Journal of Financial Econometrics, 2014, Vol. 12, No. 1, 3–46

Static Hedging of Standard Options

PETER CARR
Courant Institute, New York University

LIUREN WU
Zicklin School of Business, Baruch College, CUNY

ABSTRACT
Working in a single-factor Markovian setting, this article derives a new, static spanning relation between a given option and a continuum of shorter-term options written on the same asset. Compared to dynamic delta hedge, which breaks down in the presence of large random jumps, the static hedge works well under both continuous and discontinuous price dynamics. Simulation exercises show that under purely continuous price dynamics, discretized static hedges with as few as three to five options perform similarly to the dynamic delta hedge with the underlying futures and daily updating, but the static hedges strongly outperform the daily delta hedge when the underlying price process contains random jumps. A historical analysis using over 13 years of data on S&P 500 index options further validates the superior performance of the static hedging strategy in practical situations. (*JEL*: G12, G13, C52)

KEYWORDS: Static hedging, jumps, option pricing, Monte Carlo, S&P 500 index options, stochastic volatility

Over the past two decades, the derivatives market has expanded dramatically. Accompanying this expansion is an increased urgency in understanding and managing the risks associated with derivative securities. In an ideal setting under which the price of the underlying security moves continuously (such as in a diffusion with known instantaneous volatility) or with fixed and known size steps (such as in a binomial tree), derivatives pricing theory provides a framework in

---

We thank Eric Renault (the editor), the anonymous associate editor, two anonymous referees, Yacine Aït-Sahalia, Alexey Polishchuk, Steven Posner, Jingyi Zhu, and the participants of finance workshops at Vanderbilt University, Princeton University, and Oak Hill Asset Management for comments and suggestions. We also thank David Hait and OptionMetrics for providing the options data and Alex Mayus for clarifying trading practices. Address correspondence to Liuren Wu, Zicklin School of Business, Baruch College, One Bernard Baruch Way, B10-225, New York, NY 10010, or e-mail: liuren.wu@baruch.cuny.edu.

doi:10.1093/jjfinec/nbs014 Advanced Access publication April 5, 2013

which the risks inherent in a derivatives position can be eliminated via frequent trading in only a small number of securities.

In reality, however, large and random price movements happen much more often than typically assumed in the above ideal setting. The past two decades have repeatedly witnessed turmoil in the financial markets such as the 1987 stock market crash, the 1997 Asian crisis, the 1998 Russian default and the ensuing hedge fund crisis, the tragic event of September 11, 2001, and the most recent financial market meltdown. Juxtaposed between these large crises are many more mini-crises, during which prices move sufficiently fast so as to trigger circuit breakers and trading halts. When these crises occur, a dynamic hedging strategy based on small or fixed size movements often breaks down. Worse yet, strategies that involve dynamic hedging in the underlying asset tend to fail precisely when liquidity dries up or when the market experiences large moves. Unfortunately, it is during these financial crises such as liquidity gaps or market crashes that investors need effective hedging the most dearly.

Perhaps in response to the known deficiencies of dynamic hedging, Breeden and Litzenberger (1978) (henceforth BL) pioneered an alternative approach, which is foreshadowed in the work of Ross (1976) and elaborated on by Green and Jarrow (1987) and Nachman (1988). These authors show that a path-independent payoff can be hedged using a portfolio of standard options maturing with the claim. This strategy is completely robust to model mis-specification and is effective even in the presence of jumps of random size. Its only real drawback is that the class of claims that this strategy can hedge is fairly narrow. First, the BL hedge of a standard option reduces to a tautology. Second, the hedge can neither deal with standard options of different maturities, nor can it deal with path-dependent options. Therefore, the BL strategy is completely robust but has limited range. By contrast, dynamic hedging works for a wide range of claims, but is not robust.

In this article, we propose a new approach for hedging derivative securities. This approach lies between dynamic hedging and the BL static hedge in terms of both range and robustness. Relative to BL, we place mild structure on the class of allowed stochastic processes of the underlying asset in order to expand the class of claims that can be robustly hedged. In particular, we work in a one-factor Markovian setting, where the market price of a security is allowed not only to move diffusively, but also to jump randomly to any nonnegative value. In this setting, we can robustly hedge both vanilla options and more exotic, potentially path-dependent options, such as discretely monitored Asian and barrier options, Bermudan options, passport options, cliquets, ratchets, and many other structured notes. In this article, we focus on a simple spanning relation between the value of a given European option and the value of a continuum of shorter-term European options. The required position in each of the shorter-term options is proportional to the gamma (second price derivative) that the target option will have at the expiry of the short-term option if the security price at that time is at the strike of this short-term option. As this future gamma does not vary with the passage of time or the change in the underlying price, the weights in the portfolio of shorter-term options are static over the life of these options. Given this static spanning result, no arbitrage implies that the target option and the replicating portfolio have the same value for all times until the shorter term options expire. As a result, one can effectively hedge a long-term option even in the presence of large random jumps in the underlying security price movement. Furthermore, given the static nature of the strategy, portfolio rebalancing is not necessary until the shorter-term options mature. Therefore, one does not need to worry about market shutdowns and liquidity gaps in the intervening period. The strategy remains viable and can become even more useful when the market is in stress.

As transaction costs and illiquidity render the formation of a portfolio with a continuum of options physically impossible, we develop an approximation for the static hedging strategy using only a finite number of options. This discretization of the ideal trading strategy is analogous to the discretization of a continuous-time dynamic trading strategy. To discretize the static hedge, we choose the strike levels and the associated portfolio weights based on a Gauss–Hermite quadrature method. We use Monte Carlo simulation to gauge the magnitude and distributional characteristics of the hedging error introduced by the quadrature approximation. We compare this hedging error to the hedging error from a delta-hedging strategy based on daily rebalancing with the underlying futures. The simulation results indicate that the two strategies have comparable hedging effectiveness when the underlying price dynamics are continuous, but the performance of the delta hedge deteriorates dramatically in the presence of random jumps. As a result, a static strategy with merely three options can outperform delta hedging with daily updating when the underlying security price can jump randomly.

To gauge the impact of model uncertainty and model misspecification, we also perform the hedging exercise assuming that the hedger does not know the true underlying price dynamics but simply computes the delta and the static hedge portfolio weight using the Black and Scholes (1973) formula with the observed option implied volatility on the target option as the volatility input. The hedging performance shows no visible deterioration. Furthermore, we find that increasing the rebalancing frequency in the delta-hedging strategy does not rescue its performance as long as the underlying asset price can jump by a random amount. By contrast, the static hedging performance can be improved further by increasing the number of strikes used in the portfolio and by choosing maturities for the hedge portfolio closer to the target option maturity. Taken together, we conclude that the superior performance of static hedging over daily delta hedging in the jump model simulation is not due to model misspecification, nor is it due to the approximation error introduced via discrete rebalancing. Rather, this outperformance is due to the fact that delta hedging is inherently incapable of dealing with jumps of random size in the underlying security price movement. Our static spanning relation can handle random jumps and our approximation of this spanning relation performs equally well with and without jumps in the underlying security price process.

This article also examines the historical performance of the hedging strategies in hedging S&P 500 index options over a 13-year period. The historical run shows that a static hedge using no more than five options outperforms daily delta hedging with the underlying futures. The consistency of this result with our jump model simulations lends empirical support for the existence of jumps of random size in the movement of the S&P 500 index.

For clarity of exposition, this article focuses on hedging a standard European option with a portfolio of shorter-term options; however, the underlying theoretical framework extends readily to the hedging of more exotic, potentially path-dependent options. We use a globally floored, locally capped, compounding cliquet as an example to illustrate how this option contract with intricate path-dependence can be hedged with a portfolio of European options. The hedging strategy is semi-static in the sense that trades occur only at the discrete monitoring dates.

In related literature, the effective hedging of derivative securities has been applied not only for risk management, but also for option valuation and model verification (Bates, 2003). Bakshi, Cao, and Chen (1997), Bakshi and Kapadia (2003), and Dumas, Fleming, and Whaley (1998) use hedging performance to test different option pricing models. He et al. (2006) and Kennedy, Forsyth, and Vetzal (2009) set up a dynamic programming problem in minimizing the hedging errors under jump-diffusion frameworks and in the presence of transaction cost. Branger and Mahayni (2006, 2011) propose robust dynamic hedging strategies in pure diffusion models when the hedger knows only the range of the volatility levels but not the exact volatility dynamics. Bakshi and Madan (2000) propose a general option-valuation strategy based on effective spanning using basis characteristic securities. Carr and Chou (1997) consider the static hedging of barrier options and Carr and Madan (1998) propose a static spanning relation for a general payoff function by a portfolio of bond, forward, European options maturing at the same maturity with the payoff function. Starting with such a spanning relation, Takahashi and Yamazaki (2009a,b) propose a static hedging relation for a target instrument that has a known value function. Balder and Mahayni (2006) start with our spanning result in this article and consider discretization strategies when the strikes of the hedging options are pre-specified and the underlying price dynamics are unknown to the hedger. In a recent working paper, Wu and Zhu (2011) propose a new, completely model-free strategy of statically hedging options with nearby options, in which the hedge portfolio is formed not based on the spanning of certain pre-specified risks but rather based on the payoff characteristics of the target and hedging option contracts.

The remainder of the article is organized as follows. Section 1 develops the theoretical results underlying our static hedging strategy on a European option. Section 2 uses Monte Carlo simulation to enact a wide variety of scenarios under which the market not only moves diffusively, but also jumps randomly, with or without stochastic volatility. Under each scenario, we analyze the hedging performance of our static strategy and compare it with dynamic delta hedging with the underlying futures. Section 3 applies both strategies to the S&P 500 index options data. Section 4 shows how the theoretical framework can be applied to hedge exotic options. Section 5 concludes.

1 SPANNING OPTIONS WITH OPTIONS

Working in a continuous-time one-factor Markovian setting, we show how the risk of a European option can be spanned by a continuum of shorter-term European options. The weights in the portfolio are static as they are invariant to changes in the underlying security price or the calendar time. We also illustrate how we can use a quadrature rule to approximate the static hedge using a small number of shorter-term options.

1.1 Assumptions and Notation

We assume frictionless markets and no arbitrage. To fix notation, let S_t denote the spot price of an asset (say, a stock or stock index) at time $t \in [0, \mathcal{T}]$, where $\mathcal{T}$ is some arbitrarily distant horizon. For realism, we assume that the owners of this asset enjoy limited liability, and hence $S_t \geq 0$ at all times. For notational simplicity, we further assume that the continuously compounded riskfree rate r and dividend yield q are constant. No arbitrage implies that there exists a risk-neutral probability measure $\mathbb{Q}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ such that the instantaneous expected rate of return on every asset equals the instantaneous riskfree rate r. We also restrict our analysis to the class of models in which the risk-neutral evolution of the stock price is Markov in the stock price S and the calendar time t. Our class of models includes local volatility models, e.g., Dupire (1994), and models based on Lévy processes, e.g., Barndorff-Nielsen (1997), Bates (1991), Carr et al. (2002), Carr and Wu (2003), Eberlein, Keller, and Prause (1998), Madan and Seneta (1990), Merton (1976), and Wu (2006), but does not include stochastic volatility models such as Bates (1996, 2000), Bakshi, Cao, and Chen (1997), Carr and Wu (2004, 2007), Heston (1993), Hull and White (1987), Huang and Wu (2004), and Scott (1997).

We use $C_t(K,T)$ to denote the time-t price of a European call with strike K and maturity T. Our assumption implies that there exists a call pricing function $C(S,t;K,T;\Theta)$ such that

$$C_t(K,T) = C(S_t,t;K,T;\Theta), \quad t \in [0,T], K \geq 0, T \in [t,\mathcal{T}]. \tag{1}$$

The call pricing function relates the call price at t to the state variables (S_t,t), the contract characteristics (K,T), and a vector of fixed model parameters Θ.

We use $g(S,t;K,T;\Theta)$ to denote the probability density function of the asset price under the risk-neutral measure $\mathbb{Q}$, evaluated at the future price level K and the future time T and conditional on the stock price starting at level S at some earlier time t. Breeden and Litzenberger (1978) show that this risk-neutral density relates to the second strike derivative of the call pricing function by

$$g(S,t;K,T;\Theta) = e^{r(T-t)} \frac{\partial^2 C}{\partial K^2}(S,t;K,T;\Theta). \tag{2}$$

1.2 Spanning Vanilla Options with Vanilla Options

The main theoretical result of the article comes from the following theorem:

Theorem 1: *Under no arbitrage and the Markovian assumption in (1), the time-t value of a European call option maturing at a fixed time $T \geq t$ relates to the time-t value of a continuum of European call options at a shorter maturity $u \in [t, T]$ by*

$$C(S,t;K,T;\Theta) = \int_0^\infty w(\mathcal{K}) C(S,t;\mathcal{K},u;\Theta) d\mathcal{K}, \quad u \in [t,T], \tag{3}$$

for all possible nonnegative values of S and at all times $t \leq u$. The weighting function $w(\mathcal{K})$ does not vary with S or t, and is given by

$$w(\mathcal{K}) = \frac{\partial^2}{\partial \mathcal{K}^2} C(\mathcal{K},u;K,T;\Theta). \tag{4}$$

Proof. Under the Markovian assumption in (1), we can compute the initial value of the target call option by discounting the expected value it will have at some future date u,

$$\begin{aligned} C(S,t;K,T;\Theta) &= e^{-r(u-t)} \int_0^\infty g(S,t;\mathcal{K},u;\Theta) C(\mathcal{K},u;K,T;\Theta) d\mathcal{K} \\ &= \int_0^\infty \frac{\partial^2}{\partial \mathcal{K}^2} C(S,t;\mathcal{K},u;\Theta) C(\mathcal{K},u;K,T;\Theta) d\mathcal{K}. \end{aligned} \tag{5}$$

The first line follows from the Markovian property. The call option value at any time u depends only on the underlying security's price at that time. The second line results from a substitution of Equation (2) for the risk-neutral density function.

We integrate Equation (5) by parts twice and observe the following boundary conditions,

$$\begin{aligned} &\tfrac{\partial}{\partial K} C(S,t;K,u;\Theta)\big|_{K\to\infty} = 0, \quad C(S,t;K,u;\Theta)|_{K\to\infty} = 0, \\ &\tfrac{\partial}{\partial S} C(0,u;K,T;\Theta) = 0, \quad C(0,u;K,T;\Theta) = 0. \end{aligned} \tag{6}$$

The final result of these operations is Equation (3). ■

A key feature of the spanning relation in (3) is that the weighting function $w(\mathcal{K})$ is independent of S and t. This property characterizes the static nature of the spanning relation. Under no arbitrage, once we form the spanning portfolio, no rebalancing is necessary until the maturity date of the options in the spanning portfolio. The weight $w(\mathcal{K})$ on a call option at maturity u and strike $\mathcal{K}$ is proportional to the gamma that the target call option will have at time u, should the underlying price level be at $\mathcal{K}$ then. Since the gamma of a call option typically shows a bell-shaped curve centered near the call option's strike price, greater weights go to the options with strikes that are closer to that of the target option. Furthermore, as we let the common maturity u of the spanning portfolio approach the target call option's maturity T, the gamma becomes more concentrated around K. In the limit when $u = T$, all of the weight is on the call option of strike K. Equation (3) reduces to a tautology.

