Static Hedging of Exotic Options

PETER CARR, KATRINA ELLIS, and VISHAL GUPTA*

ABSTRACT

This paper develops static hedges for several exotic options using standard options. The method relies on a relationship between European puts and calls with different strike prices. The analysis allows for constant volatility or for volatility smiles or frowns.

THIS PAPER GENERALIZES A RELATIONSHIP due to Bates (1988) between European puts and calls with different strikes. We term the generalized result *put–call symmetry* (PCS) and use it to develop a method for valuation and static hedging of certain exotic options. We focus on path-dependent options that change characteristics at one or more critical price levels, for example, barrier and look-back options and their extensions. We do not examine American or Asian options.

While these options may be valued and *dynamically* hedged in a lognormal model,[1] we offer valuation and *static* hedging in a slightly more general diffusion setting. As in Bowie and Carr (1994) and Derman, Ergener, and Kani (1994), we create static portfolios of standard options whose values match the payoffs of the path-dependent option at expiration and along the boundaries. Since the path-dependent options we examine often have high gammas, static hedging using standard options will be considerably easier and cheaper than dynamic hedging. Furthermore, in contrast to dynamic hedging, our static positions in standard options are invariant to volatility, interest rates, and dividends, bypassing the need to estimate them.[2] Because

[1] For example, barrier options are valued in the Black–Scholes (1973) model in Merton (1973).

[2] However, we assume a certain structure on the price process to achieve these invariance results. In particular, we assume that the cost of carrying the underlying is zero, and that its volatility satisfies a symmetry restriction.

the path-dependent options we examine are often highly sensitive to volatility, the hedging error due to volatility misspecification may be substantial with dynamic hedging.

Our PCS relationship can be viewed as both an extension and a restriction of the widely known put–call parity (PCP) result. The generalization involves allowing the strikes of the put and the call to differ in a certain manner. The restrictions sufficient to achieve this result are essentially that the underlying price process has both zero drift and a symmetric volatility structure, which is described below.

The rest of the paper is organized as follows. Section I presents the assumptions and the intuition behind PCS, which is the foundation for our hedging strategy. Section II reviews the static replication of single barrier options. Section III focuses on exotic options involving multiple barriers, such as double knockouts, roll-down, ratchet, and lookback options. In Section IV, we relax the assumption of zero drift and provide tight bounds on the static hedges developed in the earlier sections. Section V concludes the paper and the Appendix contains the mathematical details supporting our results.

I. Put–Call Symmetry

Throughout this paper we assume that markets are frictionless and there are no arbitrage opportunities. Let $P(K)$ and $C(K)$ denote the time 0 price of an European put and call, respectively, with both options struck at K and maturing at T. Because maturity is the same for all instruments we consider in any given example, we suppress dependence on the time to maturity to ease notation. Let B denote the time 0 price of a pure discount bond paying one dollar at T. Then Put–Call Parity expressed in terms of the forward price F for time T delivery is

$$C(K) = [F - K]B + P(K). \tag{1}$$

PCP implies that if the common strike of the put and call is the current forward price, then the options have the same value. Since put values increase with increasing strikes and call values decrease, we can write inequalities for European puts and calls whose strikes are on the same side of the forward. By contrast, PCS is an equality between scaled puts and calls whose strikes are on opposite sides of the forward.

To obtain PCS, certain restrictions are imposed on the stochastic process governing the underlying asset's price. In particular, we assume that the underlying price process is a diffusion, with zero drift under any risk-neutral measure, and where the volatility coefficient satisfies a certain symmetry condition. Thus, we rule out jumps in the price process and assume that the process starts afresh at any stopping time, such as at a first passage time to a barrier.

The assumption of zero risk-neutral drift is innocuous for options written on the forward or futures price of an underlying asset. For options written on the spot price, the assumption implies zero carrying costs.[3] Thus, the no-drift restriction implies that options written on the spot price behave as if they were written on the forward price. We relax the assumption of zero drift in Section IV and obtain tight bounds on the value of options whose payoffs depend on the spot price path.

Throughout this paper, we assume that the volatility of the forward price is a known function $\sigma(F_t, t)$ of the forward price F_t and time t. We also assume the following symmetry condition:

$$\sigma(F_t, t) = \sigma(F^2/F_t, t), \qquad \text{for all } F_t \geq 0 \text{ and } t \in [0, T], \tag{2}$$

where F is the current forward price. Thus the volatility at any future date is assumed to be the same for any two levels whose geometric mean is the current forward.

This symmetry condition is satisfied in Black (1976) model where volatility is deterministic, i.e., $\sigma(F_t, t) = \sigma(t)$. The symmetry arises when the volatility is graphed as a function of $X_t \equiv \ln(F_t/F)$. Letting $v(X_t, t) \equiv \sigma(F_t, t)$, the equivalent condition is:

$$v(x, t) = v(-x, t), \qquad \text{for all } x \in \Re \text{ and } t \in [0, T]. \tag{3}$$

Thus, the symmetry condition is also satisfied in models with a symmetric smile[4] in the log of K/F. As a result, a graph of volatility against K/F will be asymmetric, with higher put volatility than call volatility for strikes equidistant from the forward. Finally, the symmetry condition also allows for volatility frowns or even for more complex patterns.

Given the above assumptions, the Appendix proves[5]:

EUROPEAN PUT-CALL SYMMETRY: *Given frictionless markets, no arbitrage, zero drift, and the symmetry condition, the following relationship holds:*

$$C(K)K^{-1/2} = P(H)H^{-1/2}, \tag{4}$$

where the geometric mean of the call strike K and the put strike H is the forward price F:

$$(KH)^{1/2} = F. \tag{5}$$

---

[3] Thus, for options written on a single stock or on a stock index, the no-drift assumption implies that the dividend yield always equals the risk-free rate. For options on spot FX, the no-drift assumption implies that the foreign interest rate always equals the domestic rate. For options on the spot price of a commodity, the assumption implies that the convenience yield is the riskless rate.

[4] Note that the assumed smile is in the local volatility as opposed to the Black (1976) model implied volatility.

[5] Bates (1988) first proves this result for constant volatility. See Bates (1991) for an excellent exposition of the implications of asymmetry for implying out crash premia.
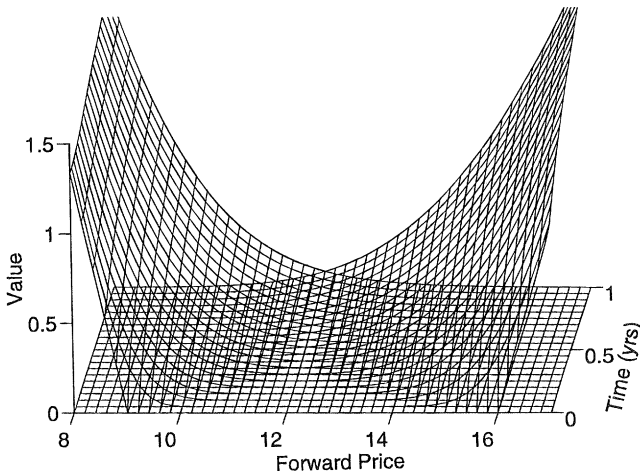
Figure 1. Illustration of Put-Call Symmetry (PCS). A call with strike 16 is equal to $\frac{4}{3}$ puts with strike 9 when the forward price is 12.

Consider a numerical illustration of the PCS result: When the current forward is \$12, a call struck at \$16 has the same value as $\frac{4}{3}$ puts struck at \$9. This example is depicted in Figure 1. The reason the call has much greater value, even though it is further out-of-the-money arithmetically, is that our diffusion process has greater *absolute* volatility[6] when prices are high than when prices are low. Because call and put payoffs are determined by the arithmetic distance between terminal price and strike, the higher absolute volatility at higher prices leads to higher call values.

One intuition for PCS arises from generalizing the following intuition for put-call parity. Imagine a graph of the "risk-neutral" density of terminal prices and suppose that the horizontal axis is placed on a wedge with the objective of finding the fulcrum. The fulcrum is found by balancing the product of density and distance from the wedge integrated across terminal prices. In other words, the fulcrum occurs at the expected value under the risk-neutral distribution, which is the current forward price. Summing the product of density and distance from the wedge on the right of the fulcrum gives the (forward) price of a European call struck at the forward. Similarly, summing the product of density and absolute distance from the wedge on the left of the fulcrum gives the forward price of an at-the-money forward European put. Because the options' forward prices coincide, their spot prices also coincide by a simple cost of carry argument.

