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Static Hedging under Time-Homogeneous Diffusions*

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Abstract. We consider the problem of semistatic hedging of a single barrier option in a model where the underlying is a time-homogeneous diffusion, possibly running on an independent stochastic clock. The main result of the paper is an analytic expression for the payoff of a European-type contingent claim, which has the same price as the barrier option up to hitting the barrier. We then consider some examples, such as the Black–Scholes, constant elasticity of variance, and zero-correlation SABR models. Finally, we investigate an approximation of the static hedge with options of at most two different strikes.

Key words. static hedging, barrier options, time-homogeneous diffusions, Sturm–Liouville theory, method of images, inverse transform

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1. Introduction.

Barrier options are some of the most popular path-dependent derivatives traded on OTC markets. An *up-and-out* option, written on the underlying S, with terminal payoff $F(S_T)$ at maturity T and (upper) barrier U pays $F(S_T)$ at time T, provided the underlying has not hit the barrier U by that time; otherwise the option expires worthless. Similarly, the *up-and-in* option pays $F(S_T)$ if and only if the underlying has hit the barrier by the time T. One can define lower barrier options analogously. The literature on pricing and hedging barrier options is vast and we, of course, cannot fully pay tribute to it. We, however, concentrate on the specific problem of *static hedging* of barrier options.

The idea of dynamic hedging of financial derivatives by trading in the underlying asset has some obvious drawbacks. These drawbacks are largely due to the presence of transaction costs. The problem can, in principle, be solved by including other, more liquid, derivatives in the hedging portfolio: if one can construct a fixed portfolio of liquid derivatives (not necessarily the underlying process alone), whose price coincides with the price of the given barrier option at all times, up to the first hitting time of the barrier, then it is natural to hedge the sale of this barrier option by taking long positions in the corresponding liquid derivatives. Such a portfolio of liquid derivatives is then called a *static hedge* of the corresponding barrier option. We choose European-type options (the ones that have some fixed payoff $G(S_T)$ at time T and no path dependence) as the liquid derivatives and try to find a static hedging portfolio for the up-and-out put (UOP). Recall that an UOP has the following payoff at the time of maturity

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T:

$$I_{\{\sup_{t\in[0,T]} S_t<U\}} \cdot (K - S_T)^+ ,$$

with $K < U$. The purpose of this work is to provide an exact analytic form of the *static hedge* of this option in all regular enough time-homogeneous diffusion models. *"Static hedge" in the present setup is identified with a function $G : [0,\infty) \to \mathbb{R}$, such that the European-type option with payoff $G(S_T)$ at maturity date T has the same price as the UOP, up to hitting the barrier (when the hedge can be liquidated at no cost, provided the price process is continuous).* Notice that our definition of static hedge involves only options of the same maturity.

Although we consider only UOPs, the static hedges for other upper barrier options can be obtained from this one in a straightforward way. First, we can find the static hedge of a digital put by differentiating with respect to strike K (and, if necessary, setting $K \nearrow U$). Then, using put-call parity, we can express the price of an up-and-out call through a linear combination of the corresponding UOP, some up-and-out digital puts, and another UOP with $K = U$, which yields a desired static hedge. Thus, given enough smoothness of the static hedge of a UOP with respect to K, we can obtain the static hedges of all other up-and-out barrier options from it. The static hedges for up-and-in options follow immediately from the fact that the sum of the corresponding up-and-out and up-and-in options results in a European-type claim. The case of lower barrier options is very similar to the upper barrier ones; however, there are certain differences resulting from the fact that "0" and "∞" are treated differently in the stochastic models for the underlying asset (in particular, we always avoid using stochastic processes which can explode, i.e., hit "∞," although we sometimes allow the underlying process to hit zero). Therefore, one can modify the results presented in this paper for lower barrier options, but a modicum of care is required to change the regularity conditions in subsection 2.3 appropriately. Notice also that, in the case of "one-touch" options, we cover the case when payment is made at expiry. However, applying similar techniques one can derive a static hedge for "one-touch" with payment at hit.

The problem of static hedging of barrier options in the Black–Scholes model turns out to be essentially equivalent to the problem of pricing. A number of works have been devoted to the study of this case (see, for example, [27], [28], [8], [10], or [21]), and the corresponding static hedges are available in closed form. In particular, in the Black (zero-drift Black–Scholes) model, the static hedge of a UOP option with strike K and barrier U is given by

$$G(x) = (K - x)^+ - \frac{K}{U}\left(x - \frac{U^2}{K}\right)^+ .$$

Notice that static hedge consists of a long position in the corresponding put option and a short position in a call option with a different strike K^*, such that the barrier U is the *geometric mean* of the two strikes K and K^*. As we will see, the same type of symmetry can be observed in other diffusion models (see the constant elasticity of variance (CEV) example in subsection 3.3).

The static hedge of the Black–Scholes model succeeds even when the underlying price follows a geometric Brownian motion running on a continuous independent stochastic clock. In addition, it was shown in [11] that the same static hedging strategy works in *symmetric local volatility* models. However, the above models are rather limited due to the respective assumptions of independence of the time-change and symmetry of the local volatility function against the barrier. In particular, these models are inconsistent with certain market phenomena, such as the well-known *skew* of the *implied smile*. Nevertheless, to the best of our knowledge, aside from the Black–Scholes model or the above-mentioned extensions, until now there has been no exact solution available for the problem of static hedging of barrier options in any other stochastic model for the underlying. The present paper provides a solution to this problem in a large class of (nonsymmetric) local volatility models and, in particular, shows that the static hedge in a model which produces a nontrivial skew of the implied smile does not coincide with the Black–Scholes hedge (see, for example, subsection 3.2).

A different approach to the problem of static hedging was pursued by Hobson, Brown, and Rogers in [18] and [9], and, more recently, by Cox and Obloj in [13] and [12], who have obtained model-free bounds on the prices of barrier options, which result in corresponding superreplicating strategies. However, most of the research done in this area is concerned with developing algorithms for numerical computation of the static hedge: such is the work of Derman, Ergener, and Kani [16], who have proposed an algorithm for static hedging of barrier options on a tree using vanilla options of all maturities. This construction may be problematic if the partition of the time interval is very fine, and, therefore, vanilla options of many different maturities are required. Many articles propose some optimization-based methods to find an approximate static hedge, providing, in several instances, a rather informal discussion of the analytic approach to the problem (see, for example, [3], [2], [17], [26], [19]). Indeed, since static hedging is essentially an inverse problem, it seems very natural, especially from the practical point of view, to formulate it as a problem of minimizing the hedging error. Such a method, in principle, could work in a very general stochastic model for the underlying. However, as shown in [4], even in diffusion-based models, an exact static hedge may not be available. Therefore, one is naturally forced to consider the approximate, optimal in some sense, static hedging strategies, as is done in [17], [26], [19]. In addition, the optimization-based methods, despite their generality, very often work as a "black box" and, hence, fail to provide an intuitive explanation for the solution.

We, on the other hand, would like to construct an *exact* static hedge *explicitly*, so that the corresponding payoff function can be analyzed in detail, and, in particular, its approximations and semirobust extensions can be constructed by the "geometric" arguments. Having an explicit formula for the exact static hedge, one can replicate the price of a barrier option, up to the time of hitting the barrier, with the price of a (fixed) European-type claim of the same maturity. As previously mentioned, Bardos, Douady, and Fursikov show in [4] that such a static hedge does not exist in some time-inhomogeneous diffusion models for the underlying. However, the above authors show that in any regular enough diffusion model, for any $\varepsilon > 0$, there exists an *approximate* static hedge, such that

$$\sup_{t\in[0,T]} |UOP(U,t) - SH^{\varepsilon}(U,t)| < \varepsilon,$$

where we denote the price of a UOP as a function of the underlying level S and time t by $UOP(S,t)$ and, similarly, $SH^{\varepsilon}(S,t)$ is the price of the "approximate" static hedge. The above result may seem, at least from a practical point of view, to be a solution to the problem of static hedging in the class of diffusion models. However, the proof of existence of the "approximate" hedge provided in [4] is not constructive. Hence, the payoff of SH^{ε} has to be computed numerically, and, due to the lack of exact solution, the resulting optimization problem, generally, becomes unstable (for example, as $\varepsilon \to 0$, the corresponding static hedge payoff may diverge, which is confirmed numerically). Since the examples of nonexistence of the exact static hedge, provided in [4], are based on the time dependence of the coefficients in the underlying diffusion equation, here we limit ourselves to the class of *time-homogeneous diffusion* models. We show that, under some regularity assumptions, an exact static hedge does exist in each of these models and obtain an explicit analytic expression for the associated payoff. The class of time-homogeneous diffusion models considered herein includes, in particular, all models with diffusion coefficients being asymptotically linear at zero and infinity (see Theorem 2.8 for the precise conditions). In other words, the *drift* and *local volatility* functions in such models have finite limits at the extreme values of the argument. The scope of our results is not limited to the aforementioned class of models (see Theorem 2.7), and it allows for a more general behavior of the local volatility (including the power function, as in the CEV model), given that the drift function decays fast enough at infinity (in order to satisfy Assumption 1).

Recall that the solution of static hedging problem in the Black–Scholes model relies on the "reflection principle," which can be viewed as a symmetry property of a Brownian motion. It turns out that we can still formulate the problem of static hedging as a "symmetry problem" for a general time-homogeneous diffusion, although the desired symmetries no longer follow immediately from the definition of the underlying stochastic process, as is the case for a Brownian motion. This idea can be explained as follows. Assume that we are looking for a payoff G, associated with the static hedge, in the form of a corresponding put payoff minus some function g with support to the right of the barrier. This is natural, since in financial terms it means that we try to represent the price of a UOP as the price of the corresponding put (which is higher than the UOP price), minus the price of some European-type claim with payoff g, which accounts for the possibility that the underlying hits the barrier (therefore this claim should not pay anything if the underlying ends up below the barrier at maturity). The function g is then obtained from the observation that the total price of the static hedge should be zero "along the barrier" (that is, whenever the underlying is at the barrier). In other words, g is "symmetric" to the put payoff, in the sense that, at the barrier, the effect from buying the put option can be canceled by short-selling the option with payoff g. Assuming a *linear pricing rule*, the above condition can be formulated as

$$\mathbb{E}\left(\left(K - S_\tau\right)^+ \middle|\, S_0 = U\right) = \mathbb{E}\left(g\left(S_\tau\right) \middle|\, S_0 = U\right) \quad \forall \tau \geq 0.$$

Of course, there is no straightforward probabilistic solution to the above problem, since, in general, the underlying process S (or any invertible function of it) is not symmetric with respect to the barrier. In the PDE language the above problem can be described as an attempt to develop the "method of images" for a generalized Black–Scholes equation for option prices (see, for example, [4]). In the Black model, the classical method of images implies that the corresponding function g is a scaled call payoff, but for general diffusions the straightforward application of the method of images is impossible. However, since we consider only time-homogeneous diffusions, we simplify the Black–Scholes PDE to an ODE by taking the *Laplace transform* of option prices with respect to "time-to-maturity." We then rewrite the resulting ODE as a *Sturm–Liouville (SL) problem* in the canonical form and unveil the desired symmetries of the problem via the properties of the fundamental solutions to the associated SL equation. The idea of using the Laplace transform to represent option prices in time-homogeneous diffusion models is not new: for example, Davydov and Linetsky expressed the Laplace transform of the price of a European or barrier option in terms of the spectral expansion of the corresponding SL problem (see [14], [15], [24]). The problem of static hedging turns out to be quite different from the problem of pricing: one cannot simply go from the pricing formulas of Davydov and Linetsky to static hedges in a general time-homogeneous diffusion model. However, we observe that a static hedge that succeeds "in Laplace space" will also succeed in the original time domain. Hence we concentrate on constructing the static hedge in the Laplace space, thus obtaining the solution to the original problem.

The paper is organized as follows. In subsection 2.1, we define precisely the class of financial models under consideration and formulate the static hedging problem in the "time" domain. We translate this problem into the "Laplace" domain in subsection 2.2 and relate it to the inversion of an integral transform associated with an SL problem, analyzed in subsection 2.3. The main result of our work is given in Theorems 2.7 and 2.8 in subsection 2.4. Theorem 2.7 provides an exact integral expression for the payoff of a European-type contingent claim which produces the exact static hedge of a UOP option in any regular enough time-homogeneous diffusion model (the hedge is, however, model-dependent). Theorem 2.8 contains the same result, formulated for a smaller class of diffusion models (that is, models with asymptotically linear diffusion coefficients), but under much simpler conditions, which are very easy to verify in practice. Finally, section 3 illustrates how the general formula for the static hedge can be applied in particular models, including the Black–Scholes and CEV models. We also consider an approximation of our static hedge with a linear combination of vanilla options of only two strikes, and discuss an extension of the proposed static hedge to the models constructed from time-homogeneous diffusions via an independent time-change, in subsections 3.3 and 3.4, respectively. Finally, we explain how the model-specific hedges constructed in this paper can be used to produce semirobust static superreplicating strategies in section 4.

2. Static hedge: Existence and explicit formulae.

2.1. Model setup and problem formulation. We assume that our financial market uses a linear pricing rule; i.e., prices of all contingent claims are given by discounted expectations of respective payoffs at time of maturity T under some risk-neutral measure. The interest rate r is assumed to be constant. We further assume that, under the risk-neutral measure, the underlying $(S_t)_{t\geq 0}$ is a *time-homogeneous diffusion* on $(0,\infty)$ with the generator

$$(\mathcal{G}f)(x) = \frac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x), \tag{1}$$

acting on the infinitely differentiable functions with compact support in $(0,\infty)$. The exact assumptions on μ and σ will be formulated in subsection 2.3 (see Assumption 1). For now, we assume that μ and σ are, respectively, once and twice continuously differentiable functions on $(0,\infty)$, and σ is strictly positive on this interval. In addition, we assume that $|\mu(x)|/\sigma(x)$ and $\sigma(x)/x$ are bounded as $x\to\infty$. This implies that ∞ is not an *exit* boundary point for S (see [20] or [7] for classification of the boundary points of a diffusion), and, in particular, since we always start S from a finite point, process S never reaches ∞. We, however, would like to allow for a very general behavior of the diffusion coefficients at zero; therefore, we have to assume (in the same way as was done, for example, in [24]) that 0 is either a *natural*, *exit*, or *regular* boundary, where, in the case when it is regular, the process is sent to a *cemetery* state ∂ after hitting 0 (in other words, we stop S after hitting zero). Notice that our assumptions on the behavior of the diffusion coefficients at ∞ also imply that $\mathbb{E}S_t^n < \infty$ for all $t \in [0, T]$ and $n \geq 1$. By a localization argument one can also show that process S can be represented as a time-homogeneous Itô diffusion process

$$dS_t = \mu(S_t)dt + \sigma(S_t)dB_t, \tag{2}$$

stopped at the first hitting time of zero.

For any "admissible" function g (the exact admissibility conditions are not important now and will be specified later) consider the price of a corresponding European-type contingent claim

$$u^g(S_t, \tau) = \mathbb{E}\left(e^{-r\tau} g(S_{t+\tau}) \middle| S_t\right).$$

Note that, formally, function u^g satisfies the Black–Scholes PDE, which, in the above model, becomes

$$\frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}u(x,\tau) + \mu(x)\frac{\partial}{\partial x}u(x,\tau) - ru(x,\tau) - \frac{\partial}{\partial \tau}u(x,\tau) = 0, \tag{3}$$

with the initial condition $u^g(x, 0) = g(x)$. For the prices of call and put options we will use the special notation $C(x, \tau, K)$ and $P(x, \tau, K)$, respectively, where the current level of underlying is x, the time to maturity is τ, and the strike is equal to K.

Consider a UOP option with strike K, barrier U, and maturity T (which, from now on, is assumed to be fixed, and it is the maturity of all contingent claims under consideration). Notice that, from the PDE point of view, the only difference between the price of a UOP option (viewed, again, as a function of the current level of underlying x and time to maturity τ) and the price of a corresponding put is that the former has to be equal to zero along the barrier $x = U$. Therefore, if we can find a payoff G such that, to the left of the barrier, G coincides with the corresponding put payoff, and the price of the European-type option with payoff G is zero along $x = U$, then its price also has to coincide with the price of the UOP option, up until the barrier is hit. This PDE argument can be made precise by considering a trading strategy which prescribes opening a long position in a European-type claim with payoff G and liquidating it (at no cost) when/if the underlying hits the barrier U. The payoff of such a strategy always coincides with the payoff of the UOP option, and therefore the prices have to coincide as well. Since G has to coincide with the put payoff below the barrier, we will look for G in the form $G(x) = (K - x)^+ - g(x)$, where g has support in $[U, \infty)$. Then the problem becomes the following: find g such that

$$P(U, \tau, K) = u^g(U, \tau) \quad \forall \tau \in [0, T], \quad \text{and} \quad g(x) = 0 \quad \forall x < U. \tag{4}$$

Of course, we also have to make sure that function g is such that the corresponding conditional expectation is well defined. It turns out that in many cases (including all "regular" ones) we can find a solution to the above problem in the class of continuous functions g of at most polynomial growth. In other cases the absolute integrability of $g(S_T)$ has to be verified by hand.