The spanning relation in (3) represents a constraint imposed by no-arbitrage and the Markovian assumption on the relation between prices of options at two different maturities. Given that the Markovian assumption is correct, a violation of Equation (3) implies an arbitrage opportunity. For example, if at time t, the market price of a call option with strike K and maturity T (left-hand side) exceeds the price of a gamma-weighted portfolio of call options for some earlier maturity u (right-hand side), conditional on the validity of the Markovian assumption (1), the arbitrage is to sell the call option of strike K and maturity T, and to buy the gamma-weighted portfolio of all call options maturing at the earlier date u. The cash received from selling the T maturity call exceeds the cash spent buying the portfolio of nearer dated calls. At time u, the portfolio of expiring calls pays off:

$$\int_0^\infty \frac{\partial^2}{\partial \mathcal{K}^2} C(\mathcal{K}, u; K, T; \Theta)(S_u - \mathcal{K})^+ d\mathcal{K}.$$

Integrating by parts twice implies that this payoff reduces to $C(S_u, u; K, T; \Theta)$, which we can use to close the short call position.

To understand the implications of our theorem for risk management, suppose that at time t there are no call options of maturity T available in the listed market. However, it is known that such a call will be available in the listed market by the future date $u \in (t, T)$. An options trading desk could consider writing such a call option of strike K and maturity T to a customer in return for a (hopefully sizeable) premium. Given the validity of the Markov assumption, the options trading desk can hedge away the risk exposure arising from writing the call option over the time period $[t, u]$ using a static position in available shorter-term options. The maturity of the shorter-term options should be equal to or longer than u and the portfolio weight is determined by Equation (3). At date u, the assumed validity of the Markov condition (1) implies that the desk can use the proceeds from the sale of the shorter-term call options to purchase the T maturity call in the listed market. Thus, this hedging strategy is semi-static in that it involves rolling over call options once. In contrast to a purely static strategy, there is a risk that the Markov condition will not hold at the rebalancing date u. We will continue to use the terser term "static" to describe this semi-static strategy; however, we warn the practically minded reader that our use of this term does not imply that there is no model risk.

The replication principle behind our static option hedge is different from dynamic delta hedging with the underlying security. At initiation of the dynamic delta hedge, a position in the underlying security and bond can match the initial level and initial slope of the target call option. However, it does not match the gamma and higher security price derivatives. If left static, a small move in the security price and time will preserve level matching, provided that the square of the small move corresponds to the variance rate used in the delta hedge. This static position no longer matches the slope. A self-financing trade is needed to rematch the slope. Thus, the success of the dynamic delta hedging relies on continuous rebalancing and the security price following a particular continuous process. If the size of the security price movement is not as expected, even the level matching cannot be achieved. As such, even continuous rebalancing cannot guarantee a successful hedge.

By contrast, at initiation of our static hedge, the option portfolio matches the level, slope, gamma, and all higher price derivatives of the target option. Thus, level matching can be preserved under a much wider range of security price movements. Furthermore, with the Markovian assumption, our options hedge matches all price derivatives at all price levels and time, thus making the portfolio static.

Theorem 1 states the spanning relation in terms of call options. The spanning relation also holds if we replace the call options on both sides of the Equation by their corresponding put options of the same strike and maturity. The relation on put options can either be proved analogously or via the application of the put–call parity to the call option spanning relation in Equation (3).

More generally, for any twice-differentiable value function $V(S_u)$ at time u, we can perform a Taylor expansion with remainder about any point F to obtain the following generic spanning relation (Carr and Madan, 1998):

$$V(S_u)=V(F)+V'(F)(S_u-F)+\int_0^F V''(K)(K-S_u)^+dK+\int_F^\infty V''(K)(S_u-K)^+dK. \quad (7)$$

In words, the value function $V(S_u)$ can be replicated by a bond position $V(F)$, a forward position $V'(K)$ with strike F, and a continuum of call and put options maturing at time u with the weights at each strike given by $V''(K)dK$. Under the one-factor Markovian setting, we know the time-u value function $V(S_u)$ of any European options maturing at a later date $T>u$. Accordingly, we can hedge these options statically up until time u using options maturing at time u. To derive the static hedging relation for the call option in (3), we choose $F=0$ so that $C(0,u)=0$ and $C'(0,u)=0$.

Equation (7) also highlights the key underlying assumption for the static hedging relation: The value of the target option at the future time u must be purely a deterministic function of the underlying stock price at that time S_u. Any other random sources (such as stochastic volatility) cannot influence the option value at time u in order for the static hedging relation to hold. Thus, the hedging effectiveness of this static strategy presents an indirect test for the presence of additional risk sources such as stochastic volatility.

1.3 Finite Approximation with Gaussian Quadrature Rules

In practice, investors can neither rebalance a portfolio continuously, nor can they form a static portfolio involving a continuum of securities. Both strategies involve an infinite number of transactions. In the presence of discrete transaction costs, both would lead to financial ruin. As a result, dynamic strategies are only rebalanced discretely in practice. The trading times are chosen to balance the costs arising from the hedging error with the cost arising from transacting in the underlying. Similarly, to implement our static hedging strategy in practice, we need to approximate it using a finite number of call options. The number of call options used in the portfolio is chosen to balance the cost from the hedging error with the cost from transacting in these options.

We propose to approximate the spanning integral in Equation (3) by a weighted sum of a finite number (N) of call options at strikes $\mathcal{K}_j, j=1,2,\cdots,N$,

$$\int_0^\infty w(\mathcal{K})C(S,t;\mathcal{K},u;\Theta)d\mathcal{K} \approx \sum_{j=1}^{N} \mathcal{W}_j C(S,t;\mathcal{K}_j,u;\Theta), \tag{8}$$

where we choose the strike points $\mathcal{K}_j$ and their corresponding weights based on the Gauss–Hermite quadrature rule. The Gauss–Hermite quadrature rule is designed to approximate an integral of the form $\int_{-\infty}^{\infty} f(x)e^{-x^2}dx$, where $f(x)$ is an arbitrary smooth function. After some rescaling, the integral can be regarded as an expectation of $f(x)$ where x is a normally distributed random variable with zero mean and variance of two. For a given target function $f(x)$, the Gauss–Hermite quadrature rule generates a set of weights w_i and nodes x_i, $i=1,2,\cdots,N$, that approximate the integral with the following error representation (Davis and Rabinowitz, 1984),

$$\int_{-\infty}^{\infty} f(x)e^{-x^2}dx = \sum_{j=1}^{N} w_j f(x_j) + \frac{N!\sqrt{\pi}}{2^N}\frac{f^{(2N)}(\xi)}{(2N)!} \tag{9}$$

for some $\xi \in (-\infty,\infty)$. The approximation error vanishes if the integrand $f(x)$ is a polynomial of degree equal or less than $2N-1$.

To apply the quadrature rules, we need to map the quadrature nodes and weights $\{x_i, w_j\}_{j=1}^N$ to our choice of option strikes $\mathcal{K}_j$ and the portfolio weights $\mathcal{W}_j$. One reasonable choice of a mapping function between the strikes and the quadrature nodes is given by

$$\mathcal{K}(x) = Ke^{x\sigma\sqrt{2(T-u)}+(q-r-\sigma^2/2)(T-u)}, \tag{10}$$

where σ^2 denotes the annualized variance of the log asset return. This choice is motivated by the gamma weighting function under the Black–Scholes model, which is given by

$$w(\mathcal{K})=\frac{\partial^2 C(\mathcal{K},u;K,T;\Theta)}{\partial \mathcal{K}^2}=e^{-q(T-u)}\frac{n(d_1)}{\mathcal{K}\sigma\sqrt{T-u}}, \tag{11}$$

where $n(\cdot)$ denotes the probability density of a standard normal and d_1 is defined as

$$d_1 \equiv \frac{\ln(\mathcal{K}/K)+(r-q+\sigma^2/2)(T-u)}{\sigma\sqrt{T-u}}.$$

We can then obtain the mapping in (10) by replacing d_1 with $\sqrt{2}x$.

Given the Gauss–Hermite quadrature $\{w_j, x_j\}_{j=1}^N$, we choose the strike points as

$$\mathcal{K}_j = Ke^{x_j\sigma\sqrt{2(T-u)}+(q-r-\sigma^2/2)(T-u)}, \tag{12}$$

with the portfolio weights given by

$$\mathcal{W}_j=\frac{w(\mathcal{K}_j)\mathcal{K}_j'(x_j)}{e^{-x_j^2}}w_j=\frac{w(\mathcal{K}_j)\mathcal{K}_j\sigma\sqrt{2(T-u)}}{e^{-x_j^2}}w_j. \tag{13}$$

Different practical situations call for different finite approximation methods. The Gauss–Hermite quadrature method chooses both the strike levels and the associated weights. In a market where options are available at many different strikes, such as the S&P 500 index options market at the Chicago Board of Options Exchange (CBOE), this quadrature approach provides guidance in choosing both the appropriate strikes and the appropriate weights to approximate the static hedge. On the other hand, in some over-the-counter options markets where only a few fixed strikes are available, it would be more appropriate to use an approximation method that takes the available strike points as fixed and solves for the corresponding weights. The latter approach has been explored in Balder and Mahayni (2006), Carr and Mayo (2007), and Wu and Zhu (2011).

2 MONTE CARLO ANALYSIS BASED ON POPULAR MODELS

Consider the problem faced by the writer of a call option on a certain stock. For concreteness, suppose that the call option matures in 1 year and is written at-the-money. The writer intends to hold this short position for a month, after which the option position will be closed. During this month, the writer can hedge the risk using various exchange traded liquid assets such as the underlying stock, futures, and/or options on the same stock.

We compare the performance of two types of strategies: (i) a static hedging strategy using 1-month vanilla options, and (ii) a dynamic delta-hedging strategy using the underlying stock futures. The static strategy is based on the spanning relation in Equation (3) and is approximated by a finite number of options, with the portfolio strikes and weights determined by the quadrature method. The dynamic strategy is discretized by rebalancing the futures position daily. The choice of using futures instead of the stock itself for the delta hedge is intended to be consistent with our empirical study in the next section on S&P 500 index options. For these options, direct trading in the 500 stocks comprising the index is impractical. Practically all delta hedging is done in the very liquid index futures market.

We compare the performance of the above two strategies based on Monte Carlo simulation. For the simulation, we consider four data-generating processes: the Black–Scholes model (BS), the Merton (1976) jump-diffusion model (MJ), the Heston (1993) stochastic volatility model (HV), and a special case of this model proposed by Heston and Nandi (2000) (HN). Under the objective measure, $\mathbb{P}$, the stock price dynamics are governed by the following stochastic differential equations,

$$\begin{aligned} \text{BS:}\quad dS_t/S_t &= \mu dt + \sigma dW_t, \\ \text{MJ:}\quad dS_t/S_t &= \left(\mu - \lambda g\right) dt + \sigma dW_t + dJ(\lambda), \\ \text{HV:}\quad dS_t/S_t &= \mu dt + \sqrt{v_t} dW_t, \\ dv_t &= \kappa(\theta - v_t) dt - \sigma_v \sqrt{v_t} dZ_t, \quad \mathbb{E}[dZ_t dW_t] = \rho dt, \\ \text{HN:}\quad \text{HV} &\quad \text{with } \rho = -1. \end{aligned} \tag{14}$$

where W denotes a standard Brownian motion that drives the stock price movement in all models. Under the MJ model, $J(\lambda)$ denotes a compound Poisson jump process with constant intensity λ. Conditional on a jump occurring, the MJ model assumes that the log price relative is normally distributed with mean μ_j and variance σ_j^2, with the mean percentage price change induced by a jump being $g = e^{\mu_j + \frac{1}{2}\sigma_j^2} - 1$. Under the Heston (HV) model, Z_t denotes another standard Brownian motion that governs the randomness of the instantaneous variance rate. The two Brownian motions have an instantaneous correlation of ρ. Heston and Nandi derive a special case of this model with $\rho = -1$ as a continuous time limit of a discrete-time GARCH model. With perfect correlation, the stock price is essentially driven by one source of uncertainty under the HN model.

The four data-generating processes cover four different scenarios. Under the BS model, the stock price process is both purely continuous and Markovian. Hence, both the dynamic hedging strategy and the static strategy work perfectly in the theoretical limit when we ignore transaction costs and allow continuous rebalancing of the futures and trading of a continuum of options. The hedging errors from our simulation exercise come from discrete rebalancing in the dynamic hedging case and from the choice of a discrete number of options in the static hedging portfolio.

Under the MJ model, the static spanning relation in (3) remains valid because the stock price process remains Markovian. Thus, we expect the static hedging errors from the simulation to be of similar magnitude to those in the BS case, when the hedging exercises are performed using comparable number of options in the hedging portfolio. However, the presence of random jumps renders the dynamic hedging strategy ineffective even in the theoretical limit of continuous rebalancing.

Even within infinitesimal intervals, the stock price movement can have random magnitudes due to the random jumps. Thus, two instruments (the underlying stock and riskfree bonds) are not enough to span all the different movements. From our simulation exercise, we gauge the degree to which the dynamic hedging performance deteriorates.

The HN model represents the exact opposite of the MJ case. The stock price process is purely continuous with one source of uncertainty. The dynamic hedging strategy works perfectly in the theoretical limit of continuous rebalancing. Thus, we expect the dynamic hedging error in our simulation exercise to be of similar magnitude to that under the BS model. However, due to the historical dependence of the volatility process, the evolution of the stock price is no longer Markovian in the stock price and calendar time. Therefore, the static spanning relation in (3) no longer holds. In particular, at time t, we do not know the variance rate level at time $u>t, v_u$. Hence, we do not know the gamma of the target call option at time u, which determines the weighting function of the static hedging portfolio. We investigate the degree to which this violation of the Markovian assumption degenerates the static hedging performance.

Finally, neither hedging strategy works perfectly under the Heston model with $|\rho|\neq 1$. The two instruments in the dynamic hedging strategy are not enough to span the two sources of uncertainty under the HV model. The non-Markovian property also invalidates the static spanning relation in (3). The presence of stochastic volatility has been documented extensively. Our simulation exercise gauges the degree of performance deterioration for both hedging strategies.

