

The Stop-Loss Start-Gain Paradox and Option Valuation: A New Decomposition into Intrinsic and Time Value

Peter P. Carr
Robert A. Jarrow
Cornell University

The downside risk in a leveraged stock position can be eliminated by using stop-loss orders. The upside potential of such a position can be captured using contingent buy orders. The terminal payoff to this stop-loss start-gain strategy is identical to that of a call option, but the strategy costs less initially. This article resolves this paradox by showing that the strategy is not self-financing for continuous stock-price processes of unbounded variation. The resolution of the paradox leads to a new decomposition of an option's price into its intrinsic and time value. When the stock price follows geometric Brownian motion, this decomposition is proven to be mathematically equivalent to the Black-Scholes (1973) formula.

The key insight of the celebrated Black-Scholes (1973) model is that an option's payoff can be replicated by continuously trading in bonds and the underlying stock. In the Black-Scholes economy,

We are grateful for comments and assistance from Kaushik Amin, Warren Bailey, Michael Brennan, Phil Dybvig, Sheldon Gao, David Heath, Jonathan Ingersoll, Ravi Myneni, Zeynep Onder, Lawrence Robinson, Stephen Ross, Chester Spatt, the referee Kerry Back, and an anonymous referee. We also thank the participants of the Cornell Finance Workshop and the Workshop on Mathematical Theory of Modern Financial Markets. They are not responsible for any errors. Address reprint requests to Peter P. Carr, S. C. Johnson Graduate School of Management, Cornell University, Ithaca, NY 14853.

The Review of Financial Studies 1990 Volume 3, number 3, pp. 469–492
© 1990 The Review of Financial Studies 0893-9454/90/\$1.50

however, there is a far simpler trading rule in these assets, which also duplicates the payoff to a European call option. For simplicity, suppose that the option is initially out-of-the-money (i.e., the stock price starts below the *present value* of the exercise price). Using the simple trading rule, the investor only trades when the option is at-the-money. When the investor trades, he buys one share of stock using borrowed funds every time the call option goes in-the-money. Conversely, the investor sells the share and repays the loan whenever the option subsequently falls out-of-the-money. When liquidated at expiration, this trading rule pays the difference between the stock and the strike price if the option finishes in-the-money, and zero otherwise. Consequently, the trading rule's liquidation value replicates the payoff of a European call option.

This trading rule has previously been recognized in the finance literature. Seidenverg (1988) terms it a stop-loss start-gain strategy,¹ and analyzes it using a binomial model for stock-price movements. Carr (1989) generalizes Seidenverg's results in allowing for positive dividends and interest rates. Ingersoll (1987) uses the strategy to argue that the diffusion limit of this binomial process is "infinitely variable." Carr (1989) and Omberg (1988) study the diffusion limit of the binomial model. Asay and Edelsburg (1986) and Hull and White (1987) test the effectiveness of the strategy in duplicating a call option by simulating this diffusion limit.

When the option to be replicated starts out-of-the-money, the above replicating strategy is initially costless. Furthermore, the strategy *appears* to be self-financing when the stock-price process is continuous.² If the strategy is self-financing, then the trading strategy represents an arbitrage opportunity. A paradox then arises because the Black-Scholes model appears to allow arbitrage opportunities, and yet relies on their absence to price options.

The literature cited above has argued that the stop-loss start-gain strategy is not self-financing. We provide a formal proof of this result, thereby rigorously resolving the foregoing stop-loss start-gain paradox. We relate the external financing costs to a concept called "local time," which has long been in the stochastic-processes literature. By valuing these external financing costs, we develop a new decomposition of an option's price into its intrinsic and time value.

¹ Cox and Rubinstein (1985) analyze a stop-loss strategy without a start-gain provision. They show that a leveraged stock position, hedged by a stop-loss order, provides a different kind of insurance than a call option.

² Livingston (1987) implicitly assumes that a variant of the trading rule is self-financing under all continuous processes. This leads him to conclude mistakenly that the quality option in forward contracts has zero value for any continuous process. Kane and Marcus (1988) argue that this quality option must have positive value to avoid arbitrage. The results of Harrison, Pitbladdo, and Schaefer (1984) and of this article can be used to show that the quality option cannot be priced by arbitrage when the continuous price process is of bounded variation, and has positive value otherwise.

In independent and parallel work, Omberg (1988) also relates the external financing costs to local time. He considers a wider class of strategies, termed binary strategies, which contains the above stop-loss start-gain strategy as a special case. Working in a discrete time binomial framework, he shows that binary strategies are not self-financing, even in the diffusion limit. This work differs from Omberg's in that a continuous time framework is assumed throughout. We also work with a much wider class of continuous stochastic processes than geometric Brownian motion. Finally, we show that our decomposition is mathematically equivalent to the Black–Scholes (1973) formula, when the stock price follows geometric Brownian motion.

The structure of this article is as follows. In Section 1, the Black–Scholes model, some terminology, and the stop-loss start-gain strategy are described. In Section 2, we show that this strategy is not self-financing and relates the external financing costs to local time. Section 3 uses these results to develop a new pricing formula for options. In Section 4, we widen the class of stochastic process considered from geometric Brownian motion to the class of continuous semimartingales of unbounded variation. Proofs of some lemmas and technical results are relegated to the Appendix.

1. The Black–Scholes Model, Terminology, and the Stop-Loss Start-Gain Strategy

We adopt the Black–Scholes (1973) model for simplicity.

Assumption 1. *Perfect Bond and Stock Markets: There are no short sale restrictions, transactions costs, taxes, or other frictions. Investors can trade bonds and stocks continuously in time.*

Assumption 2. *Constant Interest Rate: The yield to maturity on a pure discount bond paying one dollar at the option's expiration date T is a positive constant r . Consequently, the bond's price, $P(t)$, obeys $dP(t) = rP(t) dt$, for all $t \in [0, T]$, and $P(T) = 1$, that is,*

$$P(t) = e^{-r(T-t)}. \quad (1)$$

Assumption 3. *No Payouts: The stock underlying the option has no payouts over the option's life.*

Assumption 4. *Geometric Brownian Motion: The stock price moves continuously through time according to the stochastic process $\{S(t); t \in [0, T]\}$ defined by*

$$dS(t) = [\alpha dt + \sigma dW(t)]S(t),$$

where the initial stock price $S(0)$, the expected rate of return α , and the return volatility σ are positive constants.

The term $dW(t)$ is the differential of a (standard) Weiner process $\{W(t); t \in [0, T]\}$ defined³ on a probability space (Ω, \mathcal{F}, Q) . We relax the assumption of geometric Brownian motion in Section 5.