The PCS result (equation (4)) implies that a call struck at twice the current forward has twice the value of a put struck at half the current forward.

---

[6] Absolute volatility is defined as the standard deviation of price changes, i.e., $\mathrm{Std}(dF)$. In contrast, the usual volatility is defined as the standard deviation of relative price changes, i.e., $\mathrm{Std}(dF/F)$.

To extend the above balancing intuition to these "winger" options, we now imagine that the horizontal axis is placed on two wedges, one located at half the current forward price and the other at twice the current forward price. Then the summed product of density and distance above twice forward gives the winger call's (forward) value. Similarly, the summed product of density and absolute distance below half forward gives the winger put's (forward) value. If the density between wedges is removed, then the axis will tip right-side down because the call is more valuable than the put. However, doubling the density to the left of half forward will restore balance. In other words, two such puts have the same value as one call.

II. Single Barrier Options

In this section we represent path-dependent options with a single barrier[7] in terms of path-independent standard options. The key to providing this result is put-call symmetry, which is assumed to hold when the underlying first reaches the barrier price. Thus, the axis of symmetry for volatility is the barrier price.

Without loss of generality, we concentrate on valuing and hedging knock-out calls. Such calls behave like regular calls except that they are knocked out the first time the underlying hits a prespecified barrier. In contrast, knock in calls become standard calls when the barrier is hit and otherwise expire worthless. Given a valuation result and hedging strategy for knock-out calls, the corresponding results for knock in calls can be recovered using the following parity relation[8]:

$$OC(K,H) = C(K) - IC(K,H), \tag{6}$$

where $IC(K,H)(OC(K,H))$ is an in-call (out-call) with strike K and barrier H.

A. Down-and-Out Calls

By definition, a down-and-out call (DOC) with strike K and barrier $H < K$ becomes worthless if H is hit at any time during its life. If the barrier has not been hit by the expiration date, the terminal payoff is that of a standard call struck at K.

To hedge a down-and-out call we need to match the terminal payoff and the payoff along the barrier. Thus a first step in constructing a hedge is to match the terminal payoff, which is done by purchasing a standard call, $C(K)$. Now let us consider option values along the barrier. When $F = H$, the DOC is worthless, while our current hedge $C(K)$ has positive value. Thus we

---

[7] We focus on call options, leaving analogous results for puts as an exercise to the reader.
[8] This result does not hold for American options. See Chriss (1996) for a lucid discussion.
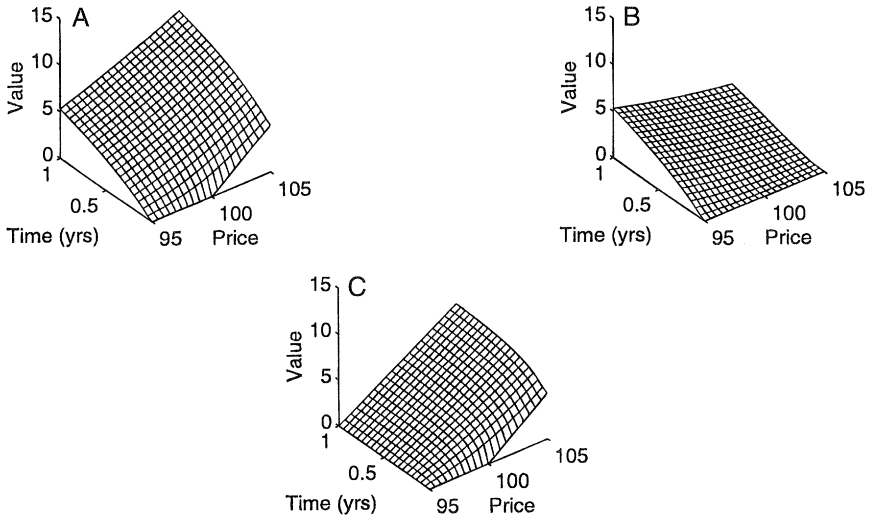
Figure 2. Static hedge for a down-and-out call ($K = 100$, $H = 95$). Panel A shows the value of a European call struck at 100 for stock prices ranging from 95 to 105 and times to maturity up to a year. The axis labeled Time indicates time to maturity. Panel B shows the value of 1.0526 puts struck at 90.25. Note that the graphs are identical along the barrier of 95. Panel C shows the difference between the first two graphs, indicating that the replicating portfolio value vanishes along the barrier.

need to sell off an instrument that has the same value as the European call when the forward price is at the barrier. Using PCS when $F = H$, we obtain[9]

$$C(K) = KH^{-1}P(H^2K^{-1}).$$

Thus, we need to write KH^{-1} European puts struck at H^2K^{-1} to complete the hedge.

Thus the complete replicating portfolio for a DOC is a buy-and-hold strategy in standard options which is purchased at the initiation of the option

$$DOC(K,H) = C(K) - KH^{-1}P(H^2K^{-1}), \qquad H < K. \tag{7}$$

If the barrier is hit before expiration, the replicating portfolio should be liquidated with PCS guaranteeing that the proceeds from selling the call are exactly offset by the cost of buying back the puts. If the barrier is not hit before expiration, then the long call gives the desired terminal payoff and the written puts expire worthless, as $H^2K^{-1} < H$ when $H < K$.

Figure 2 illustrates the replication of a down-and-out call with strike $K = \$100$, barrier $H = \$95$, and an initial maturity of one year. Panel A is of a

---

[9] The required put strike is $K_p = H^2K^{-1}$, from equation (5), and substituting this into equation (4) gives the result.

standard call with the same strike and maturity as the down-and-out. Along the barrier $F =$ \$95, the call has positive value. Panel B is of $KH^{-1} = 1.0526$ puts struck at $H^2K^{-1} =$ \$90.25. Notice that the value of these puts along the barrier $F =$ \$95 matches that of the standard call. When Panel B is subtracted from Panel A, the result is Panel C. Panel C shows that the replicating portfolio has zero value along the barrier $F =$ \$95 and the payoff of a standard call struck at \$100 at expiration.

B. Up-and-Out Calls

An up-and-out call (UOC) has a knockout barrier set above the current forward price. When the barrier is at or below the strike $(H \leq K)$, the UOC is worthless as it is always knocked out before it can have a positive payoff. Thus we need only consider barriers set above the strike $(H > K)$, which implies that the UOC has intrinsic value before it knocks out.

We again start our replicating portfolio with an European call struck at K as this matches our payoff at expiration. To get zero value along the barrier H, we could sell KH^{-1} puts struck at H^2K^{-1}, but this would give us problems at expiration if the barrier has not been hit. Instead, our replicating portfolio for an UOC uses equations (6) and (1) with up-and-in securities:

$$UOC(K,H) = C(K) - UIP(K,H) - (H-K)UIB(H), \qquad H > K, F, \qquad (8)$$

where, by definition, the up-and-in bond $UIB(H)$ pays \$1 at expiration if the barrier H has been hit before then.

To see that the portfolio matches the payoffs of the UOC, consider the payoff of the UOC if the barrier is never touched—the required payoff is that of a standard call struck at K. In the replicating portfolio, the up-and-in put and bonds expire worthless, while the standard call provides the desired payoff. Conversely, at the first passage time to the barrier, the up-and-out call knocks out just as the up-and-in put and bonds knock in. Since the forward price is at H, put–call parity implies that the replicating portfolio can be liquidated at zero cost. The up-and-in put struck at K with barrier H matches the time value of the standard call $C(K)$ at the barrier and the $(H-K)$ up-and-in bonds match its intrinsic value.

