*Mathematical Finance,* Vol. 20, No. 4 (October 2010), 527–569

# TIME-CHANGED MARKOV PROCESSES IN UNIFIED CREDIT-EQUITY MODELING

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This paper develops a novel class of hybrid credit-equity models with state-dependent jumps, local-stochastic volatility, and default intensity based on time changes of Markov processes with killing. We model the defaultable stock price process as a time-changed Markov diffusion process with state-dependent local volatility and killing rate (default intensity). When the time change is a Lévy subordinator, the stock price process exhibits jumps with state-dependent Lévy measure. When the time change is a time integral of an activity rate process, the stock price process has local-stochastic volatility and default intensity. When the time change process is a Lévy subordinator in turn time changed with a time integral of an activity rate process, the stock price process has state-dependent jumps, local-stochastic volatility, and default intensity. We develop two analytical approaches to the pricing of credit and equity derivatives in this class of models. The two approaches are based on the Laplace transform inversion and the spectral expansion approach, respectively. If the resolvent (the Laplace transform of the transition semigroup) of the Markov process and the Laplace transform of the time change are both available in closed form, the expectation operator of the time-changed process is expressed in closed form as a single integral in the complex plane. If the payoff is square integrable, the complex integral is further reduced to a spectral expansion. To illustrate our general framework, we time change the jump-to-default extended constant elasticity of variance model of Carr and Linetsky (2006) and obtain a rich class of analytically tractable models with jumps, local-stochastic volatility, and default intensity. These models can be used to jointly price equity and credit derivatives.

KEY WORDS: default, credit spread, corporate bonds, equity derivatives, credit derivatives, implied volatility skew, CEV model, JDCEV model, Lévy Subordinators, time change, jump-diffusion process, state dependent Lévy measures, credit-equity model.

## 1. INTRODUCTION

Until recently, equity derivatives pricing models and credit derivatives pricing models have developed more or less independently of each other. Equity derivatives models

This research was supported in part by the grants from the Federal Deposit Insurance Corporation, Moody's Investor Services, and the National Science Foundation under grant DMS-0802720.
*Manuscript received April 2008; final revision received December 2008.*
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largely concentrated on modeling the implied volatility smile by introducing jumps and stochastic volatility into the stock price process (see Gatheral 2006 for a survey), and ignored the possibility of default of the firm underlying the option contract. At the same time, credit models focused on modeling the default event and ignored the information available in the equity derivatives market (see Bielecki and Rutkowski 2004, Duffie and Singleton 2003, and Lando 2004 for surveys of credit risk models). Recently, market practitioners have realized that equity derivatives markets and credit markets are closely related, and a variety of cross-market trading and hedging strategies have emerged in the industry under such names as equity-to-credit and credit-to-equity. Indeed, a deep-out-of-the-money put on a firm's stock that has little chance to be exercised unless the firm goes bankrupt and its stock price drops to zero or near zero is effectively a credit derivative that pays the strike price in the event of bankruptcy. Over the past several years, every time the credit markets become seriously concerned about the possibility of bankruptcy of a firm, the open interest, daily volume of trading, and the implied volatility of deep-out-of-the-money puts on the firm's stock explode many times over their historical average. In late 2005 and early 2006, the credit markets were concerned about the possibility of a General Motors (GM) bankruptcy. While the GM stock traded between \$18 and \$22 in the December 2005–January 2006 period, January 2007 puts with strikes of \$10, \$7.50, \$5, and even \$2.50 all had very substantial open interest, large daily trading volumes, and implied volatilities of between 100% and 140%. In August and September of 2007, a similar story took place with deep-out-of-the-money puts on Countrywide Financial based on Countrywide's bankruptcy concerns due to its substantial exposure to subprime mortgages.

In this paper, we propose a flexible analytically tractable modeling framework which unifies the valuation of all credit derivatives and equity derivatives related to a given firm. We model the firm's stock price as the fundamental state variable that is assumed to follow a *time-changed Markov process with killing.* Our model architecture is to start with an analytically tractable Markov process with killing (e.g., a one-dimensional diffusion with killing) and subject it to a stochastic time change (clock) with an analytically tractable Laplace transform. If the resolvent (the Laplace transform of the transition semigroup) of the Markov process and the Laplace transform of the time change are both known in closed form, then the expectation operator of the time-changed process, and hence the corresponding pricing operator, can be recovered via the Laplace transform inversion. Moreover, if the spectral representation of the transition semigroup is known in closed form, then the Laplace inversion for the time-changed process can also be accomplished in closed form, leading to analytical pricing of credit and equity derivatives.

Many properties of the clock are inherited by the time-changed process, allowing us to produce desired behavior in the stock price process modeled as a time-changed Markov process. To introduce jumps, we add a jump component into the clock. To introduce stochastic volatility, we add an absolutely continuous component into the clock. By composing the two types of time changes, we construct models that exhibit both state-dependent jumps and stochastic volatility. The time-changed process also inherits many properties of the original process. If the original process is a Markov process with killing, then the time-changed process also has killing with the state-dependent killing rate, leading to models with the default intensity dependent on the stock price. *Thus, our modeling framework incorporates diffusive dynamics, state-dependent jumps, stochastic volatility, and state-dependent default intensity in an analytically tractable way.*

Our modeling framework can parsimoniously capture many fundamental empirical observations in equity and credit markets, including the well-known positive relationship between credit default swap (CDS) spreads and corporate bond yields and implied volatilities of equity options, the leverage effect (the negative relationship between the realized volatility of a stock and its price level), the volatility skew/smile effects, and jumps in the stock price process. As such, the class of models we propose is very general, nesting many of the models already in the credit and equity derivatives literatures as special cases corresponding to a particular choice of the Markov process and the time change.

The class of models developed in this paper can be thought of as a far-reaching generalization of the hybrid credit-equity models that describe the stock price dynamics as a one-dimensional diffusion with the local volatility and default intensity specified to be some functions of the stock price. In this class of models, in the event of default the stock price is assumed to drop to zero. Along these lines, Linetsky (2006) recently solved in closed form an extension of the Black–Scholes–Merton (BSM) model with bankruptcy, where the hazard rate of bankruptcy (default intensity) is a negative power of the stock price. The limitation of this model is that, while the default intensity is a function of the stock price, the local volatility of the diffusive stock price dynamics is constant, as in the original BSM model. To relax this restriction, Carr and Linetsky (2006) proposed and solved in closed form a *jump-to-default extended constant elasticity of variance* model (JDCEV). This model introduces stock-dependent default intensity into Cox's CEV model. This model features state-dependent local volatility and default intensity. Moreover, the default intensity is specified to be a linear function of the local variance. This specification provides a direct link between the stock price volatility and default intensity. However, the JDCEV is still a one-dimensional diffusion model, with all the attendant limitations. In particular, the stock price volatility does not have an independent stochastic component, and there are no jumps in the stock price process. By appropriately time changing one-dimensional diffusions with killing, such as the Brownian motion with killing in Linetsky (2006) and the JDCEV diffusion in Carr and Linetsky (2006), we obtain models with jumps, stochastic volatility, and default.

The class of models developed in this paper can also be thought of as a far-reaching generalization of the framework of *time-changed Lévy processes with stochastic volatility* of Carr et al. (2003). Clark (1973) introduced into finance the notion of stochastic time changes, in which the observed price process is assumed to arise by running a time-homogeneous process on a second process called a clock. A clock is an increasing process which is normalized to start at zero and which can have a stochastic component. The requirement that time increases precludes the modeling of the clock as a diffusion, although it is frequently modeled as a time integral of a positive diffusion. Alternatively, the clock is often modeled as a *Lévy subordinator*, a Lévy process with positive jumps and nonnegative drift. Time changing (subordinating) with Lévy subordinators goes back to the pioneering work of Bochner (1949, 1955) and is often called Bochner's subordination. It is well known that, if we subordinate a Lévy process, we obtain another Lévy process (see Sato 1999). In fact, many Lévy processes popular in finance can be represented as subordinate Brownian motions with drift with appropriately chosen subordinators (see Geman, Madan, and Yor 2001 for a survey). The variance gamma (VG) model of Madan and Milne (1991), Madan and Seneta (1990), and Madan, Carr, and Chang (1998); the normal inverse Gaussian (NIG) model of Barndorff-Nielsen (1998); and the Carr et al. (2002) model (CGMY) can all be represented as subordinate Brownian motions (for the latter, see Madan and Yor 2008). On the other hand, if one time changes Brownian motion with a time change that is a time integral of a CIR diffusion, one obtains Heston's (1993) stochastic volatility model. Building on this idea, Carr et al. (2003) time change general Lévy processes with time changes that are time integrals of other positive processes (e.g., CIR processes) and introduce a class of models termed *Lévy processes with stochastic volatility*. If the time change is an integral of another process, called the *activity rate* process, then the Lévy measure of the time-changed process scales with the activity rate process. Thus, the activity rate speeds up or slows down jumps in the time-changed process, in addition to speeding up or slowing down diffusive dynamics when time changing a Brownian motion (see also Barndorff-Nielsen, Nicolato, and Shephard 2002 for related work on time changes and stochastic volatility).

However, there are two significant limitations in the framework of Carr et al. (2003). First, the process to be time changed is a space-homogeneous Lévy process with state-independent Lévy measure and constant volatility. Through the time change, both the volatility and the Lévy measure scale with the activity rate process, but there is no explicit dependence of the volatility and the Lévy measure on the stock price. This space homogeneity contradicts the accumulated empirical evidence. In the context of pure diffusion models, the so-called *local-stochastic volatility* models take the volatility process to be a product of a function of the stock price (such as the power function in the CEV model) and the stochastic volatility component (see Hagan et al. 2002; Lipton 2002; Lipton and McGhee 2002). These models generalize stochastic volatility models such as Heston's to introduce explicit stock price dependence into the local volatility. In the context of jump models, we would like the Lévy measure to include both some explicit state dependence on the stock price as well as on the stochastic volatility. This is not addressed in the framework of Carr et al. (2003). The second limitation of Carr et al. (2003) is that they do not include default in their models. The original process is a Lévy process with infinite lifetime. As a result, the time-changed Lévy process with stochastic volatility also has infinite lifetime. Thus, these are pure equity derivatives models that do not capture the possibility of default of the firm.

Several interesting recent papers also exploit time changes in derivatives pricing. Albanese and Kuznetsov (2004) apply time changes to construct equity derivatives pricing models with stochastic volatility and jumps, Boyarchenko and Levendorskiy (2007) apply time changes to construct interest rate models with jumps, and Ding, Gieseckey, and Tomecekz (2009) apply time changes to birth processes to generate multiple defaults processes for multiname credit derivatives. However, in contrast to the focus of this paper, neither of these references model equity derivatives and credit derivatives in a unified fashion.

This paper develops the next generation of *hybrid credit-equity models with state-dependent jumps, local-stochastic volatility, and default intensity* based on time changes of Markov processes with killing. The class of models proposed here remedy a number of limitations of the previous generations of models. By starting from a one-dimensional diffusion with killing and time changing it with a composite time change that can be represented as a subordinator in turn time changed with a time integral of another process (a *subordinator with stochastic volatility*), we construct processes with state-dependent jumps, local-stochastic volatility, and state-dependent default intensity. Moreover, due to special properties of one-dimensional diffusions, we retain analytical tractability in this general framework. This is in contrast with the previous generations of analytically tractable jump diffusion and pure jump models based on Lévy processes with space homogeneous jumps (Merton 1976; Kou 2002; Kou and Wang 2004; Barndorff-Nielsen 1998; Eberlein, Keller, and Prause 1998; Madan et al. 1998; Carr et al. 2002). The state dependence of the Lévy measure in our approach is inherited from the state dependence of the local volatility of the original diffusion subject to time change. At the same time, many existing models, including local volatility models (e.g., CEV), stochastic volatility models (e.g., Heston), local-stochastic volatility models (e.g., SABR), Lévy processes with stochastic volatility, and diffusion models with state-dependent default intensity are all nested as special cases in our general framework. Advantages of our hybrid credit-equity modeling framework include the ability to consistently price the entire book of credit as well as equity derivatives, in addition to the ability to incorporate a rich assortment of empirically relevant features.

The rest of this paper is organized as follows. In Section 2, we present our model architecture. We define the defaultable stock price process as a time-changed Markov diffusion process with killing. In Section 3, we describe the four major classes of time changes studied in this paper: subordinators, absolutely continuous time changes (time integrals of an activity rate process), sums of subordinators and absolutely continuous time changes, and composite time changes (subordinators with stochastic volatility). In Section 4, we prove a series of key theorems on the martingale and Markov properties of our time-changed stock price processes. In Section 5, we apply our defaultable stock model to set up the general framework for the unified valuation of credit derivatives and equity derivatives. In Section 6, we present our Laplace transform approach to the valuation of contingent claims on time-changed Markov processes with the known resolvent (Laplace transform of the transition semigroup) and the known Laplace transform of the time change. In Section 7, we present our spectral expansion approach that works in the special case of symmetric Markov processes and contingent claims with square-integrable payoffs. In this case, the Laplace transform inversion is accomplished in closed form and results in a spectral expansion for the contingent claim value function. To illustrate our general approach, in Section 8, we present a detailed study of time changing the JDCEV process of Carr and Linetsky (2006). Section 8.1 presents explicit expressions for the resolvent kernel, the spectral expansion of the transition probability density, the survival probability for the JDCEV process, and the spectral expansion for put options under the JDCEV process (call options are obtained via the call-put parity). In Section 8.2, we introduce jumps and stochastic volatility into the JDCEV process and construct and numerically illustrate the time-changed JDCEV by calculating default probabilities, term structures of credit spreads, and implied volatility skews in a JDCEV time changed with an Inverse Gaussian subordinator in turn time changed with a time integral of a CIR process (subordinator with stochastic volatility). The resulting stock price process is a pure jump process with state-dependent Lévy measure, stochastic volatility, and default intensity dependent both on the stock price and on the stochastic volatility. The computations are done by applying our analytical methods based on the Laplace transform and on the spectral expansion. Section 9 summarizes our results and discusses avenues for further research and applications. The Appendix contains the proofs. The paper also has an online companion appendix available from the authors upon request.

## 2. MODEL ARCHITECTURE

We assume frictionless markets and no arbitrage and take an equivalent martingale measure (EMM) $\mathbb{Q}$ chosen by the market on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbb{Q})$ as given. All stochastic processes defined in the following live on this probability space, and all expectations are with respect to $\mathbb{Q}$ unless stated otherwise.

We model the stock price dynamics under the EMM as a stochastic process $\{S_t, t \geq 0\}$ defined by

$$S_t = \mathbf{1}_{\{t<\tau_d\}} e^{\rho t} X_{T_t} \equiv \begin{cases} e^{\rho t} X_{T_t}, & t < \tau_d, \\ 0, & t \geq \tau_d. \end{cases} \tag{2.1}$$

We now describe the ingredients in our model.

(i) *Background Markov process* $X$. $\{X_t, t \geq 0\}$ is a time-homogeneous Markov diffusion process starting from a positive value $X_0 = x > 0$ and solving a stochastic differential equation (SDE)

$$dX_t = [\mu + h(X_t)]X_t\,dt + \sigma(X_t)X_t\,dB_t, \tag{2.2}$$

where $\sigma(x)$ and $\mu + h(x)$ are the state-dependent instantaneous volatility and drift rate, $\mu \in \mathbb{R}$ is a constant parameter, and $\{B_t, t \geq 0\}$ is a standard Brownian motion. We assume that $\sigma(x)$ and $h(x)$ are Lipschitz continuous on $[\epsilon, \infty)$ for each $\epsilon > 0$, $\sigma(x) > 0$ on $(0, \infty)$, $h(x) \geq 0$ on $(0, \infty)$, and $\sigma(x)$ and $h(x)$ remain bounded as $x \to \infty$. We do not assume that $\sigma(x)$ and $h(x)$ remain bounded as $x \to 0$. Under these assumptions, the process $X$ does not explode to infinity (infinity is a *natural boundary* for the diffusion process; see Borodin and Salminen 2002, p. 14, for boundary classification of diffusion processes), but, in general, may reach zero, depending on the behavior of $\sigma(x)$ and $h(x)$ as $x \to 0$. The SDE (2.2) has a unique solution up to the first hitting time of zero, $H_0 = \inf\{t \geq 0 : X_t = 0\}$. If the process can reach zero, we kill it at $H_0$ and send it to an isolated state $\Delta$ called the *cemetery state* in the terminology of Markov processes (see Borodin and Salminen 2002, p. 4), where it remains for all $t \geq H_0$ (zero is a *killing boundary*). If the process cannot reach zero (zero is an inaccessible boundary), we set $H_0 = \infty$ by convention. We call the process $X$ the *background Markov process*. We could have included jumps in the process $X$, thus starting from a jump-diffusion process, rather than a pure diffusion as is done here. Instead, we start from a diffusion process and introduce jumps through time changing the diffusion with a Lévy subordinator. By introducing jumps via time changes we gain some important analytical tractability as will be seen later. After the jump-inducing time change, we have a Markov jump-diffusion process, which we can again time change to introduce stochastic volatility.