We next define some terms used frequently in the paper. A *trading strategy* is a stochastic process $\{m(t), n(t); t \in [0, T]\}$, satisfying certain technical conditions⁴ that rule out doubling strategies and other arbitrage opportunities. These processes specify the number of bonds, $m(t)$, and stocks, $n(t)$, held in the portfolio. Thus, the value of this portfolio at time t , $V(t)$, is

$$V(t) \equiv m(t)P(t) + n(t)S(t), \quad \text{for all } t \in [0, T]. \quad (2)$$

The trading strategy that will be analyzed in detail in this paper is the *stop-loss start-gain strategy*, defined below, for an arbitrary floor X and time period T . Let

$$m(t) = -1_{\{S(t) > XP(t)\}} X,$$

and

$$n(t) = 1_{\{S(t) > XP(t)\}}, \quad \text{for all } t \in [0, T], \quad (3)$$

where $1_{\{A\}}$ is the indicator function of the set A . This strategy involves holding one share of stock and maintaining a short position in X bonds whenever the stock price is above the present value of an arbitrarily chosen floor X . If the stock price subsequently returns to the critical level, the leveraged stock position is liquidated. Note that no stocks or bonds are held at the critical level. If the strategy were altered so that a levered stock position is held at-the-money, then the propositions to be presented in the following sections would remain the same.

Substituting the stop-loss start-gain strategy (3) into (2) implies that the resulting portfolio value at time t is the lower bound for a call option with exercise price X and maturity date T ,

³ Technically, we require that the stock-price process be adapted to the Brownian filtration $\{\mathcal{F}_t; t \in [0, T]\}$, mapping $[0, T] \times \Omega$ into the real line \mathbb{R} . The Brownian filtration $\{\mathcal{F}_t; t \in [0, T]\}$ is augmented such that \mathcal{F}_0 contains all the Q -negligible events and satisfies $\mathcal{F}_T = \mathcal{F}$.

⁴ The technical conditions are (i) $m(t), n(t)$ are adapted to $\{\mathcal{F}_t; t \in [0, T]\}$, and (ii)

$$\tilde{E}_0 \left[\int_0^T m^2(t) dt \right] < +\infty, \quad \tilde{E}_0 \left[\int_0^T n^2(t) S^2(t) dt \right] < +\infty,$$

where $\tilde{E}_0[\cdot]$ is a shorthand notation for $\tilde{E}[\cdot | \mathcal{F}_0] = \int[\cdot] d\tilde{Q}$. The first condition ensures that trading strategies are only based on available information. The second condition says that the position in each asset is integrable in mean square, where expectations are calculated using an equivalent martingale measure defined in Lemma A4 of the Appendix. Condition (ii) excludes the doubling strategy, as discussed in Duffie and Huang (1985), Heath and Jarrow (1987), and Dybvig and Huang (1988).

$$\begin{aligned}
 V(t) &= -1_{\{S(t) > XP(t)\}}XP(t) + 1_{\{S(t) > XP(t)\}}S(t) \\
 &= \max[0, S(t) - XP(t)], \quad \text{for all } t \in [0, T],
 \end{aligned} \tag{4}$$

and, because $P(T) = 1$, the terminal portfolio replicates this call's payoff:

$$V(T) = \max[0, S(T) - X]. \tag{5}$$

A trading strategy is called *self-financing* if the resulting portfolio value satisfies

$$\begin{aligned}
 V(t) &= V(0) + \int_0^t m(v) dP(v) + \int_0^t n(v) dS(v), \\
 &\text{for all } t \in [0, T].
 \end{aligned}$$

Intuitively, the strategy is self-financing if the portfolio's value arises only from the initial investment, $V(0)$, and from capital gains or losses experienced from the assets held. In particular, intermediate injections or withdrawals of funds are proscribed. The self-financing condition simplifies if the bond is made numeraire:

$$\frac{V(t)}{P(t)} = \frac{V(0)}{P(0)} + \int_0^t n(v) d\left(\frac{S(v)}{P(v)}\right), \quad \text{for all } t \in [0, T]. \tag{6}$$

To simplify notation, define

$$F(t) \equiv \frac{S(t)}{P(t)} = S(t)e^{r(T-t)}, \quad \text{for all } t \in [0, T], \tag{7}$$

from (1). We refer to $F(t)$ as the stock price in the bond numeraire, although it is also the forward price of the stock in dollars under Assumptions 1 and 3. Using this notation, the stop-loss start-gain strategy (3) is self-financing if and only if, for all $t \in [0, T]$,

$$\begin{aligned}
 &\max[0, F(t) - X] \\
 &= \max[0, F(0) - X] + \int_0^t 1_{\{F(v) > X\}} d(F(v)).
 \end{aligned} \tag{8}$$

An *arbitrage opportunity* is defined to be a self-financing trading strategy requiring no initial investment, having no probability of negative value at expiration, and yet having some chance of a positive payoff:

- (i) $V(0) = 0$,
- (ii) $Q(V(T) \geq 0) = 1$,
- (iii) $Q(V(T) > 0) > 0$.

Consider the value of the portfolio arising from following the stop-loss start-gain strategy (3) with floor X above the initial stock price in the bond numeraire $F(0)$. This value satisfies the three defining conditions of an arbitrage opportunity. The strategy also appears to be self-financing since the assumed geometric Brownian motion process is continuous. Consequently, it appears that there are arbitrage opportunities in the Black–Scholes model. Fortunately, the next section proves that the strategy is not self-financing, and consequently resolves this stop-loss start-gain paradox.

2. Resolution of the Paradox

Sections 2 and 3 present two propositions that we⁵ term “ex post” and “ex ante.” This section presents and proves the ex post proposition, which states that the stop-loss start-gain strategy is not self-financing. We also determine the extent of the external financing costs.

Proposition 1 (Ex Post Proposition). *Resolution of the Paradox: Given the Black–Scholes economy (Assumptions 1–4), the stop-loss start-gain strategy (3) is not self-financing.*

Proof. The value process in the bond numeraire $V(t)/P(t) = \max[0, F(t) - X]$ has a discontinuous first derivative. However, a generalized form of Itô’s lemma⁶ holds for convex functions $g(y)$, which are not necessarily twice continuously differentiable:

$$g(F(t)) = g(F(0)) + \int_0^t D^-g(F(v)) dF(v) + \int_{-\infty}^{\infty} \Delta_t(y) \mu(dy), \tag{9}$$

where

$$D^-g(y) \equiv \lim_{\epsilon \downarrow 0} \frac{g(y) - g(y - \epsilon)}{\epsilon}$$

is the left derivative of $g(\cdot)$, and $\mu(a, b) \equiv D^-g(b) - D^-g(a)$ is a (signed) second-derivative measure. Letting $g(y) \equiv \max[0, y - X]$ implies $D^-g(y) = 1_{\{y > X\}}$ and $\mu(dy) = 1_{\{y=X\}}$. Substitution in (9) yields the *Tanaka–Meyer formula*⁷

$$\max[0, F(t) - X]$$

⁵ We thank an anonymous referee for this terminology.

⁶ See Karatzas and Shreve [1988, p. 218, (7.4)].

⁷ See Karatzas and Shreve [1988, p. 220, (7.7)].

$$= \max[0, F(0) - X] + \int_0^t 1_{\{F(v) > X\}} d(F(v)) + \Lambda_t(X), \quad (10)$$

for all $t \in [0, T]$.

Stripped of the final term, Equation (10) is the self-financing condition (8) when the bond is numeraire. Lemma A4 of the Appendix proves that the final term is positive with positive probability for any positive t . Consequently, the self-financing condition does not hold. Q.E.D.