We specify the data-generating processes in Equation (14) under the objective measure $\mathbb{P}$. To price the relevant options and to compute the weights in the hedge portfolios, we also need to specify their respective risk-neutral $\mathbb{Q}$-dynamics,

$$\begin{aligned}
&\text{BS: } dS_t/S_t = (r-q)dt+\sigma dW_t^*,\\
&\text{MJ: } dS_t/S_t = (r-q-\lambda^* g^*)dt+\sigma dW_t^* + dJ^*(\lambda^*),\\
&\text{HV: } dS_t/S_t = (r-q)dt+\sqrt{v_t}dW_t^*, \quad dv_t=\kappa^*(\theta^*-v_t)dt-\sigma_v\sqrt{v_t}dZ_t^*,
\end{aligned} \tag{15}$$

where W^* and Z^* denote standard Brownian motions under the risk-neutral measure $\mathbb{Q}$, and $(\kappa^*,\theta^*,\lambda^*,\mu_j^*,\sigma_j^*)$ denote the corresponding parameters under this measure. Option prices under the BS model can be readily computed using the Black–Scholes option pricing formula. Under the MJ model, option prices can be computed as a Poisson probability-weighted sum of the Black–Scholes formulae. Under the Heston model and its HN special case, we can price options using Heston's (1993) Fourier transform method, Carr and Madan's (1999) Fast Fourier transform method, or the expansion formulae of Lewis (2000).

For the simulation and option pricing exercise, we benchmark the parameter values of the three models to the S&P 500 index. We set $\mu=0.10, r=0.06$, and $q=0.02$ for all three models. We further set $\sigma=0.27$ for the BS model, $\sigma=0.14$, $\lambda=\lambda^*=2.00$, $\mu_j=\mu_j^*=-0.10$, and $\sigma_j=\sigma_j^*=0.13$ for the MJ model, and $\theta=\theta^*=0.27^2$, $\kappa=\kappa^*=1$, and $\sigma_v=0.1$ for the HV and HN models. We set $\rho=-.5$ for the HV model.

In each simulation, we generate a time series of daily stock prices according to an Euler approximation of the respective data-generating process. The starting value for the stock price is set to \$100. Under the HV/HN model, we set the starting value of the instantaneous variance rate to its long-run mean: $v_0 = \theta$.[1] We consider a hedging horizon of 1 month and simulate paths over this period. We assume that there are 21 business days in a month. To be consistent with the empirical study on S&P 500 index options in the next section, we think of the simulation as starting on a Wednesday and ending on a Thursday 4 weeks later, spanning a total of 21 week days and 29 actual days. The hedging performance is recorded and, when needed, updated only on week days, but the interest earned on the money market account is computed based on actual/360 day-count convention.

At each week day, we compute the relevant option prices based on the realization of the security price and the specification of the risk-neutral dynamics. For the dynamic delta hedge, we also compute the delta each day based on the risk-neutral dynamics and rebalance the portfolio accordingly. For both strategies, we monitor the hedging error (profit and loss) at each week day based on the simulated security price and the option prices. The hedging error at each date t is defined as the difference between the value of the hedge portfolio and the value of the target call option being hedged,

$$e_t^D = B_{t-h}e^{rh} + \Delta_{t-h}\left(F_t - F_{t-h}\right) - C(S_t, t; K, T);$$

$$e_t^S = \sum_{j=1}^{N} \mathcal{W}_j C(S_t, t; \mathcal{K}_j, u) + B_0 e^{rt} - C(S_t, t; K, T), \tag{16}$$

where the superscripts D and S denote the dynamic and static strategies, respectively, Δ_t denotes the delta of the target call option with respect to the futures price at time t, h denotes the time interval between stock trades, and B_t denotes the time-t balance in the money market account. The balance includes the receipts from selling the 1-year call option, less the cost of initiating and possibly changing the hedge portfolio.

For the delta-hedging strategy, the hedge portfolio is self-financing and hence the error e^D would be zero if (i) the underlying dynamics follow the BS dynamics or some other known one-factor diffusion process and (ii) the portfolio is updated continuously without incurring any transaction cost. In practice, hedging errors can come from (a) discreteness in the portfolio rebalancing frequency and (b) deviation of the underlying dynamics from a known one-factor diffusion process. The simulation exercise reveals the behavior of the hedging errors from these sources.

For the static hedging strategy, under no arbitrage, the value of the portfolio of the continuum of shorter-term options is equal to the value of the long-term target option. As a result, B_0 is zero and there will be no hedge error at any time

[1] We have also experimented with different starting values for the variance rate. The hedging results are very similar and hence not reported.

($e_t^S=0$). However, since we use a finite number of call options in the static hedge to approximate the continuum, the initial money market account B_0 captures the value difference due to the approximation error, which is normally very small. No rebalancing is needed in the static strategy. Over time, hedging error can occur when the value of target option deviates from the discrete hedge portfolio. The simulation exercise reveals the behavior of this discretization–approximation error.

Under each model, the delta is given by the partial derivative $\partial C(S,t;K,T;\Theta)/\partial F$, with $F=Se^{(r-q)(T-t)}$ denoting the forward/futures price. If an investor could trade continuously, this delta hedge removes all of the risk in the BS model and the HN model. The hedge does not remove all risks in the MJ model because of the random jumps, nor in the HV model because of a second source of diffusion risk. The hedge portfolio for the static strategy is formed based on the weighting function $w(\mathcal{K})$ in Equation (4) implied by each model, the Gauss–Hermite quadrature nodes and weights $\{x_i, w_i\}$, and the mapping from the quadrature nodes and weights to the option strikes and weights, as given in Equations (12) and (13).

Under the HV/HN model, since the stock price is non-Markovian, the static spanning relation in (3) is no longer valid. Furthermore, when we use the spanning relation to form an approximate hedging portfolio, the weighting function in (4) is no longer known at time t because option price at time $u>t$ is also a function of the instantaneous variance rate at time u, which is not known at time t. To implement the static strategy under these two models, we replace $C(K,u;K,T;\Theta)$ in Equation (4) by its conditional expected value at time t under the risk-neutral measure $\mathbb{Q}$,

$$C(\mathcal{K},u;K,T;\Theta)\equiv\mathbb{E}_t[C(S_u,v_u;K,T)|S_u=\mathcal{K}]. \quad (17)$$

In computing the strike points for the quadrature approximation of the spanning relation, the annualized variance is $v=\sigma^2$ for the BS model, $v=\theta$ for the HV/HN model, and $v=\sigma^2+\lambda\left(\mu_j^2+\sigma_j^2\right)$ for the MJ model. Given the chosen model parameters, we have $\sqrt{v}\doteq 27\%$ for all models.

2.1 Hedging Comparison under the Diffusive Black–Scholes World

Table 1 reports the summary statistics of the simulated hedging errors, from 1000 simulated sample paths. Panel A in Table 1 summarizes the results based on the BS model. Entries are the summary statistics of the hedging errors at the last step (at the end of the 21 business days) based on both strategies. For the dynamic strategy (the last column), we perform daily updating. For the static strategy, we consider hedge portfolios with $N=3,5,9,15,21$ 1-month options.

If the transaction cost for a single option is comparable to the transaction cost for revising a position in the underlying security, we would expect that the transaction cost induced by buying 21 options at one time is close to the cost of rebalancing a

Table 1 Simulated hedge performance comparisons of static and dynamic strategies

Hedge error	Static with options					Dynamic with underlying
No. of assets	3	5	9	15	21	
Panel A. The Black–Scholes model						
Mean	−0.00	0.01	0.02	0.02	0.01	0.10
Std Deviation	1.00	0.66	0.36	0.20	0.14	0.10
RMSE	1.00	0.66	0.36	0.20	0.14	0.14
Minimum	−1.62	−1.13	−0.67	−0.38	−0.25	−0.43
Maximum	1.86	0.93	0.43	0.24	0.17	0.32
Skewness	0.01	−0.26	−0.59	−0.61	−0.48	−0.84
Kurtosis	1.87	1.79	2.01	1.96	1.78	4.68
Call value	11.72	12.20	12.36	12.37	12.36	12.35
Panel B. The Merton jump-diffusion model						
Mean	−0.01	0.00	0.02	0.02	0.02	0.07
Std Deviation	0.72	0.47	0.28	0.16	0.12	1.05
RMSE	0.72	0.47	0.28	0.16	0.12	1.05
Minimum	−1.73	−1.28	−0.90	−0.58	−0.41	−12.12
Maximum	2.84	1.48	0.48	0.20	0.14	0.37
Skewness	0.56	−0.16	−1.26	−1.77	−1.65	−6.82
Kurtosis	5.23	4.07	4.30	5.74	5.15	59.79
Call value	9.52	11.14	12.02	12.09	12.06	11.99
Panel C. The HN non-Markvian diffusion model						
Mean	−0.02	−0.00	0.01	0.01	0.00	0.09
Std Deviation	0.79	0.52	0.29	0.19	0.15	0.15
RMSE	0.79	0.52	0.29	0.19	0.15	0.18
Minimum	−1.38	−0.97	−0.80	−0.49	−0.32	−0.50
Maximum	1.21	0.65	0.35	0.23	0.18	0.38
Skewness	−0.17	−0.44	−0.68	−0.71	−0.51	−0.83
Kurtosis	1.81	1.87	2.12	2.24	1.92	3.70
Call value	11.39	11.94	12.17	12.23	12.24	12.33
Panel D. The Heston stochastic volatility model						
Mean	−0.03	−0.01	−0.00	−0.01	−0.01	0.07
Std Deviation	0.84	0.57	0.38	0.31	0.29	0.27
RMSE	0.84	0.57	0.38	0.31	0.29	0.28
Minimum	−2.13	−1.71	−1.32	−1.07	−0.97	−0.81
Maximum	1.74	1.30	0.96	0.91	0.86	0.94
Skewness	−0.14	−0.33	−0.28	−0.16	−0.09	−0.07
Kurtosis	1.94	2.18	2.69	2.84	2.87	3.07
Call value	11.31	11.87	12.10	12.15	12.16	12.33

Entries report the summary statistics from 1000 simulated paths on the hedging errors of a 1-year call option. The hedging error is defined as the difference between the value of the hedge portfolio and the value of the target call at the closing of the month-long exercise. The hedging portfolios are formed assuming that the hedger knows the exact model. The last row of each panel reports the value of the target call approximated by the quadrature method, with the theoretical value given under the dynamic hedging column.

position in the underlying stock 21 times. Hence, it is interesting to compare the performance of daily delta hedging with the performance of the static hedge with 21 options. The results in Panel A of Table 1 show that the daily updating strategy and the static strategy with 21 options have comparable hedging performance in terms of the root mean squared error (RMSE). Since the stock market is much more liquid than the stock options market, the simulation results favor the dynamic delta strategy over the static strategy, if indeed stock prices move as in the BS world.

The hedging errors from the two strategies show different distributional properties. The kurtosis of the hedging errors from the dynamic strategy is larger than that from all the static strategies. The kurtosis is 4.68 for the dynamic hedging errors, but is below two for errors from all the static hedges. Even for the static hedge with three strikes, the maximum absolute error is less than twice as big as the RMSE whereas the maximum absolute error from the delta hedge is more than three times larger than the corresponding RMSE, thus leading to the larger kurtosis for the delta-hedging error. The maximum profit and loss from the static strategy with 21 options are also smaller in absolute magnitudes. Therefore, when an investor is particularly concerned about avoiding large losses, the investor may prefer the static strategy.

The last row shows the accuracy of the Gauss–Hermite quadrature approximation of the integral in pricing the target options. Under the BS model, the theoretical value of the target call option is \$12.35, which we put under the dynamic hedging column. The approximation error is about one cent when applying a 21-node quadrature. The approximation error increases as the number of quadrature nodes declines in the approximation.

2.2 Hedging Comparison in the Presence of Random Jumps

Panel B of Table 1 shows the hedging performance under the Merton (1976) jump-diffusion model. For ease of comparison, we present the results in the same format as in Panel A for the BS model. The performance of all the static strategies are comparable to their corresponding cases under the BS world. If anything, most of the performance measures for the static strategies become slightly better under the Merton jump-diffusion case. By contrast, the performance of the dynamic strategy deteriorates dramatically as we move from the diffusion-based BS model to the jump-diffusion Merton model. The RMSE is increased by a factor of seven for the dynamic strategy. As a result, the performance of the dynamic strategy is worse than the static strategy with only three options.

The distributional differences between the hedging errors of the two strategies become even more pronounced under the Merton model. The kurtosis of the static hedge errors remains small (below six), but the kurtosis of the dynamic hedge errors explodes from 4.68 in the BS model to 59.79 in the MJ model. The maximum loss from the dynamically hedged portfolio is \$12.12, even larger than the initial revenue from writing the call option ($11.99). By contrast, the maximum loss is less than two dollars from the static hedge with merely three options.

2.3 Hedging Comparison under the Non-Markovian Diffusive HN/HV Models

Panel C of Table 1 shows the hedging performance under the non-Markovian but purely diffusive HN model. In theory, the dynamic delta strategy works perfectly under this model, the same as under the BS model. The last column in Panel C shows that the root mean squared hedging error from the daily delta hedging under the HN model is only slightly larger than that under the BS model in Panel A, consistent with the theory prediction.

The static spanning relation is no longer valid under the HN model given the non-Markovian property. Nevertheless, since the instantaneous variance does not have any independent movements, over short horizons the deviation from the Markovian assumption is small. As a result, the hedging performances of the static strategies under the HN model are comparable to those under the BS model.

Panel D of Table 1 shows the hedging performance under the Heston stochastic volatility model, where neither strategy works in theory. We observe performance deteriorations across all strategies. For the dynamic delta-hedging strategy, the RMSE increases from 0.18 under the HN model to 0.28 under the Heston model. For the static strategies, the performance deterioration becomes more pronounced when more options are used to approximate the continuum. The RMSE difference is 0.14 when 21 options are used for the hedge and it reduces to 0.05 when three to five options are used. With only three to five options, the discretization error becomes the dominating source of the hedging error.

Figure 1 plots the simulated sample paths and the corresponding hedging errors under the four data-generating processes, from top to bottom, BS, MJ, HN, and HV. The four panels in the first (left) column plot the simulated sample paths of the underlying security price under the four models. The daily movements under the BS, HN, and HV models are usually small, but the MJ model (second row) generates both small and large movements.

Panels in the second (middle) column in Figure 1 compare the sample paths of the hedging errors from the static hedging strategy using nine options. We apply the same scale for ease of comparison. Consistent with theory, the Heston stochastic volatility model generates moderately larger hedging errors due to its non-Markovian nature.