The advantage of representing an up-and-out call in terms of up-and-in puts and bonds is that equation (8) holds for any continuous process for the underlying's price. The disadvantage is that the up-and-in securities may not trade or may only trade with heavy frictions. We can apply PCS to show that the $UIP(K,H)$ can be replicated with KH^{-1} calls struck at H^2K^{-1}. The Appendix shows that an $UIB(H)$ can be replicated by buying two *binary* calls (BC) struck at H and H^{-1} European calls struck at H:

$$UIB(H) = 2BC(H) + H^{-1}C(H), \qquad H > F. \qquad (9)$$

By definition, the binary calls pay \$1 at expiry if the underlying finishes above H then. The intuition for the replication of the UIB is that when the

Table I

Convergence of Vertical Spreads to Binary Call

This table calculates the value of a vertical spread (*VS*) with parameter n, where n is the number of call spreads and its reciprocal is the spread between strikes. C is an European call. As n increases the vertical spread converges to a binary call (*BC*) with strike \$105, with analytical value \$0.292384.

n	$VS(n)$	$= n\left[C(105) - C\left(105 + \dfrac{1}{n}\right)\right]$
1	0.276446	$= C(105) - C(106)$
2	0.284331	$= 2[C(105) - C(105.50)]$
3	0.286997	$= 3[C(105) - C(105.33)]$

forward is at the barrier, each binary call is valued at approximately the probability of finishing in-the-money. If this probability were exactly 0.5, then the two binary calls alone would suffice. The positive skew of the terminal price distribution implies that the probabilities are slightly less than 0.5, entailing a minor correction using calls as in equation (9).

Rewriting equation (8) in terms of standard and binary calls gives:

$$UOC(K,H) = C(K) - KH^{-1}C(H^2K^{-1}) - (H - K)[2BC(H) + H^{-1}C(H)],$$

$$H > K, F. \quad (10)$$

It is well known that binary calls can be synthesized using an infinite number of vertical spreads of standard calls[10]

$$BC(H) = \lim_{n \uparrow \infty} n[C(H) - C(H + n^{-1})]. \tag{11}$$

As a result, the up-and-out call can be replicated using standard calls alone.

Clearly, the binary call replicating strategy in equation (11) is impractical. To remedy this, we use a technique called Richardson extrapolation which has been previously employed for option pricing (see, e.g., Geske and Johnson (1984)). Given a set of approximations indexed by a parameter (e.g., step size), Richardson extrapolation is a technique for guessing the value when the parameter is infinitesimal. We illustrate the approach for binary calls with the following example for FX options assuming constant interest rates and volatility. Suppose $F = S = \$100$, $K = \$105$, $r = r_f = 4\%$, $\sigma = 20\%$, and $T = 0.25$ years. Then the exact Black (1976) model value of the binary call is \$0.292384. Define $VS(n)$ as the value of n vertical call spreads involving strikes 105 and $105 + n^{-1}$, $n = 1,2,3$. Again using Black's model, Table I indicates the speed of convergence of $VS(n)$ to the correct value.

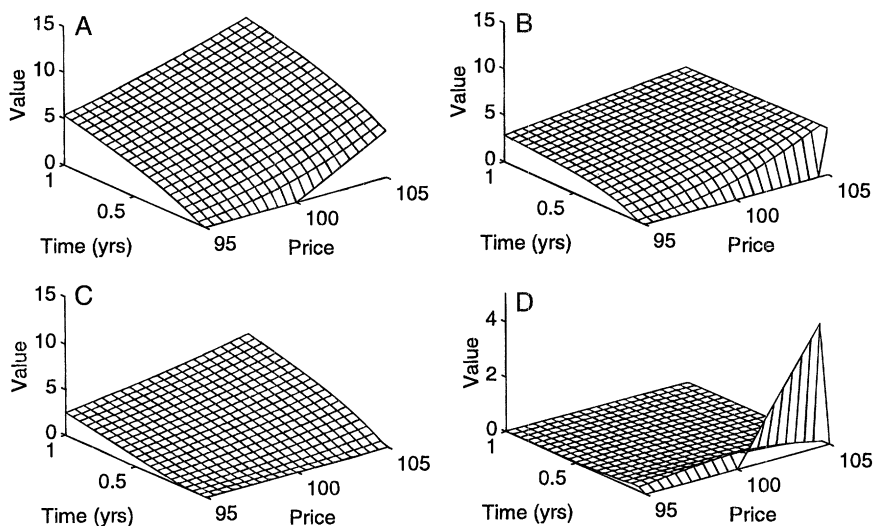[10] See Chriss and Ong (1995) for a discussion of this result.

Figure 3. Static hedge for an up-and-out call ($K = 100$, $H = 105$). Panel A shows a European call struck at 100. Panel B shows five up-and-in bonds with strike 105, which pay \$1 each at expiration if the barrier 105 has been hit by then. Panel C shows 0.9524 call struck at 110.25. The total hedge is shown in Panel D, which has been scaled up.

While the vertical spread values are slowly converging, five-decimal-place accuracy can be obtained by using the following three-point Richardson extrapolation[11]:

$$VS^{1-2-3} \equiv 0.5 \times VS(1) - 4 \times VS(2) + 4.5 \times VS(3).$$

Thus the value of a binary call is well approximated[12] by the following simple portfolio of standard calls:

$$BC(105) \approx VS^{1-2-3} = 6C(105) - 0.5C(106) + 8C(105.50) - 13.5C(105.33).$$

Figure 3 shows the value of the components of the static hedge for an up-and-out call with strike $K = \$100$, barrier $H = \$105$, and initial maturity of one year. The standard call struck at \$100 shown in Panel A has both intrinsic and time value along the barrier (105). Panel B is the $H - K = 5$ up-and-in bonds, which match the intrinsic value of the call along the barrier. Panel C is of $KH^{-1} = 0.9524$ calls struck at $H^2K^{-1} = \$110.25$, which match the time value of the call along the barrier. When Panels B and C are subtracted from Panel A, the result is shown in Panel D, indicating zero value along the barrier and the call payoff at maturity.

---

[11] See Marchuk and Shaidurov (1983), p. 24, for a derivation of Richardson weights.
[12] The approximation deteriorates near expiration when prices are near the strike.

III. Multiple Barrier Options

In this section, we discuss complex barrier options involving multiple barriers.[13] Although more complex specifications are possible, we simply assume that the volatility of the underlying is henceforth a deterministic function of time, as in Black's (1976) model; i.e., $\sigma(F,t) = \sigma(t)$ for all $F > 0$ and $t \in [0,T]$.

A. Double Knockout Calls

Consider a call option that has two barriers,[14] so that the call knocks out if either barrier is hit. We assume that the initial forward price and strike are both between the two barriers. There is a parity relation between this double knock-out call (O^2C) and a double knock in call (I^2C), which knocks in if either barrier is hit:

$$O^2C(K,L,H) = C(K) - I^2C(K,L,H), \tag{12}$$

where K is the strike, L is the lower barrier, and H is the higher barrier. We will again focus on replicating the payoffs of a double knock out call using static portfolios of standard options.

On its surface, a double knock out call $O^2C(K,L,H)$ appears to be a combination of a $DOC(K,L)$ and an $UOC(K,H)$. The payoff of the $O^2C(K,L,H)$ is zero if either barrier is hit and the standard call payoff at expiry if neither barrier is hit. A portfolio of a call knocking out at the lower barrier and a call knocking out at the higher barrier would give these payoffs, so long as the knock out of one option also knocked out the other. This additional specification is necessary as otherwise the surviving option contributes value at the other's barrier.

To construct the replicating portfolio for the $O^2C(K,L,H)$, we begin as before by purchasing a standard call $C(K)$ to provide the desired payoff at expiry. We will then attempt to zero out the value at each barrier separately. If we knew in advance that the forward price reaches the lower barrier L before it reaches the higher barrier H, then our previous analysis of a down-and-out call implies that the value of the call $C(K)$ can be nullified along the barrier L by initially selling KL^{-1} puts struck at L^2K^{-1}. Thus, the replicating portfolio under this assumption would be:

$$O^2C(K,L,H) \approx C(K) - KL^{-1}P(L^2K^{-1}). \tag{13}$$

Alternatively, if we knew in advance that the forward price reaches the higher barrier H first, then from equation (10) the replicating portfolio would instead be:

$$O^2C(K,L,H) \approx C(K) - KH^{-1}C(H^2K^{-1})$$
$$- (H - K)[2BC(H) + H^{-1}C(H)]. \tag{14}$$

---

[13] Partial barrier options may be statically hedged using a portfolio of standard and compound options. A discussion of this can be obtained from the authors.

[14] Double barrier calls and puts have been priced analytically in Kunitomo and Ikeda (1992).