2.2. Matching prices in the Laplace space. In order to proceed with the above approach we need to find a convenient representation of the prices of European-type options and solve (4) for g. However, as is well known, in the time-homogeneous diffusion models, it is more convenient to work with the *Laplace transforms* of option prices, since they satisfy an ODE rather than a PDE (3). Recall that the Laplace transform of function f is defined by

$$\mathcal{L}(f)(\lambda) := \int_0^\infty e^{-\lambda y} f(y) dy$$

for all $\lambda \in \mathbb{C}$ and $f : [0, \infty) \to \mathbb{C}$, such that the above integral is absolutely convergent. Here and throughout the paper, we use the standard notation "$\mathbb{C}$" to denote the *complex plane*. Using the above notation, we introduce

$$\hat{P}(x, \lambda, K) := \mathcal{L}(P(x, ., K))(\lambda), \qquad \hat{C}(x, \lambda, K) := \mathcal{L}(C(x, ., K))(\lambda), \tag{5}$$

and, more generally,

$$\hat{u}^g(x, \lambda) := \mathcal{L}(u^g(x, .))(\lambda).$$

Taking a Laplace transform of the Black–Scholes PDE (3) and making use of the initial condition, we obtain the following SL problem:

$$\frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}\hat{u}^g(x, \lambda) + \mu(x)\frac{\partial}{\partial x}\hat{u}^g(x, \lambda) - (\lambda + r)\hat{u}^g(x, \lambda) = -g(x). \tag{6}$$

Remark 1. Some of the above derivations are, indeed, heuristic. However, if we assume that the underlying is given by the process S stopped at the first exit time of a closed (finite) interval $[a, b] \subset (0, \infty)$, and if the function g is infinitely smooth, with a support inside the same interval, then the above equation does hold for all $x \in (a, b)$ and all (possibly complex) λ's with large enough real part. In fact, in this case we have $u^g(a, \tau) = u^g(b, \tau) = 0$, and it can be shown via Itô's formula that the unique classical solution to (3) in $(a, b) \times (0, \infty)$, equipped with zero boundary conditions and the initial condition g, has to coincide with u^g. Then, from the Feynman–Kac formula and the continuity of the first and second derivatives of μ and σ, respectively, we conclude that $u^g(x, .)$, $\partial_x u^g(x, .)$, and $\partial_x^2 u^g(x, .)$ are exponentially bounded. Applying the Laplace transform, we obtain (6). Alternatively, one can find the derivation of (6) in [20].

Notice that problem (4) turns into an equivalent problem in the Laplace space: find g, such that

$$\hat{u}^g(U, \lambda) = \hat{P}(U, \lambda, K) \qquad \forall \lambda \in (\hat{\lambda}, \infty), \tag{7}$$

where $\hat{\lambda}$ can be any nonnegative real number.

Since we would like to allow for a very general behavior of the diffusion coefficients at zero, it is more convenient to avoid using put prices. This can be done via the put-call parity

$$\hat{P}(x, \lambda, K) = \hat{C}(x, \lambda, K) - \hat{E}(x, \lambda) + \frac{K}{\lambda + r},$$

where

$$\hat{E}(x, \lambda) := \mathcal{L}\left(e^{-(\cdot)r}\, \mathbb{E}\left(S_\cdot \,|\, S_0 = x\right)\right)(\lambda).$$

Then, the problem (7) turns into

$$\hat{u}^g(U,\lambda) = \hat{C}(U,\lambda,K) - \hat{E}\,(U,\lambda) + \frac{K}{\lambda+r} \qquad \forall \lambda \in (\hat{\lambda},\infty). \tag{8}$$

2.3. The SL problem. The solution of (6) can be written explicitly in terms of the fundamental solutions of the corresponding homogeneous SL problem in its *canonical form*. Let us recall some of the results from [29]. Assume that μ and σ are once and twice continuously differentiable, respectively, and introduce the following change of variables (see [29, p. 63]):

$$Z(x) := \sqrt{2}\int_U^x \frac{dy}{\sigma(y)}, \qquad X(z) := Z^{-1}(z). \tag{9}$$

Next, we introduce

$$\begin{aligned} L &:= \lim_{x\to 0} Z(x), \\ M &:= \lim_{x\to\infty} Z(x), \end{aligned}$$

which can be "$-\infty$" and "∞," respectively, and the "normalization" of functions: for each function $f:(0,\infty)\to\mathbb{R}$, we define the *normalized* function $\tilde{f}:(L,M)\to\mathbb{R}$ via

$$\tilde{f}(z) := f\,(X(z))\, e^{\frac{1}{2}\int_0^z b(y)dy}, \tag{10}$$

where

$$b(z) := \sqrt{2}\frac{\mu\,(X(z))}{\sigma\,(X(z))} - \frac{1}{\sqrt{2}}\sigma'\,(X(z))\,.$$

The integration in (9), of course, can be taken from any positive lower bound—not necessarily U (in particular, the lower bound can be zero if the corresponding integral converges). However, for convenience, we choose the transform so that it maps U into zero. The integral in (10) can be written more explicitly:

$$\int_0^z b(y)dy = -2\int_U^{X(z)} \frac{\mu(x)}{\sigma^2(x)}dx - \log\frac{\sigma\,(X(z))}{\sigma(U)},$$

and therefore

$$\tilde{f}(z) = f\,(X(z))\,\frac{\sqrt{\sigma(U)}}{X(z)}w(z),$$

with

$$w(z) := \frac{X(z)}{\sqrt{\sigma\,(X(z))}}\exp\left(-\int_U^{X(z)}\frac{\mu(x)}{\sigma^2(x)}dx\right). \tag{11}$$

In the new variables, the SL problem (6) takes its canonical form

$$\frac{\partial^2}{\partial z^2}\tilde{u}^g(z,\xi) + (\xi - q(z))\tilde{u}^g(z,\xi) = -\tilde{g}(z), \quad z\in(L,M), \tag{12}$$

where

$$q(z) := \frac{1}{4}b^2(z) + \frac{1}{2}b'(z), \tag{13}$$

$$\tilde{g}(z) = g\left(X(z)\right)\frac{\sqrt{\sigma(U)}}{X(z)}w(z), \tag{14}$$

$$\xi := -\lambda - r.$$

Function q is continuous in (L, M), with possible discontinuities at L and/or M. However, for the derivations that follow, we require additional regularity of q.

Definition 2.1. *The SL problem* (12) *"is regular on the right" if $M = \infty$, function q is continuously differentiable in (L, ∞), and there exists a constant $\eta \in \mathbb{R}$ such that $q - \eta$ and q' are absolutely integrable at ∞.*

It is clear that the constant "η" appearing in Definition 2.1 is determined uniquely as

$$\eta := \lim_{z\to\infty} q(z). \tag{15}$$

In order to provide a sufficient condition for the *regularity of the problem on the right* in terms of μ and σ, and to facilitate some of the derivations that follow, we make the following assumption.

Assumption 1. We assume that $\sigma \in C^3\left((0,\infty)\right)$, $\mu \in C^2\left((0,\infty)\right)$, $\sigma(x) > 0$ for all $x > 0$, and μ is absolutely bounded in a neighborhood of zero. In addition, the asymptotic behavior of $\sigma(x)$ and $\mu(x)$, as $x \to \infty$, is such that

$$\int_U^\infty \left|\sigma''(x)\right| \int_U^x \frac{|\sigma'(y)|+1}{\sigma(y)}dydx < \infty, \tag{16}$$

$$\int_U^\infty \left|\left(\frac{\mu(x)}{\sigma(x)}\right)'\right| \int_U^x \frac{dy}{\sigma(y)}dx < \infty, \tag{17}$$

$$\int_U^\infty \left|\left(\frac{\mu(x)}{\sigma(x)}\right)''\right| + \left|\sigma'''(x)\right|\sigma(x)dx < \infty, \tag{18}$$

$$\liminf_{x\to\infty} \frac{1}{Z(x)} \log\frac{\sqrt{\sigma\left(x\right)}}{x} > -\infty, \tag{19}$$

where $Z(.)$ is defined in (9).

Throughout this subsection, we adopt the convention that all previously made assumptions are in place for each derivation we make. However, in the statements of the resulting lemmas, we reference all the necessary assumptions explicitly.

Notice that the properties of μ and σ assumed in subsection 2.1 follow from the above assumption. In particular, (16) implies that $\sigma(x)/x$ has a finite limit at infinity, and therefore

$$M = \int_U^\infty \frac{dx}{\sigma(x)} = \infty, \tag{20}$$

which is natural if, for example, the underlying is a stock and, after discounting and compensating for the dividends, we want it to be a true martingale. The inequalities (16) and (17), essentially, require that the first derivatives of σ' and μ/σ decay fast enough at infinity, and we will need them in order to prove the absolute integrability of $q-\eta$ at infinity. Notice that (16) and (17) are satisfied, for example, if the ratio $\mu(x)/\sigma(x)$ converges fast enough to some constant, as $x \to \infty$, and $x^\gamma \sigma''(x) \to c$ for some $c \geq 0$ and $\gamma > 1$, where, if $c = 0$, then, in addition, function $\sigma'(x)$ has to have a positive limit at infinity. These cases, clearly, cover the Black–Scholes model and the CEV model with fast decaying drift (where $\sigma(x) = x^{1+\beta}$, with $\beta < 0$).

Condition (18) asserts some asymptotic regularity of the higher order derivatives of the coefficients μ and σ and is needed to show the absolute integrability of q', which, in turn, is used only in Lemma 2.6. This condition is satisfied in the case of asymptotically linear diffusion coefficients. It is also satisfied by any σ having a power-type behavior at infinity, with the exponent not exceeding one, and any function μ whose second derivative decays fast enough at infinity.

The inequality in (19) is a technical condition, and it is used to verify various growth restrictions. Together with (17), condition (19) implies that there exist real constants c_1 and c_2 such that, for all $z \geq 0$, we have

$$w(z) \leq c_1 e^{c_2 z}, \tag{21}$$

with w defined in (11). Using (16) and (18), it is also easy to see that

$$\left|w'(z)\right| \leq c_3 e^{c_2 z}, \quad \left|w''(z)\right| \leq c_4 e^{c_2 z}. \tag{22}$$

Notice that, in fact, c_2 can be any number exceeding

$$\rho := -\liminf_{x\to\infty} \frac{1}{Z(x)} \log \frac{\sqrt{\sigma(x)}}{x} - \frac{1}{\sqrt{2}} \liminf_{x\to\infty} \frac{\mu(x)}{\sigma(x)}. \tag{23}$$

The inequality in (19) is satisfied, for example, when σ is asymptotically linear at infinity (similar to the Black–Scholes case), or when there exist numbers $n \geq 1$ and $c_5, c_6 > 0$, such that, for all large enough x, we have $c_5 x^{-n} \leq \sigma(x) \leq c_6 x^{1-1/n}$, which covers the CEV case.

Lemma 2.2. *Under Assumption 1, the SL problem (12) is regular on the right, with the corresponding constant $\eta = \lim_{z\to\infty} q(z)$.*

The proof of Lemma 2.2 is given in Appendix A.

Let us now introduce the homogeneous version of (12):

$$\frac{\partial^2}{\partial z^2}\tilde{u}(z,\xi) + (\xi - q(z))\tilde{u}(z,\xi) = 0. \tag{24}$$

Notice that (12) and (24) can be considered for complex ξ, and hence we will treat their solutions as the complex-valued functions. Consider $\varphi(.,\xi)$ and $\theta(.,\xi)$ the solutions of (24) satisfying

$$\begin{aligned} &\varphi(0,\xi) = 0, \quad \partial_z\varphi(0,\xi) = -1, \\ &\theta(0,\xi) = 1, \quad\ \ \partial_z\theta(0,\xi) = 0. \end{aligned} \tag{25}$$

Then, it is well known (for example, from [29]) that, at least for any strictly complex ξ, there exist unique constants $m_1(\xi)$ and $m_2(\xi)$, such that the functions

$$\psi_1(z,\xi) = \theta(z,\xi) + m_1(\xi)\varphi(z,\xi), \tag{26}$$

$$\psi_2(z,\xi) = \theta(z,\xi) + m_2(\xi)\varphi(z,\xi) \tag{27}$$

solve (24) and satisfy zero boundary conditions at L and ∞, respectively. More precisely, $\psi_1(.,\xi)$ is an $\mathbb{L}^2(L,0)$ limit of the solutions to (24) on the intervals $(l,0)$, satisfying zero boundary condition at l and taking value 1 at zero, as $l \searrow L$. Similarly, $\psi_2(.,\xi)$ is an $\mathbb{L}^2(0,\infty)$ limit of the solutions to (24) on the intervals $(0,N)$, satisfying zero boundary condition at N and taking value 1 at zero, as $N \nearrow \infty$. The resulting functions $\psi_1(.,\xi)$ and $\psi_2(.,\xi)$ are then extended uniquely to the solutions of (24) on the entire interval (L,∞) in a straightforward way. See, for example, sections 2.1 and 2.18 of [29] for more details on this. Clearly, we have $\psi_1(.,\xi) \in \mathbb{L}^2(L,0)$ and $\psi_2(.,\xi) \in \mathbb{L}^2(0,\infty)$. Then, it is well known that, at least for all strictly complex ξ and any continuous function $\tilde{g}$, with compact support in (L,∞), there exists a unique solution to (12), satisfying zero boundary conditions at L and ∞ (again, in the sense that it is an $\mathbb{L}^2(L,\infty)$ limit of solutions satisfying zero boundary conditions on finite subintervals), and it is given by

$$\tilde{u}^g(z,\xi) = \frac{\psi_2(z,\xi)\int_L^z \psi_1(y,\xi)\tilde{g}(y)dy + \psi_1(z,\xi)\int_z^\infty \psi_2(y,\xi)\tilde{g}(y)dy}{m_2(\xi)-m_1(\xi)}. \tag{28}$$

In fact, it can be verified directly that the above function solves (12) and that it is a limit of solutions to (12) on finite intervals satisfying zero boundary conditions. Zero boundary condition for $\tilde{u}^g$ at infinity is chosen from the probabilistic considerations, since we assume that "∞" is not accessible and the payoff g vanishes in a neighborhood of infinity. However, in general, a boundary condition for $\tilde{u}^g$ at L may be different (see [29]), resulting, possibly, in a different function ψ_1 (and, equivalently, m_1). We choose zero boundary condition at L in the present case, since the underlying stochastic process is stopped at zero (or, in other words, sent to the cemetery state, if 0 is a regular point) and because we will deal with the European-type options whose payoffs vanish in a neighborhood of zero. We do not provide a detailed discussion on how to obtain an appropriate boundary condition for ψ_1 in the case of another type of payoff function and/or when 0 is a boundary point of a different type for the diffusion process S. The interested reader is referred to [24] and the references therein for more on this.

As mentioned above, it is well known (see, for example, [29]) that for any *potential function* q, continuous in (L,∞), the functions $m_1(\xi)$ and $m_2(\xi)$ are well defined for $\mathfrak{Im}(\xi) \neq 0$ (here and further in the paper, we denote by "$\mathfrak{Im}$" the imaginary part of a complex number and by "$\mathfrak{Re}$" its real part). In addition, these functions are holomorphic in the upper and lower half-planes (see, for example, section 2.1 in [29]). As is demonstrated below, in some cases (for example, if q is absolutely integrable) the functions m_1 and m_2 (in fact, even $1/(m_1-m_2)$) can be extended to holomorphic functions in the complex plane cut along a real half-line. In the present case, due to our assumptions on the asymptotic behavior of q at infinity, we can deduce some additional properties of function m_2.

In the following lemma, and throughout the rest of the paper, we choose a version of the "square root" which generates a continuous mapping from $\mathbb{C} \setminus [0, \infty)$ to the upper half plane. The square root of a nonnegative real number is nonnegative.

In addition, we denote

$$I := \int_0^\infty |q(z) - \eta| \, dz < \infty. \tag{29}$$

Lemma 2.3. *Let the SL problem (12) be regular on the right. Then there exists a unique analytic continuation of function m_2 to the domain $\mathbb{C} \setminus \left[\eta - 16I^2, \infty\right)$, such that $\psi_2(., \xi)$, defined by (27), is square integrable over $(0, \infty)$ for each $\xi \in \mathbb{C} \setminus \left[\eta - 16I^2, \infty\right)$ and the function $\psi_2(z, .)$ is analytic in $\mathbb{C} \setminus \left[\eta - 16I^2, \infty\right)$ for each $z \in (L, \infty)$.*

In addition, we have

$$m_2(\xi) = -i\sqrt{\xi - \eta}\left(1 + \underline{\underline{O}}\left(|\xi|^{-1/2}\right)\right)$$

whenever $|\xi| \to \infty$ satisfying $\xi \in \mathbb{C} \setminus \left[\eta - 16I^2, \infty\right)$.

Proof. Notice that if we consider a new SL problem with the potential $\tilde{q} := q - \eta$, then function m_2 of the original SL problem coincides with the corresponding function of the new problem, up to the shift of variables: $\xi \mapsto \xi - \eta$. Therefore, it is enough to prove the statement of the lemma for the case $\eta = 0$.

Denote

$$H := \mathbb{C} \setminus \left(\left\{ z \mid |z| \le 16I^2 \right\} \cup [0, \infty)\right).$$

Let $\xi \in H$, and consider the following integral equation for functions of $z \in [0, \infty)$:

$$\begin{aligned} \chi(z, \xi) = e^{iz\sqrt{\xi}} &+ \frac{1}{2i\sqrt{\xi}} \int_0^z e^{i(z-x)\sqrt{\xi}} q(x)\chi(x, \xi) dx \\ &+ \frac{1}{2i\sqrt{\xi}} \int_z^\infty e^{i(x-z)\sqrt{\xi}} q(x)\chi(x, \xi) dx. \end{aligned} \tag{30}$$

It is not hard to see that if the above equation has a solution $\chi(., \xi)$, then it is twice continuously differentiable and satisfies (24). On the other hand, it is shown in section 6.2 (page 119) of [29] that the iterative scheme

$$\chi_1(z, \xi) = e^{iz\sqrt{\xi}}$$

and

$$\begin{aligned} \chi_{n+1}(z, \xi) = e^{iz\sqrt{\xi}} &+ \frac{1}{2i\sqrt{\xi}} \int_0^z e^{i(z-x)\sqrt{\xi}} q(x)\chi_n(x, \xi) dx \\ &+ \frac{1}{2i\sqrt{\xi}} \int_z^\infty e^{i(x-z)\sqrt{\xi}} q(x)\chi_n(x, \xi) dx, \quad \text{for} \quad n \ge 1, \end{aligned}$$

converges to the solution of (30), $\chi(., \xi)$.

In particular, it is shown in formulas (6.2.5) and (6.2.6) therein that

$$|\chi_{n+1}(z,\xi) - \chi_n(z,\xi)| \leq \left(\frac{I}{2\left|\sqrt{\xi}\right|}\right)^n \left|e^{iz\sqrt{\xi}}\right|,$$

and, hence, the convergence is uniform in ξ changing on any compact in H. This, in turn, yields that the function $\chi(z,.)$ is holomorphic in H for any $z \in [0,\infty)$. Moreover, the following estimate holds:

$$|\chi(z,\xi)| \leq \frac{\left|e^{iz\sqrt{\xi}}\right|}{1 - I/(2\left|\sqrt{\xi}\right|)}. \tag{31}$$

Similarly, we can derive an iterative scheme for $\partial_z\chi(z,\xi)$ and conclude that $\partial_z\chi(z,.)$ is holomorphic in H for any $z \in [0,\infty)$.