Panels in the third (right) column show the sample paths of the dynamic hedging errors under the four models. We use the same scale for the three pure diffusion models (BS, HN, and HV). The dynamic hedging errors from the BS and HN models are similar. The hedging errors from the HV model are moderately larger due to the presence of a second source of randomness. By contrast, under

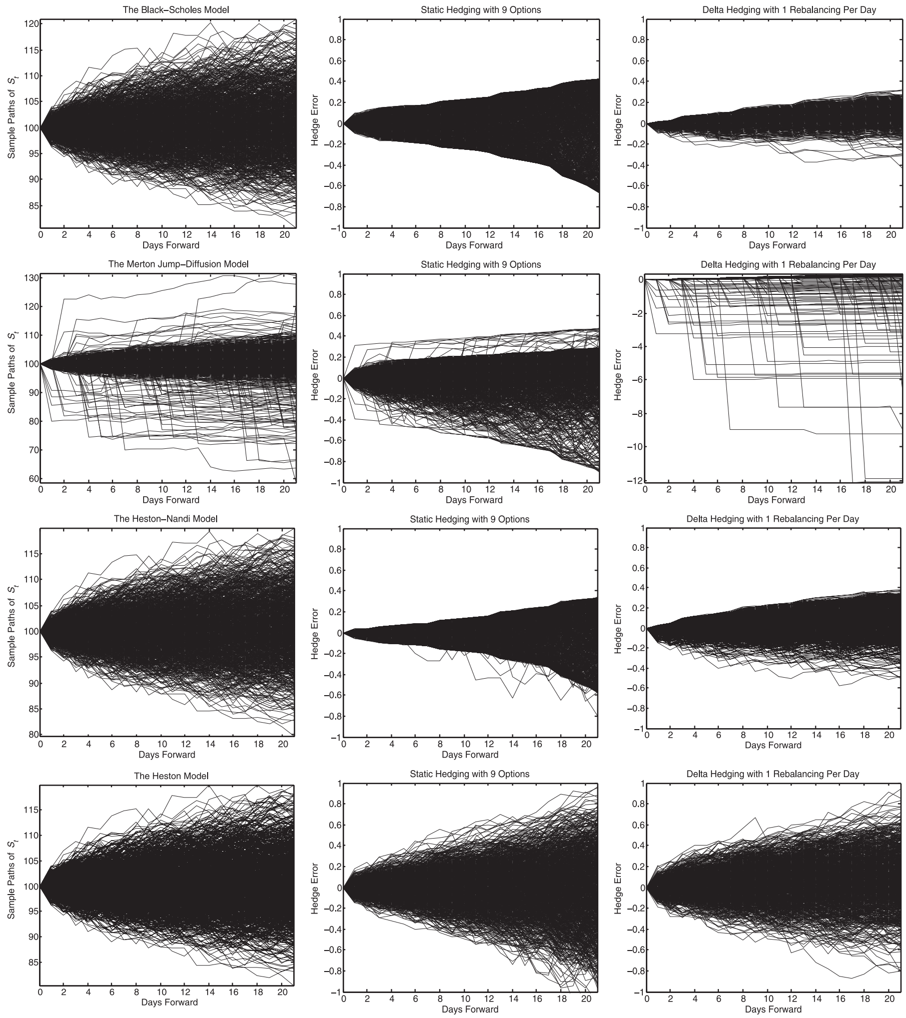

Figure 1 Hedging performance under different price dynamics. The four rows represents the four data-generating processes: BS, MJ, HN, and HV. Panels in the first column depict the simulated sample paths of the underlying security price. Panels in the second column depict the sample paths of the hedging errors from the static hedging strategy with nine option contracts. Panels in the third column depict the corresponding sample paths of the hedging errors from the dynamic delta strategy with the underlying futures and daily updating.

the MJ jump-diffusion model (second row), the dynamic hedging errors become so much larger that we have to adopt a much larger scale in plotting the error paths. The large hedging errors from the dynamic strategy correspond to the large moves in the underlying security price.

Another interesting feature is that, under the MJ model, most of the large dynamic hedging errors are negative, irrespective of the direction of the large moves in the underlying security price. The reason is that the option price function exhibits positive convexity with the underlying futures price. Under a large movement, the value of the delta-neutral portfolio is always below the value of the option contract. Therefore, most of the large hedging errors for selling an option contract are losses.

Overall, the daily delta-hedging strategy performs reasonably well under one-factor diffusion models such as the BS model and the HN model. The performance deteriorates moderately in the presence of a second source of diffusion uncertainty in return volatility. However, the strategy fails miserably when the underlying price can jump randomly. By contrast, the performance of the static hedging strategy with a few shorter-term options is much less sensitive to the nature of the underlying price processes. The static strategy takes random jumps in stride and experiences only small performance deterioration when the Markovian assumption is violated.

2.4 Effects of Model Uncertainty and Misspecification

We perform the above simulation under the assumption that the hedger knows exactly the underlying data-generating process and the model under which the options are priced. In practice, however, we can only use different models to fit the market option prices approximately. Model uncertainty is an inherent part of both pricing and hedging. To investigate the sensitivity of the hedging performance to model misspecification, we compare the performance of the hedging strategies when the hedger does not know the data-generating process and must develop a hedging approach in the absence of this information. We assume that the actual underlying asset prices and the option prices are generated from the MJ, HN, and HV models, but the hedger forms the hedge portfolios using the Black–Scholes model, using the observed option implied volatility as the model input. Specifically, for the dynamic strategy, the hedger computes the daily delta based on the Black–Scholes formula using the observed implied volatility of the target call option as the volatility input. For the static strategy, the hedger computes the weighting function $w(\mathcal{K})$ based on the Black–Scholes model, also using the observed implied volatility of the target option as the volatility input.

We summarize the hedging performance in Table 2. In all cases, we find that the impact of model misspecification is small. As in the case when the data-generating processes are known, the performance of the dynamic strategy deteriorates dramatically in the presence of random jumps, but violating the non-Markovian assumption only deteriorates the performance of the static strategy moderately. These remarkable results show that, in hedging, being able to span the right space is much more important than specifying the right parametric model. Even if an investor has perfect knowledge of the stochastic process governing the underlying asset price, and hence can compute the perfectly correct delta, a dynamic strategy in the underlying asset still fails miserably when the underlying asset price can jump by a random amount. By contrast, as long as the investor uses a few short-term call options of different strikes in the hedge portfolio, the hedging error is about the same regardless of whether jumps can occur or not. This result holds even if the investor does not know which model to use to pick the appropriate strikes and portfolio weights.

In practice, the hedger never really knows the true underlying process and thus must resort to some assumptions in coming up with the dynamic hedging ratios and the static portfolio weights. At one extreme, one specify a model (such as MJ, HN, or HV as in our simulation), estimate the model parameters based on market data, and derive the hedging ratios and portfolio weights based on the estimated dynamics. The success of this approach depends crucially on (i) the validity of model assumptions and (ii) the quality and quantity of the available market data, which determine the quality of the parameter estimates.

At the other extreme, one can use the Black–Scholes formula, with one volatility input, to derive the hedging ratios and portfolio weights. We take this simple approach and use the observed implied volatility of the target option as the volatility input for the Black–Scholes formula. Under this approach, the hedging ratio and portfolio weights calculation only relies on the availability of a known value for the target option. Its demand for data availability is at the minimum, and the hedging ratios are readily comparable across different counterparties, as long as they agree on the value of the target option. This approach is widely adopted in the industry and it is the standard approach in the over-the-counter currency options markets when the counterparties not only exchange the options but also the associated delta hedge (Carr and Wu, 2007). Under a pure diffusion stochastic volatility environment, Renault and Touzi (1996) show that the Black–Scholes delta with the implied volatility as the input can either under- or over-hedge options at different strikes and they propose a method to filter out the stochastic volatility from the option observations. However, several empirical studies, e.g., Engle and Rosenberg (2002), Jackwerth and Rubinstein (1996), and Bollen and Raisel (2003), have generally found that under practical situations when the true underlying price dynamics are unknown to the hedger, the approach of using the Black–Scholes delta with the running implied volatility works as well or better than the alternative approach of estimating a sophisticated model and delta-hedging with it.

In between the two extremes, some practitioners also try to adjust the Black–Scholes delta to accommodate the co-movements between the implied volatility and the underlying security price. The adjusted delta is often referred to as the "total delta (TD),"

$$TD = \frac{\partial B(S_t, IV_t; K, T)}{\partial S_t} + \frac{\partial B(S_t, IV_t; K, T)}{\partial IV_t} \mathbb{E}_t \left[\frac{\partial IV_t}{\partial S_t} \right], \tag{18}$$

where $B(S_t, IV_t; K, T)$ denotes the Black–Scholes value of a call option at strike T and expiry T with the current security price at S_t and the observed implied volatility

Table 2 Effect of model uncertainty on hedge performance comparisons

Hedge error	Static with options					Dynamic with underlying
No. of assets	3	5	9	15	21	
Panel A. Black–Scholes hedge under the Merton environment						
Mean	−0.00	0.01	0.01	0.02	0.01	0.06
Std Deviation	0.61	0.38	0.20	0.18	0.21	0.92
RMSE	0.61	0.38	0.21	0.18	0.21	0.92
Minimum	−1.60	−1.34	−1.28	−0.95	−0.88	−10.90
Maximum	1.57	1.06	0.76	0.66	0.63	0.68
Skewness	−0.31	−0.52	−0.39	−0.74	−0.82	−7.04
Kurtosis	3.10	4.00	8.75	9.78	5.79	64.67
Call value	11.66	12.11	12.24	12.22	12.18	11.99
Panel B. Black–Scholes hedge under the HN environment						
Mean	−0.02	−0.00	0.01	0.01	0.00	0.09
Std Deviation	1.08	0.74	0.44	0.26	0.18	0.15
RMSE	1.08	0.74	0.44	0.26	0.18	0.17
Minimum	−1.77	−1.28	−0.82	−0.52	−0.39	−0.42
Maximum	2.08	1.09	0.53	0.31	0.36	0.42
Skewness	0.02	−0.21	−0.52	−0.67	−0.70	−0.50
Kurtosis	1.87	1.78	1.91	2.13	2.26	3.06
Call value	11.73	12.20	12.36	12.35	12.34	12.33
Panel C. Black–Scholes hedge under the HV environment						
Mean	−0.03	−0.01	0.00	−0.00	−0.01	0.08
Std Deviation	1.08	0.75	0.47	0.34	0.29	0.27
RMSE	1.08	0.75	0.47	0.34	0.29	0.28
Minimum	−2.46	−1.99	−1.54	−1.25	−1.12	−0.90
Maximum	2.31	1.52	1.11	0.90	0.84	0.89
Skewness	0.00	−0.20	−0.32	−0.13	−0.02	−0.10
Kurtosis	1.91	1.96	2.40	2.83	2.96	3.17
Call value	11.71	12.19	12.35	12.35	12.34	12.33

Entries report the summary statistics of the hedging errors of a 1-year call option based on both static and dynamic strategies. The hedging error is defined as the difference between the value of the hedge portfolio and the value of the target call option being hedged at the closing of the month-long hedging exercise. The statistics are computed based on 1000 simulated paths of the MJ model (Panel A), the HN model (Panel B) and the HV model (Panel C). The hedging portfolios are formed assuming that the hedger does not know the true data-generating process and form the hedge portfolios based on the Black–Scholes formula with the observed implied volatility of the target option as the volatility input. The last row of each panel reports the value of the target call option approximated by the quadrature method, with the theoretical value given under the dynamic hedging column.

for this option at IV_t. The first term denotes the Black–Scholes delta evaluated at its implied volatility level. The second term captures the contribution of expected co-movements between the implied volatility of this option and the security price. One often tries to infer the expected co-movement $\mathbb{E}_t\left[\frac{\partial IV_t}{\partial S_t}\right]$ from the observed implied volatility smile as a function of the option strike. Unfortunately, different types of models can generate the same shape for the implied volatility smile but different implied volatility-price co-movements (Schoutens, Simons, and Tistaert, 2004). As a result, one cannot robustly infer the co-movement from the smile without knowing the type of the underlying process. Take a negatively sloped implied volatility smile as an example. If the underlying process is pure diffusion with local or stochastic volatility as in Dupire (1994) or Heston (1993), the negative skew would indicate that the expected implied volatility-price co-movement has a negative sign and hence the second term would lower the total delta. On the other hand, if the underlying process is a Lévy jump-diffusion process with constant volatility as in Merton (1976), the negative skew would indicate the presence of a negative jump on average and the expected co-movement between the implied volatility and the security price would be positive: The implied volatility smile as a function of relative strike over spot does not vary over time under the Merton model. As the spot price moves down, the relative strike (the moneyness) of the option contract increases and the implied volatility declines. In this case, the second term in (18) would raise the total delta. What this example tells us is that without knowing the type of the underlying process, it remains difficult to adjust the Black–Scholes delta based on the shape of the implied volatility smile. The same argument applies to the gamma weight calculation for our static portfolio.

Our choice of using the Black–Scholes model with the observed implied volatility of the target option as input remains the simplest and the most stable solution when one does not have any knowledge of the type of the underlying security price process. The advantage of such a simple solution becomes even more obvious when the data quality is bad and interpolating/extrapolating the implied volatility surface becomes unstable. Furthermore, our simulation analysis shows that even with this simple choice, the hedging performance deteriorates little from knowing the true model. Although one can experiment with many different methods in approximating the true hedging ratios, our analysis suggests that the room for improvement from these experiments is small. The much larger improvement comes from the switch from the delta hedge to our proposed static hedge.

2.5 Effects of Rebalancing Frequency in Delta Hedging

In the above simulations, we approximate the sample paths of the underlying stock price process using an Euler approximation with daily time steps and consider dynamic delta strategies with daily updating. We are interested in knowing how much of the failure of the delta-hedging strategy under the Merton jump-diffusion model is due to this somewhat arbitrary choice of discretization step.

Under the Black–Scholes environment, the dependence of the delta-hedging error on the discretization step has been studied extensively in, for example, Black and Scholes (1972), Boyle and Emanuel (1980), Bhattacharya (1980), Figlewski (1989), Galai (1983), Leland (1985), and Toft (1996). Several of these authors show that, under the Black–Scholes environment, the standard deviation of the hedging error arising from discrete rebalancing over a time step of length h declines to zero slowly like $O(\sqrt{h})$. Thus, doubling the trading frequency reduces the standard deviation by about 30%. By contrast, the discretization error in the Gaussian quadrature method is $(N!\sqrt{\pi})/(2^N)(f^{(2N)}(\xi))/((2N)!)$. This error drops by much more when the number of strikes N is doubled. Indeed, our simulations indicate that the standard deviation of the hedging error drops rapidly as the number of strikes increases.

This subsection focuses on relating the delta-hedging error to the rebalancing frequency under the Merton-jump diffusion model. We also simulate the Black–Scholes model as a benchmark reference. Table 3 shows the impacts of the rebalancing frequency on the hedging performance under three different cases: (A) the Black–Scholes model, (B) the Merton jump-diffusion model, assuming that the hedger knows the underlying data-generating process, and (C) Black–Scholes delta-hedging under the Merton world, assuming that the hedger does not have knowledge of the data-generating process. We consider rebalancing frequencies from once per day, to twice, five times, and ten times per day. To ease comparison, we perform all the hedging exercises on the same simulated sample paths. To accommodate the more frequent rebalancing, we now simulate the sample paths based on the Euler approximation with a time interval of one-tenth of a business day. The slight differences between the dynamic hedging with daily updating in this table and in Table 1 reflects this difference in the simulation of the sample paths.