The stop-loss start-gain strategy does not self-finance because of the “extra term” in the generalized form of Itô’s lemma (9). While the self-financing condition (8) holds when the normalized stock-price process $F(t)$ is *absolutely* continuous,⁸ it does not hold for geometric Brownian motion.

This extra term is nonzero because of the *infinite crossing property* of (geometric) Brownian motion. Once the stock price reaches the present value of the exercise price, it returns to this path infinitely, often over any succeeding interval of time, no matter how small.⁹ Each time the stock price returns, the investor must make a decision whether to hold a levered stock position or hold nothing. If the investor holds nothing, as specified by our stop-loss start-gain strategy (3), external financing is required if the stock price in the bond numeraire subsequently rises. Conversely, if the strategy were altered so that the investor holds a levered stock position at-the-money, then external financing is required if the stock price in the bond numeraire subsequently falls. Thus, irrespective of the decision made, there is some likelihood that the stock price will move in such a way that the strategy will require external financing.¹⁰ The total amount of external financing accumulates to $\Lambda_t(X)$ bonds by time t , where $\Lambda_t(X)$ is the final term in (10).

This term $\Lambda_t(X)$ is the *local time* at X by time t for the stock price in the bond numeraire. This random variable is of fundamental importance for the remainder of the paper. To understand the term, it is useful to first examine the local time of a standard Wiener process.

⁸ The proof is by the Fundamental Theorem of Calculus—see Royden (1968, p. 107) for a suitably general version.

⁹ See Ingersoll (1987, pp. 356–357 and Kane and Marcus (1988) for economic proofs of this result.

¹⁰ For a continuous stock-price process, each of the refinancings is individually infinitesimal. In addition, when the process is of bounded variation, in contrast to Assumption 4, only a finite number of refinancings are necessary, so that the total cost is infinitesimal. Consequently, under bounded variation, the stop-loss start-gain strategy is self-financing and is therefore an arbitrage opportunity. Thus, the stop-loss start-gain strategy can be used to provide a simpler proof of the result in Harrison, Pitbladdo, and Schaefer (1984) that processes of bounded variation admit arbitrage opportunities.

A standard Wiener process spends no time at any given level. This is analogous to a continuous random variable having no probability mass at any point. However, just as it is still possible to define a probability density function at a point, a density function exists for the amount of time that the standard Wiener process $W(v)$ spends in an interval A before time t . Doubled¹¹ Brownian local time, $2L_t(x)$, serves as this density function:

$$\int_A 2L_t(x) dx = \int_0^t 1_{\{W(v) \in A\}} dv, \quad \text{for any } t \in [0, T]. \quad (11)$$

The concept of local time can be generalized from a standard Wiener process to any *continuous semimartingale*.¹² Roughly speaking, a continuous semimartingale is a stochastic process that can be decomposed into a continuous process of bounded variation and a continuous local martingale.¹³ The class of continuous semimartingales includes most of the stochastic processes used in standard option pricing theory, including diffusion processes such as geometric Brownian motion. A concept of fundamental importance for studying the class of continuous semimartingales is *quadratic variation*. The quadratic variation of a continuous semimartingale over an interval of time corresponds roughly¹⁴ to the *squared* distance the process travels over a partition of this interval:

$$\langle f \rangle_t = \lim_{n \rightarrow \infty} \sum_{k=0}^{2^n-1} \left[f\left(\frac{(k+1)t}{2^n}\right) - f\left(\frac{kt}{2^n}\right) \right]^2 \text{ in probability.}$$

While a standard Wiener process has unbounded (regular) variation, its *quadratic variation* over a time interval is just the length of the time interval: $\langle W \rangle_t = t$. The quadratic variation of the stock price in the bond numeraire, $\langle F \rangle_t$, following a geometric Brownian motion¹⁵ is given by

$$\langle F \rangle_t = \sigma^2 \int_0^t [F(v)]^2 dv. \quad (12)$$

¹¹ There is no universal agreement in the stochastic processes literature as to whether $L_t(x)$ or $2L_t(x)$ is to be called Brownian local time. We follow the normalization (11) used by Karatzas and Shreve (1988).

¹² See Karatzas and Shreve (1988, pp. 217–225) for this generalization.

¹³ See Karatzas and Shreve (1988, p. 149) for an exact definition of a continuous semimartingale.

¹⁴ See Karatzas and Shreve (1988, p. 31) for an exact definition of quadratic variation.

¹⁵ See Karatzas and Shreve (1988, p. 138).

While doubled Brownian local time serves as a density function for the *time* spent in an interval A , doubled¹⁶ semimartingale local time, $2\Lambda_t(X)$, is the density function for the amount of *quadratic variation* the process experiences in A :

$$\int_A 2\Lambda_t(x) dx = \int_0^t 1_{\{F(v) \in A\}} d\langle F \rangle_v, \quad \text{for any } t \in [0, T]. \quad (13)$$

Semimartingale local time, $\Lambda_t(X)$, is a nonnegative random variable that is continuous and nondecreasing in time. Its appearance in the value process (10) can be heuristically justified as follows. The stop-loss start-gain strategy defined in (3) holds no shares at-the-money [$F(t) = X$] and one share above it. Suppose that the strategy is altered so that one share of stock is purchased each time the stock price in the bond numeraire rises from X to $X + \epsilon$, where ϵ is a small positive number. Under this altered strategy, suppose that each time the share is purchased, X bonds are shorted, so that an additional ϵ bonds worth of funds are required to finance the transaction. The levered stock position is liquidated upon each return of $F(t)$ to the level X . Let $U_t(\epsilon)$ be the number of times that the stock price in the bond numeraire F rises from X to $X + \epsilon$ by time t . Then, the total amount of external financing (measured in bonds) required by the altered strategy is given by the product of ϵ and $U_t(\epsilon)$. As ϵ becomes smaller, the altered strategy resembles (3); El Karoui (1978, p. 67) proves that the external financing converges a.s. to the semimartingale local time:

$$\lim_{\epsilon \downarrow 0} \epsilon \cdot U_t(\epsilon) = \Lambda_t(X). \quad (14)$$

This equation implies that $U_t(\epsilon)$ must approach infinity as ϵ approaches zero in order for local time not to vanish. Thus, the infinite crossing property holds for any continuous semimartingale with nonvanishing local time.

3. Valuation Results

This section provides new expressions for the values of European call and put options. These expressions decompose option prices into their intrinsic and time value. Harrison and Pliska (1981) provide sufficient conditions for the existence of an equivalent martingale

¹⁶ Semimartingale local time $\Lambda_t(X)$ can be related to Brownian local time $L_t(\cdot)$ by removing the drift of the *log* of the process F through a change of probability measure. Defining \tilde{Q} by $d\tilde{Q}/dQ = \exp[\eta W(T) - (\eta^2/2T)]$, $\eta = (\alpha - r - \sigma^2/2)/\sigma$, Carr (1989) shows that on the probability space $(\Omega, \mathcal{F}, \tilde{Q})$, $\Lambda_t(X) = \sigma X \cdot L_t(\ln(X/F(0))/\sigma)$, where $\tilde{L}_t(\cdot)$ denotes the local time of the \tilde{Q} Brownian motion $\tilde{W}(t) = W(t) - \eta t$.

measure. This measure can be used to value both the random terminal payoff and the stochastic external financing costs experienced while following a stop-loss start-gain strategy. The following proposition formalizes this approach.