Thus, function $\chi(.,\xi)$ solves (24) and is square integrable over $(0,\infty)$ for each $\xi \in H$. Clearly, $\chi(.,\xi)$ can be represented as a linear combination of $\varphi(.,\xi)$ and $\theta(.,\xi)$, and therefore it can be uniquely extended to the entire domain (L,∞).

Next, we establish that $\chi(.,\xi)$ is the unique (up to a multiplicative factor) solution to (24) which is square integrable over $(0,\infty)$. From equation (5.3.1) in section 5.3 of [29], we obtain the following representation of φ:

$$\begin{aligned}\varphi(z,\xi) = \frac{e^{-iz\sqrt{\xi}}}{2i\sqrt{\xi}}\Bigg(1 - e^{2iz\sqrt{\xi}} &+ \int_0^z e^{i(z-x)\sqrt{\xi}}e^{iz\sqrt{\xi}}\varphi(x,\xi)q(x)dx \\ &- \int_0^z e^{ix\sqrt{\xi}}\varphi(x,\xi)q(x)dx\Bigg).\end{aligned} \tag{32}$$

Using the above representation and Lemma 5.2 in section 5.2 of [29], we obtain the following estimate (given in the last equation on page 98 in section 5.3 therein):

$$\left|\varphi(z,\tilde{\lambda})\right| \leq \frac{1}{\left|\sqrt{\xi}\right|}\exp\left(\frac{I}{\left|\sqrt{\xi}\right|}\right)\left|e^{-iz\sqrt{\xi}}\right|. \tag{33}$$

The above inequality, in turn, yields, for all $z \geq 0$ and $\xi \in H$,

$$\begin{aligned}\left|\int_0^z e^{i(z-x)\sqrt{\xi}}e^{iz\sqrt{\xi}}\varphi(x,\xi)q(x)dx\right| &+ \left|\int_0^z e^{ix\sqrt{\xi}}\varphi(x,\xi)q(x)dx\right| \\ &\leq 2\frac{I}{\left|\sqrt{\xi}\right|}\exp\left(\frac{I}{\left|\sqrt{\xi}\right|}\right) < \frac{1}{2}e^{1/4} < 1.\end{aligned} \tag{34}$$

From the above estimate and (32), we conclude that, for any $\xi \in H$, we have

$$\lim_{z\to\infty}|\varphi(z,\xi)| = \infty,$$

and therefore $\varphi(.,\xi)$ is not square integrable over $(0,\infty)$. This implies that $\varphi(.,\xi)$ is linearly independent of $\chi(.,\xi)$, and, hence, the two functions span the entire space of solutions to

(24). This, in turn, yields that any solution to (24), square integrable over $(0,\infty)$, has to be proportional to χ.

Applying (31), we obtain, for all $\xi \in H$,

$$\left|\frac{1}{2i\sqrt{\xi}}\int_0^\infty e^{iy\sqrt{\xi}}q(y)\chi(y,\xi)dy\right| \leq \frac{I}{2\left|\sqrt{\xi}\right|-I} < \frac{1}{7},$$

and therefore

$$\chi(0,\xi) = 1+\frac{1}{2i\sqrt{\xi}}\int_0^\infty e^{iy\sqrt{\xi}}q(y)\chi(y,\xi)dy \neq 0.$$

Thus, for any $\xi \in H$, function $\chi(.,\xi)/\chi(0,\xi)$ is the unique solution to (24), which is square integrable over $(0,\infty)$ and is equal to one at zero. Therefore, for all strictly complex $\xi \in H$, we must have

$$\frac{\chi(z,\xi)}{\chi(0,\xi)} = \psi_2(z,\xi) = \theta(z,\xi)+m_2(\xi)\varphi(z,\xi). \tag{35}$$

In particular, the above implies that

$$m_2(\xi) = -\frac{\partial_z\chi(0,\xi)}{\chi(0,\xi)}$$

for all strictly complex $\xi \in H$. We have shown that the right-hand side of the above is holomorphic in H, and it was mentioned before that the left-hand side is holomorphic in $\mathbb{C}\setminus\mathbb{R}$. Therefore, both the left- and the right-hand sides can be analytically extended to $\mathbb{C}\setminus\left[-16I^2,\infty\right)$.

Notice also that $\varphi(z,.)$ and $\theta(z,.)$ are entire functions (holomorphic in $\mathbb{C}$) for any $z \in (L,\infty)$ (see, for example, Theorem 1.5 in section 1.5 of [29]). This implies that $\psi_2(z,.)$ is holomorphic in $\mathbb{C}\setminus\left[-16I^2,\infty\right)$ for any $z\in(L,\infty)$. And, of course, $\psi_2(.,\xi)$ is square integrable over $(0,\infty)$ for any $\xi \in \mathbb{C}\setminus\left[-16I^2,\infty\right)$, due to (35) and the definition of $m_2(\xi)$, and for all strictly complex ξ.

Finally, notice that, for all $\xi \in \mathbb{C}\setminus\left[-16I^2,\infty\right)$, we have

$$m_2(\xi) = -\frac{\partial_z\chi(0,\xi)}{\chi(0,\xi)} = -i\sqrt{\xi}\frac{1-\frac{1}{2i\sqrt{\xi}}\int_0^\infty e^{iy\sqrt{\xi}}q(y)\chi(y,\xi)dy}{1+\frac{1}{2i\sqrt{\xi}}\int_0^\infty e^{iy\sqrt{\xi}}q(y)\chi(y,\xi)dy}.$$

Considering $|\xi|\to\infty$ and applying (31), we obtain the last statement of the lemma. ■

It is clear that if the SL problem is "regular on the left" as well (namely, if $q-\eta'$ is absolutely integrable in a neighborhood of L for some $\eta' \in \mathbb{R}$), then the statement of the above lemma can be extended to m_1. However, since some of the relevant financial models imply a rather singular behavior of q at L, instead of restricting its asymptotic behavior we make the following assumption.

Assumption 2. We assume that there exists a real constant $\kappa \leq 0$, such that m_2 can be continued analytically to the domain $\mathbb{C}\setminus[\kappa,\infty)$, and the corresponding function $\psi_1(.,\xi)$, defined by (26), is square integrable over $(L,0)$ for each $\xi \in \mathbb{R}\setminus[\kappa,\infty)$.

In addition, we assume that there exists a complex number $c \neq -1$, such that

$$m_1(\xi) = i\sqrt{\xi}\left(c + \underline{\underline{O}}\left(|\xi|^{-1/2}\right)\right)$$

whenever $|\xi| \to \infty$ satisfying $\Im\left(\sqrt{\xi}\right) \geq \sqrt{|\kappa|}$.

Notice that it follows from Assumption 2 that $\psi_1(.,\xi)$ is square integrable over $(L, 0)$ for each $\xi \in \mathbb{C} \setminus [\kappa, \infty)$, and the function $\psi_1(z, .)$ is analytic in $\mathbb{C} \setminus [\kappa, \infty)$ for each $z \in (L, \infty)$. It also follows from Lemma 2.3 and Assumption 2 that function $1/(m_1 - m_2)$ is holomorphic in $\mathbb{C} \setminus \left[\kappa \wedge \left(\eta - 16I^2\right), \infty\right)$. To see this, we need only check that there are no zeros of $m_1 - m_2$ in the above domain. We know (for example, from [29]) that $\mathfrak{Im}\,(m_1)$ and $\mathfrak{Im}\,(m_2)$ have opposite signs and can have zeros only at the real axis. In addition, from the original form of (6), we know that, when $\xi \in \left(-\infty, \kappa \wedge \left(\eta - 16I^2\right)\right]$, the functions $\psi_1(.,\xi)$ and $\psi_2(.,\xi)$, after inverting the "normalization" (defined in (10)), become increasing and decreasing, respectively (see, for example, [20] or [7] for a more detailed argument), and, therefore, they cannot coincide. Thus, $m_1 - m_2$ has no zeros in $\mathbb{C} \setminus \left[\kappa \wedge \left(\eta - 16I^2\right), \infty\right)$.

From the absolute integrability of $q - \eta$ over $(0, \infty)$, we can deduce some other useful facts.

Lemma 2.4. *Let the SL problem (12) be regular on the right, and let Assumption 2 hold. Then, there exists a continuous function $f : \{\xi \in \mathbb{C} \mid \mathfrak{Im}\left(\sqrt{\xi}\right) > \sqrt{|\kappa| \vee (16I^2 + |\eta|)}\} \to (0, \infty)$, such that, for all $z \geq 0$ and all ξ in the domain of f, we have*

$$|\psi_1(z,\xi)| + |\partial_z \psi_1(z,\xi)| \leq f(\xi)\left|e^{-iz\sqrt{\xi - \eta}}\right|,$$
$$|\psi_2(z,\xi)| + |\partial_z \psi_2(z,\xi)| \leq f(\xi)\left|e^{iz\sqrt{\xi - \eta}}\right|.$$

Proof. Again, without any loss of generality, we assume that $\eta = 0$. Then the statement of the lemma follows from the derivations in the proof of Lemma 2.3.

In particular, differentiating the representation (32) and applying the estimate (33), we obtain the first inequality in the statement of the lemma, with φ in lieu of ψ_1. Using equation (5.3.1) in section 5.3 of [29] again, we obtain a similar representation of θ (defined in (25)),

$$\theta(z,\xi) = \frac{e^{-iz\sqrt{\xi}}}{2i}\left(1 + e^{2iz\sqrt{\xi}} + \frac{1}{\sqrt{\xi}}\int_0^z e^{i(z-x)\sqrt{\xi}} e^{iz\sqrt{\xi}}\theta(x,\xi)q(x)dx \right.$$
$$\left. - \frac{1}{\sqrt{\xi}}\int_0^z e^{ix\sqrt{\xi}}\theta(x,\xi)q(x)dx\right),$$

and derive an analogous estimate:

$$\left|\theta(z,\tilde{\lambda})\right| \leq \left(1 + \frac{1}{|\sqrt{\xi}|}\right)\exp\left(\frac{I}{|\sqrt{\xi}|}\right)\left|e^{-iz\sqrt{\xi}}\right|.$$

Differentiating the above representation and applying the estimate, we obtain the first inequality in the statement of the lemma, with θ in lieu of ψ_1. Collecting the above we verify the aforementioned inequality holds for ψ_1.

Applying (35) and differentiating (30), we make use of (31) to obtain the second inequality of the lemma. ∎

Next, we introduce $\tilde{K}$, the image of strike K under the change of variables $x \mapsto Z(x)$,

$$\tilde{K} := \sqrt{2} \int_U^K \frac{dy}{\sigma(y)},$$

and $\tilde{C}(\cdot)$, the normalized call option's payoff $(\cdot - K)^+$ in the new variables:

$$\tilde{C}(z) := (X(z) - K)^+ \frac{\sqrt{\sigma(U)}}{X(z)} w(z), \tag{36}$$

where w is defined in (11).

We would like to formulate the problem (8) explicitly in terms of solutions to the SL problem (12). Naturally, we need to make use of (28). However, so far we have verified only that formula (28) holds for continuous functions $\tilde{g}$ with compact support in (L, ∞). This is, clearly, not sufficient, since, for example, the modified call payoff, defined above, does not have a compact support. In addition, we have not provided a rigorous link between the solutions to (12) and the Laplace transforms of option prices with a general payoff function g. These issues are addressed in the following lemma.

Lemma 2.5. *Let Assumptions 1 and 2 hold, and let function $g : (0, \infty) \to \mathbb{R}$ be continuous, vanishing in a neighborhood of zero, polynomially bounded, and such that the corresponding $\tilde{g}$ (the normalization of g, according to (10)) is exponentially bounded. Denote by $u^g(x, T)$ the time zero price of a European-type option with payoff $g(S_T)$, given that $S_0 = x$. Denote also by $\hat{u}^g(x, .)$ the Laplace transform of $u^g(x, .)$, and by $\tilde{u}^g(., \xi)$ the normalization of $\hat{u}^g(., -\xi - r)$ according to (10). Then, for all ξ with sufficiently large (negative) real part, $\hat{u}^g(x, -\xi - r)$ and $\tilde{u}^g(z, \xi)$ are well defined for all $x > 0$ and $z \in (L, \infty)$, and $\tilde{u}^g(z, \xi)$ is given by the right-hand side of (28).*

Proof. Choose a sequence of functions $\{g_n\}$, such that each g_n is smooth and has a compact support in a (fixed) interval $[\varepsilon, \infty)$, the sequence converges to g uniformly on all compacts, and $|g_n(x)| \leq 2|g(x)| \vee 1$ for all $x > 0$. Due to the dominated convergence theorem, for all $x, T > 0$, we have

$$u^g(x, T) = \lim_{n \to \infty} u^{g_n}(x, T),$$

where u^{g_n} is the price of a European-type claim with payoff g_n. Recall that $\mu(x)/\sigma(x)$ and $\sigma(x)/x$ are bounded as $x \to \infty$; therefore, the polynomial boundedness of g implies that

$$|u^g(x, T)| \leq c_1(x) e^{c_2(x) T} \qquad \forall T > 0$$

and for some positive constants c_i depending upon x. Then, for all λ with large enough real part, $\hat{u}^{g_n}(x, \lambda)$, the Laplace transform of $u^{g_n}(x, .)$, converges to $\hat{u}^g(x, \lambda)$, the Laplace transform of $u^g(x, .)$, as $n \to \infty$, again due to the dominated convergence theorem. Introduce $\tilde{g}_n$ and $\tilde{u}^{g_n}(., \xi)$ as the normalizations of g_n and $\hat{u}^{g_n}(., -\xi - r)$ according to (10). The above implies that

$$\tilde{u}^{g_n}(z, \xi) \to \tilde{u}^g(z, \xi) \quad \text{as } n \to \infty.$$

Next, we introduce S^N, which is defined as the process S stopped at the first exit time of the interval $(1/N, N)$. Denote by $u^{g_n, N}(x, \tau)$ the price of a European-type option with payoff g_n in a model where the underlying is given by S^N. Similarly, denote by $\hat{u}^{g_n,N}(x,\lambda)$ the Laplace transform of $u^{g_n,N}(x,.)$, and by $\tilde{u}^{g_n,N}(.,\xi)$ the normalization of $\hat{u}^{g_n,N}(.,-\xi-r)$. Clearly, $\tilde{u}^{g_n,N}(.,\xi)$ is well defined for all ξ with large enough negative real part. In addition, as discussed in Remark 1, for all large enough N, function $\tilde{u}^{g_n,N}(.,\xi)$ is a unique solution to (6) satisfying zero boundary conditions at $1/N$ and N. Recall that, as is mentioned in the paragraph preceding (28) (and discussed in detail in sections 2.1 and 2.18 of [29]), at least for all strictly complex values of ξ, $\tilde{u}^{g_n,N}(z,\xi)$ converges to the right-hand side of (28) (with g_n in lieu of g) as $N\to\infty$. On the other hand, due to the dominated convergence theorem, we have

$$\lim_{N\to\infty}\tilde{u}^{g_n,N}(z,\xi)=\tilde{u}^{g_n}(z,\xi).$$

Hence $\tilde{u}^{g_n}(z,\xi)$ is given by the right-hand side of (28) for all $z\in(L,\infty)$ and all strictly complex ξ with large enough (negative) real part.

Let us show that there exists a positive real constant M, such that, for any $z\in(L,\infty)$,

$$\int_L^z \psi_1(y,\xi)\tilde{g}^n(y)dy \quad \text{and} \quad \int_z^\infty \psi_2(y,\xi)\tilde{g}^n(y)dy \tag{37}$$

are holomorphic functions of ξ in the half plane

$$H_M:=\{\xi\in\mathbb{C}\mid \mathfrak{Re}(\xi)<-M\}.$$

It is clear that, due to Lemma 2.3 and Assumption 2, there exists $M>0$, such that both $\psi_1(z,.)$ and $\psi_2(z,.)$ are holomorphic in H_M for each $z\in(L,\infty)$. Then the Riemann sums, approximating the above integrals, are also holomorphic in this half plane. It only remains to notice that, due to Lemma 2.4, the aforementioned Riemann sums converge to the above integrals uniformly in ξ changing on any compact in H_M. Thus, the above integrals are holomorphic functions of $\xi\in H$.

Finally, notice that $\hat{u}^{g_n}(x,-\xi-r)$ (and, consequently, $\tilde{u}^{g_n}(z,\xi)$) is continuous in ξ, whenever $\xi\in H_M$, for large enough M. Therefore, by continuity, we extend the above result and conclude that $\tilde{u}^{g_n}(z,\xi)$ is given by the right-hand side of (28) for all $z\in(L,\infty)$ and all ξ with large enough (negative) real part. Notice that, since $|g_n|$'s are uniformly bounded by $2|g|$, the $|\tilde{g}_n|$'s are uniformly bounded by $2|\tilde{g}|$. Thus, since $|\tilde{g}|$ is bounded by an exponential, due to Lemma 2.4 and the dominated convergence theorem, we can pass to the limit inside the integrals in (37), as $n\to\infty$, to conclude that, at least for all ξ's with large enough (negative) real part, $\tilde{u}^g(z,\xi)$ is given by the right-hand side of (28). ■

Recall the original static hedging problem (7), and notice that in the new variables it takes the following form:

$$\tilde{u}^g(0,\xi)=\tilde{u}^{(\,\cdot\,-K)^+}(0,\xi)-\hat{E}\left(U,-\xi-r\right)-\frac{K}{\xi}\qquad\forall\xi\in(-\infty,\hat{\lambda})$$

and for some constant $\hat{\lambda}\in\mathbb{R}$.