Our simulation of the Black–Scholes model is consistent with the results in previous studies. As the updating frequency increases from once to two, five, and ten times per day, the standard error of the hedging error reduces from 0.10 to 0.07, 0.04 and to 0.03, adhering fairly closely to the $\sqrt{h}$ rule.

However, this speed of improvement in hedging performance is no longer valid when the underlying data-generating process follows the Merton jump-diffusion model, irrespective of whether the hedger knows the model or not. In the case when the process is known (Panel B), the standard error of the hedging errors remains around 1.02−1.03 as we increase the rebalancing frequency. When the process is not known to the hedger (Panel C), the standard error hovers around 0.88−0.93 and exhibits no obvious dependence on the rebalancing frequency. Therefore, we conclude that the failure of the delta-hedging strategy under the Merton model is neither due to model misspecification, nor due to infrequent updating, but due to its inherent incapability in spanning risks associated with jumps of random size.

The Achilles heel of delta hedging in jump models is not the large size of the movement *per se*, but rather the randomness of the jump size. For example, Cox and Ross (1976) and Dritschel and Protter (1999) show that dynamic delta hedging can span all risks arising in their pure jump models. Under these jump models, the jump size is known just prior to any jump. This is analogous to the binomial model where only two subsequent asset prices are possible. Under both cases, delta hedging can remove all risks. Therefore, it is the a priori randomness in the jump size that creates the difficulty in dynamic delta hedging.

Table 3 Effect of rebalancing frequencies on dynamic delta hedge

Statistics	Number of rebalancing per day			
	1	2	5	10
Panel A. The Black–Scholes model				
Mean	0.11	0.11	0.11	0.11
Std Deviation	0.10	0.07	0.04	0.03
RMSE	0.15	0.13	0.12	0.12
Minimum	−0.36	−0.15	−0.03	0.02
Maximum	0.32	0.28	0.22	0.19
Skewness	−0.77	−0.41	−0.34	−0.16
Kurtosis	4.21	3.23	3.12	2.86
Panel B. The Merton jump-diffusion model				
Mean	0.09	0.09	0.09	0.09
Std Deviation	1.02	1.03	1.03	1.02
RMSE	1.02	1.03	1.03	1.03
Minimum	−11.78	−11.84	−11.72	−11.72
Maximum	0.38	0.37	0.35	0.35
Skewness	−6.27	−6.30	−6.24	−6.20
Kurtosis	50.97	51.40	50.34	49.79
Panel C. Black–Scholes hedge under the Merton environment				
Mean	0.08	0.09	0.09	0.07
Std Deviation	0.88	0.92	0.93	0.88
RMSE	0.88	0.93	0.93	0.88
Minimum	−10.13	−10.19	−10.08	−10.07
Maximum	1.21	8.97	9.43	0.76
Skewness	−6.31	−4.74	−4.43	−6.25
Kurtosis	52.66	54.10	53.41	51.59

Entries report the summary statistics of the hedging error of a 1-year call option based on a dynamic delta hedge with different rebalancing frequencies. The hedging error is defined as the difference between the value of the hedge portfolio and the value of the target call option at the closing time of the month-long exercise. The statistics are computed based on 1000 simulated paths of the Black–Scholes model (Panel A) and the Merton jump-diffusion model (Panel B) assuming that the hedger knows the exact model in forming the portfolios. In Panel C, the sample paths and option prices are simulated based on the Merton model, but we assume that the hedger does not know this information and form the hedge portfolios based on the Black–Scholes formula with the implied volatility of the target option as the volatility input.

2.6 Effects of Target and Hedging Instrument Choice

So far, the simulation exercise focuses on hedging a 1-year at-the-money call option with 1-month options in the static portfolio. This subsection examines the hedging performance when the target options are at different maturities and moneyness and when the static hedge portfolios are formed with options from different maturities. In theory, if we use a continuum of options at a certain maturity, the spanning is perfect in Markovian environments regardless of the exact maturity choice for the hedge portfolio. In practice, when we use the quadrature rule to discretize the integral in Equation (9), the discretization error depends on the higher-order derivatives of the integrant function. Choosing different target and hedging options lead to different integrant functions and hence different magnitude of approximation errors. Furthermore, the violation of the Markovian assumption under the HN and HV models may have different impacts for different target and hedging instruments. Through the simulation exercise, this subsection analyzes how the hedging errors introduced by the quadrature approximation and by the violation of the Markovian assumption vary over different choices of target and hedging options. Along the same lines, we also analyze how the dynamic delta-hedging error varies with the choice of the target option.

Table 4 summarizes the results of hedging at-the-money options at different maturities. To save space, we only report static hedges with three and five options and compare their performance with that of delta hedging with daily updating. All hedging errors are over a 1-month horizon. For the dynamic delta-hedging strategies, we consider target option maturities of 2, 4, and 12 months. For the static strategies, we consider five target-hedge option maturity combinations. The first three combinations hold the hedging option maturity fixed at 1-month while increasing the target option maturity from 2, to 4, and then to 12 months. The last three combinations have the same target option maturity at 12 months while having the hedge option maturity increasing from 1, to 2, and then to 4 months.

For the three dynamic strategies, the hedging errors are larger for hedging shorter-term options than for hedging longer-term options under all simulated environments. This deteriorating performance with declining maturity is probably linked to the gamma of the target option. The shorter the maturity, the larger is the gamma of the target option. Since the delta strategy represents a linear approximation, the hedging error increases with increasing gamma, especially in the presence of large moves.

For the static strategies, as we fix the hedging options maturity at 1 month and vary the maturity of the target option from 2, to 4 and 12 months, the hedging errors increases under the three pure diffusion models BS, HN, and HV, but they do not vary as much with the target option maturity under the MJ jump-diffusion model. As the target option maturity and hence the maturity gap between the target and the hedge options increase, the portfolio puts more weights on far out-of-the-money options. These options may not provide much hedge unless there are large price movements. The MJ model has more of such large movements and thus obtains better performance or less deterioration from using far out-of-the-money options in the hedge portfolio.

Our static spanning relation allows the use of different maturities in forming the static hedge portfolio. Thus, holding the same 1-year option as the target option, Table 4 also compares the hedging performances of static portfolios formed with options at different maturities. Under all model environments, the hedging performance improves quite significantly when the maturity of the hedging options increases. Under the Black–Scholes environment, the root mean squared hedging

Table 4 Effect of target and hedging instrument choices

Target mat	2	4	12	12	12	2	4	12	12	12	2	4	12
Strategy	Static with three options					Static with five options					Daily delta		
instruments	1	1	1	2	4	1	1	1	2	4	underlying futures		
Panel A. The Black–Scholes model													
Mean	−0.02	−0.07	−0.00	−0.02	−0.01	−0.01	−0.03	0.01	−0.00	−0.00	0.30	0.21	0.11
Std Deviation	0.28	0.56	1.00	0.50	0.15	0.14	0.33	0.66	0.25	0.04	0.26	0.18	0.10
RMSE	0.28	0.57	1.00	0.50	0.15	0.14	0.33	0.66	0.25	0.04	0.40	0.27	0.15
Minimum	−0.50	−0.90	−1.62	−0.59	−0.16	−0.26	−0.56	−1.13	−0.31	−0.04	−1.11	−0.77	−0.42
Maximum	0.32	0.79	1.86	1.22	0.43	0.18	0.41	0.93	0.44	0.07	0.97	0.63	0.33
Skewness	−0.42	−0.01	0.01	0.70	0.93	−0.31	−0.24	−0.26	0.36	0.54	−0.62	−0.76	−0.83
Kurtosis	1.73	1.60	1.87	2.37	2.90	1.81	1.61	1.79	1.75	1.99	4.14	4.45	4.64
Target call	4.68	6.60	11.72	11.94	12.20	4.70	6.77	12.20	12.28	12.34	4.71	6.81	12.35
Panel B. The Merton jump-diffusion model													
Mean	−0.10	−0.09	−0.01	−0.02	−0.02	−0.06	−0.05	0.00	−0.00	−0.01	0.23	0.16	0.09
Std Deviation	0.84	0.77	0.72	0.50	0.27	0.40	0.46	0.47	0.29	0.16	2.55	1.92	1.05
RMSE	0.85	0.77	0.72	0.50	0.27	0.40	0.46	0.47	0.29	0.16	2.56	1.93	1.05
Minimum	−1.20	−1.20	−1.73	−0.81	−0.33	−0.69	−0.74	−1.28	−0.72	−0.64	−25.22	−19.83	−12.11
Maximum	6.08	2.76	2.84	2.49	1.95	1.87	1.26	1.48	1.25	0.98	1.14	0.76	0.38
Skewness	1.29	0.51	0.56	1.76	4.76	0.11	0.66	−0.16	1.13	4.44	−5.51	−5.85	−6.81
Kurtosis	7.89	2.65	5.23	9.43	29.43	2.67	3.38	4.07	7.10	26.23	39.24	43.87	59.66
Target call	3.76	4.80	9.52	9.76	10.15	3.72	5.33	11.14	11.19	11.18	4.11	6.34	11.99

Panel C. The HN non-Markovian diffusion model													
Mean	−0.01	−0.06	−0.02	−0.02	−0.01	−0.01	−0.02	−0.00	−0.01	−0.01	0.30	0.20	0.11
Std Deviation	0.24	0.56	0.79	0.39	0.05	0.16	0.44	0.52	0.18	0.02	0.29	0.24	0.15
RMSE	0.24	0.57	0.79	0.39	0.05	0.16	0.44	0.52	0.18	0.02	0.41	0.31	0.19
Minimum	−0.41	−0.89	−1.38	−0.49	−0.10	−0.36	−0.78	−0.97	−0.26	−0.04	−0.95	−0.63	−0.49
Maximum	0.33	0.83	1.21	0.82	0.21	0.24	0.68	0.65	0.30	0.06	1.02	0.71	0.40
Skewness	−0.21	0.05	−0.17	0.59	0.54	−0.30	0.24	−0.44	0.20	0.47	−0.58	−0.63	−0.82
Kurtosis	1.70	1.66	1.81	2.12	2.25	2.25	1.63	1.87	1.61	2.31	3.39	3.15	3.69
Target call	4.84	6.84	11.39	11.94	12.12	4.51	7.43	11.94	12.29	12.43	4.70	6.81	12.33
Panel D. The Heston stochastic volatility model													
Mean	−0.02	−0.07	−0.03	−0.03	−0.01	−0.02	−0.03	−0.01	−0.02	−0.01	0.29	0.19	0.09
Std Deviation	0.27	0.61	0.84	0.46	0.18	0.17	0.53	0.57	0.29	0.21	0.29	0.26	0.27
RMSE	0.28	0.61	0.84	0.46	0.18	0.17	0.53	0.57	0.29	0.21	0.41	0.32	0.28
MAE	0.23	0.53	0.72	0.39	0.15	0.14	0.47	0.48	0.24	0.17	0.35	0.27	0.22
MSF	−0.13	−0.30	−0.38	−0.21	−0.08	−0.08	−0.25	−0.25	−0.13	−0.09	−0.03	−0.04	−0.07
Minimum	−0.68	−1.31	−2.15	−1.10	−0.58	−0.47	−1.06	−1.74	−0.91	−0.65	−0.91	−0.81	−0.81
Maximum	0.55	1.15	1.69	1.30	0.60	0.45	1.17	1.26	0.89	0.65	1.15	1.01	0.96
Skewness	−0.25	−0.02	−0.15	0.48	0.29	−0.07	0.22	−0.33	0.17	0.19	−0.48	−0.29	−0.07
Kurtosis	2.05	1.78	1.94	2.38	3.03	2.23	1.79	2.20	2.63	2.84	3.50	3.25	3.14
Target call	4.89	6.78	11.31	11.88	12.26	4.57	7.58	11.87	12.24	12.82	4.70	6.80	12.33

Entries report the summary statistics of the hedging errors when hedging different target options and using different hedging instruments. The first row denotes the maturity of the target option being hedged. The strikes of the target option are set to be equal to the spot level. The second row denotes the strategy, and the third row denotes the maturity of the options in the case of the static hedging strategy. The last row of each panel reports the value of the target call option approximated by the quadrature method, with the theoretical value given under the dynamic hedging column. The statistics are computed based on 1000 simulated paths of the corresponding model.

error is 0.66 when five 1-month options are used to form the static hedge. This performance is much worse than daily delta hedging, which generates a root mean squared hedging error of 0.10. However, as the hedging option maturity increases from 1 month to 2 months and then to 4 months, the performance of the static hedge improves quite dramatically, with the root mean squared hedging error of the five-option portfolio declining from 0.66 to 0.25 and then to 0.04. The performance improvement is just as pronounced under other model environments. Under all four models, the static hedging errors using five 4-month options are all smaller than the corresponding dynamic-hedging errors with daily updating.

Intuitively, when we use 1-month options to hedge a 12-month option, we are using a piece-wise linear function to approximate the smooth convex value function of an 11-month option a month later. On the other hand, if the hedge is composed of 4-month options, we are using smooth, convex 3-month option value functions to approximate the 11-month option value function. The latter tends to do a much better job in the approximation as the shapes of the curve match better in between the approximation points.

Tables 5 and 6 report the correpsonding hedging results on in-the-money and out-of-the-money call options, respectively. Table 5 sets the target option strike at 90% of the initial spot level, whereas Table 6 sets the target option strike at 110% of the spot level. The maturity choices for the target and hedging options are the same as in Table 4. Overall, the hedging errors for in-the-money, out-of-the-money, and at-the-money options are similar in magnitudes for each strategy and under each model environment, despite the fact that the target option values vary greatly with moneyness.

More careful comparison shows that under the dynamic delta-hedging strategy, the hedging errors are smaller for hedging 110% and 90% strike than for hedging the at-the-money options, especially at shorter maturities. We contribute this hedging error difference again to the different gamma of the target options. At the same maturity, an at-the-money option has larger gamma than an in-the-money or out-of-the-money option. The hedging error is larger as a result for the at-the-money option.

For the static strategies, the hedging errors for in-the-money and out-of-the-money options are similar to those for the at-the-money options. Furthermore, the observation remains that for each target option, the hedging errors decline markedly as the hedging option maturity increases. Under all model environments and for target options of all strikes and maturities, the static hedging errors from using five 4-month options are smaller than the corresponding dynamic delta hedge with daily updating with the underlying futures.