(ii) *Time change process* $T$. The process $\{T_t, t \geq 0\}$ is a random time change (called a *directing process*) assumed independent of $X$. It is a right-continuous with left limits (RCLL) increasing process starting at zero, $T_0 = 0$. We also assume that $\mathbb{E}[T_t] < \infty$ for every $t > 0$. In this paper, we focus on two important classes of time changes: Lévy subordinators (Lévy processes with positive jumps and non-negative drift) that are employed to introduce jumps and absolutely continuous time changes $T_t = \int_0^t V_u\,du$ with a positive rate process $\{V_t, t \geq 0\}$ called *activity rate* that are employed to introduce stochastic volatility. We also consider time changes $T_t = T_t^1 + T_t^2$ with both jump and absolutely continuous components, as well as composite time changes of the form $T_t = T^1_{T_t^2}$, where $T_t^1$ is a Lévy subordinator and $T_t^2$ is an absolutely continuous time change with some activity rate process $V$. This can be thought of as first time changing the diffusion process $X$ with the Lévy subordinator $T^1$ to introduce jumps, and then time changing the resulting Markov jump-diffusion process with the absolutely continuous time change $T^2$

to introduce stochastic volatility. Alternatively, the process $T$ can be understood as a subordinator with stochastic volatility along the lines of time-changed Lévy processes of Carr et al. (2003). We describe these classes of time changes in detail in Section 3.

(iii) *Default time* $\tau_d$. The stopping time $\tau_d$ models the time of default of the firm on its debt. We assume that in default strict priority rules are followed, so that while debt holders receive some recovery, the stock becomes worthless (stock price is equal to zero in default). The time of default $\tau_d$ is constructed as follows. Let $H_0$ be the first time the diffusion process $X$ reaches zero as defined previously. Let $\mathcal{E}$ be an exponential random variable with unit mean, $\mathcal{E} \sim \text{Exp}(1)$, and independent of $X$ and $T$. Define

$$\zeta := \inf\left\{t \in [0, H_0] : \int_0^t h(X_u)\, du \geq \mathcal{E}\right\}, \tag{2.3}$$

where $h(x)$ is the function appearing in the drift of $X$ (in equation [2.3] we assume that $\inf\{\emptyset\} = H_0$ by convention). It can be interpreted as the first jump time of a doubly stochastic Poisson process with the state-dependent intensity (hazard rate) $h(X_t)$ if it jumps before time $H_0$, or $H_0$ if there is no jump in $[0, H_0]$. At time $\zeta$, we kill the process $X$ and send it to the cemetery state $\Delta$, where it remains for all $t \geq \zeta$. We note that, in general, the process $X$ may be killed either at time $H_0$ via diffusion to zero if $\zeta = H_0$ or at the first jump time $\zeta$ of the doubly stochastic Poisson process with intensity $h$ if $\zeta < H_0$ (according to our definition, $\zeta \leq H_0$). In the latter case, the process is killed from a positive value $X_{\zeta-} > 0$. The process $X$ is thus a Markov process with killing with *lifetime* $\zeta$.[1]

The drift in (2.2) includes the hazard rate $h$ to make the process $1_{\{t<\zeta\}} X_t$ with $\mu = 0$ into a martingale. The inclusion of the hazard rate in the drift compensates for the possibility of killing the process from a positive state, that is, a jump of the process $X_t$ from a positive value $X_{\zeta-} > 0$ to the cemetery state $\Delta$ and, correspondingly, a jump of the process $1_{\{t<\zeta\}} X_t$ from a positive value $X_{\zeta-} > 0$ to zero. This compensation of the jump to zero makes the process $1_{\{t<\zeta\}} X_t$ with $\mu = 0$ into a martingale (our assumptions on $\sigma(x)$ and $h(x)$ ensure that this process is a true martingale and not just a local martingale).

After applying the time change $T$ to the process $X$ with lifetime $\zeta$, the lifetime of the time-changed process $X_{T_t}$ is

$$\tau_d := \inf\{t \geq 0 : T_t \geq \zeta\}. \tag{2.4}$$

Although the process $X_t$ is in the cemetery state for all $t \geq \zeta$, the time-changed process $X_{T_t}$ is in the cemetery state for all times $t$ such that $T_t \geq \zeta$ or, equivalently, $t \geq \tau_d$ with $\tau_d$ defined by equation (2.4). That is, $\tau_d$ defined by equation (2.4) is the first time the time-changed process $X_{T_t}$ is in the cemetery state. We take $\tau_d$ to be the time of default. Because we assume that the stock becomes worthless in default, we set $S_t = 0$ for all $t \geq \tau_d$, so that $S_t = 1_{\{t<\tau_d\}} e^{\rho t} X_{T_t}$.

[1]The process killed at $\zeta \leq H_0$ is a subprocess of the process killed at $H_0$. We could have used different notation for the process killed at $\zeta$ to distinguish it from the process killed at $H_0$. To simplify notation, we denote both processes by $X$. It should not lead to any confusion as it should be clear from the context whether we are working with the process killed at $H_0$ or its subprocess killed at $\zeta \leq H_0$.

(iv) *Scaling factor* $e^{\rho t}$. To gain some additional modeling flexibility, we also include a scaling factor $e^{\rho t}$ with some constant $\rho \in \mathbb{R}$ in our definition of the stock price process (2.1).

(v) *The Martingale condition.* For the model (2.1) to be well defined, the functions $\sigma(x)$, $h(x)$, the time-change process $T$, and the constant parameters $\mu$ and $\rho$ must be such that the discounted stock price process with the dividends reinvested is a nonnegative martingale under the EMM $\mathbb{Q}$, that is,

$$\mathbb{E}[S_t] < \infty \quad \text{for every} \quad t \tag{2.5}$$

and

$$\mathbb{E}[S_{t_2} | \mathcal{F}_{t_1}] = e^{(r-q)(t_2-t_1)} S_{t_1} \quad \text{for every} \quad t_1 < t_2, \tag{2.6}$$

where $r \geq 0$ is the risk-free interest rate and $q \geq 0$ is the dividend yield (in this paper, we assume $r$ and $q$ are constant). The martingale condition (2.5–6) imposes important restrictions on the model parameters. In Section 3, we describe the classes of time changes we work with and, in Section 4, prove key theorems that give the necessary and sufficient conditions for the martingale condition (2.5–6) to hold.

## 3. TIME-CHANGE PROCESSES

### 3.1. Lévy Subordinators

Let $\{T_t, t \geq 0\}$ be a Lévy *subordinator*, that is, a nondecreasing Lévy process with positive jumps and nonnegative drift with the Laplace transform

$$\mathbb{E}[e^{-\lambda T_t}] = e^{-t\phi(\lambda)} \tag{3.1}$$

with the Laplace exponent given by the Lévy–Khintchine formula

$$\phi(\lambda) = \gamma\lambda + \int_{(0,\infty)} (1 - e^{-\lambda s})\nu(ds) \tag{3.2}$$

with the Lévy measure $\nu(ds)$ satisfying $\int_{(0,\infty)}(s \wedge 1)\nu(ds) < \infty$, with nonnegative drift $\gamma \geq 0$, and the transition probability $\mathbb{Q}(T_t \in ds) = \pi_t(ds)$, $\int_{[0,\infty)} e^{-\lambda s}\pi_t(ds) = e^{-t\phi(\lambda)}$. The standard references on subordinators include Bertoin (1996, 1999) and Sato (1999) (see also Geman et al. 2001 for finance applications). A subordinator starts at zero, $T_0$, drifts at the constant nonnegative drift rate $\gamma$, and experiences positive jumps controlled by the Lévy measure $\nu(ds)$ (we exclude the trivial case of constant time changes with $\nu = 0$ and $\gamma > 0$). The Lévy measure $\nu$ describes the arrival rates of jumps so that jumps of sizes in some Borel set $A$ bounded away from zero occur according to a Poisson process with intensity $\nu(A) = \int_A \nu(ds)$. If $\int_{\mathbb{R}^+} \nu(ds) < \infty$, the subordinator is of compound Poisson type with the Poisson arrival rate $\alpha = \int_{\mathbb{R}^+} \nu(ds)$ and the jump size probability distribution $\alpha^{-1}\nu$. If the integral $\int_{\mathbb{R}^+} \nu(ds)$ is infinite, the subordinator is of infinite activity. Subordinators are processes of finite variation and, hence, the truncation of small jumps is not necessary in the Lévy–Khintchine formula (3.2).

Consider an exponential moment $\mathbb{E}[e^{\mu T_t}]$ of a subordinator $T$ with Lévy measure $\nu$. When $\mu < 0$, it is always finite and is given by the Lévy–Khintchine formula with $\lambda = -\mu$. We will also need to consider the case $\mu \geq 0$. Generally, we are interested in the set $\mathcal{I}_\nu$ of all $\mu \in \mathbb{R}$ such that $\mathbb{E}[e^{\mu T_t}] < \infty$. As a corollary of Theorem 25.17 of Sato (1999), $\mathbb{E}[e^{\mu T_t}] < \infty$ for all $t \geq 0$ if and only if

$$\int_{[1,\infty)} e^{\mu s} \nu(ds) < \infty. \tag{3.3}$$

For a given subordinator with Lévy measure $\nu$, the set $\mathcal{I}_\nu$ of all $\mu$ such that (3.3) holds is an interval $(-\infty, \bar{\mu})$ or $(-\infty, \bar{\mu}]$. The right endpoint $\bar{\mu} \geq 0$ may be finite or infinite and, if it is finite, may or may not belong to the set $\mathcal{I}_\nu$. It is also possible that $\bar{\mu} = 0$. For all $\mu \in \mathcal{I}_\nu$ we have

$$\mathbb{E}[e^{\mu T_t}] = e^{-t\phi(-\mu)}. \tag{3.4}$$

Further information on subordinators can be found in Applebaum (2004), Bertoin (1996, 1999), and Sato (1999). For applications in finance, see Geman et al. (2001), Boyarchenko and Levendorskiy (2002), Cont and Tankov (2004), and Schoutens (2003). Some examples of subordinators are listed in Appendix C in the companion appendix available from the authors upon request.

### 3.2. Absolutely Continuous Time-Change Processes

Let $\{Z_t, t \geq 0\}$ be a conservative $n$-dimensional Markov process independent of $X$ ($Z$ can have a diffusion component and a jump component, but no killing, so that $Z$ has infinite lifetime). Consider an integral process $T_t = \int_0^t V(Z_u)\,du$, where $V(z)$ is some positive function from the state space $D \subset \mathbb{R}^n$ of the process $Z$ into $(0, \infty)$ so that the *activity rate* process $\{V_t := V(Z_t), t \geq 0\}$ is positive (we exclude the trivial case of constant time changes with constant $V > 0$). The process $T_t$ is strictly increasing and starts at the origin. We are interested in such Markov processes $Z$ and such functions $V(z)$ that the Laplace transform

$$\mathcal{L}_z(t, \lambda) = \mathbb{E}_z[e^{-\lambda \int_0^t V(Z_u)\,du}] \tag{3.5}$$

is known in closed form (the subscript $z$ signifies that the Laplace transform $\mathcal{L}_z(t, \lambda)$ explicitly depends on the initial state $Z_0 = z$ of the Markov process $Z$).

A key example is given by the CIR activity rate process ($V(z) = z$ so that $V_t = Z_t$)

$$dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dW_t,$$

where the standard Brownian motion $W$ is independent of the Brownian motion $B$ driving the SDE (2.2), the activity rate process starts at some positive value $V_0 = v > 0$, $\kappa > 0$ is the rate of mean reversion, $\theta > 0$ is the long-run activity rate level, $\sigma_V > 0$ is the activity rate volatility, and it is assumed that the Feller condition is satisfied $2\kappa\theta \geq \sigma_V^2$ to ensure that the process never hits zero (zero is an inaccessible boundary for the CIR process when the Feller condition is satisfied). Due to the Cox, Ingersoll, and Ross (1985) result giving the closed-form solution for the zero-coupon bond in the CIR interest rate model (note that the Laplace transform (3.5) can be interpreted as the price of a unit face value zero-coupon bond with maturity at time $t$ when the short rate process is $r_t = \lambda V_t$), we have

$$\mathcal{L}_v(t, \lambda) = A(t, \lambda) e^{-B(t,\lambda)v},$$

where $V_0 = v$ is the initial value of the activity rate process and (here $\varpi = \sqrt{2\sigma_V^2\lambda + \kappa^2}$)

$$A(t,\lambda) = \left(\frac{2\varpi e^{(\varpi+\kappa)t/2}}{(\varpi+\kappa)(e^{\varpi t}-1)+2\varpi}\right)^{\frac{2\kappa\theta}{\sigma_V^2}}, \quad B(t,\lambda) = \frac{2\lambda(e^{\varpi t}-1)}{(\varpi+\kappa)(e^{\varpi t}-1)+2\varpi}.$$

Heston's stochastic volatility model is based on Brownian motion time changed with the integral of the CIR process. The CIR activity rate process has been used more generally in Carr et al. (2003) to time change Lévy processes to introduce stochastic volatility in the popular Lévy models, such as VG, NIG, CGMY, etc.

More generally, there are several known classes of Markov processes that yield closed-form expressions for the Laplace transform (3.5). The first class are affine jump-diffusion processes with the affine function $V(z)$ (Duffie, Pan, and Singleton 2000; Duffie, Filipovic, and Schachermayer 2003). In this class, the Laplace transform of the time change is the exponential of an affine function of the initial state $Z_0 = z$ of the Markov process $Z$ driving the activity rate process. The CIR example is a particular representative of the affine class. The second class are the so-called quadratic models (Leippold and Wu 2002), where the function $V(z)$ is quadratic in the state vector, and the state vector follows an $n$-dimensional Gaussian Markov process (an $n$-dimensional Ornstein–Uhlenbeck process). In this case, the Laplace transform of the time change is the exponential of a quadratic function of the initial state $Z_0 = z$ of the Markov process. The third class are Ornstein–Uhlenbeck processes driven by Lévy processes used by Carr et al. (2003) to time change Lévy processes. Explicit expressions for the Laplace transforms of these time changes can be found in this reference. Carr and Wu (2004) use all three of these classes of absolutely continuous time changes to time change Lévy processes (a listing of closed form expressions for Laplace transforms of these time changes can be found in tables 1 and 2 in this reference). Here we use them to time change Markov processes. We note that, while the Laplace transforms are known in closed form for these three classes of absolutely continuous time changes, in general the transition probability distributions $\mathbb{Q}_z(T_t \in ds) = \pi_t(z, ds)$ can only be obtained numerically by Laplace transform inversion (note that they explicitly depend on the initial state $Z_0 = z$ of the Markov process $Z$ driving the activity rate $V$).

REMARK 3.1. Time changing Brownian motion with the integral of the CIR process leads to the zero-correlation Heston's model. Carr and Wu's (2004) complex-valued measure change approach extends it to Heston's model with nonzero correlation. More generally, Carr and Wu's approach is applicable to time changing general Lévy processes. However, so far we have not been able to extend their approach to general Markov processes. This is an interesting problem for future research.

### 3.3. Combining and Composing Time Changes

We can also combine the two types of time changes $T_t = T_t^1 + T_t^2$, where $T_t^1$ is a subordinator with Laplace exponent $\phi$ and $T_t^2$ is an integral of some positive function of a Markov process with analytically tractable Laplace transform $\mathcal{L}_z(t,\lambda)$. The combined time change has a jump component, as well as an absolutely continuous component.

The Laplace transform of the combined time change is simply a product of the Laplace transforms for its components

$$\mathbb{E}[e^{-\lambda T_t}] = e^{-t\phi(\lambda)}\mathcal{L}_z(t,\lambda).$$

Alternatively, we can compose the two types of time changes and consider a composite time-change process

$$T_t = T^1_{T^2_t}, \tag{3.6}$$

where $T^1_t$ is a subordinator with Laplace exponent $\phi$ and $T^2_t$ is an integral of some positive function of a Markov process with analytically tractable Laplace transform $\mathcal{L}_z(t,\lambda)$. That is, the process $T$ is obtained by time changing a Lévy subordinator $T^1$ with an absolutely continuous time change $T^2$. The process $T$ is in the class of Lévy processes time changed with an integral of an activity rate process studied by Carr et al. (2003). By conditioning on $T^2$, the Laplace transform of the composite time change is

$$\mathbb{E}[e^{-\lambda T_t}] = \mathbb{E}\big[\mathbb{E}\big[\exp\big(-\lambda T^1_{T^2_t}\big)\,\big|\,T^2_t\,\big]\big] = \mathbb{E}[e^{-T^2_t\phi(\lambda)}] = \mathcal{L}_z(t,\phi(\lambda)). \tag{3.7}$$

We note that, after we have done the absolutely continuous time change $T^2_t$, further time changes will no longer have analytically tractable Laplace transforms, because, in contrast to subordinators with the Laplace transform $e^{-t\phi(\lambda)}$ that depends on time exponentially, the Laplace transform $\mathcal{L}_z(t,\lambda)$ may have a complicated general dependence on time.

