetric matrix:

$$\begin{bmatrix} \sqrt{\frac{1}{1}}\sqrt{\frac{1}{2}}\sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{2}}\sqrt{\frac{2}{2}}\sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}}\sqrt{\frac{2}{3}}\sqrt{\frac{3}{3}} \end{bmatrix}.$$

By induction, the value of the general pseudo-American exchange option E_n is:

$$E_n = V w_1(\delta_v) - D w_2(\delta_d) \tag{12}$$

where:

$$\begin{aligned}w_1(\delta_v) \;&\equiv e^{-\delta_v \triangle t} N_1(d_1(\tfrac{Pe^{-\delta \triangle t}}{P_1^*})) \\ &+e^{-\delta_v 2\triangle t} N_2(-d_1(\tfrac{Pe^{-\delta \triangle t}}{P_1^*}), d_1(\tfrac{Pe^{-\delta 2\triangle t}}{P_2^*}); -\sqrt{\tfrac{1}{2}}) \\ &+e^{-\delta_v 3\triangle t} N_3(-d_1(\tfrac{Pe^{-\delta \triangle t}}{P_1^*}), -d_1(\tfrac{Pe^{-\delta 2\triangle t}}{P_2^*}), d_1(\tfrac{Pe^{-\delta 3\triangle t}}{P_3^*}); \Omega_3) \\ &+\ldots + e^{-\delta_v \tau} N_n(-d_1(\tfrac{Pe^{-\delta \triangle t}}{P_1^*}), \ldots, -d_1(\tfrac{Pe^{-\delta(n-1)\triangle t}}{P_{n-1}^*}), d_1(Pe^{-\delta\tau}); \Omega_n)\end{aligned}$$

and

$$
\begin{aligned}
w_2(\delta_d) &\equiv e^{-\delta_d \triangle t} N_1(d_2(\tfrac{Pe^{-\delta \triangle t}}{P_1^*})) \\
&+e^{-\delta_d 2\triangle t} N_2(-d_2(\tfrac{Pe^{-\delta \triangle t}}{P_1^*}), d_2(\tfrac{Pe^{-\delta 2\triangle t}}{P_2^*}); -\sqrt{\tfrac{1}{2}}) \\
&+e^{-\delta_d 3\triangle t} N_3(-d_2(\tfrac{Pe^{-\delta \triangle t}}{P_1^*}), -d_2(\tfrac{Pe^{-\delta 2\triangle t}}{P_2^*}), d_2(\tfrac{Pe^{-\delta 3\triangle t}}{P_3^*}); \Omega_3) \\
&+\ldots + e^{-\delta_d \tau} N_n(-d_2(\tfrac{Pe^{-\delta \triangle t}}{P_1^*}), \ldots, -d_2(\tfrac{Pe^{-\delta(n-1)\triangle t}}{P_{n-1}^*}), d_2(Pe^{-\delta \tau}); \Omega_n),
\end{aligned}
$$

where $\triangle t = \frac{\tau}{n}$, N_k is the standard k-variate normal distribution function with correlation matrix Ω_k:

$$\int_{\infty}^{x_1} \int_{\infty}^{x_2} \cdots \int_{\infty}^{x_k} (2\pi)^{-k/2} |\Omega_k|^{-1/2} \exp\left\{-\frac{1}{2} z' \Omega_k^{-1} z\right\} dz_1 dz_2 \cdots dz_k,$$

with z as the $k \times 1$ vector:

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{bmatrix},$$

Ω_k as the $k \times k$ symmetric matrix whose $i - j$th element is:

$$\sqrt{\frac{i}{j}} \quad \begin{array}{l} i = 1 \ldots j \\ j = 1 \ldots k, \end{array}$$

and where P_k^* is the critical value of P at $k\triangle t,\ k = 1 \ldots n-1$.

Since the discrete exercise policy employed above is not strictly optimal, the pseudo-American exchange option value, E_n, is actually a lower bound on the true American exchange option value. However, arbitrary accuracy can be achieved for sufficiently large values of n. Unfortunately, the formula involves n-variate normal distribution functions which are not tabulated for large values of n. This problem can be solved by extrapolating for E_n from its lower order values. The three-point Richardson extrapolation which achieves reasonable accuracy is[3]:

$$E_n \approx \frac{1}{2} E_1 - 4E_2 + \frac{9}{2} E_3. \tag{13}$$

IV. SPECIAL CASES

In this section, the parameters of the general valuation formula (12) are restricted to yield various known special cases. In particular, the valuation formulae for standard American put and call options are easily derived. The valuation formulae for the European options given in section II also arise as special cases. Throughout this section, the riskless rate is assumed to be (a positive) constant.

A. American Put Option

Recall that a put is an exchange option whose optioned asset's value is constant over time. As in section II, constant value is achieved ($\frac{dV}{V} = 0$) by "zeroing out" asset V's variance rate ($\sigma_v^2 = 0$) and equating its dividend yield to the riskless rate ($\delta_v = r$). Making these substitutions yields the formula for an American put on a dividend-paying stock:

$$P_n = Vw_1(r) - Dw_2(\delta_d), \tag{14}$$

where:

V is the exercise price of the put option,

D is the current price of the underlying asset,

$\delta = r - \delta_d$, and

$\sigma^2 = \sigma_d^2$.