Because we don't know in advance which barrier will be hit first, we try combining the two portfolios:

$$O^2C(K,L,H) \approx C(K) - DIC(K,L) - UIC(K,H)$$
$$= C(K) - KL^{-1}P(L^2K^{-1}) - KH^{-1}C(H^2K^{-1})$$
$$- (H - K)[2BC(H) + H^{-1}C(H)]. \tag{15}$$

The problem with this portfolio is that each written-in call contributes (negative) value at the other's barrier. For example, if the forward price reaches H first, then the $DIC(K,L) = KL^{-1}P(L^2K^{-1})$ contributes (negative) value along H. Thus, we need to add securities to the portfolio in an effort to zero out value along each barrier. Along the barrier H, the negative influence of the KL^{-1} puts struck at L^2K^{-1} can be offset by buying LH^{-1} calls struck at H^2KL^{-2}. To cancel the negative influence of the $UIC(K,H)$ along the barrier L, we will need to extend PCS to binary calls.

Recall that a binary call (put) is a cash-or-nothing option that pays \$1 if the stock price is above (below) a strike price K, and zero otherwise. Similarly, a gap call (put) is an asset-or-nothing option that pays the stock price S if it is above (below) a strike price K, and zero otherwise. The following parity relations are easily shown:

$$GC(K) = K \cdot BC(K) + C(K), \qquad GP(K) = K \cdot BP(K) - P(K).$$

Since binary options may be synthesized out of standard options, these parity relations imply that the same is true for gap options. The Appendix proves the following symmetry result, relating values of binary options to gap options.

BINARY PUT–CALL SYMMETRY: *Given frictionless markets, no arbitrage, zero drift, and deterministic volatility, the following relationships hold:*

$$K^{1/2}BC(K) = GP(H)H^{-1/2} \qquad H^{1/2}BP(H) = GC(K)K^{-1/2}, \tag{16}$$

where the geometric mean of the binary call strike K and the binary put strike H is the forward price F:

$$(KH)^{1/2} = F.$$

Armed with this result, we can cancel the negative influence of the $UIC(K,H)$ in equation (15) along the barrier L. Thus, our first layer approximation for the double knock out call value is:

$$O^2C(K,L,H) = C(K) - L^{-1}(KP(L^2K^{-1}) - HP(L^2KH^{-2}))$$
$$- H^{-1}(KC(H^2K^{-1}) - LC(H^2KL^{-2}))$$
$$- (H - K)[2BC(H) + H^{-1}C(H) - 2L^{-1}GP(L^2H)$$
$$- L^{-1}P(L^2H^{-1})]. \tag{17}$$

Although equation (17) is a better approximation than equation (15), the added options still contribute value at the other's barrier. Thus, we need to continue to subtract or add options, noting that each additional layer of hedge at one barrier creates an error at the other barrier. As a result, the replicating portfolio for a double knock-out call can be written as an infinite sum:

$$
\begin{aligned}
O^2C(K,L,H) = C(K) - \sum_{n=0}^{\infty} \Big[& L^{-1}(HL^{-1})^n(KP(L^2K^{-1}(LH^{-1})^{2n}) \\
& - HP(K(LH^{-1})^{2(n+1)})) \\
& + H^{-1}(LH^{-1})^n(KC(H^2K^{-1}(HL^{-1})^{2n}) \\
& - LC(K(HL^{-1})^{2(n+1)})) + 2(H-K)(HL^{-1})^n \\
& \times [BC(H(HL^{-1})^{2n}) - L^{-1}GP(L(LH^{-1})^{2n+1})] \\
& + (H-K)[H^{-1}(LH^{-1})^nC(H(HL^{-1})^{2n}) \\
& \qquad - L^{-1}(HL^{-1})^nP(L(LH^{-1})^{2n+1})] \Big].
\end{aligned}
$$
(18)

Note that the options in the infinite sum are all initially out-of-the-money. Furthermore, as n increases, the number of options held and the options' moneyness both decrease exponentially. As a result, for large n, the options' contribution to the infinite sum becomes minimal. Thus we can get a good approximation to the option value with a small value of n. Table II shows a typical example. With $F = K = 100$, barriers at $L = 95$ and $H = 105$, $r = r_f = 4$ percent, $\sigma = 20$ percent, and $T = 0.25$, five-decimal-place accuracy has occurred by summing the values for $n = 0, 1, 2$. The value for $n = \infty$ is obtained from the analytic solution by Kunitomo and Ikeda (1992).

Figure 4 graphs the value of the second-layer hedge, i.e., $n = 1$ in equation (18), for a double knock out call option. Notice that the value along both barriers is very close to zero.

In general, bringing in the barriers of a double knock out call reduces both its value and the number of options needed to achieve a given accuracy.

B. Roll-Down Calls

A double knock out call involves two barriers that straddle the initial spot. In contrast, a roll-down call (RDC)[15] involves two barriers, both below the initial spot and strike. If the nearer barrier is not hit prior to maturity, then a roll-down call has the same terminal payoff as a standard call struck at K_0. However, if the nearer barrier H_1 is hit prior to maturity, then the strike is rolled down to it, and a new out-barrier becomes active at $H_2 < H_1$. For later use, we extend the definition of a RDC as follows. We assume that if the nearer barrier H_1 is hit, then the strike rolls down to some level $K_1 \in [H_1, K_0]$,

---

[15] For a discussion of roll-down calls and roll-up puts, see Gastineau (1994).

Table II
Convergence of Replicating Portfolio to Double Knock out Call (O^2C) Value

The values are generated by using the static hedging portfolio for $O^2C(K,L,H)$ for increasing values of N.

$$O^2C(K,L,H) = C(K) - \sum_{n=0}^{N} \Big[L^{-1}(HL^{-1})^n(KP(L^2K^{-1}(LH^{-1})^{2n}) - HP(K(LH^{-1})^{2(n+1)}))$$
$$+ H^{-1}(LH^{-1})^n(KC(H^2K^{-1}(HL^{-1})^{2n}) - LC(K(HL^{-1})^{2(n+1)}))$$
$$+ 2(H-K)(HL^{-1})^n[BC(H(HL^{-1})^{2n}) - L^{-1}GP(L(LH^{-1})^{2n+1})]$$
$$+ (H-K)[H^{-1}(LH^{-1})^nC(H(HL^{-1})^{2n})$$
$$- L^{-1}(HL^{-1})^nP(L(HL^{-1})^{2n+1})]\Big].$$

C and P are European calls and puts, respectively; K is the strike price; L and H are lower and upper barriers, respectively; BC is a binary call; GP is a gap put; r_f is the foreign interest rate and r is the domestic interest rate; σ is the volatility of the underlying asset; and T is the time to maturity of the option. The parameters for the option are initial forward price $F = 100$, $K = 100$, barriers at $L = 95$ and $H = 105$, $r = r_f = 4\%$, $\sigma = 20\%$, and $T = 0.25$. The value for $N = \infty$ is given by the analytic solution of Kunitomo and Ikeda (1992).

N	Value of Replicating Portfolio
0	0.074763
1	0.007781
2	0.007746
3	0.007746
⋮	⋮
∞	0.007744

which need not equal H_1. We also assume that if the farther barrier H_2 is hit, then the strike rolls down to some level $K_2 \in [H_2, K_1]$ and a new out-barrier becomes active further down at $H_3 < H_2$. This process repeats an arbitrary number of times.

Let $H_1, \ldots, H_n$ be a decreasing sequence of positive barrier levels set below the initial forward price, $F > H_1$. Similarly, let $K_0, \ldots, K_n$ be a decreasing sequence of strikes, with $K_i \geq H_i$, $i = 1, \ldots, n$. Then at initiation, the extended roll-down call can be decomposed into down-and-out calls as

$$ERDC(K_i, H_i) = DOC(K_0, H_1) + \sum_{i=1}^{n} [DOC(K_i, H_{i+1}) - DOC(K_i, H_i)]. \quad (19)$$

This representation is model-independent. To obtain a standard roll-down call, we set $n = 1$ and $K_1 = H_1$. For any n, the hedge works as follows. If the underlying never hits the barrier H_1, then the $DOC(K_0, H_1)$ provides the desired payoff $(F_T - K_0)^+$, and the knock out calls in the sum cancel each
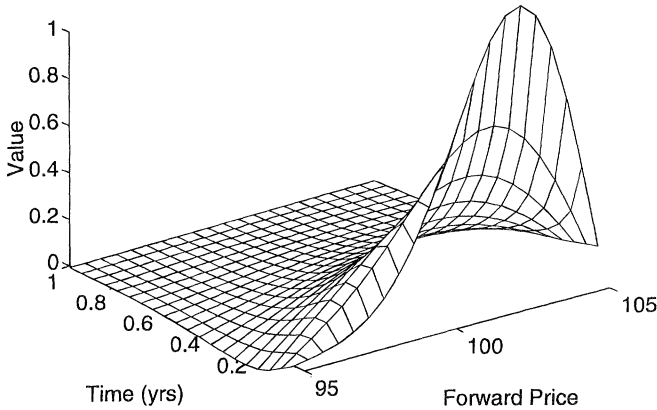
Figure 4. Second layer static hedge for a double knockout call option with barriers at 95 and 105 and strike at 100.

other. If the barrier H_1 is hit, then the $DOC(K_0, H_1)$ vanishes, as does the written $DOC(K_1, H_1)$. Thus the position when $F = H_1$ may be rewritten as

$$ERDC(K_i, H_i) = DOC(K_1, H_2) + \sum_{i=2}^{n} [DOC(K_i, H_{i+1}) - DOC(K_i, H_i)]. \quad (20)$$

This is analogous to our initial position. In between any two barriers H_i and H_{i+1}, the $DOC(K_i, H_{i+1})$ provides the desired payoff if the next barrier is never hit, but the DOCs in the sum roll down the strike to K_{i+1} if this barrier is hit.