It follows from Lemma 2.5 and the exponential bound on the normalized call payoff $\tilde{C}$, provided by (21), that the first term in the right-hand side of the above equation can be represented via (28). Assuming that the target payoff g satisfies the assumptions of Lemma

2.5 and recalling that $\psi_1(0,\xi) = \psi_2(0,\xi) = 1$, we rewrite the left-hand side of the above equation using (28) as well. Then the static hedging problem becomes the following: find an exponentially bounded continuous function $\tilde{g}$, with support in $[0,\infty)$, such that

$$\int_0^\infty \psi_2(y,\xi)\tilde{g}(y)dy = \int_{\tilde{K}}^0 \psi_1(y,\xi)\tilde{C}(y)dy + \int_0^\infty \psi_2(y,\xi)\tilde{C}(y)dy + (m_1(\xi) - m_2(\xi))\left(\hat{E}\left(U, -\xi - r\right) + \frac{K}{\xi}\right) \tag{38}$$

holds for all real $\xi < \hat{\lambda}$, with some constant $\hat{\lambda} \in \mathbb{R}$.

Notice that, as follows from Lemma 2.3 and Assumption 2, the functions $\psi_1(z,.)$, $\psi_2(z,.)$, m_1, and m_2 are well defined and analytic in

$$\mathbb{C} \setminus \left[-|\kappa| \vee \left(|\eta| + 16I^2\right), \infty\right) \supset \mathbb{C} \setminus \left[-|\kappa| \vee \left(16I^2\right) - |\eta|, \infty\right).$$

From Lemma 2.4 and the particular form of the exponential bound on $\tilde{C}(z)$, given in (21) and (23), we conclude that the integrals in the right-hand side of (38) are absolutely convergent, uniformly over ξ changing on any compact in

$$\left\{\xi \in \mathbb{C} \mid \mathfrak{Im}\left(\sqrt{\xi}\right) > \sqrt{\rho^2 \vee |\kappa| \vee (16I^2) + |\eta|}\right\} \subset \left\{\xi \in \mathbb{C} \mid \mathfrak{Im}\left(\sqrt{\xi - \eta}\right) > |\rho|\right\},$$

where ρ is defined in (23). Hence, repeating the argument in the proof of Lemma 2.5, we conclude that the integrals in the right-hand side of (38) are analytic functions of ξ in the above domain. However, since we want to allow for a rather general behavior of μ and σ around zero and do not require S to be tradable, we do not know a priori the analyticity of $\hat{E}(U, -\xi - r)$ in the above domain.

Assumption 3. We assume that $\hat{E}(U, -\xi - r)$ can be extended analytically to

$$\left\{\xi \in \mathbb{C} \mid \mathfrak{Im}\left(\sqrt{\xi}\right) > \sqrt{\zeta}\right\}$$

for some real $\zeta \geq 0$. In addition, we assume that there exists a constant $d \in \mathbb{C}$, such that, in the above domain, we have

$$\hat{E}(U, -\xi - r) = \frac{d}{\xi} + \underline{\underline{O}}\left(|\xi|^{-3/2}\right)$$

as $|\xi| \to \infty$.

Remark 2. In the case when S is tradable, $\left(e^{-rt}S_t\right)$ is a martingale, and we have

$$\hat{E}(U, -\xi - r) = -\frac{U}{\xi + r}.$$

Therefore, Assumption 3 is satisfied with $\zeta = r$.

Next, we choose arbitrary $\delta > 0$ and introduce

$$J := \sqrt{\zeta \vee (\rho^2 \vee |\kappa| \vee (16I^2) + |\eta|)} + \delta, \tag{39}$$

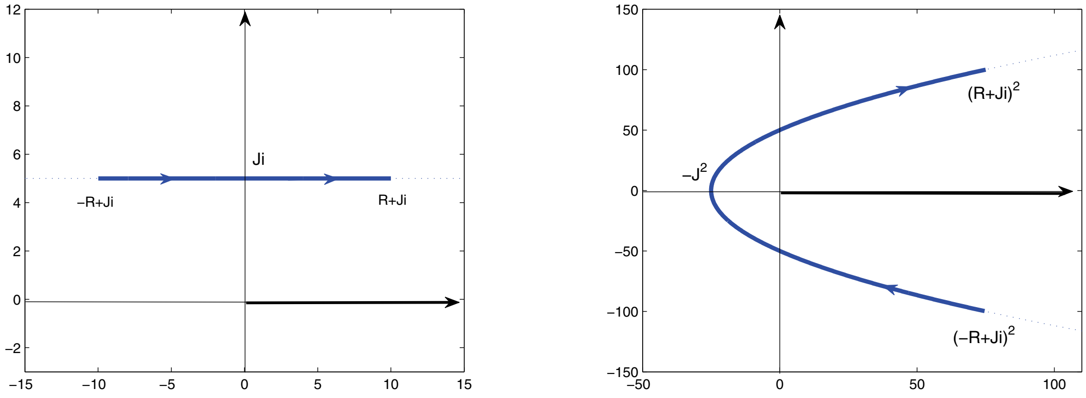

Figure 1. *Contours $\mathcal{I}_R$ (on the left) and $\mathcal{G}_R$ (on the right).*

where the constant η is given by (15), I is defined in (29), κ is given by Assumption 2, ρ is defined in (23), and ζ is given by Assumption 3.

We will find a solution to (38), with any $\hat{\lambda} < -J^2 - |\eta|$, in an integral form. For all real $R > 0$, introduce the intervals

$$\mathcal{I}_R := [-R + iJ, R + iJ], \tag{40}$$

and denote by $\mathcal{G}_R$ the image of $\mathcal{I}_R$, with the direction "from left to right," under the mapping: $\xi \mapsto \xi^2$. The contours $\mathcal{I}_R$ and $\mathcal{G}_R$ are shown in Figure 1. We also introduce

$$\mathcal{G} := \lim_{R\to\infty} \mathcal{G}_R, \quad \mathcal{G}^+ := \left\{\xi \in \mathbb{C} \mid \mathfrak{Im}\left(\sqrt{\xi}\right) > J\right\}. \tag{41}$$

Next, we denote

$$\Upsilon(\xi) := \int_{\tilde{K}}^{0} \psi_1(y,\xi)\tilde{C}(y)dy + \int_0^\infty \psi_2(y,\xi)\tilde{C}(y)dy + (m_1(\xi) - m_2(\xi))\left(\hat{E}\left(U, -\xi - r\right) + \frac{K}{\xi}\right). \tag{42}$$

Recall that, due to Assumption 3 and the preceding arguments, Υ is well defined and analytic in a domain containing $\mathcal{G}^+ \cup \mathcal{G}$. Finally, for each $R > 0$, we define function $\tilde{g}_R : (L, \infty) \to \mathbb{R}$ via

$$\tilde{g}_R(z) := \frac{1}{2\pi i} \int_{\mathcal{G}_R} \psi_1(z,\alpha) \frac{\Upsilon(\alpha)}{m_2(\alpha) - m_1(\alpha)} d\alpha,$$

whenever $z \geq 0$, and $\tilde{g}_R(z) = 0$ otherwise. Herein, we denote the conjugate of a complex number (or a set of numbers) w by "$\bar{w}$." Notice that $\overline{\mathcal{G}_R} = \mathcal{G}_R$ (with the reversed direction), and as shown, for example, in [29], we have

$$\overline{\psi_i(z,\xi)} = \psi_i(z,\bar{\xi}) \quad \text{and} \quad \overline{m_i(\xi)} = m_i(\bar{\xi}).$$

This yields

$$\overline{\tilde{g}_R(z)} = \tilde{g}_R(z),$$

and, hence, $\tilde{g}_R$ is, indeed, real-valued.

We will show that, as $R \to \infty$, the left-hand side of (38), with $\tilde{g}_R$ in lieu of $\tilde{g}$, converges to the right-hand side of (38). Fix some $\xi \in \mathcal{G}^+$, and consider

$$\begin{aligned}\int_0^\infty \psi_2(y,\xi)\tilde{g}_R(y)dy &= \frac{1}{2\pi i}\int_0^\infty \psi_2(y,\xi)\int_{\mathcal{G}_R}\psi_1(y,\alpha)\frac{\Upsilon(\alpha)}{m_2(\alpha)-m_1(\alpha)}d\alpha dy\\ &= \frac{1}{2\pi i}\int_{\mathcal{G}_R}\frac{\Upsilon(\alpha)}{m_2(\alpha)-m_1(\alpha)}\int_0^\infty \psi_1(y,\alpha)\psi_2(y,\xi)dyd\alpha.\end{aligned}$$

From section 2.1 of [29], we know that

$$(\xi-\alpha)\int_0^\infty \psi_1(y,\alpha)\psi_2(y,\xi)dy = W_0\left(\psi_1(.,\alpha),\psi_2(.,\xi)\right) - \lim_{z\to\infty} W_z\left(\psi_1(.,\alpha),\psi_2(.,\xi)\right),$$

where $W_z(h_1,h_2) = h_1(z)h_2'(z) - h_2(z)h_1'(z)$ is the Wronskian of h_1 and h_2 at point z. Notice also that

$$W_0\left(\psi_1(.,\alpha),\psi_2(.,\xi)\right) = m_1(\alpha) - m_2(\xi).$$

And from Lemma 2.4 we conclude that, for any $\xi \in \mathcal{G}^+$ and any $\alpha \in \mathcal{G}_R$, we have

$$\lim_{z\to\infty} W_z\left(\psi_1(.,\alpha),\psi_2(.,\xi)\right) = 0.$$

Thus, we obtain

$$\int_0^\infty \psi_2(y,\xi)\tilde{g}_R(y)dy = \frac{1}{2\pi i}\int_{\mathcal{G}_R}\frac{\Upsilon(\alpha)}{(\alpha-\xi)}\frac{m_1(\alpha)-m_2(\xi)}{m_1(\alpha)-m_2(\alpha)}d\alpha. \tag{43}$$

Now we close the contour of integration $\mathcal{G}_R$ with $\mathcal{C}_R$, the arch of a circle, centered at zero, which connects the end points of $\mathcal{G}_R$ counterclockwise. Notice that each $\mathcal{C}_R$ is in $\mathcal{G}^+$; hence Υ is well defined and analytic in the domain bounded by $\mathcal{G}_R \cup \mathcal{C}_R$. Therefore, by the calculus of residues, the contour integral of the integrand in the right-hand side of (43) is equal to $2\pi i\Upsilon(\xi)$ if ξ is inside the contour $\mathcal{G}_R \cup \mathcal{C}_R$, and zero otherwise. It only remains to prove that

$$\lim_{R\to\infty}\int_{\mathcal{C}_R}\frac{\Upsilon(\alpha)}{(\xi-\alpha)}\frac{m_1(\alpha)-m_2(\xi)}{m_1(\alpha)-m_2(\alpha)}d\alpha = 0. \tag{44}$$

From Lemma 2.3 and Assumption 2 we conclude that the expression

$$\frac{m_1(\alpha)-m_2(\xi)}{m_1(\alpha)-m_2(\alpha)} = 1 + \frac{\frac{m_2(\alpha)}{i\sqrt{\alpha}} - \frac{m_2(\xi)}{i\sqrt{\alpha}}}{\frac{m_1(\alpha)}{i\sqrt{\alpha}} + 1 - \left(1 + \frac{m_2(\alpha)}{i\sqrt{\alpha}}\right)}$$

is bounded for all $\alpha \in \mathcal{C}_R$, uniformly over R. Therefore, for a fixed $\xi \in \mathcal{G}_+$ and for all large enough R, we have

$$\left|\int_{\mathcal{C}_R}\frac{\Upsilon(\alpha)}{(\xi-\alpha)}\frac{m_1(\alpha)-m_2(\xi)}{m_1(\alpha)-m_2(\alpha)}d\alpha\right| \le \frac{c_7}{R^2}\int_{\mathcal{C}_R}|\Upsilon(\alpha)|\,|d\alpha| \tag{45}$$

for some positive real constant c_7. Recall that Υ is defined in (42) as a sum of three terms. Let us plug the second term into the right-hand side of (45) to obtain

$$\int_{\mathcal{C}_R} \left| \int_0^\infty \psi_2(z,\alpha)\tilde{C}(z)dz \right| |d\alpha| \leq \int_0^\infty \int_{\mathcal{C}_R} |\psi_2(z,\alpha)|\, |d\alpha|\, \tilde{C}(z)dz. \tag{46}$$

In order to proceed with the estimate, we need to study the asymptotic behavior of $\psi_2(z,\xi)$ for large $|\xi|$.

Lemma 2.6. *Let the SL problem* (12) *be regular on the right. Then the asymptotic relation*

$$\psi_2(z,\xi) = e^{iz\sqrt{\xi-\eta}} \left(1 + \frac{1}{2i\sqrt{\xi-\eta}} \int_0^z (q(y)-\eta)\, dy + \underline{\underline{O}}\left(|\xi|^{-1}\right) \right)$$

holds uniformly over z changing on $[0,\infty)$ and on any compact in $(L,0]$, as $|\xi| \to \infty$, satisfying $\mathfrak{Im}\left(\sqrt{\xi}\right) \geq \left(16I^2 - \eta \wedge 0\right)^{1/2}$.

Proof. Using the representation (30) and estimate (31) (recall that "ξ" has to be changed to "$\xi-\eta$," and "q" to "$q-\eta$," in those formulas), we conclude that

$$\chi(z,\xi) = e^{iz\sqrt{\xi-\eta}} \left(1 + \underline{\underline{O}}\left(|\xi|^{-1/2}\right) \right), \tag{47}$$

as $|\xi| \to \infty$, uniformly over $z \in [0,\infty)$. Plugging the above expression back into (30), we obtain

$$\begin{aligned}\chi(z,\xi) &= e^{iz\sqrt{\xi-\eta}} + \frac{1}{2i\sqrt{\xi-\eta}} \int_0^z e^{iz\sqrt{\xi-\eta}} \left(q(x)-\eta\right) dx \\ &\qquad + \frac{1}{2i\sqrt{\xi-\eta}} \int_z^\infty e^{i(2x-z)\sqrt{\xi-\eta}} \left(q(x)-\eta\right) dx \\ &= e^{iz\sqrt{\xi-\eta}} \left(1 + \frac{1}{2i\sqrt{\xi-\eta}} \int_0^z \left(q(x)-\eta\right) dx \right. \\ &\qquad \left. + \frac{1}{4\left(\xi-\eta\right)} \left(q(z)-\eta\right) + \frac{1}{4\left(\xi-\eta\right)} \int_z^\infty e^{2i(x-z)\sqrt{\xi-\eta}} q'(x)dx \right),\end{aligned} \tag{48}$$

where we integrated by parts and used the fact that q' is absolutely integrable at infinity. Collecting the above results, we make use of (35) to obtain that the asymptotic relation in the statement of the lemma holds uniformly over $z \in [0,\infty)$. Let us extend it to all compacts in $(L,0]$. Choose arbitrary $l \in (L,0)$, and consider the SL problem with the potential function $\tilde{q}(z) := q(z-l)$. Denote by $\tilde{\chi}(.,\xi)$ the corresponding solution to the new SL problem, which is square integrable over $(0,\infty)$. Notice that the representation (48) holds for $\tilde{\chi}$ as well, with $\tilde{q}$ in lieu of q. In particular, this implies that, for all large enough $|\xi|$, $\tilde{\chi}(-l,\xi) \neq 0$. It is also easy to deduce (recall (35)) that $\tilde{\chi}$ satisfies

$$\frac{\tilde{\chi}(z-l,\xi)}{\tilde{\chi}(-l,\xi)} = \psi_2(z,\xi).$$

Making use of (48) once more, we conclude that the asymptotic relation in the statement of the lemma holds uniformly over $z \in [l,\infty)$. Since l is arbitrary, this completes the proof. ∎

Next, we apply Lemma 2.6 to estimate the inner integral in the right-hand side of (46):

$$\int_{\mathcal{C}_R} |\psi_2(z,\alpha)|\,|d\alpha| \leq c_8 R^2 \int_0^\pi \exp\left(-\left[(\delta+\rho)\vee(R\sin(\theta))\right]z\right) d\theta,$$

where c_8 stands for a positive real constant, and ρ is defined in (23). Thus, we recall (36) and (21)–(23) to obtain, for all large enough $R > 0$,

$$\begin{aligned}(49)\qquad &\frac{1}{R^2}\int_{\mathcal{C}_R}\left|\int_0^\infty \psi_2(y,\alpha)\tilde{C}(y)dy\right||d\alpha| \\ &\leq c_9 \int_0^\pi\int_0^\infty \exp\left(-\left[(\delta+\rho)\vee(R\sin(\theta))\right]z\right)w(z)dzd\theta \\ &\qquad \leq c_{10}\int_0^\pi\int_0^\infty \exp\left(-\frac{1}{2}\left[\delta\vee(2R\sin(\theta)-2\rho-\delta)\right]z\right)dzd\theta,\end{aligned}$$

where w is given by (11) and c_i's are some positive real constants. The above integral vanishes, as $R\to\infty$, due to the dominated convergence theorem.

Next, we need to provide similar estimates for the integrals of ψ_1. However, since we do not assume absolute integrability of q' at L, the analysis of function ψ_1 is somewhat more complicated. Therefore, we make the following assumption.

Assumption 4. We assume that there exist functions $h_1, h_2 : (L,\infty)\to\mathbb{C}$, twice and once continuously differentiable, respectively, such that

$$\psi_1(z,\xi) = e^{-iz\sqrt{\xi}}\left(h_1(z) + \frac{h_2(z)}{\sqrt{\xi}} + \underline{\underline{O}}\left(|\xi|^{-1}\right)\right),$$

uniformly over z changing on any compact in (L,∞), as $|\xi|\to\infty$, satisfying $\xi\in\mathcal{G}^+\cup\mathcal{G}$.

In addition, we assume that function h_1 is absolutely bounded on $[0,\infty)$.

Using Assumption 4, we can estimate the first integral in (42) in the same way we derived (49). In order to handle the last term in the right-hand side of (42), we make use of Assumption 3. Thus, we conclude that (44) holds, and, therefore, we have

$$\begin{aligned}(50)\qquad \lim_{R\to\infty}\int_0^\infty \psi_2(y,\xi)\tilde{g}_R(y)dy &= \int_{\tilde{K}}^0 \psi_1(y,\xi)\tilde{C}(y)dy + \int_0^\infty \psi_2(y,\xi)\tilde{C}(y)dy \\ &\quad + (m_1(\xi)-m_2(\xi))\left(\hat{E}\left(U,-\xi-r\right)+\frac{K}{\xi}\right).\end{aligned}$$

2.4. Main results.

In the previous subsection we made all the constructions needed to prove the main result of the paper.