The fact that a static hedging strategy with merely three to five options can outperform a dynamic strategy with daily updating is remarkable. In addition to its superior performance, the static hedge also enjoys several other advantages. First, the much fewer transactions for the static hedge may incur smaller transaction costs. Second, the strategy is very flexible as one can choose options at different maturities to form the static hedge. The particular choice can be made based on a

Table 5 Hedging 90%-strike in-the-money call options

Target mat	2	4	12	12	12	2	4	12	12	12	2	4	12
Strategy	Static with three options					Static with five options					Daily delta		
instruments	1	1	1	2	4	1	1	1	2	4	underlying futures		
Panel A. The Black–Scholes model													
Mean	0.01	0.07	0.11	0.07	0.02	−0.00	0.02	0.08	0.04	0.01	0.16	0.14	0.09
Std Deviation	0.16	0.36	0.58	0.40	0.14	0.08	0.22	0.32	0.15	0.02	0.16	0.13	0.08
RMSE	0.16	0.36	0.59	0.41	0.14	0.08	0.23	0.33	0.16	0.02	0.22	0.19	0.12
MAE	0.12	0.30	0.47	0.34	0.12	0.06	0.20	0.27	0.13	0.02	0.18	0.16	0.11
MSF	−0.05	−0.12	−0.18	−0.14	−0.05	−0.03	−0.09	−0.09	−0.05	−0.01	−0.01	−0.01	−0.01
Minimum	−0.49	−1.13	−2.53	−1.17	−0.35	−0.23	−0.58	−1.52	−0.51	−0.08	−0.52	−0.47	−0.33
Maximum	0.25	0.38	0.60	0.46	0.18	0.16	0.30	0.33	0.17	0.03	0.65	0.46	0.28
Skewness	−0.69	−1.46	−1.87	−1.15	−0.78	0.08	−0.45	−2.12	−1.43	−1.42	−0.08	−0.55	−0.78
Kurtosis	4.01	4.71	6.77	3.50	2.65	3.25	2.18	7.97	4.49	4.38	3.85	4.01	4.47
Target call	11.42	13.13	18.32	18.09	17.89	11.40	13.03	18.19	17.98	17.85	11.40	12.99	17.82
Panel B. The Merton jump-diffusion model													
Mean	0.02	0.02	0.03	0.02	0.00	0.00	0.00	0.03	0.01	−0.00	0.12	0.11	0.07
Std Deviation	0.99	0.46	0.41	0.26	0.14	0.30	0.18	0.25	0.18	0.15	1.90	1.47	0.88
RMSE	0.99	0.46	0.41	0.27	0.14	0.30	0.18	0.26	0.18	0.15	1.91	1.47	0.88
Minimum	−3.01	−2.21	−2.35	−1.21	−0.50	−1.26	−0.95	−1.50	−0.77	−0.55	−20.42	−17.17	−10.55
Maximum	6.08	2.72	1.80	1.78	1.68	1.84	0.85	0.95	1.01	1.06	0.79	0.54	0.30
Skewness	0.21	−1.50	−2.98	−1.32	6.97	−0.20	0.13	−3.23	−0.79	1.74	−6.58	−7.11	−7.39
Kurtosis	6.02	10.49	17.06	17.17	82.57	7.62	16.00	17.55	9.43	14.56	52.33	62.97	69.59
Target call	13.37	13.00	16.43	16.33	16.27	12.04	12.64	17.45	17.24	16.96	11.76	13.26	17.86

Continued

Table 5 continued

Target mat	2	4	12	12	12	2	4	12	12	12	2	4	12
Strategy	Static with three options					Static with five options					Daily delta		
instruments	1	1	1	2	4	1	1	1	2	4	underlying futures		
Panel C. The HN non-Markovian diffusion model													
Mean	−0.02	0.06	0.08	0.04	0.01	−0.02	−0.04	0.05	0.01	−0.02	0.17	0.15	0.09
Std Deviation	0.19	0.35	0.34	0.21	0.17	0.17	0.42	0.39	0.17	0.19	0.16	0.15	0.09
RMSE	0.19	0.36	0.35	0.21	0.17	0.17	0.42	0.40	0.17	0.19	0.23	0.21	0.13
Minimum	−0.30	−0.94	−1.50	−1.00	−1.07	−0.25	−0.71	−1.74	−0.52	−0.19	−0.52	−0.50	−0.32
Maximum	0.59	0.40	0.34	0.24	0.15	0.54	2.62	0.37	0.16	1.15	0.58	0.47	0.29
Skewness	0.16	−1.01	−2.15	−1.38	−2.60	0.56	2.39	−1.91	−1.41	2.17	−0.59	−1.08	−1.04
Kurtosis	1.98	2.94	8.08	4.38	10.90	2.69	11.59	6.80	4.17	8.92	4.47	4.81	4.65
Target call	11.36	13.22	17.21	17.62	17.72	11.49	13.40	17.33	17.67	18.20	11.48	13.12	18.02
Panel D. The Heston stochastic volatility model													
Mean	−0.02	0.05	0.08	0.03	0.01	−0.03	−0.04	0.04	−0.00	−0.03	0.16	0.13	0.07
Std Deviation	0.19	0.37	0.48	0.28	0.22	0.16	0.41	0.39	0.27	0.22	0.17	0.18	0.23
RMSE	0.19	0.37	0.49	0.28	0.22	0.16	0.41	0.39	0.27	0.22	0.23	0.23	0.24
Minimum	−0.38	−1.22	−2.56	−1.24	−1.14	−0.35	−0.95	−1.71	−0.84	−0.60	−0.73	−0.69	−0.71
Maximum	0.50	0.69	0.90	0.73	0.51	0.40	2.42	0.84	0.73	1.12	0.86	0.84	0.92
Skewness	0.30	−0.96	−1.79	−0.66	−1.15	0.20	2.08	−1.22	−0.41	1.17	−0.14	−0.29	−0.08
Kurtosis	2.05	3.36	7.51	4.11	5.80	2.18	10.24	5.05	2.94	5.71	4.25	3.76	3.14
Target call	11.23	13.15	17.57	17.24	17.58	11.36	13.30	17.59	17.36	18.07	11.44	13.06	17.92

Entries report the summary statistics of the hedging errors when hedging different target options and using different hedging instruments. The first row denotes the maturity of the target option being hedged. The strikes of the target options are set to be 90% of the spot level. The second row denotes the strategy, and the third row denotes the maturity of the options in the case of the static hedging strategy. The last row of each panel reports the value of the target call option approximated by the quadrature method, with the theoretical value given under the dynamic hedging column. The statistics are computed based on 1000 simulated paths of the corresponding model.

Table 6 Hedging 110%-strike out-of-the-money call options

Target mat	2	4	12	12	12	2	4	12	12	12	2	4	12
Strategy	Static with three options					Static with five options					Daily delta		
instruments	1	1	1	2	4	1	1	1	2	4	underlying futures		
Panel A. The Black–Scholes model													
Mean	0.01	−0.02	−0.18	−0.09	−0.02	−0.00	0.01	−0.12	−0.04	−0.01	0.23	0.19	0.12
Std Deviation	0.23	0.55	0.84	0.34	0.09	0.11	0.31	0.60	0.19	0.03	0.22	0.17	0.10
RMSE	0.23	0.55	0.86	0.35	0.10	0.11	0.31	0.61	0.19	0.03	0.32	0.25	0.15
Minimum	−0.59	−1.14	−1.29	−0.40	−0.11	−0.29	−0.66	−0.94	−0.23	−0.03	−1.04	−0.77	−0.45
Maximum	0.32	0.69	2.07	1.17	0.38	0.20	0.41	1.29	0.57	0.09	0.85	0.60	0.35
Skewness	−1.01	−0.74	0.47	1.33	1.55	−0.40	−0.80	0.31	1.08	1.24	−0.47	−0.75	−0.85
Kurtosis	3.39	2.15	2.18	4.13	5.05	2.93	2.38	1.92	3.19	3.71	4.83	4.83	4.79
Target call	1.38	2.97	7.08	7.56	8.02	1.37	3.06	7.79	8.06	8.23	1.37	3.06	8.26
Panel B. The Merton jump-diffusion model													
Mean	0.02	0.00	−0.17	−0.10	−0.05	0.03	0.02	−0.11	−0.06	−0.03	0.13	0.14	0.09
Std Deviation	0.42	0.88	1.16	0.78	0.47	0.27	0.63	0.75	0.44	0.24	1.08	1.38	1.07
RMSE	0.43	0.88	1.17	0.79	0.47	0.27	0.63	0.75	0.45	0.25	1.09	1.39	1.07
Minimum	−1.46	−2.02	−1.26	−0.57	−0.27	−1.10	−1.53	−0.90	−0.36	−0.53	−13.27	−13.74	−11.81
Maximum	3.81	1.23	3.83	3.10	2.13	0.91	0.67	2.13	1.60	0.99	0.89	0.72	0.41
Skewness	−0.57	−0.73	1.75	2.73	3.38	−2.57	−1.04	1.28	2.32	2.89	−7.17	−5.77	−6.35
Kurtosis	14.75	2.54	5.89	10.15	14.20	9.50	3.00	4.10	7.91	11.00	66.73	43.11	52.05
Target call	0.38	1.09	4.13	4.72	5.48	0.53	1.70	6.05	6.38	6.68	0.63	2.09	7.49

Continued

Table 6 continued

Target mat	2	4	12	12	12	2	4	12	12	12	2	4	12
Strategy	Static with three options					Static with five options					Daily delta underlying futures		
instruments	1	1	1	2	4	1	1	1	2	4			
Panel C. The HN non-Markovian diffusion model													
Mean	0.01	−0.03	−0.18	−0.09	−0.03	0.00	−0.00	−0.12	−0.05	−0.02	0.22	0.18	0.11
Std Deviation	0.21	0.51	0.79	0.31	0.08	0.11	0.32	0.54	0.16	0.12	0.26	0.25	0.20
RMSE	0.21	0.51	0.81	0.32	0.08	0.11	0.32	0.56	0.16	0.12	0.34	0.31	0.23
Minimum	−0.50	−1.06	−1.22	−0.38	−0.09	−0.29	−0.69	−0.88	−0.21	−0.27	−0.93	−0.56	−0.55
Maximum	0.47	0.85	1.78	0.99	0.39	0.18	0.34	1.11	0.54	0.35	1.01	0.82	0.55
Skewness	−0.50	−0.75	0.46	1.31	1.87	−0.33	−0.85	0.27	1.09	0.42	0.11	−0.07	−0.43
Kurtosis	2.67	2.19	2.15	3.99	7.00	2.83	2.44	1.86	3.36	2.62	3.04	2.53	2.91
Target call	1.34	2.84	6.81	7.44	8.10	1.27	2.83	7.57	7.96	7.59	1.26	2.89	7.99
Panel D. The Heston stochastic volatility model													
Mean	0.00	−0.03	−0.20	−0.10	−0.03	−0.00	−0.01	−0.14	−0.06	−0.03	0.22	0.18	0.09
Std Deviation	0.23	0.53	0.86	0.37	0.19	0.14	0.36	0.62	0.27	0.17	0.26	0.26	0.28
RMSE	0.23	0.53	0.88	0.38	0.19	0.14	0.36	0.64	0.27	0.17	0.34	0.31	0.30
Minimum	−0.67	−1.46	−1.96	−0.89	−0.55	−0.47	−1.10	−1.61	−0.77	−0.50	−0.89	−0.70	−0.86
Maximum	0.60	0.95	2.23	1.44	0.89	0.42	0.69	1.73	1.10	0.50	0.98	0.90	0.95
Skewness	−0.54	−0.76	0.42	0.93	0.81	−0.04	−0.79	0.25	0.46	0.06	−0.03	−0.04	−0.02
Kurtosis	3.04	2.46	2.29	3.84	4.54	3.55	2.93	2.25	3.56	2.81	3.34	3.08	3.12
Target call	1.39	2.96	6.95	7.62	8.43	1.33	2.92	7.69	8.11	7.79	1.32	2.97	8.11

Entries report the summary statistics of the hedging errors when hedging different target options and using different hedging instruments. The first row denotes the maturity of the target option being hedged. The strikes of the target options are set to be 110% of the spot level. The second row denotes the strategy, and the third row denotes the maturity of the options in the case of the static hedging strategy. The last row of each panel reports the value of the target call option approximated by the quadrature method, with the theoretical value given under the dynamic hedging column. The statistics are computed based on 1000 simulated paths of the corresponding model.

joint consideration of contract availability, transaction cost, order flow, and relative hedging performance. Third, since the static hedge employs neither short stock positions nor substantial borrowing, it is not subject to either short sales restrictions or leverage constraints. By contrast, delta hedges of options always involve a short position in either the risky asset or a riskfree bond, and hence always face one of these restrictions. Finally, the use of a static hedge also allows one to economize on the monitoring costs (e.g., paying for traders and real-time data feeds) associated with dynamic rebalancing. These costs are much larger in practice than typically assumed in theory and potentially explain the current situation that dynamic hedging is usually only performed by specialized institutions.

3 HEDGING S&P 500 INDEX OPTIONS: AN APPLIED EXAMPLE

The simulation study in the previous section compares the performance of the two different types of hedging strategies under controlled environments. In this section, we investigate the historical performance of the strategies in hedging the sale of S&P 500 index options. While simulation allows us to benchmark the magnitude of the approximation error in various models, the empirical study measures the likely effectiveness of the hedging strategies in practice.

3.1 Data and Estimation

The data on S&P 500 index options are obtained from OptionMetrics. The data sample is from January 1996 to March 2009. The S&P 500 index options are standard European options on the spot index and are listed at the CBOE. The data set includes, among other information, the closing quotes on each options contract (bid and ask) and implied volatilities based on the mid quote. Also included in the data set is a unique option contract identifier to facilitate the tracking of an option contract over time. The underlying index level at close, the interest rate curve, and the projected dividend yield for the calculation of implied volatility are also supplied by OptionMetrics. Our hedging exercises are based on the mid option price quotes.

In parallel with the hedging exercises in the simulation studies, we perform month-long hedging exercises on the index options. The S&P 500 index options expire on the Saturday following the third Friday. Since the terminal payoff is computed based on the opening price on that Friday morning, trades and quotes on the expiring options effectively stop on the preceding Thursday. Hence, we start the hedging exercise each month 30 days prior to the expiring Friday, which is a Wednesday. From these starting dates, we can perform month-long hedging exercises for 158 nonoverlapping months from January 17, 1996 to February 18, 2009. Sampling properties of the hedging errors are computed from the 158 hedging experiments. To be comparable with the simulations, we normalize the option prices and hedging errors as percentages of the underlying index level at the starting date of each hedging exercise.

At each starting date, we classify options into four maturity groups, matching those used in the simulation exercise: (i) 1-month options (31 days), (ii) 2-month options (59–66 days), (iii) options with maturities 4–6 months (115–185 days), and (iv) options with maturities 12–17 months (360–521 days). The variations in maturities in the last two maturity groups are necessary to obtain a monthly series because we do not have 4- and 12-month options in all months. As in the simulation, we use the first three maturity groups (1-, 2-, and 4-month options) to form the static hedge portfolios and the last three maturity groups (2-, 4-, and 12-month options) as the target option being hedged. For each target option maturity, we choose three strikes that are closest to 90, 100, and 110% of the spot level, respectively. The available number of option contracts at each of the starting dates ranges from 48 to 372 at 1-month maturity, from 30 to 342 at 2-month maturity, from 33 to 132 at 4 to 6-month maturity, and from 12 to 98 at the 12–17 month maturity. About half of these options are calls and the other half are puts. We report our results on call options to match the simulation exercise. Hedging put options generates similar results.