Proposition 2. (*Ex Ante Proposition*). *An Alternate Call Valuation Formula: Consider a European call option with market value $c(0)$, maturity date T , and exercise price X . Assuming the Black–Scholes economy (Assumptions 1 to 4), no arbitrage opportunities exist if and only if*

$$c(0) = \max[0, S(0) - XP(0)] + P(0) \cdot \tilde{E}_0 \Delta_T(X), \quad (15)$$

where \tilde{E}_0 denotes the expectation calculated at time 0 as if $F(t)$ followed a martingale process.¹⁷

Proof. From martingale pricing theory, no arbitrage opportunities exist if and only if the call price in the bond numeraire follows a martingale process:

$$\begin{aligned} \frac{c(0)}{P(0)} &= \tilde{E}_0 \{ \max[0, F(T) - X] \}, \\ c(0) &= P(0) \tilde{E}_0 \left\{ \max[0, F(0) - X] + \int_0^T 1_{\{F(v) > X\}} d(F(v)) \right. \\ &\quad \left. + \Delta_T(X) \right\} \quad [\text{by (10)}] \\ &= \max[0, S(0) - XP(0)] + P(0) \cdot \tilde{E}_0 \Delta_T(X) \end{aligned}$$

since the expectation of the middle term, a martingale, vanishes. Q.E.D.

The first term on the right side of (15) is the investment required in the stop-loss start-gain strategy. Merton (1973) describes this term as the option's intrinsic value. The difference between an option's price and its intrinsic value is sometimes referred to as the option's *time value*. Consequently, this time value is given by the second term on the right side of (15). The terminology is surprisingly accurate since the term is the expected local *time's* present *value*. It is also the expected external financing cost of the strategy (when the stock price in the bond numeraire follows a martingale).

Both terms in Equation (15) are nonnegative. Since the option's intrinsic value exceeds its exercise value, (15) implies that American

¹⁷ Under Assumptions 1–4, expectations are calculated using the probability measure \tilde{Q} , whose Radon–Nikodym derivative with respect to the equivalent probability measure Q is given in Lemma A4 of the Appendix.

call options are never exercised early. The option's time value is maximized when the option starts at-the-money, since expected local time is greatest when the stock price starts on the critical path. Seidenberg (1988) describes the option's time value as the extra value accruing to an option because it anticipates when it is optimal to hold the levered stock position at-the-money.

Lemma A5 of the Appendix proves that the expected local time may be represented as

$$\tilde{E}_0 \Lambda_T(X) = \frac{\sigma^2 X^2}{2} \int_0^T \frac{1}{\sigma X \sqrt{t}} N' \left(\frac{\ln(F(0)/X) - \sigma^2 t/2}{\sigma \sqrt{t}} \right) dt, \quad (16)$$

where $N'(z) \equiv \exp(-z^2/2)/\sqrt{2\pi}$ is the standard normal density function. The Appendix shows that the integrand in (16) is the lognormal density function for $F(t)$ evaluated at X , while $\sigma^2 X^2$ is the instantaneous variance also evaluated at X . These results lead to the following heuristic interpretation of (16). The integrand gives the "risk-neutral" probability (density) that the stock price in the bond numeraire F reaches X at time t . The instantaneous variance $\sigma^2 X^2$ is interpreted as the average movement of F from X at t before the investor can react. Since only upward movements require external financing, the instantaneous variance is halved. The product of the probability and half the variance is then integrated over all possible dates. This interpretation is given solely as a mnemonic device and is not meant to be interpreted literally.

Substituting (1) and (16) back into (15) provides a new way of relating option value to the exogenous variables:

$$c(0) = \max[0, S(0) - Xe^{-rT}] + e^{-rT} \frac{\sigma X}{2} \int_0^T \frac{1}{\sqrt{t}} N' \left(\frac{\ln(S(0)/Xe^{-rT}) - \sigma^2 t/2}{\sigma \sqrt{t}} \right) dt. \quad (17)$$

This decomposition is mathematically equivalent to the Black-Scholes (1973) formula for a call. To see this, write the Black-Scholes formula as an explicit function of the volatility σ :

$$c(\sigma) = S(0)N(d_1(\sigma)) - Xe^{-rT}N(d_2(\sigma)), \quad (18)$$

where $N(\cdot)$ is the standard normal distribution function,

$$d_1(\sigma) \equiv \frac{\ln(S(0)/Xe^{-rT}) + \sigma^2 T/2}{\sigma \sqrt{T}},$$

and $d_2(\sigma) \equiv d_1(\sigma) - \sigma \sqrt{T}$. Differentiating with respect to volatility

yields

$$c'(\sigma) = Xe^{-rT} N'(d_2(\sigma)) \sqrt{T}.$$

Integrating back over volatility yields

$$c(\sigma) = Xe^{-rT} \int_0^\sigma N'(d_2(\eta)) \sqrt{T} d\eta + \max[0, S(0) - Xe^{-rT}], \quad (19)$$

where $\max[0, S(0) - Xe^{-rT}]$ is the constant of integration determined by the boundary condition $c(0) = \max[0, S(0) - Xe^{-rT}]$. Performing a change of variables $t \equiv (T/\sigma^2)\eta^2$ then leads to our decomposition (17).

Note that (19) is useful for determining the implied standard deviation. Other equivalent valuation results can similarly be determined by differentiating and integrating the Black-Scholes formula (18) with respect to other variables. For example, differentiating and integrating with respect to time to maturity T leads to a decomposition into $\max[0, S - X]$ and the residual. Carr, Jarrow, and Myneni (1990) develop this decomposition using local time.

Put-call parity implies that puts and calls have the same time value. Consequently, our ex ante Proposition 2 and (16) can be used to value European puts.

Corollary. Put Valuation: Consider a European put with market price $p(0)$, maturity date T , and exercise price X . Assuming the Black-Scholes economy (Assumptions 1 to 4), no arbitrage opportunities exist if and only if

$$p(0) = \max[0, Xe^{-rT} - S(0)] + e^{-rT} \frac{\sigma X}{2} \int_0^T \frac{1}{\sqrt{t}} N' \left(\frac{\ln(S(0)/Xe^{-rT}) - \sigma^2 t/2}{\sigma \sqrt{t}} \right) dt.$$

The Ex Ante Proposition and its corollary indicate that prices of out-of-the-money European calls and puts are equal to the present value of expected local time under martingale pricing. Equation (16) shows that this expected local time is a function of the same variables which affect option value as in the Black-Scholes (1973) model. However, given the expected local time under the martingale measure, it is not necessary to know the volatility¹⁸ of the stock-price process.

The stop-loss start-gain strategy may be extended to boundaries

¹⁸ From our decomposition, it is clear that the volatility parameter serves two roles in the Black-Scholes (1973) model. It affects the quadratic variation experienced at-the-money; it also affects the probability of a path to-the-money.

other than the present value of the exercise price and to other contingent claims. For example, Carr, Jarrow, and Myneni (1990) value European options using an arbitrary¹⁹ boundary, and use their results to decompose *American* option prices into intrinsic and time value.