Theorem 2.7. *Let Assumptions 1–4 hold, with the corresponding constants κ and ζ. Then there exists an exponentially bounded continuous function $\tilde{g}$, with support in $[0,\infty)$, satisfying (38), and it is given by*

$$(51)\qquad \tilde{g}(z) = \frac{1}{2\pi i}\int_{\mathcal{G}} \psi_1(z,\xi)\frac{\Upsilon(\xi)}{m_2(\xi)-m_1(\xi)}d\xi,$$

where Υ *is defined in* (42)*, and the contour of integration* $\mathcal{G}$ *is constructed in* (39)–(41)*, with the above constants* κ *and* ζ *and the constants* η*,* ρ*, and* I*, given by* (15)*,* (23)*, and* (29)*, respectively.*

If, in addition,

$$\liminf_{x\to\infty} \sigma(x)/x \neq 0,$$

then the static hedge of a UOP option with strike K *and barrier* U *is given by a European-type contingent claim with the polynomially bounded payoff*

$$G(x) = (K-x)^+ - \tilde{g}\left(Z(x)\right)\sqrt{\frac{\sigma(x)}{\sigma(U)}}\exp\left(\int_U^x \frac{\mu(y)}{\sigma^2(y)}dy\right), \tag{52}$$

where the function $Z(.)$ *is defined in* (9)*. Otherwise, the above-mentioned European-type claim provides a static hedging strategy if* G *is of at most polynomial growth.*

Proof. We need to check the convergence of $\tilde{g}_R$ and verify that the corresponding limit $\tilde{g}$ is a continuous and exponentially bounded function. First, we notice that Lemma 2.3, Assumption 2, and the asymptotic relation

$$\sqrt{\xi-\eta} = \sqrt{\xi} + \underline{\underline{O}}\left(|\xi|^{-1/2}\right), \tag{53}$$

for $|\xi|\to\infty$, yield that

$$\frac{1}{m_2(\xi)-m_1(\xi)} = -\frac{1}{(1+c_1)i\sqrt{\xi}}\left(1+\underline{\underline{O}}\left(|\xi|^{-\frac{1}{2}}\right)\right), \tag{54}$$

where $c_1\neq -1$ and $|\xi|\to\infty$. Throughout this proof, the c_i's denote some (possibly complex-valued) constants.

Consider $\xi\in\mathcal{G}$. Applying Lemma 2.6 and making use of (36), (21), and (22), we obtain, as $|\xi|\to\infty$,

$$\begin{aligned}\int_0^\infty \psi_2(y,\xi)\tilde{C}(y)dy &= \int_0^\infty e^{iy\sqrt{\xi-\eta}}\tilde{C}(y)dy + \frac{c_2}{\sqrt{\xi-\eta}}\int_0^\infty e^{iy\sqrt{\xi-\eta}}\int_0^y q(z)dz\tilde{C}(y)dy\\ &+\underline{\underline{O}}\left(|\xi|^{-1}\right) = -\frac{|c_3|}{i\sqrt{\xi}}+\underline{\underline{O}}\left(|\xi|^{-1}\right),\end{aligned} \tag{55}$$

where we used integration by parts (twice in the first integral and once in the second one). In the above, we have also made use of the estimates of the first two derivatives of $\tilde{C}$, which, again, follow easily from (36), (21), and (22). Similarly, using Assumption 4, we obtain, as $|\xi|\to\infty$,

$$\int_{\tilde{K}}^0 \psi_1(y,\xi)\tilde{C}(y)dy = \frac{c_4}{\sqrt{\xi}}+\underline{\underline{O}}\left(|\xi|^{-1}\right). \tag{56}$$

Collecting the above results and making use of Assumption 3, we obtain, as $|\xi|\to\infty$,

$$\frac{\Upsilon(\xi)}{m_2(\xi)-m_1(\xi)} = \frac{c_5}{\xi}+\underline{\underline{O}}\left(|\xi|^{-3/2}\right). \tag{57}$$

Next, we consider arbitrary $R' > R$ and introduce $s := \sqrt{\xi}$. Then, using Assumption 4, (57), and the asymptotic relation

$$e^{iz\sqrt{\xi-\eta}} = e^{iz\sqrt{\xi}}\left(1 - \frac{1}{2}\eta z\xi^{-1/2} + \underline{\underline{O}}\left(|\xi|^{-1}\right)\right), \tag{58}$$

for $|\xi| \to \infty$, we obtain

$$\begin{aligned}|\tilde{g}_{R'}(z) - \tilde{g}_R(z)| &= \frac{1}{\pi}\left|\int_{\mathcal{I}_{R'}\setminus\mathcal{I}_R} \psi_1(z,s^2)\frac{\Upsilon(s^2)}{m_2(s^2)-m_1(s^2)} s ds\right| \\ &= \left|\int_{\mathcal{I}_{R'}\setminus\mathcal{I}_R} \frac{e^{-izs}}{s}\left(c_6(z) + \underline{\underline{O}}\left(|s|^{-1}\right)\right) ds\right| \to 0,\end{aligned}$$

as $R \to \infty$, uniformly over z changing on any compact in $[0,\infty)$, with $c_6(.)$ being absolutely bounded on any compact in $[0,\infty)$.

The above yields that $\tilde{g}(z) := \lim_{R\to\infty} \tilde{g}_R(z)$ exists and is a continuous function with support in $[0,\infty)$. Repeating, essentially, the above estimates and making use of Lemma 2.4, one can verify that

$$|\tilde{g}(z)| \le c_7 e^{c_8 z},$$

where c_8 can be any real number (strictly) greater than $\sqrt{J^2 + |\eta|}$ (see (39) for the definition of J), and c_7 is a positive real number depending upon c_8. Notice also that $\tilde{g}$ satisfies (38) for all $\xi \in \mathcal{G}^+$, due to (50) and the dominated convergence theorem.

Finally, if $\liminf_{x\to\infty} \sigma(x)/x > 0$, then, using Assumption 1, we deduce that, as $x \to \infty$,

$$Z(x) = \underline{\underline{O}}(\log x), \qquad \int_U^x \frac{\mu(y)}{\sigma^2(y)} dy = \underline{\underline{O}}(\log x).$$

Undoing the normalization, defined in (10), we obtain function $g : (0,\infty) \to \mathbb{R}$ and conclude that it is polynomially bounded (otherwise, according to the statement of the theorem, we have to assume it). Function $\tilde{g}$ is exponentially bounded, and the corresponding g is continuous, polynomially bounded, and vanishing in a neighborhood of zero. Hence, Lemma 2.5 can be applied, and we conclude that, given that the underlying is at the barrier, the Laplace transform of the price of a European-type contingent claim with payoff $g(S_\tau)$ (as a function of "τ") is well defined and coincides with the Laplace transform of the price of a corresponding put option for any λ (the variable of a "Laplace space") with large enough real part. From the uniqueness of the Laplace transform, we conclude that the above prices themselves have to coincide for all $T > 0$. Therefore, the price of a European-type contingent claim with payoff G defined in (52) is zero whenever the level of the underlying is equal to U. It only remains to notice that if we open a long position in this contingent claim and liquidate it (at no cost) when, or if, the underlying hits the barrier, the payoff of this trading strategy is exactly that of a UOP option. ∎

Remark 3. Recall that, in fact, Υ can be expressed through the Laplace transform (in the "time-to-maturity" variable) of a put price along the barrier. In other words,

$$\frac{\Upsilon(\xi)}{m_2(\xi) - m_1(\xi)} = \hat{P}\left(U, -\xi - r\right), \tag{59}$$

where $\hat{P}$ is defined in (5). One, naturally, expects the Laplace transform of a put price along the barrier to be given by the right-hand side of (28), with $\tilde{g}$ being the *modified put payoff*

$$\tilde{P}(z) := (K - X(z))^+ \frac{\sqrt{\sigma(U)}}{X(z)} w(z), \tag{60}$$

where w is defined in (11). When this is the case, Υ can be expressed through one of the fundamental solutions to the SL equation in a somewhat simpler form (see (62) in Theorem 2.8). We, however, used the put-call parity to obtain an (equivalent) representation of Υ in (42). This was done in order to avoid dealing with put prices: studying them would require knowledge of the asymptotic behavior of the underlying diffusion at zero. Recall that, as shown in Lemma 2.5, formula (28) provides an expression for the Laplace transform of the price of a European-type contingent claim with payoff g vanishing in a neighborhood of zero. In some cases, the argument in the proof of Lemma 2.5 can be applied even if the payoff function does not vanish at zero, so that, in particular, the Laplace transform of a put price can be computed via the right-hand side of (28). However, this requires certain assumptions on the asymptotic behavior of the coefficients μ and σ at zero (see, for example, the assumptions of Theorem 2.8). The statement of Theorem 2.7 is designed to hold without any additional assumptions on the asymptotic behavior of the diffusion coefficients at zero, and this is why Υ is defined via (42). It turns out that, in the models satisfying the assumptions of Theorem 2.8, which, in particular, guarantee that the underlying S has strictly positive paths (and include the Black–Scholes model), the Laplace transform of a put price can indeed be represented via (28). However, such a representation, generally, fails if S can hit zero with positive probability, and one has to use (42) to compute Υ, as is done in the CEV example below.

Notice that Assumptions 2, 3, and 4 are not formulated explicitly in terms of the diffusion coefficients μ and σ, which, of course, can be viewed as a shortcoming of Theorem 2.7. This shortcoming is due to the fact that we assumed only the "regularity of the problem on the right" (see Definition 2.1), and not "*on the left*" (which can be defined analogously). Clearly, if we assume that the problem is both "regular on the right" and "on the left," then Assumptions 2 and 4 become lemmas, whose proofs are the repetitions of the proofs of Lemmas 2.3 and 2.6, respectively, and Assumption 3 follows easily as well.

The "regularity of the problem on the left" results in a restriction on the asymptotic behavior of μ and σ at zero (see Lemma 2.2 for a similar connection at infinity). Such a restriction reduces the class of financial models under consideration, excluding some of the models of interest, such as the CEV model. Therefore, we chose to avoid it in the formulation of Theorem 2.7. However, the following theorem shows that the diffusion models in which the coefficients μ and σ are *asymptotically linear* at zero and infinity are both regular "on the left" and "on the right," and the assumptions of Theorem 2.7 are satisfied for such models. From the modeling perspective, the assumption of linear asymptotic behavior of the diffusion coefficients at infinity is natural, since we would like the underlying process to be a martingale (after an appropriate discounting). The linear asymptotic behavior of the coefficients at zero, essentially, means that the process stays strictly positive at all times, which may or may not be a natural assumption, depending on whether we want to incorporate the default event in the model.

Theorem 2.8. *Assume that the coefficients μ and σ are given by*

$$\mu(x) = x\nu(x), \quad \sigma(x) = x\gamma(x),$$

where ν and γ (the drift and local volatility functions, respectively) are three times continuously differentiable functions on $(0,\infty)$; ν, γ and their first three derivatives have finite (right) limits at zero; $\gamma(x)$ is strictly positive for $x \geq 0$; the first three derivatives of ν and γ behave at infinity as $\underline{\underline{O}}\left(x^{-3}\right)$. Then the following hold:

(i) *Assumption 1 is satisfied, and, hence, the SL problem (12) is regular on the right, with a corresponding constant η, given by (15).*

(ii) *$L = -\infty$, and there exists a constant $\eta' := \lim_{z\to-\infty} q(z)$, such that*

$$I' := \int_{-\infty}^{0} \left|q(z) - \eta'\right| dz < \infty.$$

(iii) *Assumption 2 is satisfied, with*

$$\kappa = -\left|\eta'\right| - 16{I'}^2.$$

(iv) *In the notation*

$$\rho' := \lim_{x\to 0}\left|\frac{\log\left(x\gamma\left(x\right)\right)}{2Z(x)} + \frac{\nu(x)}{\sqrt{2}\gamma(x)}\right| \vee \lim_{x\to\infty}\left|\frac{\log\left(\gamma\left(x\right)/x\right)}{2Z(x)} + \frac{\nu(x)}{\sqrt{2}\gamma(x)}\right|, \tag{61}$$

Assumption 3 is satisfied, with

$$\zeta = {\rho'}^2 \vee |\kappa| \vee \left(16I^2\right) + |\eta| \vee |\eta'|,$$

where I is given by (29).

(v) *Finally, if we construct the contour of integration $\mathcal{G}$ according to (39)–(41), with the constants κ and ζ as above, then Assumption 4 holds, and there exists a static hedging strategy for a UOP option, given by a polynomially bounded payoff function G, defined in (51) and (52), where Υ can be computed via*

$$\Upsilon(\xi) = \int_{-\infty}^{0} \psi_1(y,\xi)\tilde{P}(y)dy, \tag{62}$$

with $\tilde{P}$ given by (60).

Proof. It is easy to check by direct computation that Assumption 1 is verified. Also, using the properties of μ and σ at zero, together with the explicit expression for q in terms of μ and σ, given in (72), we deduce that

$$L = -\infty, \quad c_1 e^{c_2 z} \leq X(z) \leq c_3 e^{c_4 z}, \quad X'(z) = \frac{1}{\sqrt{2}}\sigma\left(X(z)\right),$$

and there exists a constant $\eta' \in \mathbb{R}$, such that

$$\int_{-\infty}^{0} \left(|q(z) - \eta'| + |q'(z)|\right) dz < \infty.$$

Here, and throughout the rest of the proof, c_i's denote some positive real constants, Naturally, we call the above property the "regularity of the SL problem on the left." Given the above property, we notice that Assumption 2 is satisfied, with κ as given in the statement of the theorem, and its proof is an exact repetition of the proof of Lemma 2.3, with the additional use of (53), and with η' in lieu of η. Similarly, if we define the contour $\mathcal{G}$ as described in the statement of the theorem, then Assumption 4 follows from the proof of Lemma 2.6 and (58). Notice also that, in the present case, since the corresponding SL problem is "regular on the left," we can easily extend the statement of Lemma 2.4 to hold for all $z \in \mathbb{R}$.

In addition, a stronger version of Lemma 2.5 can be obtained. Namely, in the present setup, the statement of the lemma holds for functions g which do not necessarily vanish in a neighborhood of zero but only have to be continuous in $[0, \infty)$. In order to see this, we need to modify the very first argument in the proof of Lemma 2.5. Notice that process S never reaches zero—this follows in a straightforward way from the fact that $\mu(x)/x$ and $\sigma(x)/x$ are bounded from above and away from zero. Then, since g is bounded in a neighborhood of zero, the expectation of $e^{-rT}g(S_T)$ can still be approximated by the expectations of $e^{-rT}g^n(S_T)$, where $\{g^n\}$ is a sequence of functions with compact supports in $(0, \infty)$, converging to g uniformly on any compact. It only remains to notice that

$$\tilde{P}(z) \le (K - X(z))^+ \sqrt{\frac{U\gamma(U)}{X(z)\gamma\left(X(z)\right)}} \exp\left(-\int_U^{X(z)} \frac{\nu(x)}{x\gamma^2(x)}dx\right) \le c_5 \exp\left(-c_6 z\right)$$

for all $z \in \mathbb{R}$, where c_6 is any constant greater than ρ', defined in (61), and the rest of the proof of Lemma 2.5 goes through without changes. Hence, the Laplace transform of a put price along the barrier can be computed via the right-hand side of (28), which, together with (59), implies (62).

With the contour $\mathcal{G}$ defined via (39)–(41), and with the constants κ and ζ as given in the statement of the theorem, we apply the above estimate and the aforementioned extension of Lemma 2.4 to conclude that the integral in the right-hand side of (62) is absolutely convergent, uniformly over ξ changing on any compact in $\mathcal{G}^+ \cup \mathcal{G}$. This implies (see the discussion preceding Assumption 3) that Υ, defined by (62), is analytic in $\mathcal{G}^+ \cup \mathcal{G}$. Recall that, due to the put-call parity, at least for all ξ with large enough (negative) real part, both definitions of Υ, (42) and (62), should coincide. Therefore, we have

$$\hat{E}\left(U, -\xi - r\right) = \frac{1}{m_1(\xi) - m_2(\xi)} \left[\Upsilon(\xi) - \int_{\tilde{K}}^{0} \psi_1(y,\xi)\tilde{C}(y)dy - \int_0^{\infty} \psi_2(y,\xi)\tilde{C}(y)dy\right] - \frac{K}{\xi}.$$

Notice that we have shown that the right-hand side of the above equation can be extended analytically to $\mathcal{G}^+$. This, in turn, implies that the left-hand side of the above can be understood as an analytic function of ξ in $\mathcal{G}^+$. Finally, notice that Lemmas 2.3 and 2.6, and Assumptions 2 and 4, imply the asymptotic relations (54)–(56) in the proof of Theorem 2.7. From the definition of Υ, given by (62), we can obtain (57) along the same lines as (55). Equations (55)–(57) and the above representation of $\hat{E}$ yield the asymptotic relation of Assumption 3. Applying Theorem 2.7, we conclude the proof.[1] ■

[1] Notice that, in this case, we do not need Assumption 3 in order to construct the static hedge; however, for the sake of completeness, we show that this assumption is satisfied as well.

Remark 4. From a mathematical point of view, the results of subsection 2.3, finalized in Theorems 2.7 and 2.8, can be viewed as a construction of a new class of function transforms associated with the SL problems. In the above, we considered the integral transform with kernel ψ_2, which maps function $f : (0,\infty) \to \mathbb{R}$ into a (possibly complex-valued) function ϕ of a complex argument ξ, given by

$$\phi(\xi) = \int_0^\infty \psi_2(z,\xi) f(z) dz, \tag{63}$$

whenever the above integral is well defined. We have shown that if Assumptions 1–4 are satisfied (or, alternatively, if the coefficients μ and σ satisfy the assumptions of Theorem 2.8), then there exists $\epsilon > 0$, such that, for any analytic function $\phi : \{z \in \mathbb{C} |\, \mathfrak{Im}\left(\sqrt{z}\right) > \epsilon\} \to \mathbb{C}$, satisfying

$$\overline{\phi(\xi)} = \phi\left(\overline{\xi}\right) \quad \text{and} \quad \lim_{R\to\infty} \int_{\arcsin(\epsilon/R)}^{\pi-\arcsin(\epsilon/R)} \left|\phi\left(R^2 \exp(2i\theta)\right)\right| d\theta = 0,$$

there exists a continuous exponentially bounded function $f : (0,\infty) \to \mathbb{R}$ such that (63) holds, and we have

$$f(z) = \frac{1}{2\pi i} \int_{\mathfrak{Im}\left(\sqrt{\xi}=\epsilon\right)} \frac{\psi_1(z,\xi)}{m_2(\xi) - m_1(\xi)} \phi\left(\xi\right) d\xi.$$

Notice that a lot of the effort in subsection 2.3 was to verify that function ϕ (in that case, Υ) has the above properties.