When PCS holds at each barrier, the extended roll-down call value at initiation, for $F > H_1$, is given by

$$ERDC(K_i, H_i) = C(K_0) - K_0 H_1^{-1} P(H_1^2 K_0^{-1})$$

$$+ \sum_{i=1}^{n} [K_i H_i^{-1} P(H_i^2 K_i^{-1}) - K_i H_{i+1}^{-1} P(H_{i+1}^2 K_i^{-1})]. \quad (21)$$

The replicating strategy is as follows. At any time, we are always holding a call struck at or above the highest untouched barrier and puts struck at or below this barrier. Thus, if the forward price never reaches this barrier, the call provides the desired payoff at expiry, and the puts expire worthless. Each time the forward price touches a barrier H_i for the first time, we sell the call struck at K_{i-1} and buy back $K_{i-1} H_i^{-1}$ puts struck at $H_i^2 K_{i-1}^{-1}$; sell $K_i H_i^{-1}$ puts struck at $H_i^2 K_i^{-1}$ and buy the call struck at K_i. PCS guarantees that both transitions are self-financing.

As previously mentioned, the standard roll-down call is the special case of equation (21) with $n = 1$ and $K_1 = H_1$. Figure 5 illustrates the replication
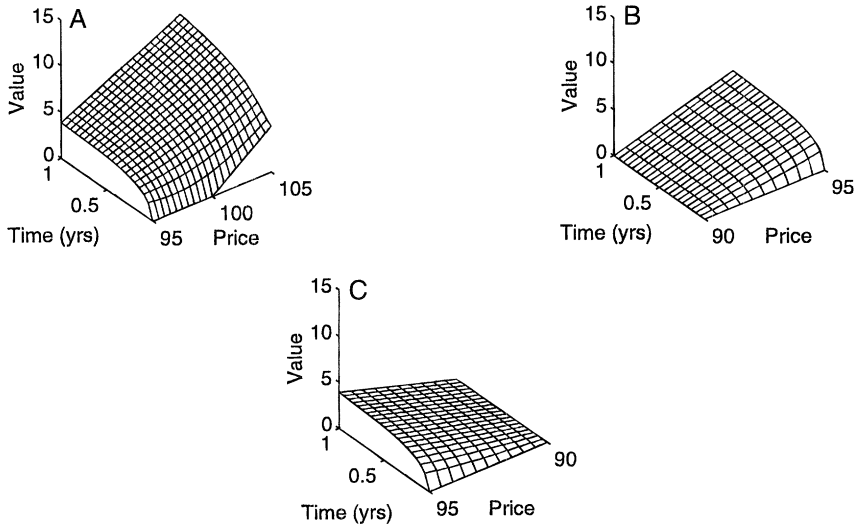
Figure 5. The two part static hedge for a roll-down call ($K_0 = 100$, $K_1 = H_1 = 95$, $H_2 = 90$). Panel A shows the replicating portfolio when the price is above the first barrier, 95. Panel B shows the replicating portfolio after the barrier 95 has been hit. The two portfolios are identical along the barrier of 95, and the second is worth zero along the lower barrier 90. Panel C is a different perspective of Panel B, showing more clearly value of the portfolio at 95.

procedure for a standard roll-down call with initial strike $K_0 = \$100$, rolled-down strike $K_1 = H_1 = \$95$, and final out-barrier $H_2 = \$90$. Panel A shows the value of the replicating portfolio before the first barrier is hit. If the forward hits the first barrier H_1, then the portfolio is costlessly revised to $C(H_1) - H_1 H_2^{-1} P(H_2^2 H_1^{-1})$. Panel B shows the value of this new portfolio for prices below \$95. The revised portfolio has zero value along the knock out barrier $H_2 = 90$ as required. Panel C is just Panel B with a different orientation, showing that the value of the two portfolios match along the first barrier $H_1 = 95$.

C. Ratchet Calls

A ratchet call is an extended roll-down call, with strikes set at the barriers, which never knocks out completely. This is accomplished by having the only purpose of the lowest barrier be to ratchet down the strike. This suggests that we can create a static hedge for a ratchet call once we account for this difference.

To synthesize a ratchet call with initial strike K_0, we set the strikes K_i in the $ERDC(K_i, H_i)$ equal to the barriers H_i, $i = 1, \ldots, n-1$. To deal with the fact that an extended roll-down call knocks out completely if the forward reaches H_n, while the ratchet call rolls down the strike to H_n, we replace the

last spread of down-and-out calls $[DOC(H_n,H_{n+1}) - DOC(H_n,H_n)]$ in equation (19) with a down-and-in call $DIC(H_n,H_n)$. Thus, a model-independent valuation of a ratchet call, using barrier calls, is

$$RC(K_0,H_i) = DOC(K_0,H_1) + \sum_{i=1}^{n-1} [DOC(H_i,H_{i+1}) - DOC(H_i,H_i)]$$
$$+ DIC(H_n,H_n), \quad F > H_1. \tag{22}$$

Substituting in the model-free results, $DOC(K,H) = C(K) - DIC(K,H)$ and $DIC(H,H) = P(H)$ simplifies the result to:

$$RC(K_0,H_i) = DOC(K_0,H_1) + \sum_{i=1}^{n-1} [P(H_i) - DIC(H_i,H_{i+1})]$$
$$+ P(H_n), \quad F > H_1. \tag{23}$$

When PCS holds at each barrier, a ratchet call can be represented in terms of standard options as

$$RC(K_0,H_i) = C(K_0) - K_0 H_1^{-1} P(H_1^2 K_0^{-1}) + \sum_{i=1}^{n-1} [P(H_i) - H_i H_{i+1}^{-1} P(H_{i+1}^2 H_i)]$$
$$+ P(H_n), \quad F > H_1. \tag{24}$$

Hedging with this replicating portfolio is analogous to the extended roll-down call hedge: the position held is changed at every barrier, and the transitions are self-financing. Comparing equation (24) with its counterpart for an extended roll-down call allows us to capture the value of removing an out-barrier at H_{n+1}. Setting $K_i = H_i$ in equation (21) and comparing with equation (24) implies that the value of removing this barrier is given by $H_n H_{n+1}^{-1}$ puts struck at $H_{n+1}^2 H_n^{-1}$.

D. Lookback Calls

A floating strike lookback call (LC) is similar to a ratchet call, except that there is a continuum of rolldown barriers extending from the initial forward price to the origin, so that the strike price is the minimum price over the option's life. A ratchet call with $K_0 = F$, $H_n = 0$, undervalues a lookback because the active strike is always at or above that of a lookback. Thus, a model-free lower bound for a lookback call is

$$LC \geq RC(F,H_i) = DOC(F,H_1) + \sum_{i=1}^{n-1} [P(H_i) - DIC(H_i,H_{i+1})] + P(H_n).$$

$$\tag{25}$$

When PCS holds at each barrier, this lower bound can be expressed in terms of standard options:

$$LC \geq C(F) - FH_1^{-1}P(H_1^2 F^{-1}) + \sum_{i=1}^{n-1}[P(H_i) - H_i H_{i+1}^{-1}P(H_{i+1}^2 H_i^{-1}) + P(H_n).$$

$$(26)$$

The portfolio of standard options undervalues the lookback because the call held is always struck at or above the lookback. By adding more strikes, we obtain a tighter bound. Since the underlying's prices are actually discrete, one possibility is to set the barriers at each possible level.