The strategy can also be applied to other stochastic processes. For continuous processes other than geometric Brownian motion, it may be easier to work with the proper time integral, analogous to the one in (17), rather than the usual improper spatial integral, analogous to the one in (18). In the next section, we use our approach to value options under a more general stochastic process for the stock price.

4. Generalizing the Stock-Price Process

The preceding analysis is robust to generalizations of the Black–Scholes (1973) economy. In this section, we maintain Assumptions 1 to 3 and relax²⁰ Assumption 4 of a geometric Brownian motion process for the stock price.

Assumption 4. *Continuous Semimartingale of Unbounded Variation: The stock price $\{S(t); t \in [0, T]\}$ follows a continuous semimartingale process of unbounded²¹ variation, with initial value $S(0)$ a positive constant.*

Since the stock price in dollars, $S(t)$, follows a continuous semimartingale, so does the stock price in bonds, $F(t) \equiv S(t)/P(t)$. Consequently, by the definition of a continuous semimartingale,²² this process can be uniquely decomposed into a local martingale and a process of bounded variation, Q a.s.:

$$F(t) = F(0) + M(t) + B(t), \quad \text{for all } t \in [0, T], \quad (20)$$

where $\{M(t); t \in [0, T]\}$ is a continuous local martingale²³ with $M(0) = 0$, and $\{B(t); t \in [0, T]\}$ is the difference of continuous, nondecreasing processes with $B(0) = 0$. Royden (1968, p. 100) shows that $\{B(t); t \in [0, T]\}$ is a continuous process of bounded variation.

¹⁹ The boundary satisfies certain regularity conditions and equals the exercise price at expiration.

²⁰ Implicitly, we are also assuming that (Ω, \mathcal{F}, Q) is the underlying probability space with $\{F(t); t \in [0, T]\}$ adapted to the filtration $\{\mathcal{F}_t; t \in [0, T]\}$ satisfying the usual conditions with $\mathcal{F}_T = \mathcal{F}$. The usual conditions are that the filtration is right-continuous, with \mathcal{F}_0 containing all the Q -negligible events.

²¹ We assume a process of unbounded variation because Lemma A1 of the Appendix proves that the ex post Proposition 2 does not hold otherwise.

²² See Karatzas and Shreve (1988, p. 149).

²³ $\{B(t); t \in [0, T]\}$ and $\{M(t); t \in [0, T]\}$ are adapted to the filtration $\{\mathcal{F}_t; t \in [0, T]\}$.

From (4) and (7), the value of the portfolio (in bonds) achieved when following a stop-loss start-gain strategy²⁴ is given by

$$\frac{V(t)}{P(t)} = \max[0, F(t) - X], \quad \text{for all } t \in [0, T]. \quad (21)$$

Since $F(t)$ follows a continuous semimartingale, (21) implies that the value of the stop-loss start-gain strategy (in bonds), $V(t)/P(t)$, also follows a continuous semimartingale. The Tanaka–Meyer formula [Karatzas and Shreve, 1988, p. 220, (7.7)] provides the unique decomposition of this process:

$$\begin{aligned} & \max[0, F(t) - X] \\ &= \max[0, F(0) - X] + \int_0^t 1_{\{F(v) > X\}} dM(v) \\ & \quad + \int_0^t 1_{\{F(v) > X\}} dB(v) + \Lambda_t(X) \\ &= \max[0, F(0) - X] + \int_0^t 1_{\{F(v) > X\}} dF(v) \quad [\text{from (20)}] \\ & \quad + \Lambda_t(X), \quad \text{for all } t \in [0, T] \text{ and } X \in \mathfrak{R}, \end{aligned} \quad (22)$$

where $\Lambda_t(X)$ is the semimartingale local time for the process $\{F(t); t \in [0, T]\}$. Stripped of the final local-time term, Equation (22) is recognized as the self-financing condition in the bond numeraire (8).

Assumptions 1–4 of this section are not sufficient to prove that the local time is nonzero with positive probability. Consequently, we append two assumptions. The first states that at any future date, the probability that the option will be in-the-money is never 1, and the probability that it is out-of-the money is also never 1.

Assumption 5. Positive Probability of Boundary Crossings:

$$Q(F(t) > X) > 0 \text{ and } Q(F(t) < X) > 0, \quad \text{for all } t \in (0, T).$$

The second additional assumption is that an equivalent martingale measure exists. The rationale is based upon the strong relationship between the existence of equivalent martingale measures and the absence of arbitrage opportunities (see Cox and Ross, 1976; Harrison and Kreps, 1979; Kreps, 1981).

²⁴ When the stock price follows a continuous semimartingale, trading strategies must be progressively measurable. Using Karatzas and Shreve (1988), it is easy to see that the stop-loss start-gain strategy (3) is progressively measurable (in fact, predictable). We thank the referee for pointing this out.

Assumption 6. Existence of an Equivalent Martingale Measure: There exists a probability measure \tilde{Q} on (Ω, \mathcal{F}) , equivalent to Q [i.e., $\tilde{Q}(A) = 0$ if and only if $Q(A) = 0$] such that

$$\tilde{E}(F(t) | \mathcal{F}_s) = F(s) \text{ a.e. } \tilde{Q}, \text{ for all } s, t, \text{ such that } 0 \leq s \leq t \leq T.$$

Note that we do not assume that the equivalent martingale measure is unique. Consequently, our ex post Proposition 1 will generate a range²⁵ of rational option prices. Bolstered with these assumptions, we next prove our ex post proposition.

Proposition 3. Given Assumptions 1–6 of this section, the stop-loss start-gain strategy (3) is not self-financing.

Proof. Expression (22) shows that the local-time term $\Lambda_t(X)$ disrupts the self-financing condition. When Assumptions 1 to 6 hold, Lemma A2 in the Appendix proves that this term is positive with positive probability (for positive t). Consequently, the self-financing condition is violated. Q.E.D.

Since the local time will probably not be zero, the investor will probably have to finance externally to implement the stop-loss start-gain strategy. This stochastic external financing can be valued using an equivalent martingale measure (our ex ante proposition).

Proposition 4. Given Assumptions 1 to 6, then

$$c(0) = \max[0, S(0) - XP(0)] + P(0)\tilde{E}_0\Lambda_T(X), \quad (23)$$

where \tilde{E}_0 denotes expectation with respect to a martingale measure \tilde{Q} , whose existence was maintained in Assumption 6.

Proof. From martingale pricing,

$$\begin{aligned} \frac{c(0)}{P(0)} &= \tilde{E}_0 \{ \max[0, F(T) - X] \} \\ c(0) &= P(0) \cdot \tilde{E}_0 \left\{ \max[0, F(0) - X] + \int_0^T 1_{\{F(v) > X\}} d(F(v)) \right. \\ &\quad \left. + \Lambda_T(X) \right\} \quad [\text{from (22)}] \\ &= \max[0, F(0) - XP(0)] + P(0)\tilde{E}_0\Lambda_T(X), \end{aligned}$$

since the expectation of the middle term, a martingale, vanishes. Q.E.D.

Equation (23) provides a decomposition of an option's price into

²⁵ If uniqueness is desired, the assumption of market completeness can be imposed.

its intrinsic and time value. An option's price equals its intrinsic value if the stock price can never cross the present value of the exercise price. The second term is the additional value that arises because of our Assumption 5 that such crosses are always possible.