Since we do not know the true data-generating process nor the option pricing model underlying the market prices, we resort to the simple method of computing the hedging ratios and static portfolio weights based on the Black–Scholes model using the observed implied volatility of the target option as the volatility input. We use the quadrature rule to generate the appropriate strikes and weights for the static hedge. Since the quadrature-generated strikes do not necessarily match the strikes of the available option contracts, we use the available strikes closest to the quadrature-generated strikes to form the static portfolio.

We follow all strategies for 29 actual days, running from the starting date to the Thursday of the fourth following week, the last day of trading for the 1-month options used in the static hedge. For the static strategy, we only need to track the price of the options at each date and record the difference between the price of the hedge portfolio and the price of the target call option. When there is a discrepancy between the price of the target call option and the cost of the hedge portfolio at the starting date, we also monitor the typically small money market account balance. For the dynamic strategy, we need to compute a new delta at each date based on the newly observed underlying price level and option implied volatility, and perform the appropriate delta rebalancing.

3.2 Static versus Dynamic Hedging in Practice

Table 7 presents the summary statistics of the hedging errors for the various hedging exercises on S&P 500 index options. To ease comparison, we present the results in a similar format to those from the simulation exercises summarized in Tables 4–6.

The three panels are for the three sets of relative strikes for the target options. As in the simulation exercise, we represent the option prices and hedging errors as percentages of the underlying index level at the starting date of each exercise.

Panel A reports the hedging results on at-the-money options. For a 2-month at-the-money call option, daily delta hedging with the underlying futures generates a root mean squared hedging error of 0.63. The corresponding statistic for the static strategy with three 1-month options is 0.33, about half of the RMSE from the dynamic strategy. Using five 1-month options makes the hedging errors even smaller at 0.30. Therefore, a static hedge with just three 1-month options significantly outperforms daily delta hedging in reducing the risks associated with writing 2-month call options.

In hedging the sale of a 4 to 6-month call option, the dynamic hedging strategy generates a root mean squared error of 0.63, compared to 0.57 from the static strategy with three 1-month call options. Hence, the performances from the two strategies are on par in hedging the sale of a 4 to 6-month call option.

When hedging the sale of a call option with a time-to-maturity of 12 months or longer, the dynamic strategy generates a RMSE of 0.70. This performance is better than the static strategy with three 1-month call options, but on par with the static strategy with five 1-month call options. Consistent with the results observed in the simulation exercise, the performance of the static strategy improves if we increase the time-to-maturity of the options in the hedge. In hedging the sale of a 12-month or longer call option, the RMSE from the static strategy with three call options declines from 0.97 to 0.65 and then to 0.39, as the time-to-maturity of the three call options in the hedge portfolio increases from 1 month to 2 months, and then to 4–6 months. We also observe a similar reduction when using five call options in the static hedge portfolio.

The results from Panels B and C on hedging in-the-money and out-of-the-money options are similar. In all cases, the performance of static hedging with three to five call options is on par with or better than the performance of daily delta hedging. In addition, the performance of our static strategy can be further improved by choosing slightly longer maturities for the options in the hedge portfolio. Therefore, the static strategy not only works in theory and in simulation, but it also works on historical data.

Comparing the hedging results in Table 7 on the S&P 500 index options to that from the simulation in Tables 4–6, we observe several differences. For the dynamic strategies, the performance on the index options is worse than that from the three pure diffusion models (BS, HN, and MJ) but better than that from the MJ jump-diffusion model, suggesting that during our sample period, the S&P 500 index did not move purely continuously, nor had it generated jumps as large as those in the MJ model simulation. Furthermore, ranking of the dynamic hedging errors for the three target options is opposite to that from the simulation. The hedging errors are the largest on the 12-month options on the S&P 500 index, but they are the smallest under the simulation. On the exchange, options transactions are concentrated at short maturities. The option quotes at longer maturities can be more susceptible to

Table 7 Static and dynamic hedging of S&P 500 index options

Target mat	2	4	12	12	12	2	4	12	12	12	2	4	12
Strategy	Static with three options					Static with five options					Daily delta underlying futures		
instruments	1	1	1	2	4	1	1	1	2	4			
Panel A. Target option strike ≈ 100% of spot													
Mean	0.01	−0.11	−0.14	−0.10	−0.02	0.03	−0.08	−0.10	−0.07	−0.01	0.16	0.06	0.00
Std Deviation	0.33	0.56	0.96	0.64	0.39	0.30	0.48	0.80	0.57	0.39	0.62	0.63	0.70
RMSE	0.33	0.57	0.97	0.65	0.39	0.30	0.48	0.80	0.57	0.39	0.63	0.63	0.70
Minimum	−0.98	−2.19	−3.23	−2.67	−1.54	−1.02	−1.93	−2.95	−2.43	−1.58	−3.03	−3.77	−4.33
Maximum	1.65	1.27	1.90	1.33	0.99	1.63	1.29	1.76	1.09	0.91	2.29	1.72	1.66
Skewness	0.31	−0.58	−0.52	−0.75	−0.69	0.79	−0.75	−0.72	−1.02	−0.59	−1.08	−1.84	−1.95
Kurtosis	7.13	4.11	3.58	4.74	4.36	8.98	5.11	4.26	5.45	4.33	8.38	12.06	12.10
Target call	3.47	5.51	9.36	9.43	10.27	3.50	5.72	9.81	9.80	10.31	3.38	5.62	10.26
Panel B. Target option strike ≈ 90% of spot													
Mean	0.01	−0.01	−0.03	−0.02	0.00	−0.02	−0.00	−0.02	−0.01	0.01	0.11	0.05	−0.00
Std Deviation	0.32	0.60	0.90	0.60	0.38	0.30	0.50	0.73	0.52	0.36	0.48	0.55	0.64
RMSE	0.32	0.59	0.90	0.60	0.38	0.30	0.49	0.72	0.52	0.36	0.49	0.55	0.64
Minimum	−1.46	−3.89	−5.25	−2.99	−1.31	−1.56	−3.02	−4.18	−2.34	−1.38	−2.20	−3.38	−3.92
Maximum	1.39	1.83	2.11	1.15	1.07	0.81	1.50	1.81	0.97	1.21	1.97	1.54	1.56
Skewness	−1.48	−2.57	−2.10	−1.18	−0.41	−1.97	−1.91	−1.87	−0.97	−0.11	−1.54	−2.30	−2.05
Kurtosis	11.05	15.69	11.18	6.66	3.76	11.64	11.95	10.27	5.54	4.31	11.29	14.65	12.48
Target call	11.00	12.68	16.45	15.73	16.68	10.99	12.55	16.06	15.56	16.53	11.17	12.84	16.81

Panel C. Target option strike ≈ 110% of spot													
Mean	−0.01	−0.01	−0.15	−0.10	−0.04	−0.02	−0.03	−0.12	−0.08	−0.02	0.06	0.04	−0.00
Std Deviation	0.25	0.50	0.84	0.58	0.33	0.23	0.44	0.69	0.52	0.34	0.47	0.55	0.70
RMSE	0.25	0.50	0.85	0.59	0.34	0.23	0.44	0.70	0.52	0.33	0.47	0.55	0.69
Minimum	−0.92	−1.99	−3.57	−2.61	−1.39	−1.05	−1.99	−3.26	−2.45	−1.46	−3.08	−3.87	−4.31
Maximum	0.82	1.74	2.04	1.56	0.78	0.78	1.55	1.52	1.15	0.73	1.70	1.62	1.79
Skewness	−0.14	−0.33	−0.48	−0.56	−0.68	0.17	−0.60	−0.85	−1.03	−0.84	−1.26	−2.17	−1.80
Kurtosis	5.13	5.71	4.36	4.95	4.49	6.59	7.22	5.92	6.08	4.70	16.36	18.26	12.18
Target call	0.64	1.90	4.78	5.00	5.53	0.63	1.97	5.38	5.44	5.76	0.46	1.62	5.48

Entries report the summary statistics of the hedging errors for the hedging exercises on S&P 500 index options. The maturities (in months) of target options being hedged are given in the first row. The hedging strategy is either static with a portfolio of three options, five options, or dynamic with the underlying futures and daily updating. The maturity of the options in the static hedge portfolio (in months) are given in the third row. The three panels are for three sets of strike choices for the target options, with the strikes set to be close to the spot level in Panel A, 90% of the spot level in Panel B, and 110% of the spot level in Panel C. The statistics are computed based on the 158 nonoverlapping month-long hedging exercises over the sample period January 1996 to February 2009. The hedging errors are computed in percentages of the spot index level at the starting date of each exercise. The hedging ratios are computed using the Black–Scholes model with the observed implied volatility of target option as the volatility input. The last row reports the sample average of the value of the target call option approximated by the quadrature-based hedge portfolio. Numbers under the dynamic hedging columns are the sample average of the observed target call option price, all in percentages of the underlying spot index level at the starting date of each month.

data error. We conjecture that part of the larger hedging errors on the long-term options are due to data noise.

For the static strategies, the hedging errors on 2-month and 4-month S&P 500 index options are comparable or smaller than that from the simulation under all model environments, but the hedging errors on 12-month options are in general larger than those from the simulation. This larger error again may be contributed by data noise on long-term options.

Compared to the simulation, our hedging exercise on the S&P 500 index options contains one important difference. The simulation exercise chooses strikes for the hedge portfolio based exactly on the quadrature rule, but the hedging exercise on the S&P 500 index options can only use available strikes, with the quadrature rule used only as an approximation. The similar magnitudes for the hedging errors from the two types of exercises suggest that the available option strikes from the options exchange are dense enough for the approximation to be reasonably accurate. The average adjacent strike spacing on the S&P 500 index options is about 1% of the spot level at short maturities and about 3% of the spot level at long maturities.

4 SEMI-STATIC HEDGING OF PATH-DEPENDENT OPTIONS

For ease of exposition, the focus of this article thus far has been on static hedging of standard European options. In this section, we show that we can also form semi-static hedges of path-dependent options with European options, provided that the path is discretely monitored. Hedging path-dependent options is not possible under the BL framework. Dynamically delta-hedging path-dependent options is plausible in theory, but for many path-dependent claims, the reality of jumps often destroys the effectiveness of these hedges in practice. Our semi-static hedging theory takes jumps in stride.

We consider the wide class of contingent claims whose single payoff at the fixed time T depends on a finite number ($n < \infty$) of points of the price path of a single underlying asset

$$V_T = f(S_{t_0}, S_{t_1}, \ldots, S_{t_n}), \tag{19}$$

where $t_0 = 0$ and $t_n = T$. We label the times $t_0, t_1, \ldots, t_n$ as monitoring times. The payoff structure in Equation (19) excludes various continuously monitored Asian and barrier options, or American claims. Although we can always discretize a continuous problem, the analysis of this section assumes that we can trade at each fixed monitoring time t_i in options maturing at t_{i+1}.

To simplify the discretely monitored payoff function in Equation (19), we note that for many claims, we can capture the path-dependence by one or more summary statistics. In what follows, we will work with a single summary statistic, but it should be clear how to extend the analysis to multiple such statistics. A single summary statistic captures the path-dependence of a claim if we can write the final payoff of the claim recursively as follows,

$$V_T = \phi(H_T), \tag{20}$$

where

$$H_{t_i} = g_i(H_{t_{i-1}}, S_{t_{i-1}}, S_{t_i}), \quad i = 1, \ldots, n, \tag{21}$$

where $\phi(\cdot)$ and $g_i(\cdot)$ are known functions, H is the single summary statistic, and H_0 and S_0 are known constants. Examples in this class include discretely monitored Asian and barrier options, Bermudan, passport, and cliquet options, and many structured notes. A concrete example which we will focus on is a globally floored, locally capped, compounding cliquet with discrete monitoring,

$$V_T = S_0 \max[L, H_T], \tag{22}$$

with

$$H_{t_i} = H_{t_{i-1}} \left[\left(\frac{S_{t_i}}{S_{t_{i-1}}} \right) \wedge U \right], \quad i = 1, \ldots, n, \tag{23}$$

where L is the global floor, $U > 1$ is the local cap, and n denotes the number of monitoring periods. Here, $H_0 = 1$, and S_0 is known. In practice, L is typically chosen to be one so that the annualized return is always positive. A typical value of the local cap U is 1.35 so that the maximum return for any year cannot exceed 35 percent.

We assume the same one-factor Markovian setting as in Equation (1). To hedge the discretely monitored options as described by the payoff function in (20) and (21), we assume that at each discrete time t_i, we can take static positions in European options of all strikes and maturing at t_{i+1}, for $i = 0, 1, \ldots, n-1$. Given this access to markets, the algorithm for valuing a path-dependent option of the specified type is as follows.

At time t_{n-1}, conditioning on the history to that time $H_{t_{n-1}}$ and the contemporaneous stock price $S_{t_{n-1}}$, and from (20) and (21) with $i = n$, the final payoff becomes a known function of only the final stock price,

$$V_T = \phi(H_T) = \phi(g_n(H_{t_{n-1}}, S_{t_{n-1}}, S_T)) \equiv f_n(S_T; H_{t_{n-1}}, S_{t_{n-1}}), \tag{24}$$

where the last two arguments of f_n are known due to the conditioning. We can span the final payoff using options maturing at time T,

$$\begin{aligned} f_n(S_T; H_{t_{n-1}}, S_{t_{n-1}}) = {} & f_n(\kappa_n; H_{t_{n-1}}, S_{t_{n-1}}) + f_n'(\kappa_n; H_{t_{n-1}}, S_{t_{n-1}}) \\ & [(S_T - \kappa_n)^+ - (\kappa_n - S_T)^+] + \int_0^{\kappa_n} f_n''(\mathcal{K}; H_{t_{n-1}}, S_{t_{n-1}})(\mathcal{K} - S_T)^+ d\mathcal{K} \\ & + \int_{\kappa_n}^{\infty} f_n''(\mathcal{K}; H_{t_{n-1}}, S_{t_{n-1}})(S_T - \mathcal{K})^+ d\mathcal{K}, \end{aligned} \tag{25}$$

where the expansion point $\kappa_n \geq 0$ can be any convenient constant separating the put options from the call options. A common choice is the forward price $\kappa_n = F_0(T)$.