To obtain an upper bound on the value of a lookback call, we may use an *extended* ratchet call, which ratchets the strike down to the next barrier each time a new barrier is crossed. When the last positive barrier is touched, the strike is ratcheted down to zero. Thus, a model-free upper bound in terms of down-and-in bonds is

$$LC \leq C(H_1) - P(H_1) + \sum_{i=1}^{n-1}[(H_i - H_{i+1})DIB(H_i)] + H_n DIB(H_n). \qquad (27)$$

Intuitively, when each barrier H_i is reached for the first time, the down-and-in bonds ratchet down the delivery price of the synthetic forward $C(H_1) - P(H_1)$ by $H_i - H_{i+1}$ dollars.

When PCS holds at each barrier, it can be used to represent the down-and-in bonds in terms of standard options. In particular, using an argument analogous to that in the Appendix for an up-and-in bond, a down-and-in bond can be replicated using the following static portfolio of binary and standard puts: $DIB(H) = 2BP(H) - H^{-1}P(H)$. Richardson extrapolation may again be used to efficiently represent the binary puts in terms of standard puts.[16]

We can modify the above bounds for both a forward-start and a backward-start lookback call. Let 0 be the valuation date and let T_1 be the start date of the lookback period. In the backward-start case ($T_1 < 0$), the underlying has some minimum-to-date, m, which is in between two barriers H_i and H_{i+1} for some $\hat{\imath}$. The lower bound is thus a ratchet call with initial strike H_i and barriers H_i where $i = \hat{\imath} + 1, \ldots, n - 1$. Similarly, the upper bound is an extended ratchet call with initial strike H_{i+1} and barriers H_i, $i = \hat{\imath} + 1, \ldots,$ $n - 1$. Because ratchet calls and extended ratchet calls can be replicated with standard options, we have bounded the lookback call in terms of static portfolios of standard options.

---

[16] Richardson extrapolation may also be used to enhance convergence of the lower and upper bounds of a lookback call by extrapolating down the distance between barriers. A discussion of this can be obtained from the authors.

In the forward-start case ($T_1 > 0$), we use the fact that the formula for a backward-start lookback call is linearly homogeneous in the current spot/forward price and the minimum to date. At T_1, the minimum is S_{T_1} so the lookback call value at T_1 may be written as $c(\cdot)S_{T_1}$, for some function $c(\cdot)$ independent of S_{T_1}. Thus, for a forward-start LC, we should initially hold $ce^{-\delta T_1}$ units of the underlying. Moving forward through time, the dividends received are reinvested back into the security, bringing the number of units held up to c by time T_1. At T_1, the c shares can be sold for proceeds just sufficient to initiate the approximating strategy described above.

IV. Nonzero Carrying Costs

The previous results were derived assuming that the drift of the underlying was zero (under the martingale measure). This assumption is natural for options on futures, but strained somewhat for options on the spot. In this section, we relax the assumption of zero drift. Although we are no longer able to obtain exact static hedges for options on the spot, we can develop tight bounds on option values using static hedges. Bowie and Carr (1994) give the bounds for single barrier options, so we concentrate on multiple barrier calls. For concreteness, we deal with options on spot foreign exchange (FX), assuming constant interest rates for simplicity. Then, interest rate parity links forward prices ($F(t)$) and spot prices ($S(t)$) of FX by

$$F(t) = S(t)e^{(r-r_f)(T-t)}, \qquad t \in [0, T], \tag{28}$$

where r is the domestic rate and r_f is the foreign rate. Thus, when the spot hits a flat barrier H, the forward hits a time-dependent barrier $H(t) \equiv He^{(r-r_f)(T-t)}$.

A. Double Knock out Calls

When the drift of the underlying is not zero, a double knock out call on the spot with flat barriers L and H is equivalent to a double knock out call on the forward price with time-dependent barriers $L(t) = Le^{(r-r_f)(T-t)}$ and $H(t) = He^{(r-r_f)(T-t)}$ with $t \in [0, T]$. We can give flat upper and lower bounds on these time-dependent barriers. If $r > r_f$[17]:

$$L \leq L(t) \leq \bar{L} \equiv Le^{(r-r_f)T} \qquad H \leq H(t) \leq \bar{H} \equiv He^{(r-r_f)T}.$$

Double knock out options increase in value as the out-barriers are moved farther apart. Thus for the double knock out call on the forward,[18] we can write

$$O^2C_f(K,\bar{L},H) \leq O^2C_f(K,L(t),H(t)) \leq O^2C_f(K,L,\bar{H}). \tag{29}$$

---

[17] The details for hedging multiple barrier calls when $r < r_f$ are left as an exercise for the reader.

[18] Since interest rates are constant, results for options on forwards also hold for options on futures.
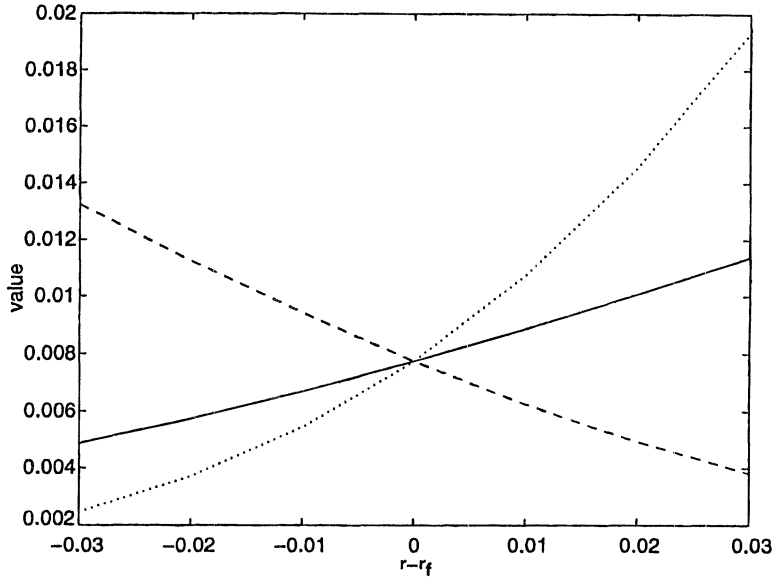
Figure 6. Synthesizing a double knockout call with cost of carry. Value of the upper (dashed line) and lower (dotted line) bound static hedges for a double knock-out call ($K = 100$, lower barrier 95, and upper barrier 105) compared with the analytical value (solid line). The foreign interest rate (r_f) is fixed at 4 percent and the domestic interest rate (r) varies from 1 percent to 7 percent. For $r < r_f$ the lower bound is the upper bound and vice versa.

Furthermore, by definition, the double knock out on the forward with time-dependent barriers is the same as the double knock out on the spot with flat barriers:

$$O^2 C_f(K, L(t), H(t)) = O^2 C_s(K, L, H). \tag{30}$$

Combining equations (29) and (30) allows us to bound the value for a double knock-out call on the spot between the values of two double knock out calls on the forward:

$$O^2 C_f(K, \bar{L}, H) \le O^2 C_s(K, L, H) \le O^2 C_f(K, L, \bar{H}). \tag{31}$$

As we know how to replicate each of the two bounds with a static portfolio, we have upper and lower bounds on the double knock out call on the spot. Figure 6 indicates how the bounds vary with the interest rate differential.

B. Roll-Down Calls

Recall that under zero drift and with PCS holding at every barrier, an extended roll-down call (ERDC) was synthesized out of standard call and put options in equation (21).

When the drift of the underlying is not zero, an ERDC on the spot with flat strikes K_i, $i = 0, \ldots, n$ and barriers H_i, $i = 1, \ldots, n$ is equivalent to an ERDC on the forward with time-dependent strikes $K_{it} \equiv K_i e^{(r-r_f)(T-t)}$ and barriers $H_{it} \equiv H_i e^{(r-r_f)(T-t)}$:

$$ERDC_s(K_i, H_i) = ERDC_f(K_{it}, H_{it}).$$

We can give flat upper and lower bounds on these time-dependent quantities. If $r > r_f$,

$$K_i \le K_{it} \le \overline{K}_i \equiv K_i e^{(r-r_f)T}, \qquad t \in [0, T], i = 0, \ldots, n$$

$$H_i \le H_{it} \le \overline{H}_i \equiv H_i e^{(r-r_f)T}, \qquad t \in [0, T], i = 1, \ldots, n.$$

The $ERDC_f$ value is decreasing in the level of the strikes and increasing in the level of the barriers. Thus, we can place bounds on $ERDC_f(K_{it}, H_{it})$:

$$ERDC_f(\overline{K}_i, H_i) \le ERDC_f(K_{it}, H_{it}) \le ERDC_f(K_i, \overline{H}_i). \tag{32}$$

From Section III.B we can create static hedge portfolios for the upper and lower bounds, and the tightness of these bounds for a standard roll-down call ($n = 1$, $K_1 = H_1$) is shown in Figure 7.