A European put can be valued using the ex ante Proposition 4 and put-call parity.

Corollary. Put Valuation: *Given Assumptions 1–6, then*

$$p(0) = \max[0, XP(0) - S(0)] + P(0) \tilde{E}_0 \Lambda_T(X), \quad (24)$$

for all $t \in [0, T]$.

To relate the expected local time $\tilde{E}_0 \Lambda_T(X)$ to the exogenous variables, we first assume that under an equivalent martingale measure, the continuous semimartingale F_t has²⁶ a transition density $p(F(t), t; F(0), 0)$. We also assume that the time derivative of the quadratic variation of the process can be written as a function $q(\cdot)$ of the level of the process; that is,

$$\frac{d\langle F \rangle_t}{dt} = q(F(t)),$$

where $q: \mathfrak{R} \rightarrow \mathfrak{R}$ is a strictly positive Borel-measurable function. Then Lemma A3 of the Appendix proves that expected local time is given by

$$\tilde{E}_0 \Lambda_T(X) = \frac{q(X)}{2} \int_0^T p(X, t; F(0), 0) dt. \quad (25)$$

To apply these results to diffusion processes, suppose that the martingale $F(t)$ follows a time-homogeneous diffusion process

$$dF(t) = \sigma(F(t)) d\tilde{W}(t).$$

Under this process, the time derivative of the quadratic variation is just the instantaneous variance

$$\frac{d\langle F \rangle_t}{dt} = \sigma^2(F(t)) \equiv q(F(t)).$$

Consequently, European call and put prices are given by

$$c(0) = \max[0, S(0) - Xe^{-rT}] + e^{-rT} \frac{\sigma^2(X)}{2} \int_0^T p(X, t; F(0), 0) dt, \quad (26)$$

²⁶ See Gihman and Skorohod (1972, pp. 96–97) for sufficient conditions for existence, as well as the formula for the transition density of an arbitrary diffusion process.

$$p(0) = \max[0, Xe^{-rT} - S(0)] + e^{-rT} \frac{\sigma^2(X)}{2} \int_0^T p(X, t; F(0), 0) dt.$$

Lemma A6 of the Appendix shows that (26) is mathematically equivalent to the initial investment in a self-financing replicating strategy:

$$\begin{aligned} c(0) &= e^{-rT} \tilde{E}_0 c(T) \\ &= e^{-rT} \int_0^\infty \max[0, F - X] p(F, T; F(0), 0) dF. \end{aligned}$$

5. Conclusions

A stop-loss start-gain strategy with a floor above the stock's forward price is initially costless. Yet the strategy has the same terminal payoff as a European call option whose strike price is that floor. When the forward price process follows a continuous semimartingale process of unbounded variation, the stop-loss start-gain strategy does not self-finance. Recognizing this result resolves the stop-loss start-gain paradox. Furthermore, when an equivalent martingale measure exists, option values can be related to the initial investment in a replicating strategy, and to the expected external financing requirement. This insight leads to a new decomposition of an option's price, which is equivalent to extant valuation results.

Appendix

This appendix is subdivided into three sections. The first section proves lemmas needed in the proofs of our propositions for continuous semimartingales. These results are then specialized to geometric Brownian motions in the second section. The third section examines the equivalence of our results to extant valuation results when the stock price in the bond numeraire follows a time-homogeneous diffusion process.

Continuous semimartingales

Lemma A1. *Assumptions 1–4 of Section 4 hold, except that the stock-price process is of bounded variation. Then $Q(\Lambda, (X) > 0) = 0$ for all $t \in [0, T]$.*

Proof. Recall that in the decomposition (22) of $F(t)$, $\{B(t); t \in [0, T]\}$ was of bounded variation. Consequently, $\{M(t); t \in [0, T]\}$ must also be of bounded variation since $\{F(t); t \in [0, T]\}$ is of bounded

variation. However, $\{M(t); t \in [0, T]\}$ of bounded variation implies $M(t) \equiv 0$ for all $t \in [0, T]$ (Karatzas and Shreve, 1988, p. 37, ex. 5.21). Furthermore $M(t) \equiv 0$ implies $\langle M \rangle_t = 0$ for all $t \in [0, T]$. Recall that $\Lambda_t(X)$ measures the quadratic variation of this process:

$$0 = \langle M \rangle_t = 2 \int_{-\infty}^{\infty} \Lambda_t(X) dX, \quad \text{for } t \in [0, T] \text{ a.s. } Q.$$

Hence, the local time vanishes, since (Karatzas and Shreve, 1988, p. 224) $\Lambda_t(X) \geq 0$ for all $t \in [0, T]$, $X \in \mathfrak{R}$ a.e. Q , and $\Lambda_t(X)$ is right-continuous with left limits existing in X . Q.E.D.

Lemma A2. *Given Assumptions 1–6 of Section 4, $Q(\Lambda_t(X) > 0) > 0$ for all $t \in (0, T]$.*

Proof. Under \tilde{Q} , $\{F(t); t \in [0, T]\}$ has the decomposition $F(t) = F(0) + \tilde{M}(t)$. Since the process $\{F(t); t \in [0, T]\}$ is of unbounded variation, $\tilde{M}(t)$ is not identically zero. Taking expectations of Equation (22) yields

$$\tilde{E}_0 \max[0, F(t) - X] = \max[0, F(0) - X] + \tilde{E}_0 \Lambda_t(X),$$

for all $t \in [0, T]$.

We are done if we show $\tilde{E}_0(\Lambda_t(X)) > 0$; that is, $\tilde{E}_0(\max[0, F(t) - X]) > \max[0, F(0) - X]$. However, Assumption 5 and the equivalence of \tilde{Q} and Q imply $\tilde{Q}(F(t) > X) > 0$, $\tilde{Q}(F(t) < X) > 0$. Since $g(y) \equiv \max[0, y - X]$ is strictly convex over an interval containing X , Jensen's inequality is strict; that is,

$$\tilde{E}_0[g(F(t))] > g(\tilde{E}_0[F(t)]) = g(F(0)).$$

The result follows. Q.E.D.