As we show in Section 4.3, diffusion processes time changed with a combined time change acquire stochastic volatility in the diffusion component, but do not have stochastic volatility in the jump component. In contrast, diffusions time changed with a composed time change acquire stochastic volatility both in the diffusion and in the jump component.

## 4. MARTINGALE AND MARKOV PROPERTIES OF THE DEFAULTABLE STOCK MODEL

We now prove key theorems that establish when our stock price model (2.1) satisfies the martingale condition (2.5)–(2.6) and when it is a Markov process.

### 4.1. Time Changing with Lévy Subordinators

THEOREM 4.1. *Let $X$ be a background diffusion process as described in Section 2(i) with $\mu \in \mathbb{R}$ and $h(x)$ and $\sigma(x)$ satisfying the assumptions listed there, let $T$ be a Lévy subordinator with drift $\gamma \geq 0$ and Lévy measure $\nu$ with the characteristic exponent $\phi(\lambda)$ and with the interval $\mathcal{I}_\nu$ as described in Section 3.1, and let $\tau_d$ be the default time as described in Section 2 (iii). Then the stock price process (2.1) satisfies the martingale condition (2.5)–(2.6) if and only if*

$$\mu \in \mathcal{I}_\nu \tag{4.1}$$

*and*

$$\rho = r - q + \phi(-\mu). \tag{4.2}$$

*Proof.* The proof is by conditioning on the time change $T$ that is independent of $X$ and using Equation (3.4) to compute the expectation and is given in Appendix A. □

Thus, when the time change $T$ is a Lévy subordinator, our model (2.1) is characterized by the local volatility function $\sigma(x)$, hazard rate $h(x)$, Lévy measure $\nu$ and drift $\gamma \geq 0$ of the Lévy subordinator, and a constant $\mu \in \mathcal{I}_\nu$. Depending on the Lévy measure, it may or may not be possible to select $\mu \in \mathcal{I}_\nu$ so that

$$\rho = r - q + \phi(-\mu) = 0. \tag{4.3}$$

From (3.2) we see that $-\phi(-\mu)$ is a strictly increasing function on $\mathcal{I}_\nu$. Thus, the equation (4.3) has at most one solution in $\mathcal{I}_\nu$. If it exists, we denote it $\mu_0$ and call the corresponding model (2.1) with $\mu = \mu_0$ and $\rho = 0$ the *zero-$\rho$ model*. If the equation (4.3) has no solution in $\mathcal{I}_\nu$, one possible choice is to set $\mu = 0$ so that $\rho = r - q$. We call this choice the *zero-$\mu$ model*. For this choice, the process $\mathbf{1}_{\{t<\zeta\}} X_t$ and the time-changed process $\mathbf{1}_{\{t<\tau_d\}} X_{T_t}$ are both martingales, and the desired mean for the stock price process $S_t = \mathbf{1}_{\{t<\tau_d\}} e^{(r-q)t} X_{T_t}$ is achieved by including the factor $e^{\rho t} = e^{(r-q)t}$. We now establish when equation (4.3) has a solution.

THEOREM 4.2. *Equation (4.3) has at most one solution in $\mathcal{I}_\nu$. If $r < q$, then equation (4.3) has no solution in $\mathcal{I}_\nu$ if and only if $\gamma = 0$ and the subordinator is of finite activity with finite Lévy measure with Poisson intensity $\alpha = \int_{(0,\infty)} \nu(ds)$ such that $-\alpha > r - q$. If $r > q$, then equation (4.3) has no solution in $\mathcal{I}_\nu$ if and only if $\bar{\mu}$ is included in $\mathcal{I}_\nu$ (i.e., $\int_{[1,\infty)} e^{\bar{\mu}s} \nu(ds) < \infty$) and $r - q > -\phi(-\bar{\mu})$. If $r = q$, equation (4.3) has a unique solution $\mu = 0$ in $\mathcal{I}_\nu$.*

*Proof.* The proof follows from the analysis of equation (3.2) and is given in Appendix A. □

We now turn to the question of whether the model (2.1) is Markovian. It turns out that when $T$ is a Lévy subordinator, the time changed process $X_{T_t}$ is again a Markov process.

THEOREM 4.3. *Let $X$ be a background diffusion process with lifetime $\zeta$ as described in Section 2(i) with assumptions listed there, and let $T$ be a Lévy subordinator with drift $\gamma \geq 0$ and Lévy measure $\nu(ds)$ as described in Section 3.1. Then the time changed process (the superscript $\phi$ refers to the subordinate quantities with the subordinator with the Laplace exponent $\phi$)*

$$X_t^\phi := X_{T_t} = \begin{cases} X_{T_t}, & T_t < \zeta \\ \Delta, & T_t \geq \zeta \end{cases} \equiv \begin{cases} X_{T_t}, & t < \tau_d \\ \Delta, & t \geq \tau_d \end{cases} \tag{4.4}$$

*is a Markov jump-diffusion process with lifetime $\tau_d$ and with the Lévy-type infinitesimal generator $\mathcal{G}^\phi$ that for any twice continuously differentiable function with compact support $f \in C_c^2((0,\infty))$ can be written in the form*

$$\begin{aligned} \mathcal{G}^\phi f(x) = {} & \frac{1}{2}\gamma\sigma^2(x)x^2\frac{d^2 f}{dx^2}(x) + b(x)\frac{df}{dx}(x) - k(x)f(x) \\ & + \int_{(0,\infty)} \left( f(y) - f(x) - \mathbf{1}_{\{|y-x|\leq 1\}}(y-x)\frac{df}{dx}(x) \right) \Pi(x,dy) \end{aligned} \tag{4.5}$$

*with the jump measure (state-dependent Lévy measure)*

$$\Pi(x, dy) = \pi(x, y)dy \tag{4.6}$$

*with the density defined for all* $x, y > 0$, $x \neq y$, *by*

$$\pi(x, y) = \int_{(0,\infty)} p(s; x, y)\nu(ds), \tag{4.7}$$

*killing rate*

$$k(x) = \gamma h(x) + \int_{(0,\infty)} P_s(x, \{\Delta\})\nu(ds), \tag{4.8}$$

*and drift with respect to the truncation function* $\mathbf{1}_{\{|y-x|\leq 1\}}$

$$b(x) = \gamma[\mu + h(x)]x + \int_{(0,\infty)} \left( \int_{\{y>0:|y-x|\leq 1\}} (y - x)p(s; x, y)\, dy \right) \nu(ds). \tag{4.9}$$

*Here* $p(t; x, y)$ *is the transition probability density of the background Markov process* $X$ *with lifetime* $\zeta$, *so that the probability to find the process in a Borel set* $A \subset (0, \infty)$ *at time* $t$ *if the process starts at* $X_0 = x$ *at time zero is* $P_t(x, A) = \int_A p(t; x, y)\, dy$, *and*

$$P_t(x, \{\Delta\}) = 1 - \int_{(0,\infty)} p(t; x, y)\, dy \tag{4.10}$$

*is the transition probability of the background process* $X$ *with lifetime* $\zeta$ *from the state* $x > 0$ *to the cemetery state* $\Delta$ *by time* $t$.

*The transition density* $p^\phi(t; x, y)$ *of the time-changed Markov process* $X^\phi$ *with lifetime* $\tau_d$ *is given by*

$$p^\phi(t; x, y) = \int_{[0,\infty)} p(s; x, y)\pi_t(ds), \tag{4.11}$$

*where* $p(s; x, y)$ *is the transition density of the background Markov process* $X$ *with lifetime* $\zeta$ *and* $\pi_t(ds)$ *is the transition measure of the subordinator* $T$. *The transition probability of the process* $X^\phi$ *with lifetime* $\tau_d$ *from the state* $x > 0$ *to the cemetery state* $\Delta$ *by time* $t$ *is given by*

$$P_t^\phi(x, \{\Delta\}) = 1 - \int_{(0,\infty)} p^\phi(t; x, y)\, dy = \int_{[0,\infty)} P_s(x, \{\Delta\})\pi_t(ds). \tag{4.12}$$

*Proof.* The proof relies on R.S. Phillips' theorem on subordination of Markov semigroups and is given in Appendix A. □

The theorem asserts that when the background process is Markov and the time change is a Lévy subordinator, the time-changed process is again Markov and gives explicitly its local characteristics (volatility, drift with respect to the truncation function, killing rate, and jump measure). Intuitively, for any $x > 0$ and a Borel set $A \subset (0, \infty)\backslash\{x\}$ bounded away from $x$, the Lévy measure $\Pi(x, A)$ gives the arrival rate of jumps from the state $x$ into the set $A$, that is, the transition probability from the state $x$ into the set $A$ bounded away from $x$ has the following asymptotics: $P_t(x, A) \sim \Pi(x, A)t$ as $t \to 0$. The truncation function in the integral in (4.5) is only needed when jumps are of infinite variation. When

$$\int_{\{y>0:|x-y|\le 1\}} |y-x|\Pi(x, dy) < \infty \tag{4.13}$$

for all $x > 0$, jumps of the time-changed process are of finite variation, the truncation is not needed, and the infinitesimal generator (4.5) of the time-changed Markov process simplifies to

$$\begin{aligned} \mathcal{G}^{\phi} f(x) = {} & \frac{1}{2}\gamma\sigma^2(x)x^2\frac{d^2 f}{dx^2}(x) + \gamma[\mu + h(x)]x\frac{df}{dx}(x) - k(x)f(x) \\ & + \int_{(0,\infty)} (f(y) - f(x))\,\Pi(x, dy). \end{aligned} \tag{4.14}$$

If $\Pi$ is a finite measure with $\lambda(x) := \Pi(x, (0, \infty)) < \infty$ for every $x > 0$, then the process has a finite number of jumps in any finite time interval, and $\lambda(x)$ is the (state-dependent) jump arrival rate. The subordinated process $X^{\phi}$ has finite activity jumps if and only if the subordinator $T$ has finite activity jumps. Note that, while subordinators are jump processes of finite variation, the subordinated processes $X^{\phi}$ may have jumps of either finite or infinite variation, depending on whether the Lévy measure (4.6)–(4.7) satisfies the integrability condition (4.13).

From equations (4.5)–(4.8), we see that time changing the process $X$ with a Lévy subordinator with drift $\gamma \ge 0$ and Lévy measure $\nu$ scales volatility and drift with $\gamma$, introduces jumps with state-dependent Lévy measure with Lévy density $\pi(x, y) = \int_{(0,\infty)} p(s; x, y)\nu(ds)$ determined by the Lévy measure of the subordinator and the transition density of the diffusion process $X$, and modifies the killing rate by scaling the original killing rate with $\gamma$ and adding the term $\int_{(0,\infty)} P_s(x, \{\Delta\})\nu(ds)$ determined by the Lévy measure of the subordinator and the killing probability of the Markov process $X$. If $\gamma > 0$, we can set $\gamma = 1$ without loss of generality. Then the effect of the time change is to introduce jumps into the original diffusion process, so that the resulting process is a jump diffusion with the same diffusion as the original process $X$ plus jumps induced by the time change, and to modify the killing rate. Thus, the subordination procedure allows us to introduce jumps into any diffusion process. If $\gamma = 0$, then the time-changed process has no diffusion component and is a pure jump process with killing.

Thus, we have a complete characterization of the time-changed process $X_t^{\phi}$ as a Markov process with killing. The stock price process (2.1) can be written as $S_t = \mathbf{1}_{\{t<\tau_d\}} e^{\rho t} X_t^{\phi}$. The stock price process stays positive prior to the default time $\tau_d$ (lifetime of $X_t^{\phi}$) and jumps into zero at $\tau_d$. We call this *jump to default*. It is thus a Markov jump-diffusion process with zero specified as an absorbing state.

### 4.2. Absolutely Continuous Time Changes

We now turn to absolutely continuous time changes.

THEOREM 4.4. *Let $X$ be a background diffusion process as described in Section 2(i) with $\mu \in \mathbb{R}$ and $h(x)$ and $\sigma(x)$ satisfying the assumptions listed there, let $T$ be an absolutely continuous time change with a positive activity rate process $V_t$ as described in Section 3.2,* *and let $\tau_d$ be the default time as described in Section 2(iii). Then the stock price process (2.1) satisfies the martingale condition (2.5)–(2.6) if and only if*

$$\mu = 0, \quad \rho = r - q. \tag{4.15}$$

*Proof.* The proof is given in Appendix A.

Because the time-change process $\{T_t, t \geq 0\}$ is continuous and strictly increasing (we assume the activity rate process $V$ is strictly positive), the *inverse process* $\{A_t, t \geq 0\}$ defined by $T_{A_t} = t$ is also continuous and strictly increasing and $A_{T_t} = t$. To understand the effect of the absolutely continuous time change on the process $X$, we write for $T_t < \zeta$ (equivalently $t < \tau_d$)

$$\begin{aligned} X_{T_t} &= x + \int_0^{T_t} h(X_u) X_u \, du + \int_0^{T_t} \sigma(X_u) X_u \, dB_u \\ &= x + \int_0^t h(X_{T_s}) X_{T_s} \, dT_s + \int_0^t \sigma(X_{T_s}) X_{T_s} \, dB_{T_s} \\ &= x + \int_0^t h(X_{T_s}) X_{T_s} V(Z_s) \, ds + \int_0^t \sigma(X_{T_s}) X_{T_s} \sqrt{V(Z_s)} \, d\tilde{B}_s. \end{aligned} \tag{4.16}$$

In the first equality, we did a change of variable in the integral, $u = T_s$ (with the inverse $s = A_u$). In the second equality, we observed that $dT_s = V_s ds$ and $dB_{T_s} = \sqrt{V_s} d\tilde{B}_s$, where $\tilde{B}_t = \int_0^t \frac{dB_{T_s}}{\sqrt{V_s}}$ is a standard Brownian motion (it is a continuous martingale with quadratic variation $t$ and, hence, is a standard Brownian motion by Lévy's theorem). The process $X_t$ is killed at time $\zeta = \inf\{t \in [0, H_0] : \int_0^t h(X_u) \, du \geq \mathcal{E}\}$. Then the time-changed process $X_{T_t}$ is killed at time

$$\tau_d = \inf\{t \in [0, A_{H_0}] : \int_0^{T_t} h(X_u) \, du \geq \mathcal{E}\} = \inf \left\{ t \in [0, A_{H_0}] : \int_0^t h(X_{T_s}) V(Z_s) \, ds \geq \mathcal{E} \right\}, \tag{4.17}$$

where we did a change of variable $u = T_s$ in the integral. From equations (4.16) and (4.17), we observe that the time-changed process $Y_t = X_{T_t}$ has the local volatility

$$\sigma(x, z) = \sqrt{V(z)} \sigma(x) \tag{4.18}$$

and killing rate

$$k(x, z) = V(z) h(x) \tag{4.19}$$

so that for $t < \tau_d$ the process $Y$ solves the SDE

$$dY_t = V(Z_t) h(Y_t) Y_t dt + \sqrt{V(Z_t)} \sigma(Y_t) Y_t d\tilde{B}_t. \tag{4.20}$$

Thus, the time change scales the volatility with the square root of the activity rate and scales the killing rate with the activity rate. The activity rate plays a role of stochastic volatility that both drives the instantaneous volatility of the time-changed process and the killing rate (default intensity). Thus, by construction, this class of models possesses a natural built-in connection between the stock price volatility and the firm's default intensity. This manifests itself in the connection between the implied volatility skew in the stock options market and the credit spreads in the credit markets. The linkages between credit spreads and equity volatility (both realized and implied in options prices) have been widely documented in the empirical literature (see the discussion and the references in the introduction of Carr and Linetsky 2006). Our class of models based on time changing a diffusion with killing with an integral of an activity rate (stochastic volatility) process is ideally suited to the task of modeling the linkages between equity volatility and credit spreads, as the activity rate drives both the local-stochastic volatility of the stock price and the default intensity. See Carr and Wu (2009) for the empirical support of the linkage between the volatility and default intensity in the framework of affine models.

We thus conclude that the time-changed process $Y$ is no longer a one-dimensional Markov process. However, the process $(Y, Z)$ is an $(n+1)$-dimensional Markov process with lifetime $\tau_d$ and with the infinitesimal generator $\mathcal{G}$ that for any twice continuously differentiable function with compact support $f \in C_c^2((0,\infty) \times D)$ (where $D \subset \mathbb{R}^n$ is the state space of the process $Z$) can be written in the form

$$\mathcal{G} f(x,z) = V(z)\mathcal{G}^X f(x,z) + \mathcal{G}^Z f(x,z), \tag{4.21}$$

where $\mathcal{G}^X$ is the infinitesimal generator of the background process $X$ with lifetime $\zeta$,

$$\mathcal{G}^X f(x) = \frac{1}{2}\sigma^2(x)x^2\frac{\partial^2 f}{\partial x^2}(x) + h(x)x\frac{\partial f}{\partial x}(x) - h(x)f(x) \tag{4.22}$$

and $\mathcal{G}^Z$ is the infinitesimal generator of the $n$-dimensional Markov process $Z$ driving the activity rate $V_t = V(Z_t)$.