We can value this contingent-claim at time t_{n-1} by taking conditional expectations on both sides of Equation (25) under the risk-neutral measure $\mathbb{Q}$ and then discounting the expectation by the constant risk-free rate. We can represent the value of this claim in terms of the risk-free rate and the contemporaneous option prices,

$$V_{t_{n-1}}^{f_n} = e^{-r(T-t_{n-1})} f_n(\kappa_n; H_{t_{n-1}}, S_{t_{n-1}}) + f_n'(\kappa_n; H_{t_{n-1}}, S_{t_{n-1}})[C_{t_{n-1}}(\kappa_n, T) - P_{t_{n-1}}(\kappa_n, T)] + \int_0^{\kappa_n} f_n''(\mathcal{K}; H_{t_{n-1}}, S_{t_{n-1}}) P_{t_{n-1}}(\mathcal{K}, T) d\mathcal{K} + \int_{\kappa_n}^{\infty} f_n''(\mathcal{K}; H_{t_{n-1}}, S_{t_{n-1}}) C_{t_{n-1}}(\mathcal{K}, T) d\mathcal{K}. \quad (26)$$

Therefore, at the last time step t_{n-1}, we can replicate the contingent claim using a portfolio of standard European options maturing at the same time. This result is the same as in Breeden and Litzenberger (1978) and does not need the Markovian assumption.

However, to be able to replicate the claim at any other time steps, we need the one-factor Markovian assumption. Substitution of the Markovian property (1) into Equation (26) implies that the time-t_{n-1} value of this contingent claim is a known function of $H_{t_{n-1}}$ and $S_{t_{n-1}}$,

$$V_{t_{n-1}}^{f_n} = e^{-r(T-t_{n-1})} f_n(\kappa_n; H_{t_{n-1}}, S_{t_{n-1}}) + f_n'(\kappa_n; H_{t_{n-1}}, S_{t_{n-1}})[C(S_{t_{n-1}}, t_{n-1}; \kappa_n, T; \Theta) - P(S_{t_{n-1}}, t_{n-1}; \kappa_n, T; \Theta)] + \int_0^{\kappa_n} f_n''(\mathcal{K}; H_{t_{n-1}}, S_{t_{n-1}}) P(S_{t_{n-1}}, t_{n-1}; \mathcal{K}, T; \Theta) d\mathcal{K} + \int_{\kappa_n}^{\infty} f_n''(\mathcal{K}; H_{t_{n-1}}, S_{t_{n-1}}) C(S_{t_{n-1}}, t_{n-1}; \mathcal{K}, T; \Theta) dK \equiv V(H_{t_{n-1}}, S_{t_{n-1}}, t_{n-1}). \quad (27)$$

Now, we step back to time t_{n-2} and condition on the history to that time $H_{t_{n-2}}$ and the contemporaneous stock price $S_{t_{n-2}}$. From the Markovian representation in (27) and the definition of the history summary statistic in (21) with $i = n-1$, we can write the time-t_{n-1} value of this claim as a known function of only the contemporaneous stock price at t_{n-1},

$$V_{t_{n-1}}^{f_n} = V(H_{t_{n-1}}, S_{t_{n-1}}, t_{n-1}) = V(g_{n-1}(H_{t_{n-2}}, S_{t_{n-2}}, S_{t_{n-1}}), S_{t_{n-1}}, t_{n-1}) \equiv f_{n-1}(S_{t_{n-1}}; H_{t_{n-2}}, S_{t_{n-2}}), \quad (28)$$

where $H_{t_{n-2}}$ and $S_{t_{n-2}}$ are known through the conditioning. Therefore, at time t_{n-2}, we can simply regard $f_{n-1}(S_{t_{n-1}}; H_{t_{n-2}}, S_{t_{n-2}})$ as the terminal payoff of a one-step claim, expressed as a function of the terminal stock price $S_{t_{n-1}}$. We can again replicate this payoff using options maturing at t_{n-1}, analogous to the steps in Equations (25) and (26). Furthermore, we can again exploit the Markovian assumption in (1) and derive the new value function $V(H_{t_{n-2}}, S_{t_{n-2}}, t_{n-2})$ and the new target payoff function $f_{n-2}(S_{t_{n-2}}; H_{t_{n-3}}, S_{t_{n-3}})$ by performing operations analogous to (27) and (28). We repeat the procedure until we obtain the value function at time 0. For this final iteration, we only need to condition on the known values of H_0 and S_0.

Therefore, the semi-static hedging of this path-dependent claim goes as follows. At time 0, we use a portfolio of European options maturing at time t_1 to span the value function of the claim. At time t_1, we collect the receipts from the expiring options in the hedge portfolio and form another hedge portfolio maturing at time t_2. This procedure continues until time $T = t_n$, when the payoff from the hedge portfolio formed at time t_{n-1} matches the payoff from the path-dependent claim. The hedging is static and no portfolio rebalancing is needed in between monitoring times. At each monitoring step, the options in the hedge portfolio expire and a new hedge portfolio needs to be formed. Thus, the rebalancing frequency matches the monitoring frequency, reflecting the semi-static nature of the strategy.

5 CONCLUSION

Dynamic hedging has been widely used due to its flexibility in hedging a wide class of contingent claims. However, the performance of this strategy deteriorates dramatically in the presence of jumps of random size. The static hedging strategy introduced by Breeden and Litzenberger (1978) addresses this model risk, but can only be applied to a narrow range of payoffs. In this article, we propose a new approach that is more robust than dynamic hedging and covers a much wider class of claims than BL. For simplicity, we illustrate our theory when the target claim is a European option. Since a perfect static hedge requires a continuum of options in the hedge portfolio, we develop a discrete approximation of the static hedge and test its effectiveness using Monte Carlo simulation and historical data.

The simulation results indicate that the static hedge approximation has about the same effectiveness as delta hedging with daily rebalancing in the Black–Scholes environment. When the simulated underlying price process can also experience jumps of random size, the performance of the delta hedge deteriorates dramatically, but the performance of our static option hedge is relatively insensitive to the change from a purely diffusive process to a jump diffusion. The conclusions are unchanged when the hedger does not know the driving process and must resort to the Black–Scholes model with the observed implied volatility as input for computing hedging ratios and hedge portfolio weights. Further simulation indicates that increasing the rebalancing frequency cannot improve the inferior performance of the delta hedge in the presence of random jumps, but the superior performance of the static hedging strategy can be further enhanced by using more strikes or by optimizing on the common maturity in the hedge portfolio. As a result, the static hedge can achieve superior risk reduction with as few as three options in the hedge portfolio.

To investigate how our static strategy performs in a realistic setting, we analyze its effectiveness in hedging S&P 500 index options and compare its performance with daily delta hedging with the index futures. We find that the superior performance of our static hedge found in the simulations of the Merton model also extends to the index options data. This finding lends indirect support to the existence of jumps of random size as part of the S&P 500 index dynamics.

REFERENCES

Bakshi, G., C. Cao, and Z. Chen. 1997. Empirical Performance of Alternative Option Pricing Models. *Journal of Finance* 52: 2003–2049.

Bakshi, G., and N. Kapadia. 2003. Delta-Hedged Gains and the Negative Market Volatility Risk Premium. *Review of Financial Studies* 16: 527–566.

Bakshi, G., and D. Madan. 2000. Spanning and Derivative-Security Valuation. *Journal of Financial Economics* 55: 205–238.

Balder, S., and A. Mahayni. 2006. Robust Hedging with Short-Term Options. *Wilmott Magazine* 9: 72–78.

Barndorff-Nielsen, O. E. 1997. Processes of Normal Inverse Gaussian Type. *Finance and Stochastics* 2: 41–68.

Bates, D. S. 1991. The Crash of 87: Was It Expected? The Evidence From Option Markets. *Journal of Finance* 46: 1009–1044.

Bates, D. S. 1996. Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options. *Review of Financial Studies* 9: 69–107.

Bates, D. S. 2000. Post-'87 Crash Fears in the S&P 500 Futures Option Market. *Journal of Econometrics* 94: 181–238.

Bates, D. S. 2003. Empirical Option Pricing: A Retrospection. *Journal of Econometrics* 116: 387–404.

Bhattacharya, M. 1980. Empirical Properties of the Black–Scholes Formula under Ideal Conditions. *Journal of Financial and Quantitative Analysis* 15: 1081–1105.

Black, F., and M. Scholes. 1972. The Valuation of Option Contracts and a Test of Market Efficiency. *Journal of Finance* 27: 399–417.

Black, F., and M. Scholes. 1973. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy* 81: 637–654.

Bollen, N. P., and E. Raisel. 2003. The Performance of Alternative Valuation Models in the OTC Currency Option Market. *Journal of International Money and Finance* 22: 33–64.

Boyle, P. P., and D. Emanuel. 1980. Discretely Adjusted Option hedges. *Journal of Financial Economics* 8: 259–282.

Branger, N., and A. Mahayni. 2006. Tractable Hedging: An Implementation of Robust Hedging Strategies. *Journal of Economic Dynamics and Control* 30: 1937–1962.

Branger, N., and A. Mahayni. 2011. Tractable Hedging with Additional Hedge Instruments. *Review of Derivatives Research* 14: 85–114.

Breeden, D. T., and R. H. Litzenberger. 1978. Prices of State-Contingent Claims Implicit in Option Prices. *Journal of Business* 51: 621–651.

Carr, P., and A. Chou. 1997. Breaking Barriers. *Risk* 10: 139–145.

Carr, P., H. Geman, D. Madan, and M. Yor. 2002. The Fine Structure of Asset Returns: An Empirical Investigation. *Journal of Business* 75: 305–332.

Carr, P., and D. Madan. 1998. Towards a Theory of Volatility Trading. In R. Jarrow (ed.), *Risk Book on Volatility*, New York. Risk: pp. 417–427.

Carr, P., and D. Madan. 1999. Option Valuation Using the Fast Fourier Transform. *Journal of Computational Finance*, 2: 61–73.

Carr, P., and A. Mayo. 2007. On the Numerical Evaluation of Option Prices in Jump Diffusion Processes. *European Journal of Finance* 13: 353–372.

Carr, P., and L. Wu. 2003. Finite Moment Log Stable Process and Option Pricing. *Journal of Finance* 58: 753–777.

Carr, P., and L. Wu. 2004. Time-Changed Lévy Processes and Option Pricing. *Journal of Financial Economics* 71: 113–141.

Carr, P., and L. Wu. 2007. Stochastic Skew in Currency Options. *Journal of Financial Economics* 86: 213–247.

Cox, J. C., and S. A. Ross. 1976. The Valuation of Options for Alternative Stochastic Processes. *Journal of Financial Economics* 3: 145–166.

Davis, P. J., and P. Rabinowitz. 1984. *Methods of Numerical Integration*. New York: Academic Press.

Dritschel, M., and P. Protter. 1999. Complete Markets with Discontinuous Security Price. *Finance and Stochastics* 3: 203–214.

Dumas, B., J. Fleming, and R. E. Whaley. 1998. Implied Volatility Functions: Empirical Tests. *Journal of Finance* 53: 2059–2106.

Dupire, B. 1994. Pricing with a Smile. *Risk* 7: 18–20.

Eberlein, E., U. Keller, and K. Prause. 1998. New Insights into Smile, Mispricing, and Value at Risk: The Hyperbolic Model. *Journal of Business* 71: 371–406.

Engle, R. F., and J. V. Rosenberg. 2002. Empirical Pricing Kernels. *Journal of Financial Economics* 64: 341–372.

Figlewski, S. 1989. Options Arbitrage in Imperfect Markets. *Journal of Finance* 44: 1289–1311.

Galai, D. 1983. The Components of the Return from Hedging Options Against Stocks. *Journal of Business* 56: 45–54.

Green, R. C., and R. A. Jarrow. 1987. Spanning and Completeness in Markets with Contingent Claims. *Journal of Economic Theory* 41: 202–210.

He, C., J. S. Kennedy, T. F. Coleman, P. A. Forsyth, Y. Li, and K. R. Vetzal. 2006. Calibration and Hedging under Jump Diffusion. *Review of Derivatives Research* 9: 1–35.

Heston, S. L. 1993. Closed-Form Solution for Options with Stochastic Volatility, with Application to Bond and Currency Options. *Review of Financial Studies* 6: 327–343.

Heston, S. L., and S. Nandi. 2000. A Closed-Form GARCH Option Valuation Model. *Review of Financial Studies* 13: 585–625.

Huang, J., and L. Wu. 2004. Specification Analysis of Option Pricing Models Based on Time-Changed Lévy Processes. *Journal of Finance* 59: 1405–1440.

Hull, J., and A. White. 1987. The Pricing of Options on Assets with Stochastic Volatilities. *Journal of Finance*, 42: 281–300.

Jackwerth, J. C., and M. Rubinstein. 1996. Recovering Probability Distributions from Contemporary Security Prices. *Journal of Finance* 51: 347–369.

Kennedy, J. S., P. A. Forsyth, and K. R. Vetzal. 2009. Dynamic Hedging under Jump Diffusion with Transaction Cost. *Operations Research* 57: 541–559.

Leland, H. E. 1985. Option Pricing and Relation with Transaction Costs. *Journal of Finance* 40: 1283–1301.

Lewis, A. L. 2000. *Option Valuation under Stochastic Volatility*. Newport Beach, CA: Finance Press.

Madan, D., and E. Seneta. 1990. The Variance Gamma (V.G.) Model for Share Market Returns. *Journal of Business* 63: 511–524.

Merton, R. C. 1976. Option Pricing When Underlying Stock Returns Are Discontinuous. *Journal of Financial Economics* 3: 125–144.

Nachman, D. 1988. Spanning and Completeness with Options. *Review of Financial Studies* 3: 311–328.

Renault, E., and N. Touzi. 1996. Option Hedging and Implied Volatilities in a Stochastic Volatility Model. *Mathematical Finance* 6: 279–302.

Ross, S. A. 1976. Options and Efficiency. *Quarterly Journal of Economics* 90: 75–89.

Schoutens, W., E. Simons, and J. Tistaert. 2004. A Perfect Calibration! Now What?. *Wilmott Magazine*, 3, 66–78.

Scott, L. O. 1997. Pricing Stock Options in a Jump-Diffusion Model with Stochastic Volatility and Interest Rates: Applications of Fourier Inversion Methods. *Mathematical Finance* 7: 413–426.

Takahashi, A., and A. Yamazaki. 2009a. Efficient Static Replication of European Options under Exponential Lévy Models. *Journal of Futures Markets* 29: 1–15.

Takahashi, A., and A. Yamazaki. 2009b. A New Scheme for Static Hedging of European Derivatives under Stochastic Volatility Models. *Journal of Futures Markets* 29: 397–413.

Toft, K. B. 1996. On the Mean-Variance Tradeoff in Option Replication with Transactions Costs. *Journal of Financial and Quantitative Analysis* 31: 233–263.

Wu, L. 2006. Dampened Power Law: Reconciling the Tail Behavior of Financial Asset Returns. *Journal of Business* 79: 1445–1474.

Wu, L., and J. Zhu. 2011. Simple Robust Hedging with Nearby Contracts. Working paper, Baruch College and University of Utah.