C. Ratchet Calls

A ratchet call on the spot is a special case of an extended roll-down call on the spot created by setting the strikes K_i equal to the barriers H_i and removing the last knock-out barrier. As a result, the bounds for a ratchet call on the spot are determined similarly to an extended roll-down call. The lower bound is a ratchet call that ratchets every time the flat barrier H_i is hit to a strike of $\overline{H}_i$. The higher bound is an extended ratchet call on the forward that ratchets every time the barrier $\overline{H}_i$ is hit to a strike of H_i:

$$RC_f(\overline{H}_i, H_i) \le RC_s(H_i \le RC_f(H_i, \overline{H}_i). \tag{33}$$

D. Lookback Calls

Consider a ratchet call on spot with the initial strike K_0 set at the initial spot price and the final rung H_n set at the origin. As in the case of zero carrying costs, this ratchet call undervalues a lookback due to the discreteness of the rungs. As a result, the lower bound for a ratchet call on spot is also a lower bound for a lookback call on spot. An upper bound for a lookback call on spot can be obtained from an extended ratchet call on spot, which ratchets the strike to the next lower barrier. However, an extended ratchet call on the spot with flat barriers is equivalent to an extended ratchet call on forward with time-dependent barriers. A lower bound can be obtained from a generalization of equation (27). For each time-dependent barrier, H_{it}, each
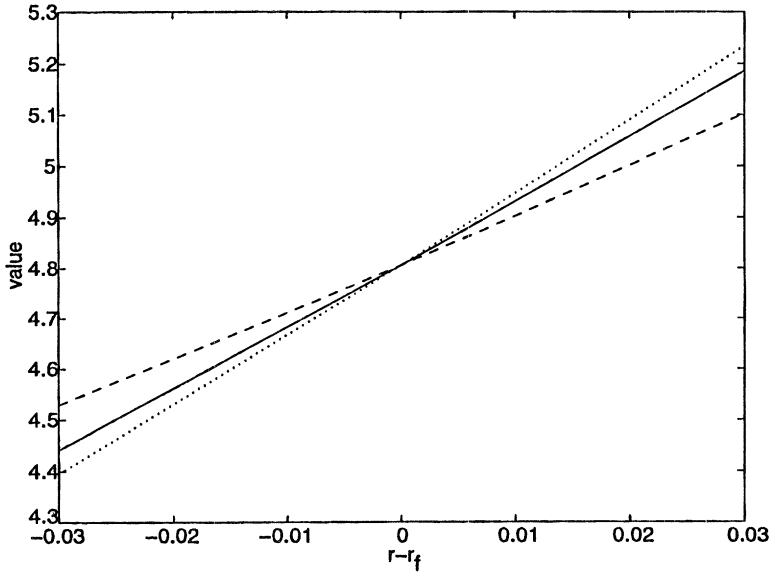
Figure 7. Synthesizing a roll-down call with cost of carry. Value of the upper (dashed line) and lower (dotted line) bound static hedges for a roll-down call ($K_0 = 100$, $K_1 = H_1 = 95$, $H_2 = 90$) compared with the analytical value (dotted line). The foreign interest rate is fixed at 4 percent and the domestic interest rate varies between 1 percent and 7 percent. When the domestic rate is below the foreign rate, the lower bound becomes the upper bound and vice versa.

time the forward reaches the flat upper bound $\bar{H}_i$, we ratchet the strike to the flat lower bound H_i. The resulting bounds for a lookback call on spot at issuance are:

$$C(F) - FH_1^{-1}P(H_1^2\bar{H}_0^{-1}) + \sum_{i=1}^{n-1}[\bar{H}_iH_i^{-1}P(H_i^2\bar{H}_i^{-1}) - \bar{H}_iH_{i+1}^{-1}P(H_{i+1}^2\bar{H}_i^{-1})]$$

$$+ \bar{H}_nH_n^{-1}P(H_n^2\bar{H}_n^{-1})$$

$$\leq LC_s$$

$$\leq C(H_1) - P(H_1) + \sum_{i=1}^{n-1}[(H_i - H_{i+1})DIB(\bar{H}_i)] + H_nDIB(\bar{H}_n).$$

Using this approach, bounds for forward and backward start lookback calls can also be obtained.

V. Conclusion

The concept of hedging exotic options with a static portfolio of standard instruments simplifies the risk management of exotic options in several ways. First, when compared with dynamically rebalancing with the underlying,

the static portfolio is easier to construct initially and to maintain over time. Instead of continuously monitoring the underlying and trading with every significant price change, the hedger can place contingent buy and sell orders with start/stop prices at the barriers. Second, when compared with offsetting the risk using another path-dependent option, the investor uses instruments with which he is familiar, the risks are better understood, and the markets are more liquid.

A static hedge can exactly replicate the payoffs of the path-dependent option when carrying costs are zero; and a pair of static hedges can bracket the payoffs when nonzero carrying costs are introduced. These techniques apply to many path-dependent options, which are related in that their payoffs depend on whether one or more barriers are crossed.

The fundamental result underpinning the creation of our replicating portfolios is put–call symmetry. By using this simple formula, we can engineer simple portfolios to mimic the values of standard options along barriers. The result is an extension of put–call parity to different strike prices which provides further insight into the relation between put and call options.

The main extension to this line of research would involve relaxation of the zero drift and symmetry conditions. Just as bounds are developed when drift is nonzero, perhaps tight bounds can be developed when volatility structures display asymmetry with sufficient stationarity. In the interests of brevity, this extension is left for future research.

Appendix

Put–Call Symmetry

Let $F(t)$ be the forward price at $t \in [0, T]$ of the underlying for delivery in T years. Let $\sigma(F(t), t)$ be the local volatility rate of the forward price as a function of the forward price $F(t)$ and time t. Under the martingale measure,[19] the forward price process is

$$\frac{dF(t)}{F(t)} = \sigma(F(t), t) dW(t). \tag{A1}$$

Let $B(0)$ be the price at time 0 of a bond paying one dollar in T years and let $C(0, K, T)$ and $P(0, K, T)$ be the initial value of an European call and put struck at K and maturing in T years. Let $G_c(K, T) \equiv B(0)^{-1} C(0, K, T)$ and $G_p(K, T) \equiv B(0)^{-1} P(0, K, T)$ be the respective forward values quoted at 0 of these options for delivery in T years. We now show that both forward values satisfy the following forward partial differential equation (pde):

$$\frac{\sigma^2(K, T) K^2}{2} \frac{\partial^2 G}{\partial K^2}(K, T) = \frac{\partial G}{\partial T}(K, T), \qquad K > 0, T > 0, \tag{A2}$$

---

[19] We use a pure discount bond maturing at T as the numeraire.

In contrast to Black (1976) backward pde, this pde indicates how (forward) option values change with the strike and maturity, holding the initial time and underlying forward price fixed. The above result and its proof below are essentially due to Dupire (1994).

To prove the forward pde for a call, we begin with the standard result that the forward price of a call is given by its expected payoff under the equivalent martingale measure:

$$G_c(K,T) = \int_K^\infty (F(T) - K)p(F(T),T;F(0),0)dF(T), \tag{A3}$$

where $p(F(T),T;F(0),0)$ is the transition density of the forward price, indicating the probability density of the forward price being at $F(T)$ at time T, given that it is at $F(0)$ at time 0. The Kolmogorov forward equation governing this density is

$$\frac{1}{2}\frac{\partial^2}{\partial K^2}[\sigma^2(K,T)K^2p(K,T;F(0),0)] = \frac{\partial}{\partial T}p(K,T;F(0),0), \qquad K > 0, T > 0.$$

$$\tag{A4}$$

Differentiating (A3) twice with respect to K gives

$$\frac{\partial^2 G_c(K,T)}{\partial K^2} = p(K,T;F(0),0). \tag{A5}$$

Substituting into the Kolmogorov equation gives

$$\frac{1}{2}\frac{\partial^2}{\partial K^2}\left[\sigma^2(K,T)K^2\frac{\partial^2 G_c(K,T)}{\partial K^2}\right] = \frac{\partial}{\partial T}\frac{\partial^2 G_c(K,T)}{\partial K^2}, \qquad K > 0, T > 0. \tag{A6}$$

Integrating twice with respect to K gives the desired result. The same proof applies to European puts. It is easily verified that Black (1976) formulas for calls and puts satisfy the above equation with $\sigma^2(K,T) = \sigma^2$.