The fact that, in general, the time-changed process is not Markovian is illustrated by the Heston model. If we start with Brownian motion and do a time change with the time-change process taken to be an integral of an independent CIR process, the resulting time-changed process is no longer a one-dimensional Markov process because of the second source of uncertainty (stochastic volatility) entering through the time change. The Markov property is restored in an enlarged two-dimensional state space with both the stock price and its instantaneous volatility as two state variables.

### 4.3. Combined and Composite Time Changes

We now turn to composite time changes where we first time change the diffusion process $X$ with a Lévy subordinator to introduce jumps, and then time change the resulting Markov jump-diffusion process with an absolutely continuous time change to introduce stochastic volatility as described in Section 3.3. Equivalently, we can think of it as a single time change, where the process $T_t$ is a time-changed Lévy process with stochastic volatility as in Carr et al. (2003).

THEOREM 4.5. *Let $X$ be a background diffusion process as described in Section 2(i) with $\mu \in \mathbb{R}$ and $h(x)$ and $\sigma(x)$ satisfying the assumptions listed there, let $T_t$ be a composite time change (3.6), where $T^1$ is a Lévy subordinator with drift $\gamma \geq 0$ and Lévy measure $\nu$ and $T^2$ is an absolutely continuous time change with a positive activity rate process $V_t = V(Z_t)$ as described in Sections 3.2 and 3.3, and let $\tau_d$ be the default time as described in Section 2(iii). Then the stock price process (2.1) satisfies the martingale condition (2.5)–(2.6) if and only if $\mu = 0$, $\rho = r - q$.*

*Proof.* The proof is given in Appendix A. □

Recalling Theorem 4.2 and arguing as in Section 4.2, we conclude that the process $(Y, Z)$, where $Y_t = X_{T_t} = X_{T^1_{T^2_t}}$, is an $(n+1)$-dimensional Markov jump-diffusion process with the infinitesimal generator $\mathcal{G}$ that for any twice continuously differentiable function with compact support $f \in C^2_c((0,\infty) \times D)$ (where $D \subset \mathbb{R}^n$ is the state space of the process $Z$) can be written in the form (we set $\mu = 0$ in the drift of $X$ according to Theorem 4.5; here $\mathcal{G}^\phi$ is the infinitesimal generator (4.5) after the first time change with the Lévy subordinator and $\mathcal{G}^Z$ is the infinitesimal generator of the $n$-dimensional Markov process $Z$):

$$
\begin{aligned}
(4.23)\quad \mathcal{G}f(x,z) &= V(z)\mathcal{G}^\phi f(x,z) + \mathcal{G}^Z f(x,z) \\
&= \frac{1}{2}\gamma V(z)\sigma^2(x)x^2 \frac{\partial^2 f}{\partial x^2}(x,z) + b(x,z)\frac{\partial f}{\partial x}(x,z) - k(x,z)f(x,z) \\
&\quad + \int_{(0,\infty)} \left( f(y,z) - f(x,z) - \mathbf{1}_{\{|y-x|\le 1\}}(y-x)\frac{\partial f}{\partial x}(x,z) \right) \\
&\quad \times \Pi(x,z;dy) + \mathcal{G}^Z f(x,z)
\end{aligned}
$$

with the jump measure (state-dependent Lévy measure)

$$
(4.24)\qquad \Pi(x,z;dy) = \pi(x,z;y)dy
$$

with the density defined for all $x, y > 0$, $x \neq y$, $z \in D$ by

$$
(4.25)\qquad \pi(x,z;y) = V(z)\int_{(0,\infty)} p(s;x,y)\nu(ds),
$$

killing rate

$$
(4.26)\qquad k(x,z) = V(z)\left( \gamma h(x) + \int_{(0,\infty)} P_s(x,\{\Delta\})\nu(ds) \right),
$$

and drift with respect to the truncation function $\mathbf{1}_{\{|y-x|\le 1\}}$.

$$
(4.27)\qquad b(x,z) = V(z)\left[ \gamma h(x)x + \int_{(0,\infty)} \left( \int_{\{y>0:|y-x|\le 1\}} (y-x)p(s;x,y)dy \right) \nu(ds) \right].
$$

Here $p(t;x,y)$ is the transition probability density of the process $X$ with lifetime $\zeta$ and $P_t(x,\{\Delta\})$ is the transition probability of the process $X$ from the state $x > 0$ to the cemetery state $\Delta$ by time $t$ given by equation (4.10).

The first time change $T^1$ scales the volatility with $\gamma$, introduces jumps with the Lévy measure (4.6–7), and modifies the killing rate by scaling the old killing rate $h$ with $\gamma$ and adding the term to it as in (4.8). The second time change introduces stochastic volatility by scaling the volatility with $\sqrt{V(z)}$, and scaling the Lévy measure (4.25) and the killing rate (4.26) with $V(z)$.

As an alternative to composing the time changes, we can also consider combined time changes $T_t = T^1_t + T^2_t$ as discussed in Section 3.3. Theorem 4.5 carries through verbatim to the combined time-change case. However, the Markov generator has a different structure. In the combined time-change case, equation (4.23) is replaced with

$$\begin{aligned}\mathcal{G}f(x,z) &= \mathcal{G}^{\phi}f(x,z) + V(z)\mathcal{G}^{X}f(x,z) + \mathcal{G}^{Z}f(x,z)\\ &= \frac{1}{2}(\gamma + V(z))\sigma^2(x)x^2\frac{\partial^2 f}{\partial x^2}(x,z) + b(x,z)\frac{\partial f}{\partial x}(x,z) - k(x,z)f(x,z)\\ &+ \int_{(0,\infty)}\left(f(y,z) - f(x,z) - \mathbf{1}_{\{|y-x|\le 1\}}(y-x)\frac{\partial f}{\partial x}(x,z)\right)\\ &\times \Pi(x;dy) + \mathcal{G}^{Z}f(x,z)\end{aligned}$$

with the jump measure given by equations (4.6)–(4.7), killing rate

$$k(x,z) = (\gamma + V(z))h(x) + \int_{(0,\infty)} P_s(x,\{\Delta\})\nu(ds),$$

and drift with respect to the truncation function $\mathbf{1}_{\{|y-x|\le 1\}}$

$$b(x,z) = (\gamma + V(z))h(x)x + \int_{(0,\infty)}\left(\int_{\{y>0:|y-x|\le 1\}}(y-x)p(s;x,y)\,dy\right)\nu(ds).$$

In contrast with the composite time change, the combined time change does not introduce stochastic volatility into the jump component. Stochastic volatility only enters into the diffusion component of the process. If one is interested in constructing a pure jump process with stochastic volatility and killing, then the composite time change is more suitable. In applications where one would like to have both diffusion and jump components in the asset price process, whether to use combined or composite time changes is an empirical question that depends on whether or not jumps should exhibit stochastic volatility.

REMARK 4.1. The explicit expressions for the infinitesimal generators provide explicit characterizations for local characteristics of the Markov processes: volatility, drift, killing rate, and jump measure. These explicit expressions are also necessary in order to price American-style and other derivatives numerically by solving the corresponding partial differential equations (PDEs) (or partial integro-differential equations [PIDEs] in case of processes with jumps). While in this paper we study European-style securities and pursue analytical methods, American-style securities can be priced in these models by numerically solving the corresponding pricing PDEs or PIDEs defined by these infinitesimal generators.

## 5. UNIFIED VALUATION OF CORPORATE DEBT, CREDIT DERIVATIVES, AND EQUITY DERIVATIVES

We assume that the stock price follows the process (2.1). We view the stock price as the fundamental observable state variable and, within the framework of our reduced-form model (2.1), view all securities related to a given firm, such as corporate debt, credit derivatives, and equity derivatives, as contingent claims written on the stock price process (2.1). Before proceeding with the valuation of contingent claims, we first consider the calculation of the (risk-neutral) *survival probability*—the probability of no default up to time $t > 0$. Conditioning on the time change, we have

$$\mathbb{Q}(\tau_d > t) = \mathbb{Q}(\zeta > T_t) = \int_0^\infty \mathbb{Q}(\zeta > s)\pi_t(ds) = \int_0^\infty P_s(x, (0,\infty))\pi_t(ds), \tag{5.1}$$

where $P_t(x, (0,\infty)) = \mathbb{Q}(\zeta > t)$ is the survival probability for the Markov process $X$ with lifetime $\zeta$ (transition probability for the Markov process $X$ with lifetime $\zeta$ from the state $x > 0$ into $(0,\infty)$, $P_t(x, (0,\infty)) = 1 - P_t(x, \{\Delta\})$) and $\pi_t(ds)$ is the probability distribution of the time change $T_t$. If the survival probability for the process $X$ and the probability distribution of the time change $\pi_t(ds)$ are known in closed form, then we can obtain the survival probability for the stock price process (2.1) by integration (5.1).

Next, consider a European-style contingent claim with the payoff $\Psi(S_t)$ at maturity $t > 0$ given no default by time $t$, and constant recovery payment $R > 0$ if default occurs by $t$. Separating the claim into two building blocks, a claim with the payoff $\Psi$ and no recovery and the recovery payment, the valuation is done by conditioning on the time change similar to the calculation of the survival probability (5.1). For the European claim with the payoff $\Psi(S_t)$ given no default by time $t$ and with no recovery if default occurs by $t$ we have

$$e^{-rt}\mathbb{E}[\mathbf{1}_{\{\tau_d > t\}}\Psi(S_t)] = e^{-rt}\mathbb{E}[\mathbf{1}_{\{\zeta > T_t\}}\Psi(e^{\rho t}X_{T_t})] = e^{-rt}\int_0^\infty \mathbb{E}\left[\mathbf{1}_{\{\zeta > s\}}\Psi(e^{\rho t}X_s)\right]\pi_t(ds). \tag{5.2}$$

For the fixed recovery $R$ paid at time $t$ if default occurs by $t$ we have

$$Re^{-rt}[1 - \mathbb{Q}(\tau_d > t)], \tag{5.3}$$

where the survival probability is given by equation (5.1).

From equations (5.1)–(5.3), we observe that, by conditioning on the time change, the calculation of the survival probability and the valuation of contingent claims reduce to computing expressions of the form

$$\mathbb{E}[\mathbf{1}_{\{\tau_d > t\}}f(X_{T_t})] = \mathbb{E}[\mathbf{1}_{\{\zeta > T_t\}}f(X_{T_t})] = \int_0^\infty \mathbb{E}[\mathbf{1}_{\{\zeta > s\}}f(X_s)]\pi_t(ds), \tag{5.4}$$

for some function $f$ (to compute the survival probability set $f = 1$). This involves first computing the expectation $\mathbb{E}[\mathbf{1}_{\{\zeta > s\}}f(X_s)]$ for the background diffusion process $X$ and then integrating the result in time against the probability distribution of the time change $T_t$, if the probability distribution of the time change is known in closed form (e.g., the closed-form expressions for compound Poisson, Gamma, and inverse Gaussian subordinators given in Appendix C from the companion appendix). In general, if the closed-form expression for the distribution of the time change is not available, it can be recovered by inverting the Laplace transform numerically, which involves numerical integration in the complex plane by means of the Bromwich Laplace inversion formula. The second step is to compute the integral from zero to infinity in equations (5.1) and (5.2). Thus, if we can determine the expectation $\mathbb{E}[\mathbf{1}_{\{\zeta > s\}}f(X_s)]$ for the original Markov process $X$ in closed form, we still need to perform double numerical integration to compute (5.4) for the time-changed process. Fortunately, when the function $f$ satisfies an additional integrability condition, there is an alternative approach that avoids any need for Laplace transform inversion to recover $\pi_t(ds)$ and for numerical integration in $s$ in (5.4). In the next section, we will present a remarkably powerful Laplace transform approach that will effectively evaluate both of these integrals in closed form.

The two building blocks (5.2) and (5.3) can be used to value corporate debt, credit derivatives, and equity derivatives. In particular, a *defaultable zero-coupon bond* with unit face value, maturity $t > 0$, and recovery $R \in [0, 1]$ can be represented as the European claim with $\Psi(S_t) = 1$ and valued at time zero by

(5.5)
$$B_R(x, t) = e^{-rt}\mathbb{Q}(\tau_d > t) + Re^{-rt}[1 - \mathbb{Q}(\tau_d > t)] = e^{-rt}R + e^{-rt}(1 - R)\mathbb{Q}(\tau_d > t),$$

where we indicate explicitly the dependence of the bond value function on the initial stock price $S_0 = X_0 = x$. Our recovery assumption corresponds to the *fractional recovery of treasury* assumption (see, e.g., Lando 2004, p. 120). Defaultable bonds with coupons can be valued as portfolios of defaultable zeros.

A *European call option* with strike $K > 0$ with the payoff $(S_t - K)^+$ at expiration $t$ has no recovery if the firm defaults. A *European put option* with strike $K > 0$ with the payoff $(K - S_t)^+$ can be decomposed into two parts: the put payoff $(K - S_t)^+\mathbf{1}_{\{\tau_d>t\}}$, given no default by time $t$, and a recovery payment equal to the strike $K$ at expiration in the event of default $\tau_d \le t$. The pricing formulas for European-style call and put options take the form

(5.6)
$$C(x; K, t) = e^{-rt}\mathbb{E}\big[\big(e^{\rho t}X_{T_t} - K\big)^+\mathbf{1}_{\{\tau_d>t\}}\big] = e^{-rt}\int_0^\infty \mathbb{E}\big[\big(e^{\rho t}X_s - K\big)^+\mathbf{1}_{\{\zeta>s\}}\big]\pi_t(ds),$$

and

$$P(x; K, t) = P_0(x; K, t) + P_D(x; K, t), \tag{5.7}$$

where

$$P_0(x; K, t) = e^{-rt}\int_0^\infty \mathbb{E}\big[\big(K - e^{\rho t}X_s\big)^+\mathbf{1}_{\{\zeta>s\}}\big]\pi_t(ds) \tag{5.8}$$

and

$$P_D(x; K, t) = Ke^{-rt}[1 - \mathbb{Q}(\tau_d > t)], \tag{5.9}$$

respectively. One notes that the put pricing formula (5.7) consists of two parts: the present value $P_0(x; K, t)$ of the put payoff conditional on no default given by equation (5.8) (this can be interpreted as the down-and-out put with the down-and-out barrier at zero), as well as the present value $P_D(x; K, t)$ of the cash payment equal to the strike $K$ in the event of default given by equation (5.9). This recovery part of the put is a *European-style default claim*, a credit derivative that pays a fixed cash amount $K$ at maturity $t$ if and only if the underlying firm has defaulted by time $t$. Thus, the put option contains an embedded credit derivative. Generally, we emphasize that, in our model, corporate debt, credit derivatives, and equity options are all valued in an unified framework as contingent claims written on the defaultable stock.

Although we will now focus on deriving explicit closed-form expressions for European-style securities by probabilistic methods, the framework of this section can be extended to the valuation of American-style options and more complicated securities with American features, such as convertible bonds. The standard results imply that the value function solves the appropriate PIDE with the integro-differential operator $\mathcal{G}$ (the infinitesimal generator of the time-changed Markov process; one-dimensional in the case of time changes by Lévy subordinators or $(n+1)$-dimensional in the case of absolutely continuous or composite time changes) on the appropriate domain and subject to appropriate terminal and boundary conditions. The solution can be derived via numerical methods.

## 6. VALUATION OF CONTINGENT CLAIMS ON TIME-CHANGED MARKOV PROCESSES: A LAPLACE TRANSFORM APPROACH

We now present a powerful method to compute expectations of the form (5.4) needed to value contingent claims in our model. We will tackle it in two steps. First, we show how to use the Laplace transform to compute the expectation operator

$$\mathcal{P}_t f(x) = \mathbb{E}_x[\mathbf{1}_{\{\zeta > t\}} f(X_t)], \tag{6.1}$$

where $X$ is a one-dimensional diffusion process with lifetime $\zeta$ started at $x$ at time zero and the function $f$ satisfies some integrability conditions to be specified later. Second, we show how the time change can be accomplished so that the integral with respect to the time variable in the expectation (5.4) is evaluated in closed form from the knowledge of the Laplace transform representation for the expectation (6.1) for the process $X$ and the Laplace transform of the time change $T$, without any need to recover the probability distribution of the time change.