A large number of function transforms, including, for example, the classical Laplace and Fourier transforms, and the less known Hankel transform, can be obtained as the "eigenvalue expansions" of the corresponding SL problems. These eigenvalue (or "spectral") expansions have been studied for a very long time: the classical references include [29], [30], [22], [23], [25]; alternatively, one may consult with [5], [32] for a contemporary presentation of the results. The function transforms based on the eigenfunction expansions of the SL equations typically involve function φ, the solution of (24), defined in (25), corresponding to the real values of ξ, as a kernel for both the direct and inverse transformations. However, to the best of our knowledge, the integral transforms with kernels ψ_2 and $\psi_1/(m_2 - m_1)$, for the direct and the inverse transformations, respectively, have not been considered before in this generality (even though a particular case of the "K-transform" was known: see, for example, [31]). The key difference of the transforms considered herein from the previously known function transforms, associated with the SL problems, is that, in the present case, the inverse transformation is given by an integral over a curve *outside of the spectrum* of the corresponding SL problem. Thanks to that, we are able to characterize explicitly a family of functions, with a domain outside of the spectrum of the SL problem, for which the inverse transformation is well defined (which is impossible in the case of classical SL transforms, since their inversion requires integration along the spectrum). The latter property was necessary to solve the static hedging problem, since the function Υ, used in Theorems 2.7 and 2.8, in general, is not well defined at the spectrum.

3. Examples and further extensions.

3.1. Black–Scholes model. Recall that, in the Black–Scholes case, we have $\mu(x) = (r - d)x$ and $\sigma(x) = \sigma x$, with some scalars $r, d \geq 0$ and $\sigma > 0$. It is easy to see that the conditions of Theorem 2.8 are satisfied. In the notation of section 2, we obtain the corresponding SL problem

$$\frac{\partial^2}{\partial z^2}\tilde{u}(z,\xi) + (\xi - q(z))\tilde{u}(z,\xi) = 0,$$

where

$$q \equiv \eta = \eta' = \frac{\left(r - d - \sigma^2/2\right)^2}{2\sigma^2},$$

and other elements of the model are given by

$$Z(x) = \frac{\sqrt{2}}{\sigma}\left(\log\frac{x}{U}\right), \quad X(z) = Ue^{\frac{\sigma}{\sqrt{2}}z}, \quad w(z) = \frac{X(z)}{\sqrt{\sigma}}e^{-\frac{r-d+\sigma^2/2}{\sqrt{2}\sigma}z},$$

$$\tilde{K} = \frac{\sqrt{2}}{\sigma}\left(\log\frac{K}{U}\right), \quad L = -\infty, \quad I = I' = 0, \quad \rho' = \frac{\left|r - d - \sigma^2/2\right| \vee \left|r - d + \sigma^2/2\right|}{\sqrt{2}\sigma}.$$

The above SL equation can be solved explicitly, and we obtain

$$\psi_1(z,\xi) = e^{-iz\sqrt{\xi-\eta}}, \quad \psi_2(z,\xi) = e^{iz\sqrt{\xi-\eta}},$$
$$m_1(\xi) = i\sqrt{\xi-\eta}, \quad m_2(\xi) = -i\sqrt{\xi-\eta}.$$

Applying Theorem 2.8, we construct the contour of integration $\mathcal{G}$ according to (39)–(41), with some $\delta > 0$ and

$$\kappa = -\eta, \qquad \zeta = \rho'^2 \vee \eta + \eta,$$

where the constants η and ρ' are given above. Then, we make use of (62) to obtain

$$\begin{aligned}\Upsilon(\xi) &= \int_{-\infty}^{\tilde{K}} e^{-iz\sqrt{\xi-\eta}}\left(K - Ue^{\frac{\sigma}{\sqrt{2}}z}\right)e^{-\frac{r-d+\sigma^2/2}{\sqrt{2}\sigma}z}dz\\ &= -\frac{Ke^{-\left(i\sqrt{\xi-\eta}+\frac{r-d+\sigma^2/2}{\sqrt{2}\sigma}\right)\tilde{K}}}{i\sqrt{\xi-\eta}+\frac{r-d+\sigma^2/2}{\sqrt{2}\sigma}} + \frac{Ue^{-\left(i\sqrt{\xi-\eta}+\frac{r-d-\sigma^2/2}{\sqrt{2}\sigma}\right)\tilde{K}}}{i\sqrt{\xi-\eta}+\frac{r-d-\sigma^2/2}{\sqrt{2}\sigma}}.\end{aligned}$$

It is worth mentioning that we could have equivalently used (42) to compute Υ; however, here we chose to illustrate how Remark 3 and Theorem 2.8 can simplify the computations. Finally, according to Theorem 2.8, we have

$$\tilde{g}(z) = -\frac{1}{4\pi i}\int_{\mathcal{G}} e^{-iz\sqrt{\xi-\eta}}\frac{\Upsilon(\xi)}{i\sqrt{\xi-\eta}}d\xi.$$

Let us make the change of variables in the above integral, from ξ to $s := -i\sqrt{\xi-\eta}$, to obtain

$$\begin{aligned}\tilde{g}(z) = -\frac{1}{2\pi i}\int_{\mathcal{G}'} e^{zs}\Upsilon(\eta - s^2)ds &= \frac{Ke^{-\frac{r-d+\sigma^2/2}{\sqrt{2}\sigma}\tilde{K}}}{2\pi i}\int_{\mathcal{G}'}\frac{e^{s\left(z+\tilde{K}\right)}}{\frac{r-d+\sigma^2/2}{\sqrt{2}\sigma} - s}ds\\ &\quad - \frac{Ue^{-\frac{r-d-\sigma^2/2}{\sqrt{2}\sigma}\tilde{K}}}{2\pi i}\int_{\mathcal{G}'}\frac{e^{s\left(z+\tilde{K}\right)}}{\frac{r-d-\sigma^2/2}{\sqrt{2}\sigma} - s}ds,\end{aligned}$$

where $\mathcal{G}' = -i\sqrt{\mathcal{G} - \eta}$. Recall (53), and deduce that $|\mathfrak{Re}(s)|$ is bounded over all $s \in \mathcal{G}'$. Thus, we can apply Jordan's lemma to compute the above integrals as the sums of corresponding residues. In particular, when $z < -\tilde{K}$, we close the contour of integration, $\mathcal{G}'$, by the "right" arch of a circle centered at zero. We easily deduce (this is, essentially, Jordan's lemma) that integrals over the arch vanish as the radius goes to infinity. And since the corresponding integrands have no poles to the right of $\mathcal{G}'$, we conclude that both of the above integrals are zero. When $z \geq -\tilde{K}$, we use the "left" arch of the circle to close the contour of integration and, computing the residues, obtain

$$\tilde{g}(z) = \left(Ke^{\frac{r-d+\sigma^2/2}{\sqrt{2}\sigma}z} - Ue^{\frac{r-d-\sigma^2/2}{\sqrt{2}\sigma}z} \right)^+,$$

which, after changing the variables back, from z to $x = X(z)$, becomes

$$g(x) = e^{\frac{r-d+\sigma^2/2}{\sqrt{2}\sigma}Z(x)}\tilde{g}\left(Z(x)\right) = \frac{K}{U}\left(\frac{x}{U}\right)^{\frac{2r-2d}{\sigma^2}}\left(x - \frac{U^2}{K}\right)^+,$$

and we recognize the well-known formula for the price (and static hedge) of a UOP option in the Black–Scholes model.

3.2. CEV model. The constant elasticity of variance (CEV) model is given by $\mu(x) = \mu x$ and $\sigma(x) = x^{1+\beta}$, with $\beta < 0$. The constant "2β" is called the *elasticity* of the variance of returns. Recall that, in this case, $\sigma(x)$ is not asymptotically linear at infinity (instead, it exhibits a power-type decay), and, therefore, $\mu(x)/x$ has to vanish at infinity, in order to satisfy Assumption 1. Thus, we assume that $\mu = 0$. We introduce $\nu = 1/(2|\beta|)$ and denote by J_ν and $H_\nu^{(1)}$ the Bessel and Hankel functions, respectively, and by I_ν and K_ν the modified Bessel functions of the first and second kind, respectively (see, for example, [1] for the definitions and properties of these functions). Then, applying the results of the previous section, we obtain the static hedging strategy for a UOP option in the CEV model.[2]

Proposition 3.1. *In a CEV model with zero drift and elasticity 2β, the price of a UOP option with strike K, maturity T, and barrier $U(> K)$, given that the current level of underlying is x and the barrier has not been hit, is equal to the price of a European-type contingent claim with maturity T and the payoff $G(x) = (K - x)^+ - g(x)$, where g is given by*

$$g(x) = \frac{2\nu}{\pi i}\sqrt{Kx}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \frac{I_\nu(sx^{-\beta})I_\nu\left(sK^{-\beta}\right)K_\nu(sU^{-\beta})}{sI_\nu(sU^{-\beta})}ds \tag{64}$$

for any $\varepsilon > 0$. In addition, as $x \to \infty$, we have $g(x)/x \to \frac{K}{U}$.

Proof. As was mentioned in the previous section, the integral in (9) can start from zero rather than U. In the present case, it will greatly simplify the notation without affecting the results (different formula for $Z(x)$ will result only in a shift of the "z" variable in every

[2] We would like to mention that our search for a solution to the static hedging problem was originally restricted to the CEV model, and we managed to obtain (64) using a somewhat different technique. However, later it became clear that the solution can be generalized to any time-homogeneous diffusion model, with the help of the SL theory.

derivation in section 2 following (9)). Therefore, we define $Z(x)$ as an integral starting from zero:

$$Z(x) = \sqrt{2}\int_0^x \frac{dy}{\sigma(y)}, \quad X(z) = Z^{-1}(z).$$

Notice that, with the above choice of the diffusion coefficients, Assumption 1 is satisfied (see the discussion after Assumption 1); hence, the corresponding SL problem is regular on the right, with some constant $\eta \in \mathbb{R}$, and we have

$$L = 0, \quad Z(x) = \sqrt{2}(-\beta)^{-1}x^{-\beta}, \quad X(z) = 2^{\frac{1}{2\beta}}(-\beta)^{-\frac{1}{\beta}}z^{-\frac{1}{\beta}}, \quad q(z) = (\nu^2 - 1/4)/z^2,$$
$$\tilde{K} = Z(K) = \sqrt{2}(-\beta)^{-1}K^{-\beta}, \quad \tilde{U} := Z(U) = \sqrt{2}(-\beta)^{-1}U^{-\beta}, \quad \eta = 0, \quad \rho = 0.$$

Consequently, in the definitions of θ and φ (see (25)), we use the usual initial conditions, but specified at $\tilde{U}$ rather than at zero. From section 4.11 of [29] we obtain

$$\psi_1(z,\xi) = \frac{\sqrt{z}J_\nu\left(z\sqrt{\xi}\right)}{\sqrt{\tilde{U}}J_\nu\left(\tilde{U}\sqrt{\xi}\right)}, \qquad \psi_2(z,\xi) = \frac{\sqrt{z}H_\nu^{(1)}\left(z\sqrt{\xi}\right)}{\sqrt{\tilde{U}}H_\nu^{(1)}\left(\tilde{U}\sqrt{\xi}\right)},$$
$$m_1(\xi) = -\sqrt{\xi}\frac{J_\nu{}'\left(\tilde{U}\sqrt{\xi}\right)}{J_\nu\left(\tilde{U}\sqrt{\xi}\right)} - \frac{1}{2\tilde{U}}, \qquad m_2(\xi) = -\sqrt{\xi}\frac{H_\nu^{(1)\prime}\left(\tilde{U}\sqrt{\xi}\right)}{H_\nu^{(1)}\left(\tilde{U}\sqrt{\xi}\right)} - \frac{1}{2\tilde{U}},$$
$$w(z) = \frac{X(z)}{\sqrt{\sigma\left(X(z)\right)}}\tilde{U}^{-\frac{1}{2\beta}-\frac{1}{2}}z^{\frac{1}{2\beta}+\frac{1}{2}}, \qquad I = \int_{\tilde{U}}^\infty |q(z)|\,dz = \frac{\left|\nu^2 - 1/4\right|}{\tilde{U}}.$$

Using the asymptotic properties of Bessel functions (see, for example, equation (9.2.5) in [1]), it is easy to see that $m_1(\xi)$ and $m_2(\xi)$ are well defined and analytic in $\mathbb{C}\setminus[0,\infty)$, and we have

$$m_1(\xi) = i\sqrt{\xi}\left(1 + O\left(|\xi|^{-\frac{1}{2}}\right)\right), \quad m_2(\xi) = -i\sqrt{\xi}\left(1 + O\left(|\xi|^{-\frac{1}{2}}\right)\right),$$

as $|\xi| \to \infty$, which fulfills Assumption 2 with $\kappa = 0$. Using the asymptotic relation (see, again, [1])

$$J_\nu(z\sqrt{\xi}) \sim c_1\frac{e^{-iz\sqrt{\xi}}}{\sqrt{z}\xi^{1/4}}\left(1 + \frac{c_2}{z\sqrt{\xi}} + \underline{\underline{O}}\left(z^{-2}|\xi|^{-1}\right)\right),$$

we conclude that ψ_1 fulfills Assumption 4 (recall that, in the present example, the z variable is shifted by $\tilde{U}$; therefore, "z" and "L" in Assumption 4 have to be changed to "$z - \tilde{U}$" and "$L - \tilde{U}$," respectively).

Since, in this case, the underlying diffusion process S is a martingale, we have

$$\hat{E}(U, -\xi - r) = -\frac{U}{\xi},$$

and, hence, Assumption 3 is satisfied with $\zeta = 0$.

Next, we construct $\mathcal{G}$ via (39)–(41), with some $\delta > 0$, and the constants η, ρ, I, κ, and ζ defined above. Then, for any $\xi \in \mathcal{G}$, we have

$$\begin{aligned}\Upsilon(\xi) &= \int_{\tilde{K}}^{\tilde{U}} \psi_1(y,\xi)\tilde{C}(y)dy + \int_{\tilde{U}}^{\infty} \psi_2(y,\xi)\tilde{C}(y)dy + (m_1(\xi) - m_2(\xi))\left(-\frac{x}{\xi + r} + \frac{K}{\xi}\right)\\ &= \frac{\tilde{U}^{-\frac{1}{2\beta}-1}}{J_\nu\left(\tilde{U}\sqrt{\xi}\right)} \int_{\tilde{K}}^{\tilde{U}} J_\nu\left(y\sqrt{\xi}\right)\left((-\beta)^{-\frac{1}{\beta}} y^{-\frac{1}{\beta}} - K\right) y^{\frac{1}{2\beta}+1} dy\\ &\qquad + \frac{\tilde{U}^{-\frac{1}{2\beta}-1}}{H_\nu^{(1)}\left(\tilde{U}\sqrt{\xi}\right)} \int_{\tilde{U}}^{\infty} H_\nu^{(1)}\left(y\sqrt{\xi}\right)\left((-\beta)^{-\frac{1}{\beta}} y^{-\frac{1}{\beta}} - K\right) y^{\frac{1}{2\beta}+1} dy\\ &\qquad + (m_1(\xi) - m_2(\xi))\left(-\frac{U}{\xi} + \frac{K}{\xi}\right).\end{aligned}$$

Introduce $s := -i\sqrt{\xi}$, and recall that $I_\nu(z) = i^{-\nu} J_\nu(iz)$ and $K_\nu(z) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(iz)$. Then, we have

$$\begin{aligned}&\int_{\tilde{K}}^{\tilde{U}} J_\nu\left(y\sqrt{\xi}\right)\left((-\beta)^{-\frac{1}{\beta}} y^{-\frac{1}{\beta}} - K\right) y^{\frac{1}{2\beta}+1} dy\\ &\quad = i^\nu \int_{\tilde{K}}^{\tilde{U}} I_\nu(ys)\left((-\beta)^{-\frac{1}{\beta}} y^{-\frac{1}{\beta}} - K\right) y^{\frac{1}{2\beta}+1} dy\\ &\quad = i^\nu s^{-1}\Big[(-\beta)^{-\frac{1}{\beta}} \tilde{U}^{1-\frac{1}{2\beta}} I_{\nu+1}(\tilde{U}s) - (-\beta)^{-\frac{1}{\beta}} \tilde{K}^{1-\frac{1}{2\beta}} I_{\nu+1}(\tilde{K}s)\\ &\qquad\qquad - K\tilde{U}^{1+\frac{1}{2\beta}} I_{\nu-1}(\tilde{U}s) + K\tilde{K}^{1+\frac{1}{2\beta}} I_{\nu-1}(\tilde{K}s)\Big]\\ &\quad = i^\nu s^{-1}\left[(-\beta)^{-\frac{1}{\beta}} \tilde{U}^{1-\frac{1}{2\beta}} I_{\nu+1}(\tilde{U}s) - K\tilde{U}^{1+\frac{1}{2\beta}} I_{\nu-1}(\tilde{U}s) + K\tilde{K}^{\frac{1}{2\beta}} \frac{2\nu}{s} I_\nu(\tilde{K}s)\right],\end{aligned}$$

and

$$\begin{aligned}&\int_{\tilde{U}}^{\infty} H_\nu^{(1)}\left(y\sqrt{\xi}\right)\left((-\beta)^{-\frac{1}{\beta}} y^{-\frac{1}{\beta}} - K\right) y^{\frac{1}{2\beta}+1} dy\\ &\quad = \frac{2}{\pi} i^{-\nu-1} \int_{\tilde{U}}^{\infty} K_\nu(ys)\left((-\beta)^{-\frac{1}{\beta}} y^{-\frac{1}{\beta}} - K\right) y^{\frac{1}{2\beta}+1} dy\\ &\qquad = \frac{2}{\pi} i^{-\nu-1} s^{-1}\left[(-\beta)^{-\frac{1}{\beta}} \tilde{U}^{1-\frac{1}{2\beta}} K_{\nu+1}(\tilde{U}s) - K\tilde{U}^{1+\frac{1}{2\beta}} K_{\nu-1}(\tilde{U}s)\right],\end{aligned}$$

where we used the properties of Bessel functions (see [1]). Using, for example, the identity given in section 4.11 of [29], we obtain

$$\frac{1}{m_2(\xi) - m_1(\xi)} = \xi^{-1/2}\left(\frac{J_\nu{}'(\tilde{U}\sqrt{\xi})}{J_\nu(\tilde{U}\sqrt{\xi})} - \frac{H_\nu^{(1)\prime}(\tilde{U}\sqrt{\xi})}{H_\nu^{(1)}(\tilde{U}\sqrt{\xi})}\right)^{-1} = -\frac{\pi\tilde{U}}{2i} J_\nu\left(\tilde{U}\sqrt{\xi}\right) H_\nu^{(1)}\left(\tilde{U}\sqrt{\xi}\right).$$