If the underlying asset for the American put pays no dividends ($\delta_d = 0$), then the Geske-Johnson formula for an American put arises.[4]

B. American Call Option

If the underlying asset for an American call pays a continuous dividend at a constant yield, then the option may rationally be exercised before maturity. To value such a call option with the general valuation formula (12), the delivery asset parameters are restricted to achieve constant value. In particular, by setting $\delta_d = r$ and $\sigma_d^2 = 0$, we obtain:

$$C_n = Vw_1(\delta_v) - Dw_2(r), \tag{15}$$

where:

V is the current value of the underlying asset,

D is the exercise price of the call option,

$\delta = \delta_v - r$, and

$\sigma^2 = \sigma_v^2$.

As the dividend yield on the underlying asset gets smaller, the critical price ratios required to trigger early exercise get larger. When this dividend vanishes ($\delta_v = 0$), no finite asset price is sufficiently high so as to induce exercise at any time prior to maturity. As a result, $P_k^* = \infty, \forall k = 1 \ldots n-1$ in (12) and the Black-Scholes formula is consequently obtained.

C. European Exchange Options

Recall that early exercise of an exchange option occurs when the price ratio exceeds the critical price ratio P^*. To value an exchange option which precludes exercise on any given date prior to maturity, the critical price ratio corresponding to that date can be set to infinity. As a result, a European exchange option can again be valued by setting $P_k^* = \infty, \forall k = 1 \ldots n-1$ in (12). The general formula then reduces to McDonald and Siegel's (1985) equation (2) above for a European exchange option on dividend-paying assets. Section II demonstrated that this formula in turn contains Margrabe's (1978) solution for an exchange option on non-dividend paying assets, as well as the Merton (1973) and Black-Scholes (1973) option formulas.

V. APPLICATION TO REAL OPTIONS

In this section, the general valuation formula (12) is used to illustrate valuation of the timing option available to firms when making real investment decisions. McDonald and Siegel (1986) have valued a firm's option to invest (at time t) a random amount D_t to undertake a project whose current value to the firm is V_t. If V_t and D_t are not prices of traded assets, their expected growth rates may actually differ from the expected rate of return required for their risk in equilibrium in the financial markets. Let δ_v and δ_d be the assumed constant difference (return shortfall) between these expected rates. If the firm could invest only at a fixed time point, T, then the value of the option to invest would be given by equation (2) for a European exchange option. Using an equilibrium argument, McDonald and Siegel value this option when its life is either infinite or random.

The general valuation formula (12) can also be derived in an equilibrium model. The formula may then be used to value a timing option which expires within a fixed period of time. Concrete examples of this situation may occur when a firm has an option to buy land or to drill for oil within, say, six months. Alternatively, a patent, injunction, or a temporary competive advantage may allow a firm to exploit a production opportunity for a limited period of time.

The option to abandon a project (having current value D_t) in exchange for its salvage (or best alternative use) value (V_t) has been studied in Myers and Majd (1990) and in McDonald and Siegel (1986). This abandonment option is a mirror problem to the timing option one, and can be similarly valued with our general formula with a suitable re-interpretation of variables. Other real options, such as to switch inputs or outputs in production, could be valued similarly.

The major impediment to such real option applications appears to be the potential unobservability of the asset values, V_t and D_t. In certain situations, these values can be backed out of a valuation model which employs observable prices as inputs. For example, Brennan and Schwartz (1985) value a mine when the ore is traded in futures markets. Assuming that a geometric Brownian motion is a reasonable approximation for the dynamics of the mine's value, the American option to buy or sell the mine can be valued using the results of this paper.

Alternatively, the effect of the unobservability of the asset values, V_t and D_t, can be included in the valuation model. For example, one could assume that these quantities are observed with noise. The principal effect of the noise would be to induce suboptimal exercise. In particular, real options might be exercised when they are out-of-the-money, and deep in-the-money options may sometimes fail to be optimally exercised. These effects work to reduce option value relative to the case with perfect observability. The magnitude of mispricing would depend positively on the the amount of noise (or the variance of the error term).

VI. CONCLUSION

This paper has developed a model for valuing American exchange options on dividend-paying assets. After a brief review of the literature, a general formula was developed which was shown to encompass many earlier results under suitable parameter restrictions. In particular, this general formula values both European and American calls and puts as special cases. The general valuation formula was also applied to valuing the timing option in investment theory.

The foregoing analysis may be extended to allow for imperfect capital markets, stochastic interest rates, and/or discrete dividends. Furthermore, the formula for the American exchange option can be used to value certain financial options, such as those in exchange offers or embedded in convertible or commodity-linked bonds.

Footnotes

1. Merton (1973) also shows that the riskless rate r need not appear in the Black-Scholes formula if the present value of the exercise price is replaced by the price of a zero-coupon bond paying the strike price at expiration.

2. A function $f(x_1, x_2)$ is *linearly homgeneous* if $f(\lambda x_1, \lambda x_2) = \lambda f(x_1, x_2)$ for any $\lambda > 0$.

3. See the Appendix of Geske and Johnson (1984) for a derivation of this formula. The formulae for E_1, E_2, and E_3 are given by earlier equations (2), (9), and (11) respectively.

4. To express the formula in the Geske-Johnson (1984) notation, make the following substitutions in (14): $X = V$, $S = D$, $\frac{S_t^*}{S} = \frac{V}{DP_t^*}$, and use the identities given by equation (5) in Section I.

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