Taking the Laplace transform of the expectation operator, we define the *resolvent operator* $\mathcal{R}_s$ (e.g., Ethier and Kurtz 1986):

$$\mathcal{R}_s f(x) := \int_0^\infty e^{-st}\mathcal{P}_t f(x)\,dt = \frac{1}{s}\mathbb{E}_x[\mathbf{1}_{\{\zeta > \tau_s\}} f(X_{\tau_s})],$$

where $\tau_s$ is an independent exponential time with mean $1/s$. For one-dimensional diffusions, it is well known (see Borodin and Salminen 2002) that resolvent operator can be represented as an integral operator

$$\begin{aligned}\mathcal{R}_s f(x) &= \int_\ell^r f(y) G_s(x,y)\,dy \\ &= \frac{\phi_s(x)}{w_s}\int_\ell^x f(y)\psi_s(y)\mathfrak{m}(y)\,dy + \frac{\psi_s(x)}{w_s}\int_x^r f(y)\phi_s(y)\mathfrak{m}(y)\,dy,\end{aligned} \tag{6.2}$$

where the *resolvent kernel* or *Green's function* $G_s(x, y)$ is the Laplace transform of the transition probability density, $G_s(x, y) = \int_0^\infty e^{-st} p(t; x, y)\,dt$, that admits the following explicit representation (Borodin and Salminen 2002, p. 19; note that we define the Green's function with respect to the Lebesgue measure, while Borodin and Salminen define it with respect to the speed measure $\mathfrak{m}(y)dy$, where $\mathfrak{m}(y)$ is the speed density, and so $\mathfrak{s}(y)$ does not appear in their expression)

$$G_s(x,y) = \frac{\mathfrak{m}(y)}{w_s}\begin{cases}\psi_s(x)\phi_s(y), & x \le y,\\ \psi_s(y)\phi_s(x), & y \le x.\end{cases} \tag{6.3}$$

For $s > 0$, the functions $\psi_s(x)$ and $\phi_s(x)$ can be characterized as the unique (up to a multiplicative factor independent of $x$) solutions of the *Sturm-Liouville equation* associated with the infinitesimal generator $\mathcal{G}$ of the one-dimensional diffusion process,

$$\mathcal{G}u(x) = \frac{1}{2}a^2(x)\frac{d^2u}{dx^2}(x) + b(x)\frac{du}{dx}(x) - c(x)u(x) = su(x), \tag{6.4}$$

by first demanding that $\psi_s(x)$ is increasing in $x$ and $\phi_s(x)$ is decreasing and, second, posing boundary conditions at accessible boundary points. For $\psi_s(x)$, the boundary condition is only imposed at $\ell$ if $\ell$ is an accessible boundary. Because in this paper we assume that accessible boundaries are specified as killing boundaries, we have a Dirichlet boundary condition at $\ell$, $\psi_s(\ell) = 0$. For $\phi_s(x)$ we have, similarly, $\phi_s(r) = 0$ if $r$ is an accessible boundary specified as a killing boundary. The functions $\psi_s(x)$ and $\phi_s(x)$ are called *fundamental solutions* of the Sturm–Liouville equation (6.4). They are linearly independent and all solutions can be expressed as their linear combinations. Moreover, the so-called *Wronskian* is independent of $x$

$$w_s = \frac{1}{\mathfrak{s}(x)}(\psi_s'(x)\phi_s(x) - \psi_s(x)\phi_s'(x)). \tag{6.5}$$

In equation (6.3), the function $\mathfrak{m}(x)$ is the so-called *speed density* of the diffusion process $X$ and is constructed from the diffusion and drift coefficients as follows (see Borodin and Salminen 2002, p. 17)

$$\mathfrak{m}(x) = \frac{2}{a^2(x)\mathfrak{s}(x)}, \quad \text{where } \mathfrak{s}(x) = \exp\left(-\int_{x_0}^{x} \frac{2b(y)}{a^2(y)}\,dy\right), \tag{6.6}$$

where $x_0 \in (\ell, r)$ is an arbitrary point in the state space. The function $\mathfrak{s}(x)$ is called the *scale density* of the diffusion process $X$.

In equation (6.2), we interchanged the Laplace transform integral in $t$ and the expectation integral in $y$. This interchange is allowed by Fubini's theorem if and only if the function $f$ is such that $\int_\ell^r |f(y)G_s(x, y)|\,dy < \infty$ or

(6.7)

$$\int_\ell^x |f(y)|\psi_s(y)\mathfrak{m}(y)\,dy < \infty \text{ and } \int_x^r |f(y)|\phi_s(y)\mathfrak{m}(y)\,dy < \infty \;\; \forall \;\; x \in (\ell, r), \;\; s > 0.$$

For $f$ satisfying this integrability condition, we can then recover the expectation (6.1) by first computing the resolvent operator (6.2) and then inverting the Laplace transform via the Bromwich Laplace transform inversion formula (see Pazy 1983) for the Laplace inversion formula for operator semigroups)

$$\mathcal{P}_t f(x) = \mathbb{E}_x[\mathbf{1}_{\{\zeta > t\}} f(X_t)] = \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{st}\mathcal{R}_s f(x)\,ds. \tag{6.8}$$

A crucial observation is that in the representation (6.8) time only enters through the exponential $e^{st}$ (the temporal and spatial variables are separated). We can thus write

$$\mathbb{E}[\mathbf{1}_{\{\zeta > T_t\}} f(X_{T_t})] = \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \mathbb{E}[e^{sT_t}]\mathcal{R}_s f(x)\,ds = \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \mathcal{L}(t, -s)\mathcal{R}_s f(x)\,ds, \tag{6.9}$$

where $\mathcal{L}(t, \lambda) = \mathbb{E}[e^{-\lambda T_t}]$ is the Laplace transform of the time change (here we require that $\mathbb{E}[e^{\varepsilon T_t}] = \mathcal{L}(t, -\varepsilon) < \infty$). This result has two significant advantages over the expression (5.4). First, it does not require the knowledge of the transition probability measure of the time change, and only requires the knowledge of the Laplace transform of the time change. Second, it does not require the knowledge of the expectation $\mathbb{E}[\mathbf{1}_{\{\zeta>t\}} f(X_t)]$ for the original process, and only requires the knowledge of the resolvent $\mathcal{R}_s f(x)$ given by equation (6.2).

The Laplace transform inversion in (6.9) can be performed by appealing to the Cauchy Residue Theorem to calculate the Bromwich integral in the complex plane. To do this, we need to analyze singularities of the function $\mathcal{R}_s f(x)$ in the complex plane $s \in \mathbb{C}$ (due to our assumption $\mathbb{E}[e^{\varepsilon T_t}] = \mathcal{L}(t, -\varepsilon) < \infty$, the Laplace transform of the time change $\mathcal{L}(t, -s)$ is analytic in the half-plane to the left of the integration contour in [6.9]).

REMARK 6.1. If the background process $X$ is a Lévy process (in particular, Brownian motion with drift), then the Laplace transform approach in this section can be shown to be equivalent to the fast Fourier transform (FFT) approach of Carr et al. (2003). In this case, we do not need to work with the resolvent and can work with the characteristic functions instead as is done in Carr et al. (2003), leading to the Fourier inversion by the FFT. For Lévy processes, the characteristic function/Fourier transform approach is more straightforward to use in application. However, the Laplace transform approach in this section is much more general, as it can be applied to time changing any Markov process, not just a Lévy process.

REMARK 6.2. Carr et al. (2003) work with Lévy processes without killing. We note that it is possible to introduce killing/default into the framework of time-changed Lévy processes in Carr et al. (2003) as follows. Start with a Lévy process with killing. Recall that the killing rate $k$ has to be constant in order for the killed process to be a Lévy process. That is, the Lévy process is killed at an independent exponential time. On time changing the Lévy process with an integral of an activity rate process $V_t$ (such as the CIR), the time-changed process acquires a stochastic default intensity $kV_t$. That is, the default intensity is the old constant killing rate scaled with the stochastic activity rate process that introduces stochastic volatility. To price contingent claims in this class of models based on Lévy processes with stochastic volatility and killing, one can directly follow the Fourier approach of Carr et al. (2003). However, the method developed here is more general and is applicable to any Markov process with killing.

REMARK 6.3. If the background Markov process $X$ is a one-dimensional diffusion and the time-change process is a Lévy subordinator with the exponential Lévy measure $\nu(ds) = \alpha\eta e^{-\eta s} ds$, then we note that the state-dependent Lévy density (4.7) of the time-changed process is the Green's function of the diffusion $X$ evaluated at $s = \eta$ and scaled with $\alpha\eta$. Indeed, from equation (4.7), we have $\pi(x, y) = \alpha\eta \int_0^\infty p(s; x, y) e^{-\eta s}\, ds = \alpha\eta G_\eta(x, y)$.

## 7. VALUATION OF CONTINGENT CLAIMS ON TIME-CHANGED MARKOV PROCESSES: A SPECTRAL EXPANSION APPROACH

Studying the Green's function as a function of the complex variable $s$, one can invert the Laplace transform and obtain a *spectral representation of the transition density* for one-dimensional diffusions originally due to McKean (1956) (see also Ito and McKean 1974; Section 4.1, Wong 1964; Karlin and Taylor 1981). Indeed, considered as a linear operator in the Hilbert space of functions square-integrable with the speed density $\mathfrak{m}(x)$, the expectation operator $\mathcal{P}_t$ is *self-adjoint*. Namely, define the inner product $(f, g) := \int_\ell^r f(x)g(x)\mathfrak{m}(x)\, dx$ and let $L^2((\ell, r), \mathfrak{m})$ be the Hilbert space of functions on $(\ell, r)$ square-integrable with the speed density, that is, with $\|f\| < \infty$, where $\|f\|^2 = (f, f)$. Then the semigroup $\{\mathcal{P}_t, t \geq 0\}$ of expectation operators indexed by time is self-adjoint in $L^2((\ell, r), \mathfrak{m})$, that is, $(\mathcal{P}_t f, g) = (f, \mathcal{P}_t g)$ for every $f, g \in L^2((\ell, r), \mathfrak{m})$ and $t \geq 0$. This follows from the symmetry property of the transition density, $p(t; x, y)\mathfrak{m}(x) = p(t; y, x)\mathfrak{m}(y)$ (note that this symmetry property is apparent from the structure of the Green's function [6.3]). The infinitesimal generator $\mathcal{G}$ of a self-adjoint semigroup, as well as the resolvent operators $\mathcal{R}_s$, are also self-adjoint, and we can appeal to the Spectral Theorem for self-adjoint operators in Hilbert space to obtain their spectral representations. One-dimensional diffusions are examples of *symmetric Markov processes* with symmetric transition semigroups and self-adjoint infinitesimal generators (the standard reference is Fukushima, Oshima, and Takeda 1994).

In the important special case when the spectrum of $\mathcal{G}$ in $L^2((\ell, r), m)$ is purely discrete, the spectral representation has the following form. Let $\{\lambda_n\}_{n=1}^\infty$, $0 \leq \lambda_1 < \lambda_2 < , \ldots, \lim_{n\uparrow\infty} \lambda_n = \infty$, be the eigenvalues of $-\mathcal{G}$ and let $\{\varphi_n\}_{n=1}^\infty$ be the corresponding eigenfunctions normalized so that $\|\varphi_n\|^2 = 1$. That is, $(\lambda_n, \varphi_n)$ solve the *Sturm-Liouville eigenvalue-eigenfunction problem* for the (negative of the) differential operator in (6.4): $-\mathcal{G}\varphi_n = \lambda_n\varphi_n$ (Dirichlet boundary condition is imposed at an endpoint if it is a killing boundary). Then the spectral representations for the transition density $p(t; x, y)$ and the expectation operator $\mathcal{P}_t f(x)$ for $f \in L^2((\ell, r), \mathfrak{m})$ take the form of *eigenfunction expansions* (for $t > 0$ the eigenfunction expansion (7.1) converges uniformly on compact squares in $(\ell, r) \times (\ell, r)$):

$$p(t; x, y) = \mathfrak{m}(y) \sum_{n=1}^{\infty} e^{-\lambda_n t}\varphi_n(x)\varphi_n(y), \tag{7.1}$$

$$\mathcal{P}_t f(x) = \mathbb{E}_x[\mathbf{1}_{\{\zeta > t\}} f(X_t)] = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t}\varphi_n(x) \tag{7.2}$$

with the expansion coefficients $c_n = (f, \varphi_n)$ satisfying the Parseval equality $\|f\|^2 = \sum_{n=1}^\infty c_n^2 < \infty$. The eigenfunctions $\{\varphi_n(x)\}_{n=1}^\infty$ form a complete orthonormal basis in the Hilbert space $L^2((\ell, r), \mathfrak{m})$, that is, $(\varphi_n, \varphi_n) = 1$ and $(\varphi_n, \varphi_m) = 0$ for $n \neq m$. They are also eigenfunctions of the expectation operator, $\mathcal{P}_t\varphi_n(x) = e^{-\lambda_n t}\varphi_n(x)$, with eigenvalues $e^{-\lambda_n t}$, and of the resolvent operator, $\mathcal{R}_s\varphi_n(x) = \varphi_n(x)/(s + \lambda_n)$, with eigenvalues $1/(s + \lambda_n)$.

More generally, the spectrum of the infinitesimal generator $\mathcal{G}$ in $L^2((\ell, r), m)$ may be continuous, in which case the sums in (7.1)–(7.2) are replaced with the integrals. We do not reproduce general results on spectral expansions with continuous spectrum here and instead refer the reader to the literature. For further details on the spectral representation for one-dimensional diffusions and their applications in asset pricing we refer the reader to Davydov and Linetsky (2003), Lewis (1998, 2000), and Linetsky (2004a,b,c, 2007). We also refer the reader to Amrein, Hinz, and Pearson (2005) for a detailed mathematical treatment of the Sturm–Liouville theory and numerous references.

A key feature of the spectral representation is that it separates the temporal and spatial variables. Moreover, time enters the expression (7.2) only through the exponentials $e^{-\lambda_n t}$, thus setting the stage for time changes. We now turn to computing expectations of the form (5.4). Let $f \in L^2((\ell, r), \mathfrak{m})$. Substituting the eigenfunction expansion (7.2) into (5.4), we have

$$\mathbb{E}[\mathbf{1}_{\{\zeta > T_t\}} f(X_{T_t})] = \sum_{n=1}^{\infty} c_n \mathbb{E}[e^{-\lambda_n T_t}]\varphi_n(x) = \sum_{n=1}^{\infty} c_n \mathcal{L}(t, \lambda_n)\varphi_n(x), \tag{7.3}$$

where $\mathcal{L}(t, \lambda)$ is the Laplace transform of the time change. In particular, for the eigenfunctions we have

$$\mathbb{E}_x[\mathbf{1}_{\{\zeta > T_t\}} \varphi_n(X_{T_t})] = \mathcal{L}(t, \lambda_n)\varphi_n(x). \tag{7.4}$$

Due to the fact that time enters the spectral expansion only through the exponentials $e^{-\lambda_n s}$, integrating this exponential against the distribution of the time change $\pi_t(ds)$, the integral in $s$ in (5.4) reduces to the Laplace transform of the time change, $\int_{[0,\infty)} e^{-\lambda_n s}\pi_t(ds) = \mathcal{L}(t, \lambda_n)$. Thus, in one shot, we both compute the integral in $s$ in (5.4) and get rid of the necessity to invert the Laplace transform to recover the distribution of the time change. In effect, the spectral expansion approach reduces the total required number of integrations by two. In general, the spectral expansion approach is tailor-made for time changes due to the exponential dependence on time (see also Chen and Song 2005, 2007; Linetsky 2007, for related results).

REMARK 7.1. We stress that the spectral expansion (7.2) is only valid for functions $f$ that are square-integrable with the speed density $\mathfrak{m}$. For those functions that are not in $L^2((\ell, r), \mathfrak{m})$ but satisfy the integrability conditions (6.7) one needs to apply the Cauchy Residue Theorem directly to the expression (6.9) because the resolvent $\mathcal{R}_s f(x)$ may have singularities that do not coincide with the singularities of the Green's function $G_s(x, y)$, and the evaluation of (6.9) has to be done case by case for each non-square-integrable $f$.

REMARK 7.2. If the process $X$ is a Lévy process (e.g., Brownian motion with drift), then the result of the spectral method can be shown to be equivalent to the Fourier transform method based on the characteristic function. The Fourier method is more straightforward in this case. However, the spectral method is much more general, as it is applicable to any symmetric Markov process (and to any one-dimensional diffusion in particular).

## 8. TIME CHANGING THE JDCEV PROCESS

### 8.1. The JDCEV Process

Carr and Linetsky (2006) recently proposed the following extension of the classical CEV model of Cox (1975). Recall that, to be consistent with the leverage effect and the implied volatility skew, the instantaneous volatility in the CEV model is specified as a power function (see Cox 1975; Schroder 1989; Davydov and Linetsky 2001, 2003; Linetsky 2004b, for background on the CEV process)

$$\sigma(x) = ax^{\beta}, \tag{8.1}$$

where $\beta < 0$ is the volatility elasticity parameter and $a > 0$ is the volatility scale parameter. The limiting case with $\beta = 0$ corresponds to the constant volatility assumption in the BSM model. To be consistent with the empirical evidence linking corporate bond yields and CDS spreads to equity volatility, Carr and Linetsky (2006) propose to specify the default intensity as an affine function of the instantaneous variance of the underlying stock price process

$$h(x) = b + c\,\sigma^2(x) = b + c\,a^2 x^{2\beta}, \tag{8.2}$$

where $b \geq 0$ is a constant parameter governing the state-independent part of the intensity and $c \geq 0$ is a constant parameter governing the sensitivity of the intensity to $\sigma^2$. In Carr and Linetsky (2006), $a$ and $b$ are taken to be deterministic functions of time. In this paper, we assume that $a$ and $b$ are constant. The infinitesimal generator of this diffusion process on $(0, \infty)$ with killing at the rate (8.2) has the form

$$\mathcal{G} f(x) = \frac{1}{2} a^2 x^{2\beta+2} \frac{d^2 f}{dx^2}(x) + (\mu + b + c\,a^2 x^{2\beta}) x \frac{df}{dx}(x) - (b + c\,a^2 x^{2\beta}) f(x). \tag{8.3}$$

This model specification introduces the possibility of a jump to default from a positive value for the CEV process and is referred to as the JDCEV process. This model nests the standard CEV model as a limiting case with vanishing default intensity $b = c = 0$. In the standard CEV model default can only occur when the stock price hits zero through diffusion. When $c = 0$, the intensity is independent of the stock price, and the model is that of the CEV process killed at an independent exponential time with mean $1/b$ (the first jump time of a Poisson process with constant intensity $b$). In this case default can occur either through hitting zero by diffusion or through a jump to zero from a positive stock price value. When $b = 0$, the intensity does not have a state-independent term and is entirely governed by the stock price process. When $b > 0$ and $c > 0$ the intensity has two parts—a state-independent part and a state-dependent part. When $c > 0$, default can only occur through a jump from a positive value, because the default intensity increases so fast as the stock falls that the jump to default will almost surely arrive prior to the diffusion process hitting zero.