Collecting the above expressions, we conclude that

$$\begin{aligned}\frac{\Upsilon(\xi)}{m_2(\xi)-m_1(\xi)} &= \frac{1}{s}\tilde{U}^{-\frac{1}{2\beta}}K_\nu\left(\tilde{U}s\right)\\ &\quad\cdot\left[(-\beta)^{-\frac{1}{\beta}}\tilde{U}^{1-\frac{1}{2\beta}}I_{\nu+1}(\tilde{U}s)-K\tilde{U}^{1+\frac{1}{2\beta}}I_{\nu-1}(\tilde{U}s)+K\tilde{K}^{\frac{1}{2\beta}}\frac{2\nu}{s}I_\nu(\tilde{K}s)\right]\\ &\quad+\frac{1}{s}\tilde{U}^{-\frac{1}{2\beta}}I_\nu\left(\tilde{U}s\right)\left[(-\beta)^{-\frac{1}{\beta}}\tilde{U}^{1-\frac{1}{2\beta}}K_{\nu+1}(\tilde{U}s)-K\tilde{U}^{1+\frac{1}{2\beta}}K_{\nu-1}(\tilde{U}s)\right]-\frac{U}{s^2}+\frac{K}{s^2}\\ &= -\frac{\sqrt{KU}}{\beta s^2}K_\nu\left(\tilde{U}s\right)I_\nu(\tilde{K}s).\end{aligned}$$

Finally, we apply Theorem 2.7 to conclude that $\tilde{g}$ is given by

$$\tilde{g}(z)=\frac{\sqrt{zK}}{\pi i}(-\beta)^{-\frac{1}{2}}U^{\frac{1}{2}+\frac{\beta}{2}}\int_{4I+\delta-i\infty}^{4I+\delta+i\infty}\frac{I_\nu(zs)\,I_\nu(\tilde{K}s)K_\nu\left(\tilde{U}s\right)}{sI_\nu\left(\tilde{U}s\right)}ds,$$

and, changing the variables from z to $x=X(z)$, we obtain (64) for all $\varepsilon>4I$. Notice now that the static hedge of a UOP option in the model where the underlying is given by a zero-drift CEV process S is the same as the static hedge in the time-changed CEV model, where the value of the underlying at time t is given by $\tilde{S}_t:=S_{t/N}$ for some fixed $N>0$. It is easy to see that $\tilde{S}$ is a diffusion process with zero drift and the diffusion coefficient $\tilde{\sigma}(x)=x^{1+\beta}/\sqrt{N}$. In the new model, given by $\tilde{S}$, the corresponding SL equation stays the same as in the original model, together with the constants η, ρ, κ, and ζ. However, the value of $\tilde{U}$ increases by a factor of $\sqrt{N}$. Hence, by choosing N large enough, we can make the constant I, corresponding to the new model, be as small as we want. Thus, we obtain (64) for all $\varepsilon>0$.

In order to study the asymptotic behavior of g at infinity, we represent it in the following form:

$$g(x)=x\frac{K}{U}2^\nu\Gamma(\nu+1)\frac{1}{\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{I_\nu(s)}{s^{\nu+1}}\frac{\tilde{I}_\nu(s,x,K)\tilde{K}_\nu(s,x,U)}{\tilde{I}_\nu(s,x,U)}ds, \tag{65}$$

where

$$\begin{aligned}\tilde{I}_\nu(s,x,y)&:=I_\nu\left(s\,(x/y)^\beta\right)s^{-\nu}\sqrt{\frac{x}{y}}2^\nu\Gamma(\nu+1),\\ \tilde{K}_\nu(s,x,y)&:=K_\nu\left(s\,(x/y)^\beta\right)s^{\nu}\sqrt{\frac{y}{x}}\frac{1}{2^{\nu-1}\Gamma(\nu)},\end{aligned}$$

and Γ is the *Gamma-function*. The above representation can be obtained by simply changing the integration variable in (64) from s to $sx^{-\beta}$ and noticing that $\varepsilon>0$ can be chosen as a function of x. The exact asymptotic behavior of g now follows from the fact that, as $x\to\infty$, $\tilde{I}_\nu(s,x,K)$, $\tilde{I}_\nu(s,x,U)$, and $\tilde{K}_\nu(s,x,U)$ converge to 1, and we can pass to the limit inside the integral due to the dominated convergence theorem (and the resulting integral is computed using the properties of the Bessel functions). ■

One could try to compute the integral in (64) as a sum of the corresponding residues, as was done in the Black–Scholes example. We, however, stick to the integral representation, since the integral under consideration is absolutely convergent and can be efficiently computed for all reasonable values of x, especially if one uses the representation in (65).

Notice also that the asymptotic behavior, for large x, of the payoff g in the CEV model is the same as in the Black–Scholes model! Figure 2 illustrates the different behavior of function g for various values of the parameter β, and different ranges of the argument. In particular, the top left graph shows the values of $g(x)$ for relatively small values of the argument, with $\beta = -0.5$. Remarkably, for small values of x, the payoff $g(x)$ is very similar to a "hockey-stick" function. This raises the idea of approximating it with the payoff of a call option, explained in the next subsection. However, it is shown in the top right graph in Figure 2 that the values of $g(x)$, for large x, are significantly different from those of a scaled call payoff, although the slope of g does seem to converge to a constant, as predicted by Proposition 3.1. The bottom right graph in Figure 2 demonstrates the results for $\beta \approx 0$, which corresponds to the Black model, and, as expected, the corresponding function g coincides with a scaled call payoff. Finally, the bottom left graph in Figure 2 corresponds to the case of $\beta = -1$, which is the case of a Bachelier model, in which the underlying is given by a Brownian motion, stopped at the first hitting time of zero. In this case, the function g is given by a piecewise linear "staircase" function, and, in fact, it can be computed directly using the "straightforward" symmetries of the Brownian motion. To see this, consider the function

$$\phi(x) := (x - 2U + K)^+ \wedge (2K),$$

and notice that function g, in the case of $\beta = -1$, is given by

$$g(x) = \sum_{n=0}^{\infty} \phi(x - Un).$$

To prove that the above function satisfies the desired properties, we consider the corresponding payoff function G, given by

$$G(x) := (K - x)^+ - g(x), \quad x \geq 0,$$

and its "symmetric" extension to $x \in \mathbb{R}$ via $G(x) = 2K - G(-x)$ for $x < 0$. It is easy to see that, due to the symmetry of the Brownian motion (negative of a Brownian motion is a Brownian motion again), we have

$$\mathbb{E}G(B_\tau) = K \quad \text{and} \quad \mathbb{E}G(B_\tau + U) = 0 \quad \forall \tau \geq 0,$$

where G is the extended payoff, and B is a Brownian motion. Making use of the first equation in the above and applying the strong Markov property of B, we obtain

$$\mathbb{E}G(B_\tau + x) = \mathbb{E}G(B_{\tau \wedge \tau_0} + x) \quad \forall (\tau, x) \in (0, \infty)^2,$$

where τ_0 is the first hitting time of zero by "$x + B$." The above implies that the function $u(x, \tau) := \mathbb{E}G(B_\tau + x)$ corresponds to the price of a European-type option with the payoff

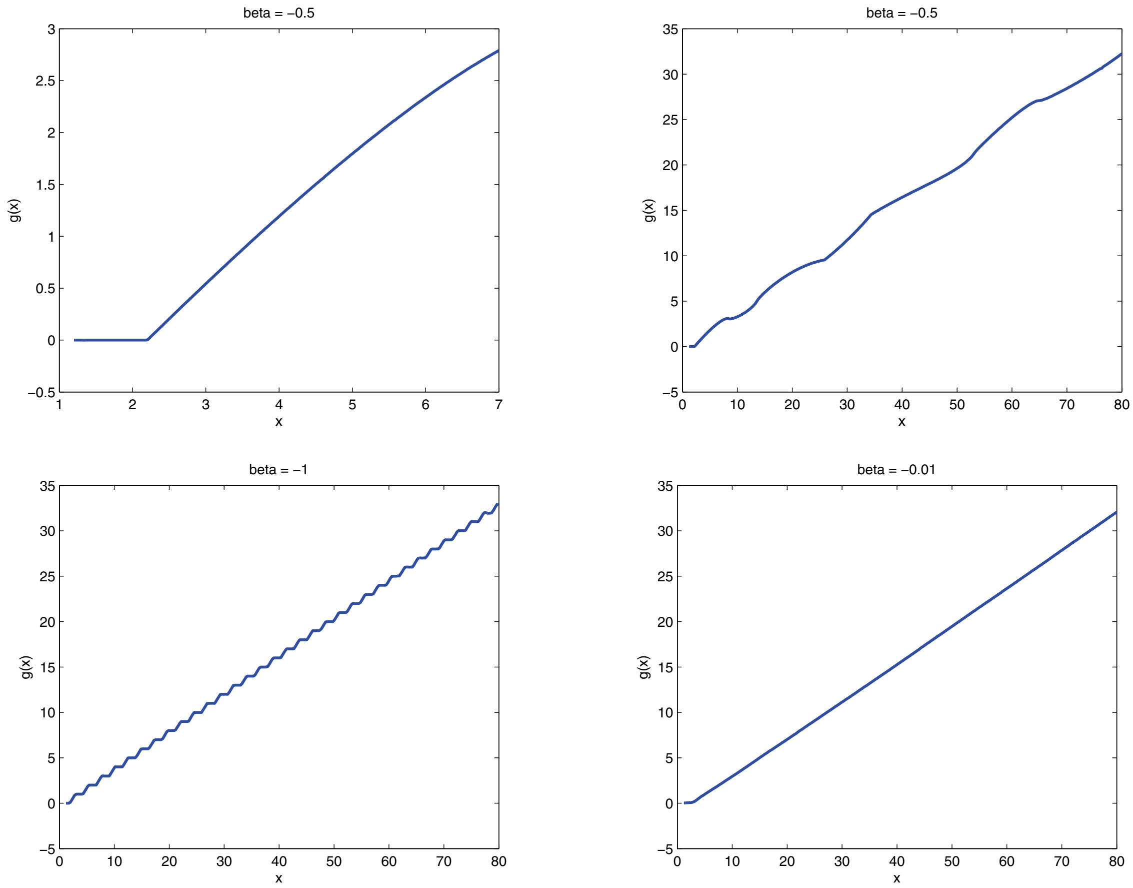

Figure 2. *Payoff $g(x)$ in the zero-drift CEV model with barrier $U = 1.2$ and strike $K = 0.5$: the case of $\beta = -0.5$, for small (top left) and large (top right) values of the argument, and the cases of $\beta = -1$ (bottom left) and $\beta = -0.01$ (bottom right).*

function $G(x)$ (restricted to $x > 0$) in a Bachelier model. Since $u(x, \tau)$ vanishes at $x = U$, we conclude that the payoff function G does, indeed, provide a static hedge of the corresponding UOP option.

We can see that, in general, the function g, which can be viewed as an "image" of the put payoff, may have either a finite or an infinite number of "kink" points, together with a smooth component. Even though we do not provide a rigorous analysis of this phenomenon here, a natural guess is that the same conclusion is true at the level of the second derivatives of the functions. In other words, it seems that the image of a Dirac delta-function, in general, may consist of a combination of either a finite or an infinite number of delta-functions and the absolutely continuous component.

3.3. Short-maturity behavior and a "single-call" hedge.

Notice that the static hedge, given by Theorem 2.7, does not necessarily have a payoff which is traded on the market (except for some special cases discussed above). Therefore, it has to be approximated with the traded (call and put) payoffs before it can be used in practice. Understanding the properties of the payoff function g, resulting from (51), is very important in order to construct such approximations. The study of various properties of function g, as well as the construction of its "tradable" approximations, in a general time-homogeneous diffusion model is the subject of a forthcoming paper. Here, we concentrate on the CEV model with zero drift, in order to illustrate the analysis that can be carried through in a more general setup.

Notice that we already know the asymptotic behavior of function g for the large values of the argument (see Proposition 3.1). However, one can wonder about the "kink" point in the top left graph in Figure 2 and what the slope of the function is at this point. Naturally, the kink should be located to the right of

$$K^* = \left(2U^{-\beta} - K^{-\beta}\right)^{-\frac{1}{\beta}},$$

since the integral in (64) is zero for all $x < K^*$ (for such values of x, the integrand decays exponentially in the right half plane and has no poles there, and, therefore, Jordan's lemma yields that the integral is zero). This observation, however, does not provide us with the slope of g at $x = K^*$.

Clearly, both the "kink" point and the slope should be determined by the asymptotic behavior of the price of a European-type option with payoff $g(S_\tau)$, as $\tau \sim 0$, or, in the Laplace space, as $\lambda \to \infty$. So, instead of searching for the location of the kink point and the slope of g at this point, we will try to solve an equivalent, but more appealing from a practical point of view, problem of approximating the static hedge at short maturities with a static portfolio of put and call options of only two strikes and one maturity. Namely, we would like to approximate the price of a UOP option at short maturities by taking a long position in a put (with strike K and maturity T, corresponding to the UOP option) and shorting δ units of a call option with strike K^* (where δ and K^* are to be determined). A solution to this problem will be referred to as a "single-call hedge."

Recall (7). We need to find an "approximate solution" to (7), function g, such that $g(x) = \delta(x - K^*)^+$, for some $\delta \in \mathbb{R}$, and such that the asymptotic behavior of both sides of (7), as $\lambda \to \infty$, is the same. Recall that the Laplace transforms (in the "time-to-maturity" variable) of out-of-the-money call and put prices in the CEV model can be computed explicitly (for example, one can consult with [15], or use the derivations in the proof of Proposition 3.1). Hence, in the notation

$$z = \frac{-i\sqrt{-2(r+\lambda)}}{|\beta|},$$

(7) becomes

$$\delta\sqrt{K^*U}\, I_\nu(zU^{-\beta})K_\nu(zK^{*-\beta}) = \sqrt{KU}\, I_\nu(zK^{-\beta})K_\nu(zU^{-\beta}). \tag{66}$$

The asymptotic expansions of the Bessel functions yield

$$I_\nu(az)K_\nu(bz) \sim \frac{e^{(a-b)z}}{2z\sqrt{ab}}\left(1 + \left(\frac{1}{4} - \nu^2\right)\left(\frac{1}{a} - \frac{1}{b}\right)\frac{1}{z} + \underline{\underline{O}}\left(|z|^{-2}\right)\right)$$

as $|z| \to \infty$. Plugging the above representation into (66), we obtain the following equation for K^* and δ:

$$\delta(K^*U)^{\frac{1}{2}(\beta+1)}\frac{\exp\left((U^{-\beta}-K^{*-\beta})z\right)}{2z}\left(1+\left(\frac{1}{4}-\nu^2\right)\left(U^{\beta}-K^{*\beta}\right)\frac{1}{z}+\underline{\underline{O}}\left(|z|^{-2}\right)\right)$$

$$= (KU)^{\frac{1}{2}(\beta+1)}\frac{\exp\left((K^{-\beta}-U^{-\beta})z\right)}{2z}\left(1+\left(\frac{1}{4}-\nu^2\right)\left(K^{\beta}-U^{\beta}\right)\frac{1}{z}+\underline{\underline{O}}\left(|z|^{-2}\right)\right). \tag{67}$$

Notice that we need only an approximate solution of the above equation, in the sense that we want to match the asymptotic behavior of both sides as $|z| \to \infty$. Thus, equating the terms of order z^{-1}, we obtain

$$K^* = \left(2U^{-\beta} - K^{-\beta}\right)^{-\frac{1}{\beta}}, \tag{68}$$

which, naturally, coincides with K^* introduced earlier. And the scale parameter (the slope) is given by

$$\delta = \left(\frac{K}{K^*}\right)^{\frac{\beta+1}{2}}. \tag{69}$$

Notice that, according to (68), in the case of $\beta = -1$ (Bachelier model), the barrier U is an arithmetic mean of K and K^*, which confirms the earlier findings. Also, when β is small, we can represent K^* in the following form:

$$K^* = \exp\left(-\frac{1}{\beta}\log(1+\beta(\log K - 2\log U) + \bar{\bar{o}}(\beta))\right),$$

which, clearly, converges to U^2/K as $\beta \to 0$. Therefore, in the case of $\beta = 0$ (Black model), the barrier is a geometric mean of the two strikes. This, again, coincides with the classical results (see, for example, subsection 3.1).

Let us now estimate the error produced by a single-call hedge along the barrier U.