Lemma A3. *Assume $F(t)$ is a continuous martingale defined on the probability space $(\Omega, \mathcal{F}, \tilde{Q})$, adapted to the filtration $\{\mathcal{F}_t; t \in [0, T]\}$. Assume the existence of a transition density $p(F(t), t; F(0), 0)$ for $F(t)$, and assume the time derivative of its quadratic variation $d\langle F \rangle_t / dt = q(F(t))$ for $q: \mathfrak{R} \rightarrow \mathfrak{R}$ a strictly positive Borel-measurable function. Then, then the expected local time for this martingale is given by*

$$\tilde{E}_0 \Lambda_T(X) = \frac{q(X)}{2} \int_0^T p(X, t; F(0), 0) dt. \quad (A1)$$

Proof. From Karatzas and Shreve (1988, p. 218), for every nonnegative Borel-measurable function $k: \mathfrak{R} \rightarrow (0, \infty)$, we have, \tilde{Q} a.e.,

$$\int_0^T k(F(t)) d\langle F \rangle_t = \int_{-\infty}^{\infty} k(X) 2\Lambda_T(X) dX. \quad (\text{A2})$$

Taking expectations using the martingale measure and employing Fubini's theorem on the right side yields

$$\begin{aligned} & \int_{-\infty}^{\infty} k(X) \int_0^T p(X, t; F(0), 0) q(X) dt dX \\ &= \int_{-\infty}^{\infty} k(X) 2\tilde{E}_0\Lambda_T(X) dX. \end{aligned}$$

Choose $k(x) = 1_{\{x \in A\}}$, where $A \in \mathcal{F}$:

$$\int_A \int_0^T p(X, t; F(0), 0) q(X) dt dX = \int_A \tilde{E}_0 2\Lambda_T(X) dX.$$

The integrals are equal for arbitrary $A \in \mathcal{F}$. Since both integrands are nonnegative, Billingsley (1986, p. 216) implies that they must be equal:

$$\int_0^T p(X, t; F(0), 0) q(X) dt = 2\tilde{E}_0\Lambda_T(X). \quad \text{Q.E.D.}$$

Geometric Brownian motion

This subsection of the Appendix assumes the Black-Scholes (1973) model as described in Assumptions 1–4 of Section 1.

Lemma A4. *Assumptions 1–4 of Section 1 imply Assumptions 4–6 of Section 4.*

Proof. It is well-known that if the stock price follows geometric Brownian motion (Assumption 4 of Section 1), then it also follows a continuous semimartingale of unbounded variation (Assumption 4 of Section 4). To show that Assumption 5 is implied, note that if the stock price $S(t)$ follows geometric Brownian motion (Assumption 4), then (7) and Itô's lemma imply that $F(t)$ also follows a different geometric Brownian motion; namely,

$$dF(t) = [\alpha - r]F(t) dt + \sigma F(t) dW(t). \quad (\text{A3})$$

The solution to (A3) is

$$F(t) = F(0) \exp\{(\alpha - r - \sigma^2/2)t + \sigma W(t)\}. \quad (\text{A4})$$

Consequently,

$$Q(F(t) > X) = Q\left(W(t) > \frac{\ln(X/F(0)) - (\alpha - r - \sigma^2/2)t}{\sigma}\right) > 0,$$

from the properties of the Brownian motion $W(t)$. Similarly, $Q(F(t) < X) > 0$.

To show that Assumption 6 is implied, define $\lambda \equiv (\alpha - r)/\sigma$ and $Z(t) \equiv \exp\{\lambda W(t) - (\lambda^2/2)t\}$. By Karatzas and Shreve (1988, p. 198, Prop. 5.12), $Z(t)$ is a martingale using the boundedness of $\alpha - r$ and σ . By Girsanov's theorem, $\tilde{W}(t) \equiv W(t) - \lambda t$ is a \tilde{Q} Brownian motion for $d\tilde{Q}/dQ = Z(T)$. Substitution into (A4) yields

$$F(t) = F(0)\exp\{(\sigma^2/2)t + \sigma\tilde{W}(t)\}. \tag{A5}$$

Taking expectations shows $\tilde{E}(F(t) | \mathcal{F}_s) = F(s)$ for all s, t such that $0 \leq s \leq t < T$. Q.E.D.

Lemma A5. *Assumptions 1-4 of Section 1 imply that expected local time satisfies*

$$\tilde{E}_0 \Lambda_T(X) = \frac{\sigma X}{2} \int_0^T \frac{1}{\sqrt{t}} N'\left(\frac{\ln(F(0)/X) - \sigma^2 t/2}{\sigma\sqrt{t}}\right) dt.$$

Proof. From Lemma A4, Assumptions 1-4 of Section 1 imply the existence of an equivalent martingale measure. Under this measure, (A5) is the solution to the following stochastic differential equation:

$$dF(t) = \sigma F(t) d\tilde{W}(t).$$

The transition density for this martingale process is lognormal:

$$\begin{aligned} p(F(t), t; F(0), 0) &= \frac{1}{F(t)\sigma\sqrt{t}\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[\frac{\ln F(t) - (\ln F(0) + \sigma^2 t/2)}{\sigma\sqrt{t}}\right]^2\right\} \\ &= \frac{1}{F(t)\sigma\sqrt{t}} N'\left(\frac{\ln(F(0)/F(t)) - \sigma^2 t/2}{\sigma\sqrt{t}}\right). \end{aligned} \tag{A6}$$

Furthermore, from Karatzas and Shreve (1988, p. 138), this process has quadratic variation

$$\langle F \rangle_t = \sigma^2 \int_0^t F^2(v) dv.$$

Consequently, the time derivative of this quadratic variation satisfies the hypothesis of Lemma A3:

$$\frac{d\langle F \rangle_t}{dt} = \sigma^2 F^2(t) \equiv q(F(t)). \tag{A7}$$

Thus, from Lemma A3, the expected local time is given by

$$\begin{aligned} \tilde{E}_0 \Lambda_T(X) &= \frac{q(X)}{2} \int_0^T p(X, t; F(0), 0) dt \\ &= \frac{\sigma^2 X^2}{2} \int_0^T \frac{1}{X \sigma \sqrt{t}} N' \left(\frac{\ln(F(0)/X) - \sigma^2 t/2}{\sigma \sqrt{t}} \right) dt, \end{aligned}$$

from evaluating (A6) and (A7) at $F(t) = X$. Q.E.D.

Time-homogeneous diffusion processes

This subsection of the Appendix accepts Assumptions 1–6 and the conditions of Lemma A3. Our analysis of the stop-loss start-gain strategy indicates that the call price in bonds is given by

$$\frac{c(0)}{P(0)} = \max[0, F(0) - X] + \frac{\sigma^2(X)}{2} \int_0^T p(X, t; F(0), 0) dt.$$

Analysis of a self-financing strategy implies

$$\frac{c(0)}{P(0)} = \int_0^\infty \max[0, F - X] p(F, T; F(0), 0) dF.$$

This subsection restricts itself to time-homogeneous diffusion processes in order to show the equivalence of the above representations by purely analytic means. While the proof is necessarily more restrictive than the probabilistic approach, the derivation sheds further light on the origins of our decomposition.