In this section, we use the general theory developed in the previous sections to construct far-reaching extensions of the original Carr–Linetsky JDCEV model. By assuming that the process $X$ in (2.1) follows a JDCEV process and time changing it as described in Section 3, we introduce jumps and stochastic volatility into the JDCEV model. To be able to value contingent claims in time-changed JDCEV models, we need to be able to compute expectations of the form (6.1) for the JDCEV process as described in Sections 6 and 7. The scale and speed densities of the JDCEV process are

$$\mathfrak{m}(x) = \frac{2}{a^2} x^{2c-2-2\beta} e^{Ax^{-2\beta}}, \quad \mathfrak{s}(x) = x^{-2c} e^{-Ax^{-2\beta}}, \quad \text{where } A := \frac{\mu + b}{a^2|\beta|}. \tag{8.4}$$

The following theorem presents the fundamental solutions $\psi_s(x)$ and $\phi_s(x)$ entering the expression for the Green's function (6.3) and their Wronskian $w_s$ (6.5) for the JDCEV process. Without loss of generality we assume that $\mu + b \geq 0$.[2] There are two distinct cases: $\mu + b > 0$ and $\mu + b = 0$.

THEOREM 8.1.

*(i) For a JDCEV diffusion process with the infinitesimal generator (8.3) with parameters $\beta < 0, a > 0, b \geq 0, c \geq 0$ and such that $\mu + b > 0$, the increasing and*

[2] For absolutely continuous and composite time changes, $\mu = 0$ by Theorems 4.4 and 4.5, while $b \geq 0$. For Lévy subordinators, by Theorem 4.1 $\mu \in \mathcal{I}_\nu$ can always be selected so that $\mu + b \geq 0$. Thus, we do not consider the case $\mu + b < 0$.

*decreasing fundamental solutions* $\psi_s(x)$ *and* $\phi_s(x)$ *are*

$$\psi_s(x) = x^{\frac{1}{2}+\beta-c} e^{-\frac{1}{2}Ax^{-2\beta}} M_{\varkappa(s),\frac{\nu}{2}}(Ax^{-2\beta}), \tag{8.5}$$

$$\phi_s(x) = x^{\frac{1}{2}+\beta-c} e^{-\frac{1}{2}Ax^{-2\beta}} W_{\varkappa(s),\frac{\nu}{2}}(Ax^{-2\beta}), \tag{8.6}$$

*where* $M_{k,m}(z)$ *and* $W_{k,m}(z)$ *are the first and second Whittaker functions with indexes*

$$\nu = \frac{1+2c}{2|\beta|},\quad \varkappa(s) = \frac{\nu-1}{2} - \frac{s+\xi}{\omega},\quad \text{where } \omega = 2|\beta|(\mu+b),\quad \xi = 2c(\mu+b)+b, \tag{8.7}$$

*and the constant* $A$ *is defined in equation (8.4). The Wronskian* $w_s$ *defined by equation (6.5) reads*

$$w_s = \frac{2(\mu+b)\Gamma(1+\nu)}{a^2\Gamma(\nu/2+1/2-\varkappa(s))}. \tag{8.8}$$

*(ii) For a JDCEV diffusion process with the infinitesimal generator (8.3) with parameters* $\beta<0, a>0, b\geq 0, c\geq 0$ *and such that* $\mu+b=0$*, the increasing and decreasing fundamental solutions* $\psi_s(x)$ *and* $\phi_s(x)$ *are*

$$\psi_s(x) = x^{\frac{1}{2}-c} I_\nu\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right),\quad \phi_s(x) = x^{\frac{1}{2}-c} K_\nu\left(\frac{x^{-\beta}\sqrt{2(s+b)}}{a|\beta|}\right), \tag{8.9}$$

*where* $I_\nu(z)$ *and* $K_\nu(z)$ *are the modified Bessel functions with index* $\nu$ *given in equation (8.7). The Wronskian* $w_s$ *defined by equation (6.5) reads*

$$w_s = |\beta|. \tag{8.10}$$

*Proof.* The proof is by reduction of the Sturm–Liouville equation (6.4) for the JDCEV operator (8.3) to the Whittaker equation when $\mu+b>0$ and to the Bessel equation when $\mu+b=0$. See Appendix A. □

Theorem 8.1 generalizes Proposition 5 in Davydov and Linetsky (2001) that gives the fundamental solutions for the standard CEV model. Their results are a special case of our Theorem 8.1 for vanishing default intensity with $b=c=0$. The Green's function is given by equation (6.3). Inverting the Laplace transform leads to the spectral representation of the transition density (7.1).

THEOREM 8.2.

*(i) When* $\mu+b>0$*, the spectrum of the negative of the infinitesimal generator (8.3) is purely discrete with the eigenvalues and eigenfunctions*

$$\lambda_n = \omega n + \xi,\quad \varphi_n(x) = A^{\frac{\nu}{2}}\sqrt{\frac{(n-1)!(\mu+b)}{\Gamma(\nu+n)}}\, x e^{-Ax^{-2\beta}} L_{n-1}^{(\nu)}(Ax^{-2\beta}),\quad n=1,2,\dots, \tag{8.11}$$

*where $L_n^{(\nu)}(x)$ are the generalized Laguerre polynomials and $\xi$ and $\omega$ are defined in (8.7). The spectral representation (eigenfunction expansion) of the JDCEV transition density is given by equation (7.1) with these eigenvalues and eigenfunctions and the speed density (8.4).*

(ii) *When $\mu + b = 0$, the spectrum of the infinitesimal generator (8.3) is purely absolutely continuous and the spectral representation for the transition density reads*

$$p(t; x, y) = \frac{1}{2|\beta|}\mathfrak{m}(y)\int_0^\infty e^{-(\lambda+b)t}(xy)^{1/2-c} J_\nu\left(\frac{x^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right) J_\nu\left(\frac{y^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right) d\lambda, \tag{8.12}$$

*where $J_\nu(x)$ is the Bessel function of the first kind with index $\nu$ given in (8.7).*

*Proof.* The proof is based on applying the Cauchy Residue Theorem to calculate the Bromwich Laplace inversion integral. See Appendix A. □

Theorem 8.2 generalizes Proposition 8(i) in Davydov and Linetsky (2003) that gives the eigenvalues and eigenfunctions for the standard CEV model. Their results are a special case of our Theorem 8.2 for vanishing default intensity with $b = c = 0$.

Carr and Linetsky (2006) present closed-form solutions for the survival probability and call and put options in the JDCEV model (Proposition 5.5, pp. 319–320). However, those expressions are not suitable for time changes because they depend on time in a complicated fashion. Here, based on the theory in Sections 6 and 7 and Theorems 8.1 and 8.2, we obtain alternative closed-form expressions for the survival probability and call and put options in the JDCEV model with time entering only through exponentials. We first present the result for the survival probability.

THEOREM 8.3.

(i) *For a JDCEV diffusion process with the infinitesimal generator (8.3) with parameters $\beta < 0, a > 0, b \geq 0, c \geq 0, \mu + b > 0$ and started at $x > 0$, the survival probability $\mathbb{Q}(\zeta > t)$ is given by*

$$\mathbb{Q}(\zeta > t) = \sum_{n=0}^{\infty} e^{-(b+\omega n)t} \frac{\Gamma\left(1+\frac{c}{|\beta|}\right)\left(\frac{1}{2|\beta|}\right)_n}{\Gamma(\nu+1)n!} A^{\frac{1}{2|\beta|}} x e^{-Ax^{-2\beta}} \, {}_1F_1\left(1-n+\frac{c}{|\beta|}; \nu+1; Ax^{-2\beta}\right), \tag{8.13}$$

*where ${}_1F_1(a; b; x)$ is the confluent hypergeometric function; $(a)_n := \Gamma(a+n)/\Gamma(a) = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol; and the constants $A$, $\nu$, and $\omega$ are as defined in Theorem 8.1.*

(ii) *For $\mu + b = 0$, the JDCEV survival probability $\mathbb{Q}(\zeta > t)$ is given by*

$$\mathbb{Q}(\zeta > t) = x^{1/2-c}(\sqrt{2}a|\beta|)^{\frac{2c-1}{2|\beta|}} \frac{\Gamma\left(1+\frac{c}{|\beta|}\right)}{\Gamma\left(\frac{1}{2|\beta|}\right)} \times \int_0^\infty e^{-(b+\lambda)t}\lambda^{-\frac{2c-1}{4|\beta|}-1} J_\nu\left(\frac{x^{-\beta}\sqrt{2\lambda}}{a|\beta|}\right) d\lambda, \tag{8.14}$$

*where $J_\nu(x)$ is the Bessel Function of the first kind.*

*Proof.* The proof is based on first computing the resolvent (6.2) with $f(x) = 1$ and then inverting the Laplace transform (6.8) analytically. Because constants are not square-integrable on $(0, \infty)$ with the speed density (8.4), we cannot use the spectral expansion approach of Section 7 and instead follow the Laplace transform approach of Section 6. See Appendix A. □

We now present the result for the put option. The put option price in our model (2.1) is given by equations (5.7)–(5.9). In particular, to compute the price of the put payoff conditional on no default before expiration, $P_0(x; K, t)$, we need to compute the expectation $\mathbb{E}[(K - e^{\rho t} X_s)^+ \mathbf{1}_{\{\zeta > s\}}] = e^{\rho t}\mathbb{E}[(e^{-\rho t} K - X_s)^+ \mathbf{1}_{\{\zeta > s\}}]$ for the JDCEV process (8.3). The survival probability entering the put pricing formula is already computed in Theorem 8.3. The pricing formula for the call option is obtained via the put-call parity.

THEOREM 8.4.

*(i) For a JDCEV diffusion process with the infinitesimal generator (8.3) with parameters $\beta < 0, a > 0, b \geq 0, c \geq 0$ and such that $\mu + b > 0$, the expectation $\mathbb{E}[(k - X_t)^+ \mathbf{1}_{\{\zeta > t\}}]$ is given by the eigenfunction expansion (7.2) with the eigenvalues $\lambda_n$ and eigenfunctions $\varphi_n(x)$ given in Theorem 8.2 and expansion coefficients*

$$c_n = \frac{A^{\nu/2+1} k^{2c+1-2\beta} \sqrt{\Gamma(\nu + n)}}{\Gamma(\nu + 1)\sqrt{(\mu + b)(n - 1)!}} \times \left\{ \frac{|\beta|}{c + |\beta|}\, {}_2F_2\left( \begin{matrix} 1 - n, & \frac{c}{|\beta|} + 1 \\ \nu + 1, & \frac{c}{|\beta|} + 2 \end{matrix} ; A k^{-2\beta} \right) - \frac{\Gamma(\nu + 1)(n - 1)!}{\Gamma(\nu + n + 1)} L_{n-1}^{(\nu+1)}\left(A k^{-2\beta}\right) \right\}, \tag{8.15}$$

*where ${}_2F_2$ is the generalized hypergeometric function.*

*(ii) For $\mu + b = 0$, the expectation has a spectral expansion with absolutely continuous spectrum*

$$\mathbb{E}\big[(k - X_t)^+ \mathbf{1}_{\{\zeta > t\}}\big] = \int_0^\infty e^{-(\lambda+b)t} c(\lambda) x^{1/2-c} J_\nu\left( \frac{x^{-\beta}\sqrt{2\lambda}}{a|\beta|} \right) d\lambda, \tag{8.16}$$

*with the expansion coefficients*

$$c(\lambda) = \frac{\lambda^{\nu/2} k^{2c+1-2\beta}}{2^{\nu/2+1}\Gamma(\nu + 1)(c + |\beta|)|\beta|^{\nu+1} a^{\nu+2}}\, {}_1F_2\left( \begin{matrix} \frac{c}{|\beta|} + 1, & \\ \nu + 1, & \frac{c}{|\beta|} + 2 \end{matrix} ; -\frac{k^{-2\beta}\lambda}{2a^2|\beta|^2} \right) - \frac{k^{c+1/2-\beta}}{\sqrt{2\lambda}|\beta| a} J_{\nu+1}\left( \frac{k^{-\beta}\sqrt{2\lambda}}{a|\beta|} \right), \tag{8.17}$$

*where ${}_1F_2$ is the generalized hypergeometric function.*

*Proof.* The put payoff $f(x) = (k - x)^+$ is in the Hilbert space $L^2((0, \infty), \mathfrak{m})$ of functions square-integrable with the speed density (8.4) and, hence, the expectation has a spectral expansion. The proof follows by applying the spectral expansion approach. See Appendix A. □

REMARK 8.1. When $b = c = 0$, all results in this section reduce to the corresponding results for the standard CEV model (without jump to default) in Davydov and Linetsky (2001, 2003). In the standard CEV model default can only occur through the stock price hitting zero via diffusion. In this case, the survival probability in Theorem 8.3 is equal to the probability of the CEV diffusion not hitting zero by time $t$.

REMARK 8.2. The series representation (8.14) for the survival probability is equivalent to the expression (5.14) in Carr and Linetsky (2006). To prove this one needs to apply the multiplication identity for the Whittaker functions given in equation (B.10) in Appendix B. Due to this identity, the series of hypergeometric functions in (8.13) collapses to the closed-form expression (5.14) in Carr and Linetsky (2006). For $\mu + b = 0$, one needs to use the integral (B.13) in Appendix B. Similarly, the eigenfunction expansion for the put in Theorem 8.4 is equivalent to the closed-form expression (5.18) in Carr and Linetsky (2006). To prove this, one needs to apply the Hille–Hardy formula for Laguerre polynomials (Erdelyi 1953, p. 189; valid for all $|t| < 1, \nu > -1, a, b > 0$)

(8.18)
$$\sum_{n=0}^{\infty} \frac{t^n n!}{\Gamma(n+\nu+1)} L_n^{\nu}(a)\, L_n^{\nu}(b) = \frac{(abt)^{-\nu/2}}{1-t} \exp\left\{-\frac{(a+b)\,t}{1-t}\right\} I_{\nu}\left(\frac{2\sqrt{abt}}{1-t}\right).$$

The closed-form formulas in Carr and Linetsky (2006) are more suitable for pricing under the original JDCEV model without time changes than the series expansions developed in this paper, as they are easier to compute. However, they are generally not suitable for time-changed models because they have complicated functional dependence on time. In contrast, the expansions in this paper explicitly depend on time only through the exponentials and are thus ideally suited for time changes with known Laplace transforms.