The forward call value $G_c(K,T)$ is the unique solution of equation (A2) subject to the following boundary conditions:

(a) $G_c(K,0) = \max[F(0) - K,0]$, $K > 0$;

(b) $\lim_{K\uparrow\infty} G_c(K,T) = 0, T > 0$;

(c) $\lim_{K\downarrow 0} G_c(K,T) = F(0), T > 0$.

Similarly, the forward put value $G_p(K,T)$ is the unique solution of equation (A2) subject to the following boundary conditions:

(a) $G_p(K,0) = \max[K - F(0),0]$, $K > 0$;

(b) $\lim\limits_{K\uparrow\infty} G_p(K,T) \sim K, T > 0$;

(c) $\lim\limits_{K\downarrow 0} G_p(K,T) = 0, T > 0$.

Let $u_c(x,T) \equiv G_c(K,T)(KF(0))^{-1/2}$ and $u_p(x,T) \equiv G_p(K,T)(KF(0))^{-1/2}$ be *normalized* call and put forward values, respectively, written as functions of $x \equiv \ln(K/F(0))$ and maturity T. Then, the normalized values both solve the following pde:

$$\frac{v^2(x,T)}{2} \frac{\partial^2 u}{\partial x^2}(x,T) - \frac{v^2(x,T)}{8} u(x,T) = \frac{\partial u}{\partial T}(x,T), \quad x \in (-\infty,\infty), T > 0,$$

$$(A7)$$

where $v(x,T) \equiv \sigma(F(0)e^x,T)$ is the volatility expressed as a function of x and T.

The normalized forward call value $u_c(x,T)$ is the unique solution of equation (A7) subject to the following boundary conditions:

(a) $u_c(x,0) = \max[e^{-x/2} - e^{x/2},0]$, $x \in \Re$;

(b) $\lim\limits_{x\uparrow\infty} u_c(x,T) = 0$, $T > 0$;

(c) $\lim\limits_{x\downarrow-\infty} u_c(x,T) = e^{-x/2}$, $T > 0$.

Similarly, the normalized forward put value $u_p(x,t)$ is the unique solution of equation (A7) subject to the following boundary conditions:

(a) $u_p(x,0) = \max[e^{x/2} - e^{-x/2},0]$, $x \in \Re$;

(b) $\lim\limits_{x\uparrow\infty} u_p(x,T) \sim e^{x/2}$, $T > 0$;

(c) $\lim\limits_{x\downarrow-\infty} u_p(x,T) = 0$, $T > 0$.

Our symmetry condition is that $v^2(x,T) = v^2(-x,T)$ for all $x \in \Re$ and for all $T > 0$. Given this condition, it is easy to see that $u_c(x,T)$ and $u_p(-x,T)$ satisfy the same boundary value problems and are therefore equal:

$$u_c(x,T) = u_p(-x,T), \quad \text{for all } x \in \Re \text{ and for all } T > 0.$$

Reverting to forward prices gives

$$G_c(K_c,T)(K_cF(0))^{-1/2} = G_p(K_p,T)(K_pF(0))^{-1/2},$$

where $(K_cK_p)^{1/2} = F(0)$. Multiplying both sides by $F(0)^{1/2}B(0)$ gives the desired result:

$$C(0,K_c,T)K_c^{-1/2} = P(0,K_p,T)K_p^{-1/2}.$$

Q.E.D.

Binary Put–Call Symmetry

Assuming that volatility is a function of time alone, the payoff and values for binary calls and puts, and gap calls and puts with time T until maturity, and strike H can be written as:

Payoff at time T	Value at time 0
$BC = 1(F(T) > H)$	$BC = B(0)N(d_2)$
$GC = F(T)1(F(T) > H)$	$GC = B(0)F(0)N(d_1)$
$BP = 1(F(T) < H)$	$BP = B(0)N(-d_2)$
$GP = F(T)1(F(T) < H)$	$GP = B(0)F(0)N(-d_1),$

where

$$N(d) \equiv \int_{-\infty}^{d} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$$

is the standard normal distribution function,

$$d_1 \equiv \frac{\ln\left(\dfrac{F(0)}{H}\right) + \dfrac{\bar{\sigma}^2 T}{2}}{\bar{\sigma}\sqrt{T}}, d_2 = d_1 - \bar{\sigma}\sqrt{T},$$

and

$$\bar{\sigma}^2 \equiv \frac{1}{T}\int_0^T \sigma^2(t)\, dt.$$

It may be verified by direct substitution that:

$$K^{1/2}BC(K) = GP(H)H^{-1/2} \qquad H^{1/2}BP(H) = GC(K)K^{-1/2}, \tag{A8}$$

where the geometric mean of the bianry (gap) call strike K and the gap (binary) put strike H is the forward price F: $(KH)^{1/2} = F$.

Up-and-In Bond

We can rewrite an up-and-in bond as a combination of an up-and-in binary call and an up-and-in binary put:

$$UIB(H) = UIBC(H) + UIBP(H). \tag{A9}$$

However, an $UIBC(H)$ is the same as a standard $BC(H)$, as it has to knock in to have positive value. We can expand the $UIBP(H)$ into its components:

$$UIB(H) = BC(H) + \lim_{n \uparrow \infty} n[UIP(H,H) - UIP(H - n^{-1}, H)]. \tag{A10}$$

We can apply PCS:

$$UIB(H) = BC(H) + \lim_{n\uparrow\infty} n[C(H) - (H - n^{-1})H^{-1}C(H^2(H - n^{-1})^{-1})],$$

(A11)

or equivalently:

$$UIB(H) \approx BC(H) + H^{-1}\lim_{n\uparrow\infty} C(H^2(H - n^{-1})^{-1})$$

$$+ \lim_{n\uparrow\infty} n[C(H) - C(H + n^{-1})].$$ (A12)

The approximation error is $O(n^{-2})$. The final term can now be rewritten as a binary call and so

$$UIB(H) = 2BC(H) + H^{-1}C(H).$$ (A13)

Q.E.D.

REFERENCES

Bates, David, 1988, The crash premium: Option pricing under asymmetric processes, with applications to options on Deutschemark futures, Working paper, University of Pennsylvania.

Bates, David, 1991, The crash of 87: Was it expected? The evidence from options markets, *Journal of Finance* 46, 1009–1044.

Black, Fischer, 1976, The pricing of commodity contracts, *Journal of Financial Economics* 3, 167–179.

Black, Fischer, and Myron Scholes, 1973, The pricing of options and corporate liabilities, *The Journal of Political Economy* 81, 637–659.

Bowie, Jonathon, and Peter Carr, 1994, Static simplicity, *Risk* 7, 45–49.

Chriss, Neil, 1996. *Black–Scholes and Beyond: Modern Option Pricing* (Irwin Professional Publishing, Burr Ridge, IL).

Chriss, Neil, and Michael Ong, 1995, Digitals diffused, *Risk* 8, 56–59.

Derman, Emanuel, Deniz Ergener, and Iraj Kani, 1994, Forever hedged, *Risk* 7, 139–145.

Dupire, Bruno, 1994, Pricing with a smile, *Risk* 7, 18–20.

Gastineau, Gary, 1994, Roll-up puts, roll-down calls, and contingent premium options, *The Journal of Derivatives* 1, 40–43.

Geske, Robert, and Herbert Johnson, 1984, The American put option valued analytically, *Journal of Finance* 39, 1511–1524.

Kunitomo, Naoto, and Masayuki Ikeda, 1992, Pricing options with curved boundaries, *Mathematical Finance*, 275–298.

Marchuk, Gurii, and Vladimir Shaidurov, 1983, *Difference Methods and Their Extrapolations* (Springer Verlag, NY).

Merton, Robert, 1973, Theory of rational option pricing, *Bell Journal of Economics and Management Science* 4, 141–183.