Lemma A6. *Suppose that under a martingale measure, the stock price in the bond numeraire follows a time-homogeneous diffusion process:*

$$dF(t) = \sigma(F(t)) d\tilde{W}(t).$$

Further suppose that the transition density $p(F, t; F(0), 0)$ satisfies the following regularity²⁷ conditions:

- (i) $\sigma^2(F) \cdot p(F, t; F(0), 0)$ is twice-differentiable in F and $p(F, t; F(0), 0)$ is once-differentiable in t .*
- (ii) $\lim_{F \rightarrow \infty} [\sigma^2(F) \cdot p(F, t; F(0), 0)] = 0$.*
- (iii) $\lim_{F \rightarrow \infty} F \cdot \frac{\partial}{\partial F} [\sigma^2(F) \cdot p(F, t; F(0), 0)] = 0$.*

Then

$$\int_0^\infty \max[0, F - X] p(F, T; F(0), 0) dF$$

²⁷ The second and third conditions require the right tail of the distribution to go to zero sufficiently fast.

$$= \max[0, F(0) - X] + \frac{\sigma^2(X)}{2} \int_0^T p(X, t; F(0), 0) dt.$$

Proof. The proof exploits the property of the stop-loss start-gain strategy that the initial investment is the same function of F as the terminal payoff. From the Fundamental Theorem of Calculus,

$$\begin{aligned} & \int_0^\infty \max[0, F - X] p(F, T; F(0), 0) dF \\ &= \int_0^\infty \max[0, F - X] \\ & \quad \cdot \left\{ p(F, 0; F(0), 0) + \int_0^T \frac{\partial}{\partial t} p(F, t; F(0), 0) dt \right\} dF \\ &= \int_0^\infty \max[0, F - X] p(F, 0; F(0), 0) dF \\ & \quad + \int_0^\infty \int_0^T \max[0, F - X] \frac{\partial}{\partial t} p(F, t; F(0), 0) dt dF. \quad (\text{A8}) \end{aligned}$$

Under the first regularity condition, the transition density satisfies the Kolmogorov forward equation,²⁸

$$\frac{\partial}{\partial t} p(F, t; F(0), 0) = \frac{\partial^2}{\partial F^2} \left[\frac{\sigma^2(F)}{2} p(F, t; F(0), 0) \right], \quad (\text{A9})$$

subject to the boundary condition

$$p(F, 0; F(0), 0) = \delta(F - F(0)), \quad (\text{A10})$$

where $\delta(\cdot)$ is the Dirac²⁹ delta function. Substituting (A10) and (A9) into (A8) implies

$$\begin{aligned} & \int_0^\infty \max[0, F - X] p(F, T; F(0), 0) dF \\ &= \int_0^\infty \max[0, F - X] \delta(F - F(0)) dF \end{aligned}$$

²⁸ See Karatzas and Shreve (1988, p. 282).

²⁹ The Dirac delta function is defined by its properties:

$$\begin{aligned} \delta(x) &= \begin{cases} 0, & \text{if } x \neq 0, \\ \infty, & \text{if } x = 0; \end{cases} \\ \int_{-a}^a \delta(x) dx &= 1, \quad \text{for any } a > 0. \end{aligned}$$

$$\begin{aligned}
 & + \int_0^\infty \int_0^T \max[0, F - X] \frac{\partial^2}{\partial F^2} \left[\frac{\sigma^2(F)}{2} p(F, t; F(0), 0) \right] dt dF \\
 & = \max[0, F(0) - X] \\
 & + \int_0^T \int_X^\infty (F - X) \frac{\partial^2}{\partial F^2} \left[\frac{\sigma^2(F)}{2} p(F, t; F(0), 0) \right] dF dt,
 \end{aligned}$$

from the properties of the Dirac delta function and Fubini's theorem. Evaluating the inner integral by parts implies

$$\begin{aligned}
 & \int_0^\infty \max[0, F - X] p(F, T; F(0), 0) dF \\
 & = \max[0, F(0) - X] \\
 & + \int_0^T \left\{ (F - X) \frac{\partial}{\partial F} \left[\frac{\sigma^2(F)}{2} p(F, t; F(0), 0) \right] \Big|_X^\infty \right. \\
 & \quad \left. - \int_X^\infty \frac{\partial}{\partial F} \left[\frac{\sigma^2(F)}{2} p(F, t; F(0), 0) \right] dF \right\} dt \\
 & = \max[0, F(0) - X] + \frac{\sigma^2(X)}{2} \int_0^T p(X, t; F(0), 0) dt,
 \end{aligned}$$

since the first term in the integrand vanishes from the third regularity condition, while the second term simplifies from the Fundamental Theorem of Calculus and the second regularity condition. Q.E.D.

References

- Asay, M., and C. Edelsburg, 1986, "Can a Dynamic Strategy Replicate the Returns of an Option?" *Journal of Futures Markets*, 6, 63-70.
- Billingsley, P., 1986, *Probability and Measure* (2d ed.), Wiley, New York.
- Black, F., and M. Scholes, 1973, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, 81, 637-659.
- Carr, P., 1989, "European Option Valuation When Dividends and Interest Rates Are Unknown: The Discrete Time Results," working paper, Cornell University.
- Carr, P., R. Jarrow, and R. Myneni, 1990, "Alternative Characterizations of American Put Options," working paper, Cornell University.
- Cox, J., and S. Ross, 1976, "A Survey of Some New Results in Financial Option Pricing Theory," *Journal of Finance*, 31, 383-402.
- Cox, J., and M. Rubinstein, 1985, *Options Markets*, Prentice-Hall, Englewood Cliffs, NJ.
- Duffie, D., and C. F. Huang, 1985, "Implementing Arrow-Debreu Equilibria by Continuous Trading of Few Long-Lived Securities," *Econometrica*, 53, 1337-1356.
- Dybvig, P., and C. F. Huang, 1988, "Nonnegative Wealth, Absence of Arbitrage, and Feasible Consumption Plans," *The Review of Financial Studies*, 1, 337-401.

- El Karoui, N., 1978, "Sur les Montées des Semimartingales," *Astérisque* 52-53, 63-72.
- Gihman, I., and A. Skorohod, 1972, *Stochastic Differential Equations*, Springer, Berlin.
- Harrison, M., and D. Kreps, 1979, "Martingales and Arbitrage in Multiperiod Securities Markets," *Journal of Economic Theory*, 20, 381-408.
- Harrison, M., and S. Pliska, 1981, "Martingales and Stochastic Integrals in the Theory of Continuous Trading," *Stochastic Processes and Their Applications*, 11, 215-260.
- Harrison, M., R. Pitbladdo, and S. Schaefer, 1984, "Continuous Price Processes in Frictionless Markets Have Infinite Variation," *Journal of Business*, 57, 353-365.
- Heath, D., and R. Jarrow, 1987, "Arbitrage, Continuous Trading, and Margin Requirements," *Journal of Finance*, 5, 1129-1142.
- Hull, J., and A. White, 1987, "Hedging Through the CAP: Implications for Market Efficiency, Hedging and Option Pricing," *International Options Journal*, 4, 17-22.
- Ingersoll, J., 1987, *Theory of Financial Decision-Making*, Rowman & Littlefield, Totowa, N.J.
- Kane, A., and A. Marcus, 1988, "The Delivery Option on Forward Contracts: A Note," *Journal of Financial and Quantitative Analysis*, 23, 337-342.
- Karatzas, I., and S. Shreve, 1988, *Brownian Motion and Stochastic Calculus*, Springer, New York.
- Kreps, D., 1981, "Arbitrage and Equilibrium in Economies with Infinitely Many Commodities," *Journal of Mathematical Economics*, 8, 15-35.
- Livingston, M., 1987, "The Delivery Option in Forward Contracts," *Journal of Financial and Quantitative Analysis*, 22, 79-88.
- Merton, R., 1973, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science*, 4, 141-183.
- Omberg, E., 1988, "Binary Trading Strategies, Brownian Local Time, and the Tanaka Formulas," working paper, Santa Clara University.
- Royden, H., 1968, *Real Analysis* (2d ed.), Macmillan, New York.
- Seidenverg, E., 1988, "A Case of Confused Identify," *Financial Analysts Journal*, 63-67.