## 8.2. Introducing Jumps and Stochastic Volatility into the JDCEV Process via Time Changes: Numerical Examples

In this section, we illustrate our approach with numerical examples. We take the background diffusion process $X$ to be a JDCEV process with $\mu = 0$ and time change it with the composite time-change process $T_t = T^1_{T^2_t}$, where $T^1$ is the Inverse Gaussian (IG) subordinator with the Lévy measure $\nu(ds) = Cs^{-3/2}e^{-\eta s}$ and the Laplace exponent $\phi(s) = \gamma s + 2C\sqrt{\pi}(\sqrt{s+\eta} - \sqrt{\eta})$ and $T^2$ is the time integral of the activity rate following the CIR process. That is, the time-change process is an IG process with stochastic volatility in the terminology of Carr et al. (2003). To satisfy the martingale condition, according to Theorem 4.5 we set $\mu = 0$ and $\rho = r - q$. The time-changed process $Y_t := X_{T_t}$ is a martingale and the process $(Y_t, V_t)$ is a two-dimensional Markov process with the infinitesimal generator

(8.19)

$$\mathcal{G}f(x,v) = \frac{1}{2}\gamma v a^2 x^{2\beta+2}\frac{\partial^2 f}{\partial x^2}(x,v) + \gamma v(b + c\,a^2x^{2\beta})x\frac{\partial f}{\partial x}(x,v) - k(x,v)\,f(x,v)$$
$$+ \int_{(0,\infty)}(f(y,v) - f(x,v))v\pi(x,y)\,dy + \frac{\sigma_V^2}{2}v\frac{\partial^2 f}{\partial v^2}(x,v) + \kappa(\theta - v)\frac{\partial f}{\partial v}(x,v),$$

where the killing rate $k(x)$ and the state-dependent Lévy density $\pi(x,y)$ are

$$k(x,v) = \gamma v(b + c\,a^2x^{2\beta}) + vC\int_{(0,\infty)}\left(1 - \frac{\Gamma\left(\frac{c}{|\beta|}+1\right)(\tau(s))^{\frac{1}{2|\beta|}}e^{-\tau(s)-bs}}{\Gamma(\nu+1)}\,{}_1F_1\left(\begin{matrix}\frac{c}{|\beta|}+1\\ \nu+1\end{matrix};\tau(s)\right)\right)$$

$$\times s^{-3/2}e^{-\eta s}\,ds, \tag{8.20}$$

where

$$\tau(s) := \frac{\omega\,x^{-2\beta}}{2|\beta|^2a^2\,(1 - e^{-\omega s})} \tag{8.21}$$

and

$$\pi(x,y) = 2|\beta|AC\left(\frac{y}{x}\right)^{c-\frac{1}{2}} y^{-(2\beta+1)} \tag{8.22}$$
$$\times \int_{(0,\infty)} \frac{s^{-3/2}e^{\left(\frac{\omega\nu}{2}-\xi-\eta\right)s}}{e^{\omega s}-1}\exp\left\{-A\left(\frac{x^{-2\beta}e^{\omega s}+y^{-2\beta}}{e^{\omega s}-1}\right)\right\}$$
$$\times I_\nu\left(\frac{A(xy)^{-\beta}}{\sinh(\omega s/2)}\right)ds.$$

The stock price process in this model is a pure jump process with a jump to default that sends the process to zero, an absorbing state.

REMARK 8.3. In equations (8.20) and (8.22) it is convenient to use the closed form expressions for the survival probability and the transition density of the JDCEV process obtained in Carr and Linetsky (2006). The spectral expansion of the JDCEV transition probability of the form (7.1) with the eigenfunctions and eigenvalues given in Theorem 8.2 collapses to the closed-form expression in terms of the Bessel function on applying the Hille–Hardy formula (8.18).

The parameter values in our numerical example are listed in Table 8.1. The JDCEV process parameter $a$ entering into the local volatility function $\sigma(x) = a\,x^\beta$ is selected so that the local volatility is equal to 20% when the stock price is equal to fifty dollars, that is, $a = 0.2 \times 50^{-\beta} = 10$ for the case of $\beta = -1$ considered here. In this example, we select $\gamma = 0$, so the time-changed process is a pure jump process with no diffusion component (recall that the diffusion component vanishes for time changes with $\gamma = 0$). For this particular choice of parameters of the IG time change and the CIR activity rate process the time change has the mean and variance $E[T_1] = 1$ and $\text{Var}[T_1] = 1/16$ at $t = 1$. If we replace the background JDCEV process with Brownian motion with drift, then

TABLE 8.1
Parameter Values

| | | | | | |
|---|---|---|---|---|---|
| *JDCEV* | S | 50 | *CIR* | V | 1 |
| | $a$ | 10 | | $\theta$ | 1 |
| | $\beta$ | $-1$ | | $\sigma_V$ | 1 |
| | c | 0.5 | | $\kappa$ | 4 |
| | b | 0.01 | *IG* | $\gamma$ | 0 |
| | r | 0.05 | | $\eta$ | 8 |
| | q | 0 | | C | $2\sqrt{2/\pi}$ |

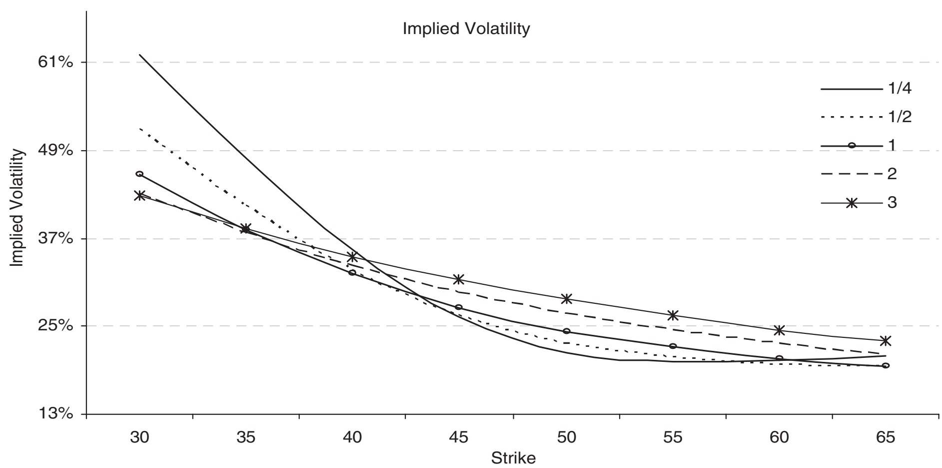


FIGURE 8.1. Implied volatility smile/skew curves as functions of the strike price for times to maturity from 0.25 to 3 years. Current stock price level is 50.

the time-changed process is a Normal Inverse Gaussian (NIG) process with stochastic volatility following the CIR process as in Carr et al. (2003). Our model extends Carr et al. (2003) in two important respects. By taking the background process to be a diffusion process with state-dependent volatility and drift, the resulting Lévy density after time change is state dependent, in contrast to the space homogeneous Lévy jumps. Second, the time-changed process has a state-dependent killing rate (default intensity) in contrast to the absence of default in Carr et al. (2003). By extending the framework of Carr et al. (2003) to state-dependent jumps and default intensity, we gain the flexibility of being able to calibrate the model jointly to options prices and CDS spreads. Moreover, the state dependence of jumps allows for more flexibility in fitting implied volatility surfaces observed in the equity options market than is available under space homogeneous Lévy models.

Figure 8.1 plots the implied volatility smile/skew curves of options priced under this model for several different maturities. The implied volatility values are shown in Table 8.2. We compute options prices in this model using Theorem 8.4 and then compute implied volatilities of these options by inverting the Black–Scholes formula. We observe that in this model shorter maturity skews are steeper and flatten out as maturity increases,

TABLE 8.2
Implied Volatilities (in %) for Different Strike Prices and Times to Maturity (Years)

| Time/strike | 30 | 35 | 40 | 45 | 50 | 55 | 60 | 65 |
|---|---|---|---|---|---|---|---|---|
| 1/4 | 62.04 | 47.94 | 35.52 | 26.19 | 21.41 | 20.09 | 20.28 | 20.88 |
| 1/2 | 51.94 | 41.47 | 32.72 | 26.39 | 22.64 | 20.72 | 19.84 | 19.46 |
| 1 | 45.74 | 38.24 | 32.14 | 27.53 | 24.30 | 22.12 | 20.65 | 19.64 |
| 2 | 43.03 | 37.68 | 33.23 | 29.61 | 26.72 | 24.45 | 22.66 | 21.25 |
| 3 | 42.80 | 38.34 | 34.55 | 31.34 | 28.64 | 26.39 | 24.52 | 22.96 |

Note: Current stock price level is 50.

consistent with empirical observations in options markets. We also observe that the short maturity skew exhibits a true volatility smile with the increase in implied volatilities both to the right and to the left of the at-the-money strike. This behavior cannot be captured in the pure diffusion JDCEV model. In JDCEV, the implied volatility skew results from the leverage effect (the local volatility is a decreasing function of stock price) and the possibility of default (the default intensity is a decreasing function of stock price). The resulting implied volatility skew is a decreasing function of strike. After the time change with jumps, the resulting jump process has both positive and negative jumps. This results in the implied volatility smile pattern with volatility "smiling" on both sides of the at-the-money level. Table 8.3 presents sample put prices for several strike and maturity combinations. The prices are computed to the accuracy of $10^{-4}$ (all of the decimals presented in the table are correct) by computing the corresponding eigenfunction expansions. All computations in this paper were performed in *Mathematica*. All the special functions appearing in the JDCEV model solution are available as built-in functions in *Mathematica*.

Figure 8.2 plots the default probability and the credit spread (assuming zero recovery in default) as functions of time to maturity for several different levels of the stock price. As the stock price decreases, the credit spreads of all maturities increase, but the shorter and intermediate maturities increase the fastest. In particular intermediate maturities of between two and seven years increase the fastest. This results in a pronounced hump in the term structure of credit spreads around these intermediate maturities. As the stock price falls, the hump becomes more pronounced and shifts toward shorter maturities. This increase in credit spreads with the decrease in the stock price is accounted for both the leverage effect through the increase in the local volatility of the original diffusion and, hence, more jump volatility for the jump process after the time change, as well as the increase in the default intensity of both the original diffusion process and the jump process after the time change. Figure 8.3 plots the default intensity (killing rate) in this model after the time change as a function of the stock price given by equation (8.20). The default intensity is a decreasing function of the stock price.

REMARK 8.4. We note the following interesting feature of our model. If we take the standard Cox's CEV diffusion process (without the jump to default introduced in Carr and Linetsky 2006) to serve as the background Markov process and time change it with a Lévy subordinator, the resulting process acquires a default intensity, even though the original CEV process does not have any killing rate. Indeed, the default event in the CEV process can only occur via hitting zero by diffusing down to the zero stock price. The default time in the original CEV process is predictable with an announcing sequence of

TABLE 8.3
Put Prices

| Strike/time | 1/4 | 1/2 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|
| 30 | (0.2227, 0.0006) | (0.4405, 0.0059) | (0.8633, 0.039) | (1.6669, 0.1334) | (2.42, 0.1855) | (3.1062, 0.189) | (3.7063, 0.1667) |
| | 0.2233 | 0.4464 | 0.9023 | 1.8003 | 2.6055 | 3.2952 | 3.8730 |
| 35 | (0.2598, 0.0053) | (0.5139, 0.0317) | (1.0072, 0.1293) | (1.9447, 0.3088) | (2.8233, 0.3779) | (3.6239, 0.3646) | (4.3241, 0.3135) |
| | 0.2650 | 0.5457 | 1.1365 | 2.2535 | 3.2012 | 3.9885 | 4.6376 |
| 40 | (0.2969, 0.0416) | (0.5873, 0.1494) | (1.1511, 0.3786) | (2.2226, 0.6489) | (3.2266, 0.7056) | (4.1416, 0.6463) | (4.9418, 0.5423) |
| | 0.3385 | 0.7368 | 1.5297 | 2.8715 | 3.9323 | 4.7879 | 5.4841 |
| 45 | (0.334, 0.2852) | (0.6607, 0.5886) | (1.295, 0.9697) | (2.5004, 1.2505) | (3.63, 1.2261) | (4.6593, 1.0714) | (5.5595, 0.8786) |
| | 0.6192 | 1.2493 | 2.2647 | 3.7509 | 4.8560 | 5.7308 | 6.4381 |
| 50 | (0.3711, 1.4541) | (0.7342, 1.8376) | (1.4389, 2.159) | (2.7782, 2.2248) | (4.0333, 2.0021) | (5.177, 1.6801) | (6.1772, 1.3499) |
| | 1.8252 | 2.5718 | 3.5978 | 5.0030 | 6.0354 | 6.8571 | 7.5271 |
| 55 | (0.4082, 4.5108) | (0.8076, 4.3813) | (1.5827, 4.1757) | (3.056, 3.6744) | (4.4366, 3.0944) | (5.6947, 2.5121) | (6.795, 1.9837) |
| | 4.9190 | 5.1889 | 5.7584 | 6.7304 | 7.5310 | 8.2068 | 8.7787 |
| 60 | (0.4453, 8.9164) | (0.881, 8.1355) | (1.7266, 7.0824) | (3.3338, 5.6661) | (4.84, 4.552) | (6.2125, 3.6029) | (7.4127, 2.8062) |
| | 9.3618 | 9.0165 | 8.8090 | 9.0000 | 9.3919 | 9.8153 | 10.2189 |
| 65 | (0.4824, 13.7265) | (0.9544, 12.5737) | (1.8705, 10.7341) | (3.6117, 8.2104) | (5.2433, 6.4037) | (6.7302, 4.9796) | (8.0304, 3.8396) |
| | 14.2090 | 13.5281 | 12.6047 | 11.8220 | 11.6470 | 11.7098 | 11.8700 |

Note: For each combination of strike and time to maturity two values are given in parenthesis. The first value is the price $P_D$ of the default claim equation (5.9). The second value is the price $P_0$ of the put payoff paid only if there is no default equation (5.8). The third value below is the put option price equal to $P_D + P_0$.

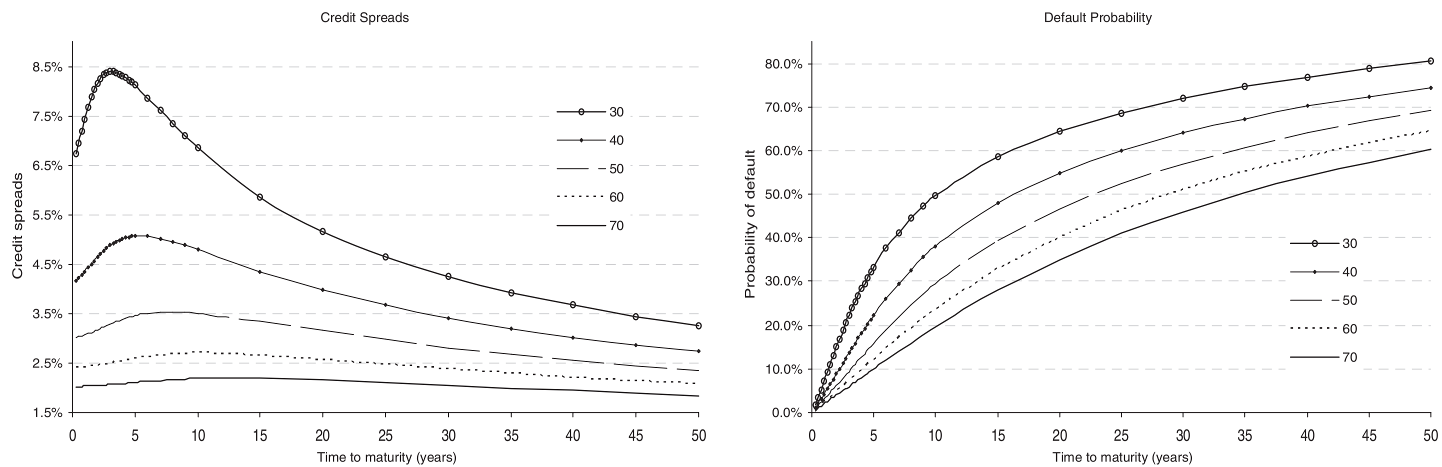


FIGURE 8.2. Credit spreads and default probabilities as functions of time to maturity for current stock price levels $S = 30, 40, 50, 60, 70$.

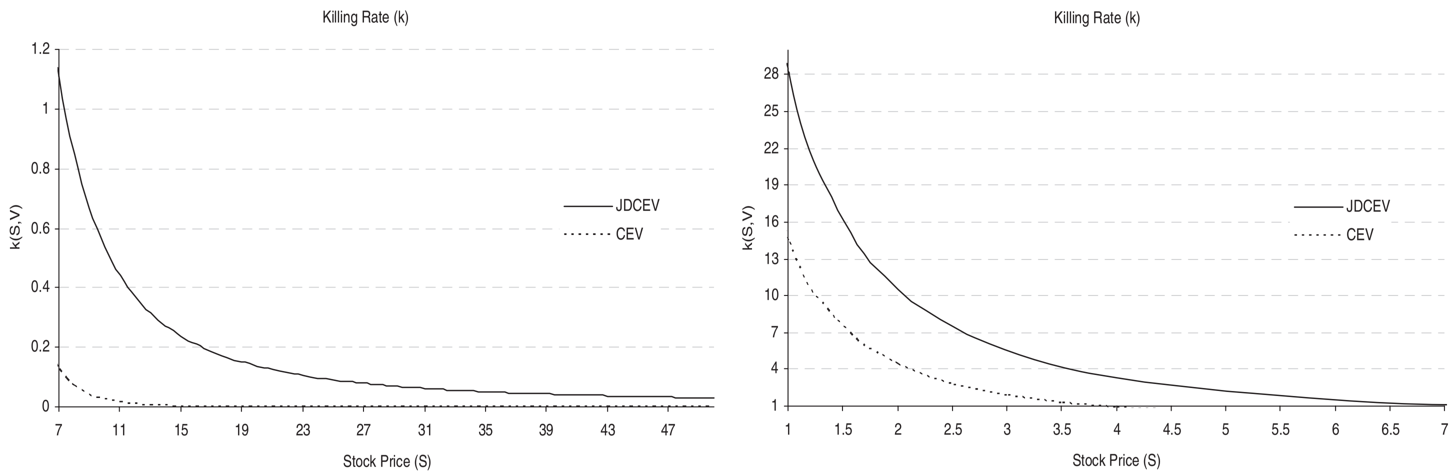


FIGURE 8.3. Killing rate (default intensity) $k$(S,V) as a function of the stock price $S$ when the activity rate is fixed at $V = 1$. The solid line is the default intensity in the model obtained by time changing the JDCEV process. The dashed line is the default intensity obtained by time changing the standard CEV process with $b = c = 0$ without the jump to default (see Remark 8.4).

hitting times of stock price levels decreasing toward zero. However, after a pure jump time change, the default time in the time-changed jump process becomes totally inaccessible with the intensity given by the integral of the default probability of the original CEV process with the Lévy measure of the subordinator in equation (8.20) (in this case $b = 0$ because the standard CEV process does not have any default intensity). This default intensity is plotted in Figure 8.3 as a dashed line. Intuitively, one can understand this as follows. Suppose one observes a sample path of a diffusion process that hits zero. Because the process is continuous, one observes the announcing sequence of the default event. When the diffusion is subjected to a pure jump time change, one can no longer observe the announcing sequence, as the hitting times of the intermediate stock price levels are left unobservable when the time jumps through them. As a result, the default event in the time-changed process looks like an unpredictable jump to default from a positive value governed by the default intensity induced by the time change.