Proposition 3.2. *Consider the zero-drift CEV model with arbitrary $\beta < 0$, and let K^* and δ be given by (68) and (69), respectively. Then, the following holds as $\tau \to 0$:*

$$\left|\frac{P(U,\tau,K) - \delta C(U,\tau,K^*)}{P(U,\tau,K)}\right| = \frac{2|c_1|}{c_2 c_3}\tau\left(1+\bar{\bar{o}}(1)\right), \tag{70}$$

where

$$c_1 = \frac{|\beta|}{4}(KU)^{\frac{\beta+1}{2}}\left(\frac{1}{4}-\nu^2\right)(K^{\beta}+K^{*\beta}-2U^{\beta}),$$

$$c_2 = (U^{-\beta}-K^{-\beta})\frac{\sqrt{2}}{|\beta|}, \quad c_3 = \frac{|\beta|}{2\sqrt{2}}(KU)^{\frac{\beta+1}{2}}.$$

The proof of Proposition 3.2 is given in Appendix B.

It seems reasonable to call the left-hand side of (70) a *relative error* of the hedge at the barrier. Notice that the above result also provides an upper bound on the hedging error at all times before the barrier is hit. Namely, if we denote by $UOP(x,\tau)$ and $SCH(x,\tau)$ the prices of a UOP option and a corresponding single-call hedge, respectively, as the functions of the level of underlying x and the time-to-maturity τ, then the absence of arbitrage implies

$$|UOP(x,\tau) - SCH(x,\tau)| \leq c_4 \varepsilon P(x,\tau,K)$$

for some fixed constant $c_4 = c_4(\varepsilon_0) > 0$, all $x \in [0,U]$, and all $0 \leq \tau \leq \varepsilon \leq \varepsilon_0 < \infty$. Thus, we have proved the following useful corollary, which shows how the above results can be used to construct exact sub- and superreplicating strategies for a UOP option, consisting of vanilla calls and puts of two strikes only.

Corollary 1. *Consider the zero-drift CEV model with arbitrary $\beta < 0$, and let the "single-call" hedge be constructed according to* (68) and (69). *Fix arbitrary $\varepsilon > 0$, and choose $c_4 = c_4(\varepsilon)$ as above. Then, the semistatic trading strategy, which consists of opening a long position in the "single-call" hedge portfolio and an additional long position of size "$c_4\varepsilon$" in the vanilla put option with strike K (and the corresponding maturity), and liquidating it whenever the underlying hits the barrier, superreplicates the price of the corresponding UOP option, provided the time to maturity does not exceed ε. Analogously, opening a short position in the "single-call" hedge portfolio and an additional short position of size "$c_4\varepsilon$" in the vanilla put option with strike K (and the corresponding maturity), and liquidating it whenever the underlying hits the barrier, provides a semistatic subreplicating strategy for the corresponding UOP option if the time to maturity does not exceed ε.*

Remark 5. Another, geometric, argument which can be used to obtain the kink location is to recall that vanilla option prices at short maturities are determined by the geodesic distance imposed by the local volatility of the model (see, for example, [6]). Since we want the effects from buying one option and selling the other one to cancel each other out at the barrier, the corresponding strikes should be located symmetrically against the barrier, with respect to this metric. In other words, in a general time-homogeneous diffusion model, K^* should be a solution to the following equation:

$$\int_K^U \frac{dy}{\sigma(y)} = \int_U^{K^*} \frac{dy}{\sigma(y)}.$$

The above considerations do not provide a general expression for the scale parameter δ. However, it can be shown, after some work, that

$$\delta = \sqrt{\frac{\sigma(K)}{\sigma(K^*)}}.$$

In the case of a CEV model, the above formulas reproduce (68) and (69). A rigorous proof of the fact that K^* and η, given by the above equations, form a "single-call" hedge in an arbitrary (regular enough) time-homogeneous diffusion model (in the sense that the statement of Proposition 3.2 holds, with some positive constants c_i in the right-hand side) will be available in the forthcoming paper.

3.4. Independent time change. It is worth mentioning that we can generalize our models by introducing a continuous stochastic change of time $\{\tau_t\}_{t\geq 0}$, independent of the Brownian motion driving the diffusion. In fact, the static hedging strategy remains the same if the underlying is given by

$$F_t = X_{\tau_t},$$

where X is defined in (2). This setup includes, for example, the zero-drift SABR model with zero correlation. More precisely, the proposed hedge will work in the model where the underlying is given by

$$\begin{cases} dF_t = \delta_t^2 \mu(F_t)dt + \delta_t \sigma(F_t)dW_t, \\ \qquad d\delta_t = \alpha \delta_t dZ_t, \end{cases}$$

where W and Z are two Brownian motions, with $dW_t \cdot dZ_t = 0$, and μ and σ satisfy the assumptions of Theorem 2.7 or 2.8. The proof of the above claim is very simple. By conditioning on $(\tau_t)_{t\geq 0}$, we reduce the problem to the original one, only with a different maturity, since the payoff becomes

$$I_{\{\sup_{t\in[0,T]} F_{\tau_t} < U\}} \left(K - X_{\tau_T}\right)^+ = I_{\{\sup_{s\in[0,\tau_T]} X_s < U\}} \left(K - X_{\tau_T}\right)^+. \tag{71}$$

Notice that the continuity of $\tau_{\cdot}$ is crucial. If we allow jumps in the time change, the equality in (71) becomes "$\geq$," and the static hedge of a UOP option in the original model becomes a subreplicating strategy in the time-changed model, which works only as long as the underlying has not exceeded the barrier (in particular, it may fail to subreplicate the price of the barrier option at the moment when the underlying jumps strictly above the barrier).

It is worth mentioning that the class of time-changed diffusion models does not include, for example, the SABR model with nonzero proportional drift "rF_t," since the change of time links the drift directly to the volatility. Therefore, it seems more natural to use the above time-changed diffusion models, with a nontrivial change of time, to describe the dynamics of the price of a forward contract, which has zero drift under the risk-neutral measure, rather than the spot price. Of course, in the case of a trivial time change, or, in other words, in the original class of diffusion processes, it is possible to allow for a proportional drift and, hence, produce a reasonable model for the spot price.

4. Conclusions and future research. In this paper, we consider the problem of hedging a barrier option with a semistatic position in a European-type contingent claim of the same maturity. We construct an exact static hedge of a UOP in all regular enough time-homogeneous diffusion models for the underlying, providing an explicit analytic expression for the payoff function of the aforementioned European-type claim. The static hedging strategies of all other classical upper barrier options follow easily from our result, and the lower barrier options can be analyzed analogously. The general form of the main result is given in Theorem 2.7. Theorem 2.8 provides a more transparent formulation of the result, albeit in a reduced class of stochastic models. In particular, it shows that our framework includes all models in which the drift and local volatility functions of the underlying diffusion process have finite limits at zero and infinity. We also consider examples of several diffusion models, including the Black–Scholes and zero-drift CEV models, in which we compute the static hedge explicitly and study its properties. Finally, we show how to construct an approximation of the static hedge of a UOP option written on a zero-drift CEV process with vanilla options of only two strikes.

Future research should address the problem of efficient approximation of the static hedge payoff given by the corresponding formulas in Theorems 2.7 and 2.8 in a general time-homogeneous diffusion model. In particular, a general expression for the "single-call" approximation of the static hedge would make the results more valuable from a practical point of view, since it would produce a payoff which is actually traded on the market (otherwise, one has to approximate the payoff function of a static hedge with the payoffs of available vanilla options). Some results in this direction are announced in Remark 5.

Finally, we believe that an important extension of the results presented in this paper is the construction of the semirobust static hedges. As we discussed in the introduction, there has been a fair amount of research done in order to obtain static sub- and superreplicating strategies, which work in all (or in a large class of) stochastic models for the underlying. The usual approach to this problem is the so-called top-down approach, when a sub- or superreplicating strategy is constructed using only the "no-arbitrage" arguments, so that it automatically works in all arbitrage-free stochastic models for the underlying (see, for example, [18], [9], [13], [12]). We suggest that one can also take the opposite, "bottom-up" point of view and, for example, construct a static superreplicating strategy, valid in a *family of models*, from the (different) exact static hedges available explicitly in each particular model. This would lead to sub- and superreplicating strategies, which are robust within a given family of models. Such strategies can provide a middle ground between the model-free but loose hedges and the model-specific hedges constructed in this paper. An important question in this context is, of course, how to choose the appropriate families of models. For example, one could consider the class of CEV models parameterized by "β." The range of admissible values of the parameter can be specified, for example, according to the historical observations of the values of "β" calibrated to the "implied skew." Then, making use of the explicit formulae for the static hedge, provided in this paper, one can construct a strategy that would sub- or superreplicate a UOP option in all such models.

Another way to construct a semirobust static hedge is to start directly with the investor's beliefs about the range of future values of the market implied volatility. Assume, for example, that the investor is interested in superreplicating the price of a UOP option, given that the implied volatility stays within a specified range. Then, we construct an "extreme" time-homogeneous diffusion model (possibly time-changed), which produces an implied volatility surface that dominates all surfaces in the prescribed range from above for the negative log-moneyness and from below for the positive log-moneyness. Then, using the exact expression for the payoff of a static hedge in the "extreme" model, we approximate it from above by a function which is convex to the left of the barrier (where, in fact, it coincides with the payoff of a put) and concave to the right of the barrier. It is then easy to see that the latter function produces a superreplicating semistatic strategy in the "extreme" model, and, therefore, due to convexity arguments, it succeeds in any (continuous) model in which the implied volatility stays within the given range. The precise description of the above idea, together with the numerical approximations and explicit constructions, is the focus of the forthcoming paper.

Appendix A. Proof of Lemma 2.2. Throughout this proof, c_i's denote some positive real constants. Recall the definition of q, given in (13), and notice that this function can be represented as

$$q(z) = \frac{1}{2}\left(\frac{\mu^2(X(z))}{\sigma^2(X(z))} - 2\frac{\mu(X(z))\,\sigma'(X(z))}{\sigma(X(z))} + \mu'(X(z)) \right.$$
$$\left. + \frac{1}{4}\sigma'^2(X(z)) - \frac{1}{2}\sigma''(X(z))\,\sigma(X(z))\right). \tag{72}$$

Clearly, it is sufficient to show that the statement of the lemma holds for each of the above terms separately. Let us show their absolute integrability at infinity (after subtracting a constant).

First, notice that (16) and the positivity of σ yield that σ'' is absolutely integrable at infinity. Therefore,

$$\int_0^\infty \left|\sigma''(X(z))\right| \sigma(X(z))\, dz = \int_U^\infty \left|\sigma''(x)\right| dx < \infty.$$

Consider now the fourth term in the right-hand side of (72). Since σ'' is absolutely integrable at infinity, function $\sigma'(x)$ converges to some constant $c_1 \in \mathbb{R}$, as $x \to \infty$, and we have

$$\sigma'(x) = c_1 - \int_x^\infty \sigma''(y)dy. \tag{73}$$

Thus, we continue

$$\int_0^\infty \left|\sigma'^2(X(z)) - c_1^2\right| dz = \int_U^\infty \left|\int_x^\infty \sigma''(y)dy\right| \frac{|\sigma'(x) + c_1|}{\sigma(x)} dx$$
$$\leq c_2 \int_U^\infty \int_x^\infty \left|\sigma''(y)\right| dy \left(\int_U^x \frac{|\sigma'(y)| + 1}{\sigma(y)} dy\right)' dx$$
$$= c_2 \int_U^\infty \left|\sigma''(x)\right| \int_U^x \frac{|\sigma'(y)| + 1}{\sigma(y)} dy dx < \infty,$$

where we integrated by parts and applied (16).

Similarly, from (17) we conclude that the derivative of μ/σ is absolutely integrable at infinity, and, therefore, μ/σ has a finite limit at infinity, say, $c_3 \in \mathbb{R}$, and we have

$$\frac{\mu(x)}{\sigma(x)} = c_3 - \int_x^\infty \left(\frac{\mu(y)}{\sigma(y)}\right)' dy. \tag{74}$$

Now we can consider the first term in the right-hand side of (72):

$$\int_0^\infty \left|\frac{\mu^2(X(z))}{\sigma^2(X(z))} - c_3^2\right| dz \leq c_4 \int_U^\infty \int_x^\infty \left|\left(\frac{\mu(y)}{\sigma(y)}\right)'\right| dy \left(\int_U^x \frac{dy}{\sigma(y)}\right)' dx$$
$$\leq c_5 \int_U^\infty \left|\left(\frac{\mu(x)}{\sigma(x)}\right)'\right| \int_U^x \frac{dy}{\sigma(y)} dx < \infty.$$

Next, making use of (73) and (74), we conclude that, in order to take care of the second term in the right-hand side of (72), it suffices to show convergence of the following integral:

$$\int_U^\infty \int_x^\infty \left| \left(\frac{\mu(y)}{\sigma(y)} \right)' \right| dy \int_x^\infty |\sigma''(y)|\, dy \left(\int_U^x \frac{dy}{\sigma(y)} \right)' dx$$
$$\leq c_6 \int_U^\infty \left| \left(\frac{\mu(x)}{\sigma(x)} \right)' \right| \int_U^x \frac{dy}{\sigma(y)} dx + c_7 \int_U^\infty |\sigma''(x)| \int_U^x \frac{dy}{\sigma(y)} dx < \infty.$$

Finally, we notice that

$$\mu'(x) = \left(\frac{\mu(x)}{\sigma(x)} \right)' \sigma(x) + \frac{\mu(x)}{\sigma(x)} \sigma'(x),$$

and the corresponding estimate of the integral of the third term in the right-hand side of (72) follows immediately.

The absolute integrability of q' can be shown in a similar manner, making use of (18). ■

Appendix B. Proof of Proposition 3.2. From (67) we have that

$$\hat{P}(U,\lambda,K) - \delta\rho(U,\lambda,K^*) = (KU)^{\frac{\beta+1}{2}} \frac{\exp\left((K^{-\beta} - U^{-\beta}) \frac{-i\sqrt{-2(\lambda+r)}}{|\beta|} \right)}{2(\lambda+r)(-i)\sqrt{-2(\lambda+r)}}$$
$$\cdot \left(\left(\frac{1}{4} - \nu^2 \right) (K^\beta + K^{*\beta} - 2U^\beta) \frac{|\beta|}{-i\sqrt{-2(\lambda+r)}} + \underline{\underline{O}}\left(|\lambda|^{-1}\right) \right)$$

as $|\lambda| \to \infty$. Recall that "$\tau \to 0$," in the Laplace space, corresponds to "$\lambda \to \infty$." Applying the inversion formula for the Laplace transform to the series expansion above, we obtain

$$P(U,\tau,K) - \delta C(U,\tau,K^*) = c_1 \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda\tau + c_2 i\sqrt{-\lambda-r}}}{(\lambda+r)^2} \left(1 + \underline{\underline{O}}\left(|\lambda|^{-1/2} \right) \right) d\lambda, \tag{75}$$

where c_i's are given in the statement of the proposition, and γ is an arbitrary positive constant. Choose

$$\gamma = \gamma(\tau) := \left(\frac{c_2}{2\tau} + 1 \right)^2.$$

Let us split the integral in (75) into two parts and analyze the first one. Differentiating twice with respect to τ, we obtain

$$\frac{\partial}{\partial\tau}\frac{\partial}{\partial\tau}\left[\frac{e^{r\tau}}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda\tau + c_2 i\sqrt{-\lambda-r}}}{(\lambda+r)^2} d\lambda \right]$$
$$= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{(\lambda+r)\tau + c_2 i\sqrt{-\lambda-r}} d\lambda = \frac{c_2}{2\sqrt{\pi\tau^3}} e^{-\frac{c_2^2}{4\tau}},$$

where the last equality is due to the fact that $f(\lambda) = e^{c_2 i\sqrt{-\lambda}}$ is the Laplace transform of a hitting time of the level c_2 by a Brownian motion, and it has a well-known density. Thus, we conclude that, as $\tau \to 0$,

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda\tau + c_2 i\sqrt{-\lambda-r}}}{(\lambda+r)^2} d\lambda = e^{-r\tau}\int_0^{\tau}\left(2\int_{\frac{c_2}{\sqrt{2s}}}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx\right) ds$$
$$= \frac{2}{\sqrt{\pi}c_2} e^{-r\tau}\int_0^{\tau}\sqrt{s}e^{-\frac{c_2^2}{4s}} ds \left(1+\overline{\overline{o}}(1)\right).$$

Next, we estimate the residual term as follows:

$$\left|\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{(\lambda+r)\tau + c_2 i\sqrt{-\lambda-r}}}{(\lambda+r)^2} \underline{\underline{O}}\left(|\lambda|^{-1/2}\right) d\lambda\right|$$
$$= e^{-\frac{c_2^2}{4\tau}}\left|\int_{\Gamma'} \frac{e^{\tau s}}{\left(\frac{c_2}{2\tau} - i\sqrt{-s}\right)^4} \frac{c_2/(2\tau) - i\sqrt{-s}}{-i\sqrt{-s}} \underline{\underline{O}}\left(\left|\frac{c_2}{2\tau} - i\sqrt{-s}\right|^{-1}\right) ds\right|$$
$$\leq c_4\tau^3 \exp\left(-\frac{c_2^2}{4\tau} + c_5\tau\right),$$

where we changed the variables from λ to $s = \left(-i\sqrt{-\lambda} - c_2/(2\tau)\right)^2$ and noticed that the real parts of all points in Γ', which is the image of the original contour of integration under the mapping $\lambda \mapsto s(\lambda)$, are bounded from above by a constant. Applying L'Hôpital's rule, we show that

$$\lim_{\tau\to 0} \frac{\tau^3 \exp\left(-\frac{c_2^2}{4\tau} + c_5\tau\right)}{\int_0^{\tau}\sqrt{s}e^{-\frac{c_2^2}{4s}} ds} = 0,$$

and hence

$$P(U,\tau,K) - \delta C(U,\tau,K^*) = \frac{2c_1}{\sqrt{\pi}c_2} e^{-r\tau}\int_0^{\tau}\sqrt{s}e^{-\frac{c_2^2}{4s}} ds \left(1+\overline{\overline{o}}(1)\right)$$

as $\tau \to 0$. Similarly, we deduce that, as $\tau \to 0$,

$$P(U,\tau,K) = \frac{c_3}{\sqrt{\pi}} e^{-r\tau}\int_0^{\tau}\frac{e^{-\frac{c_2^2}{4s}}}{\sqrt{s}} ds \left(1+\overline{\overline{o}}(1)\right).$$

Finally, applying L'Hôpital's rule again, we obtain (70). ∎

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