## 9. CONCLUSION

This paper develops a novel class of hybrid credit-equity models with state-dependent jumps, local-stochastic volatility, and default intensity based on time changes of Markov processes with killing. We model the defaultable stock price process as a time-changed Markov diffusion process with state-dependent local volatility and killing rate (default intensity). When the time change is a Lévy subordinator, the stock price process exhibits jumps with state-dependent Lévy measure. When the time change is a time integral of an activity rate process, the stock price process has local-stochastic volatility and default intensity. When the time-change process is a Lévy subordinator in turn time changed with a time integral of an activity rate process, the stock price process has state-dependent jumps, local-stochastic volatility, and default intensity. This framework offers far-reaching extensions of the framework of time-changed Lévy processes with stochastic volatility of Carr et al. (2003). By time-changing Markov processes we relax the space homogeneity assumption inherent in Lévy models. Moreover, the mechanism of killing a Markov process at a state-dependent rate is well suited to modeling the default event.

This paper develops two analytical approaches to the pricing of credit and equity derivatives in this class of models. The two approaches are based on the Laplace transform inversion and the spectral expansion approach, respectively. If the resolvent (the Laplace transform of the transition semigroup) of the diffusion process and the Laplace transform of the time change are both available in closed form, the expectation operator of the time-changed process is expressed in closed form as a single integral in the complex plane. If the payoff is square-integrable, the complex integral is further reduced to a spectral expansion. To illustrate our general framework, we time change the JDCEV model of Carr and Linetsky (2006) and obtain a rich class of analytically tractable models with jumps, local-stochastic volatility, and default intensity. These models can be used to jointly price equity and credit derivatives. In particular, we compute implied volatility surfaces, default probabilities, and credit spreads under the JDCEV process subject to the time change that is an inverse Gaussian subordinator that is itself subject to a time change with a CIR activity rate process. This process is a pure jump process with state-dependent jumps and killing (jump to default) in contrast to the pure diffusion JDCEV model of Carr and Linetsky (2006).

The contribution of this paper is in the development of a flexible modeling framework, as well as in the development of the analytical methods to solve this class of models.

A wide range of models can be constructed within this model architecture by pairing background diffusion processes with different time changes. We hope that this paper will stimulate empirical research into the joint credit-equity dynamics and the interplay between credit and equity derivatives markets.

## APPENDIX A: PROOFS

### A.1. Proof of Theorem 4.1

Because by Theorem 4.3, the process $e^{-\rho t}S_t = \mathbf{1}_{\{t<\tau_d\}}X_{T_t}$ is a time-homogeneous Markov process, it is enough to prove that

$$\mathbb{E}[\mathbf{1}_{\{t<\tau_d\}}X_{T_t}] = e^{(r-q-\rho)t}x \text{ for all } t > 0, \tag{A.1}$$

where $S_0 = X_0 = x > 0$. Let $\mathcal{F}_t^X = \sigma\{X_s, s \le t\}$ and $\mathcal{F}_t^T = \sigma\{T_s, s \le t\}$ be the filtrations generated by the background diffusion process $X$ and the time change $T$. Observing that $\mathbf{1}_{\{t<\tau_d\}} = \mathbf{1}_{\{T_t<\zeta\}} = \mathbf{1}_{\{T_t<H_0\}}\mathbf{1}_{\{T_t<\zeta\}}$ and $\mathbb{E}[\mathbf{1}_{\{t<H_0\}}\mathbf{1}_{\{t<\zeta\}}|\mathcal{F}_t^X] = \mathbf{1}_{\{t<H_0\}}e^{-\int_0^t h(X_u)\,du}$, we can write

$$\begin{aligned}\mathbb{E}[\mathbf{1}_{\{t<\tau_d\}}X_{T_t}] &= \mathbb{E}\left[\mathbf{1}_{\{T_t<H_0\}}e^{-\int_0^{T_t} h(X_u)\,du}X_{T_t}\right]\\ &= x\mathbb{E}\left[\mathbf{1}_{\{T_t<H_0\}}e^{-\int_0^{T_t} h(X_u)\,du}e^{\int_0^{T_t}[\mu+h(X_u)]du+\int_0^{T_t}\sigma(X_u)\,dB_u-\frac{1}{2}\int_0^{T_t}\sigma^2(X_u)\,du}\right]\\ &= x\mathbb{E}\left[e^{\mu T_t}\mathbf{1}_{\{T_t<H_0\}}e^{\int_0^{T_t}\sigma(X_u)\,dB_u-\frac{1}{2}\int_0^{T_t}\sigma^2(X_u)\,du}\right],\end{aligned}$$

where in the second equality we used the SDE (2.2). Because the volatility $\sigma(x)$ remains bounded as $x \to \infty$, the process $\mathbf{1}_{\{t<H_0\}}e^{\int_0^t \sigma(X_u)\,dB_u-\frac{1}{2}\int_0^t\sigma^2(X_u)\,du}$ stopped at $H_0$ is an exponential martingale starting at one (e.g., Delbaen and Shirakawa 2002).

Now suppose $\mu \in \mathcal{I}_\nu$. Then, conditioning on the time change, we have

$$x\mathbb{E}[e^{\mu T_t}\mathbb{E}]\mathbf{1}_{\{T_t<H_0\}}e^{\int_0^{T_t}\sigma(X_u)\,dB_u-\frac{1}{2}\int_0^{T_t}\sigma^2(X_u)\,du}|\mathcal{F}_t^T]] = x\mathbb{E}[e^{\mu T_t}] = xe^{-t\phi(-\mu)}.$$

Comparing the right-hand side with that of equation (A.1), we conclude that equation (A.1) holds if and only if $\rho = r - q + \phi(-\mu)$. If $\mu \notin \mathcal{I}_\nu$, then $\mathbb{E}[e^{\mu T_t}]$ is infinite and (A.1) cannot be satisfied and, hence, the process (2.1) does not satisfy the martingale condition (2.5)–(2.6). □

### A.2. Proof of Theorem 4.2

Define $f(\mu) := -\phi(-\mu)$. If $\gamma > 0$ or $\gamma = 0$ and the subordinator is of infinite activity ($\int_{(0,\infty)}\nu(ds) = \infty$), then $f(\mu)$ tends to $-\infty$ as $\mu \to -\infty$. If $\gamma = 0$ and the subordinator is of finite activity ($\int_{(0,\infty)}\nu(ds) = \alpha < \infty$), then $f(\mu)$ tends to $-\alpha$. If $\bar{\mu}$ is not included in $\mathcal{I}_\nu$, then $f(\mu)$ tends to $+\infty$ as $\mu \to \bar{\mu}$. If $\bar{\mu}$ is included in $\mathcal{I}_\nu$, then $f(\bar{\mu}) = \gamma\bar{\mu} + \int_{(0,\infty)}(e^{\bar{\mu}s} - 1)\nu(ds) < \infty$. We thus have the following alternatives for the existence of solutions of the equation $f(\mu) = r - q$. If $r < q$, then there is a unique solution $\mu_0$ for all subordinators except for subordinators with zero drift and finite activity Lévy measure with Poisson intensity $\alpha$ such that $-\alpha > r - q$. If $r > q$, then there is a unique solution if either $\bar{\mu}$ is not included in $\mathcal{I}_\nu$ or $\bar{\mu}$ is included in $\mathcal{I}_\nu$ and $r - q \le f(\bar{\mu})$. Otherwise, if $\bar{\mu}$ is included in $\mathcal{I}_\nu$ and $r - q > f(\bar{\mu})$, Equation (4.3) has no solution in $\mathcal{I}_\nu$. The statement for the case $r = q$ is immediate from the fact that $\phi(0) = 0$. □

## A.3. Proof of Theorem 4.3

The idea of time changing a Markov process with a Lévy subordinator is originally due to Bochner (1949, 1955). The following fundamental theorem due to Phillips (1952) (see Sato 1999, Theorem 32.1, p. 212) characterizes the time-changed transition semigroup and its infinitesimal generator.

THEOREM A.1 (Phillip's Theorem; Sato 1999, p. 212). *Let* $\{T_t, t \geq 0\}$ *be a subordinator with Lévy measure* $\nu$, *drift* $\gamma$, *Laplace exponent* $\phi(\lambda)$, *and transition kernel* $\pi_t(ds)$. *Let* $\{\mathcal{P}_t, t \geq 0\}$ *be a strongly continuous contraction semigroup of linear operators in the Banach space* $\mathbf{B}$ *with infinitesimal generator* $\mathcal{G}$. *Define (the superscript* $\phi$ *refers to the subordinated quantities with the subordinator with the Laplace exponent* $\phi$*)*

$$\mathcal{P}_t^{\phi} f = \int_{[0,\infty)} (\mathcal{P}_s f)\pi_t(ds), \quad f \in \mathbf{B}. \tag{A.2}$$

*Then* $\{\mathcal{P}_t^{\phi}, t \geq 0\}$ *is a strongly continuous contraction semigroup of linear operators on* $\mathbf{B}$. *Denote its infinitesimal generator by* $\mathcal{G}^{\phi}$. *Then* $\mathrm{Dom}(\mathcal{G}) \subset \mathrm{Dom}(\mathcal{G}^{\phi})$, $\mathrm{Dom}(\mathcal{G})$ *is a core of* $\mathcal{G}^{\phi}$, *and*

$$\mathcal{G}^{\phi} f = \gamma \mathcal{G} f + \int_{(0,\infty)} (\mathcal{P}_s f - f)\nu(ds), \quad f \in \mathrm{Dom}(\mathcal{G}). \tag{A.3}$$

In our case, the Banach space $\mathbf{B}$ is $C_0(0, \infty)$ (the space of continuous bounded functions on $(0, \infty)$ vanishing at infinity) and the semigroup $\{\mathcal{P}_t, t \geq 0\}$ is the Feller transition semigroup of the diffusion process $X$ with lifetime $\zeta$. Given our assumptions, the one-dimensional diffusion $X$ always has a transition density $p(t; x, y)$ with respect to the Lebesgue measure, so that $\mathcal{P}_t f(x) = \int_{(0,\infty)} f(y)p(t; s, y)\,dy$, and, moreover, $p(t; x, y)$ is continuous in all its variables. Then from equation (A.2) we obtain the density (4.11) of the subordinate process $X^{\phi} = X_{T_t}$. From equation (A.3) we can identify the infinitesimal generator of $X^{\phi}$ in equations (4.5)–(4.6) (see the online companion to this paper for the explicit calculation; for mathematical references on the Lévy characteristics of subordinate Markov processes, see Okura 2002; Theorem 2.1; Chen and Song 2005; Section 2). □

## A.4. Proof of Theorem 4.4

The proof is similar to the proof of Theorem 4.1. Because the process $(e^{-\rho t} S_t, Z_t) = (\mathbf{1}_{\{t<\tau_d\}} X_{T_t}, Z_t)$ is an $(n+1)$-dimensional time-homogeneous Markov process, it is enough to prove that equation (A.1) holds. Suppose $\mu \in \mathbb{R}$ is such that $\mathbb{E}[e^{\mu T_t}] = \mathcal{L}(t, -\mu) < \infty$. Proceeding as in the proof of Theorem 4.1 and conditioning on the time change, the left-hand side of equation (A.1) reduces to

$$x\mathbb{E}\big[e^{\mu T_t}\mathbb{E}\big[\mathbf{1}_{\{T_t<H_0\}} e^{\int_0^{T_t} \sigma(X_u)\,dB_u - \frac{1}{2}\int_0^{T_t} \sigma^2(X_u)\,du} | \mathcal{F}_t^T\big]\big] = x\mathbb{E}[e^{\mu T_t}] = x\mathcal{L}(t, -\mu).$$

We conclude that equation (A.1) holds if and only if $\mathcal{L}(t, -\mu) = e^{(r-q-\rho)t}$. However, for $\mu \neq 0$, the Laplace transform $\mathcal{L}(t, -\mu)$ is an exponential function of time if and only if the time change process has stationary and independent increments, that is, is a Lévy subordinator. The only absolutely continuous time change that is a Lévy subordinator is a trivial time change with constant activity rate $V_t = \gamma$ so that $T_t = \gamma t$.

Hence we conclude that $\mathcal{L}(t, -\mu) = e^{(r-q-\rho)t}$ cannot hold for any $\mu \neq 0$ for any nontrivial absolutely continuous time change. For $\mu = 0$, we have that $\mathcal{L}(t, 0) = 1$, and $e^{(r-q-\rho)t} = 1$ is satisfied if and only if $\rho = r - q$. □

### A.5. Proof of Theorem 4.5

The proof is completely analogous to that of Theorem 4.4. Suppose that $\mu \in \mathcal{I}_\nu$ and such that $\mathbb{E}[e^{\mu T_t}] = \mathcal{L}(t, \phi(-\mu)) < \infty$. Then arguing as in the proof of Theorem 4.4 we arrive at the following necessary and sufficient condition for the process $S$ to satisfy the martingale condition (2.5)–(2.6): $\mathcal{L}(t, \phi(-\mu)) = e^{(r-q-\rho)t}$. The only solution for a composite time change (3.6) with $T^2$ having a nonconstant activity rate process $V$ is $\mu = 0$ and $\rho = r - q$. □

### A.6. Proof of Theorem 8.1

(i) Consider the Sturm–Liouville equation (6.4) with the operator (8.3) with $\mu + b > 0$. Substitute $u(x) = x^{\frac{1}{2}-c+\beta} e^{-\frac{A}{2}x^{-2\beta}} v(y)$ with $y = Ax^{-2\beta}$, where $A$ is defined in (8.4). The Sturm–Liouville equation for the function $u(x)$ reduces to the Whittaker equation for the function $v(y)$

$$\frac{d^2 v}{dy^2}(y) + \left(-\frac{1}{4} + \frac{\varkappa(s)}{y} + \frac{1-\nu^2}{4y^2}\right) v(y) = 0, \tag{A.4}$$

with $\nu$, $\varkappa(s)$, $\xi$, and $\omega$ as defined in (8.7). The increasing and decreasing solutions of the Whittaker equation are given by the Whittaker functions $v_1(y) = M_{\varkappa(s), \frac{\nu}{2}}(y)$ and $v_2(y) = W_{\varkappa(s), \frac{\nu}{2}}(y)$, respectively. Their Wronskian is given by $\mathfrak{W}(v_1, v_2)(y) := v_1(y)v_2'(y) - v_1'(y)v_2(y) = -\Gamma(1+\nu)/\Gamma\left(\frac{1+\nu}{2} - \varkappa(s)\right)$. Thus, the increasing and decreasing solutions of the original Sturm–Liouville equation are given by (8.5) and (8.6), and the Wronskian $w_s$ with respect to the scale density is given by (8.8).

(ii) When $\mu + b = 0$, the substitution $u(x) = x^{\frac{1}{2}-c} v(y)$ with $y = \frac{x^{-\beta}}{a|\beta|}$ reduces the Sturm–Liouville equation (6.4) with the operator (8.3) to the modified Bessel equation of order $\nu$ (with $\nu$ as in (8.7))

$$y^2 \frac{d^2 v}{dy^2}(y) + y \frac{dv}{dy}(y) - (\nu^2 + 2(s+b)y^2) v(y) = 0. \tag{A.5}$$

The increasing and decreasing solutions of the modified Bessel equation are given by the modified Bessel functions $v_1(y) = I_\nu\left(y\sqrt{2(s+b)}\right)$ and $v_2(y) = K_\nu\left(y\sqrt{2(s+b)}\right)$, respectively. Their Wronskian is given by $\mathfrak{W}(v_1, v_2)(y) = -1/y$. Thus, the increasing and decreasing solutions of the original Sturm–Liouville equation are given by (8.9), and the Wronskian $w_s$ with respect to the scale density is given by (8.10). □

### A.7. Proofs of Theorems 8.2, 8.3, and 8.4

Proofs of Theorems 8.2, 8.3, and 8.4 are included in the companion appendix available from the authors upon request.

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