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SELF-DECOMPOSABILITY AND OPTION PRICING

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The risk-neutral process is modeled by a four parameter self-similar process of independent increments with a self-decomposable law for its unit time distribution. Six different processes in this general class are theoretically formulated and empirically investigated. We show that all six models are capable of adequately synthesizing European option prices across the spectrum of strikes and maturities at a point of time. Considerations of parameter stability over time suggest a preference for two of these models. Currently, there are several option pricing models with 6–10 free parameters that deliver a comparable level of performance in synthesizing option prices. The dimension reduction attained here should prove useful in studying the variation over time of option prices.

KEY WORDS: additive processes, scaling, Background Driving Lévy Processes, Ornstein–Uhlenbeck Processes

1. INTRODUCTION

The standard models for portfolio allocation (Merton 1973) and for option pricing (Black and Scholes 1973) both assume that continuously compounded returns are normally distributed. The central limit theorem is often invoked as a primary motivation for this assumption. By this theorem, the normal distribution arises as the limiting distribution for the sum of n independent random variables, when the sum is divided by $\sqrt{n}$. Hence, if returns are realized as the sum of a large number of independent influences, then one can anticipate that returns will in fact be normally distributed.

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Unfortunately, it is well documented that the assumed normality of the return distribution is violated in both the time-series data and in option prices. This has led many authors to consider jump-diffusion models (Merton 1976; Jones 1984; Naik and Lee 1990; Bates 1991), stochastic volatility models (Heston 1993), pure jump Lévy processes (Madan, Carr, and Chang 1998; Barndorff-Nielsen 1998; Eberlein, Keller, and Prause 1998), and various combinations of these alternatives (Bates 1996, 2000; Duffie, Pan, and Singleton 2000; Barndorff-Nielsen and Shephard 2001; Carr et al. 2002).The use of these processes for option pricing is now covered in a number of books and we cite by way of example, Schoutens (2003), Cont and Tankov (2004), and Applebaum (2004). To adequately explain the time-series data and the variation in option prices across both strike and maturity, these models employ between 6–10 parameters, which is a far cry from the single-parameter model originally proposed by Black and Scholes.

The reason that so many parameters are needed is usually thought to be due to the complexity of the underlying stochastic process. Bates (1996), Bakshi, Cao, and Chen (1997), and Carr et al. (2003) all argue that stochastic volatility is needed to explain option prices at long maturities while jumps are needed to simultaneously explain the short maturity option prices.

In order to see if more parsimonious models of option prices can be obtained, this paper examines if there are alternatives to the Gaussian distribution as a limit law. We note that there is no compelling economic motivation for the scaling function to be $\sqrt{n}$ as opposed to some other function of n. As a result, we are lead to the so-called laws of class L, which were defined by Khintchine (1938) and Lévy (1937) as limit laws for sums of n independent variables when centered and scaled by functions of n, not necessarily $\sqrt{n}$. These laws were subsequently found to be identical to the so-called class of self-decomposable laws, which loosely speaking describe random variables that decompose into the sum of a scaled down version of themselves and an independent residual term. For interesting examples we refer the reader to Knight (2001) who also observes that this class of laws stands between the stable laws and the infinitely divisible laws. Sato (1991) showed that the self-decomposable laws are associated with the unit time distribution of self-similar additive processes, whose increments are independent, but need not be stationary. Jeanblanc, Pitman, and Yor (2001) recently show how one may easily pass between these additive self-similar representations and stationary solutions to OU equations driven by Lévy processes (Barndorff-Nielsen and Shephard 2001). In this paper, we investigate and report on the effectiveness of these self-similar processes as models for a risk-neutral process.

We restrict our attention to several four parameter models as it was felt that the volatility smile at each term requires at least three parameters to explain cross-sectional variations in level, slope, and curvature. A four parameter model provides just a single additional parameter to govern the variation of these smiles across maturity. Based on the higher dimensional parameterizations used in the previous literature, our expectation was that these models would not succeed in providing an effective synthesis of option prices across both strike and maturity. Much to our surprise, our empirical results indicate that these four parameter models have an average relative pricing error of below 3%, when describing the prices of over 150 options written on the S&P 500. Furthermore, the low parametric dimension of these models induces calibrations which are much quicker than those of models with comparable pricing error and more parameters. While we did find that some higher parametric structures provided marginal improvements in accuracy, we restrict our attention here to the relatively parsimonious class of models associated with the self-decomposable laws at unit time and the associated self-similar additive processes.

Nine specific models are described in the paper, although the three models with relatively poor performance are not described in detail. The remaining six models were found to synthesize the option price surface equally well. However, considerations of parameter stability over time indicate a preference for two of the constructions, based on the variance gamma process and on the Meixner process.

The outline of the paper is as follows. Section 2 introduces the laws of class L and the concept of self-decomposable laws, and outlines their association with self-similar processes and with stationary solutions to OU equations. In Section 3, we introduce the six self-decomposable laws studied in this paper, and their associated stochastic processes. Section 4 summarizes the data and describes the design of the study. Results are presented in Sections 5 and 6 while Section 7 concludes.

2. SELF-DECOMPOSABLE LAWS AND ASSOCIATED PROCESSES

This section introduces the laws of class L and the self-decomposable laws, which are known to be identical. The connections of these laws to various stochastic processes is then described. Finally, we indicate the financial relevance of the resulting stochastic processes for option pricing.

2.1. Laws of Class L and Self-Decomposable Laws

Consider a sequence $(Z_k : k = 1, 2, \ldots)$ of independent random variables and let $S_n = \sum_{k=1}^{n} Z_k$ denote their sum. Suppose that there exist centering constants c_n and scaling constants b_n such that the distribution of $b_n S_n + c_n$ converges to the distribution of some random variable X. Then the random variable X is said to have the class L property. In other words, a random variable has a distribution of class L if the random variable has the same distribution as the limit of some sequence of normalized sums of independent random variables. These laws were studied by Lévy (1937) and Khintchine (1938) who coined the term class L. By the central limit theorem, a random variable with the standard normal distribution has the class L property, as does a random variable with a stable distribution. However, the laws of class L represent an important generalization of Gaussian and stable laws as they describe limit laws with more general scaling constants than $1/\sqrt{n}$. In a financial context, the increased flexibility may be required if the independent influences being summed are of different orders of magnitude. For example, suppose that the sum of a set of independent influences on returns diverges when divided by $\sqrt{n}$, but that convergence arises for an exponentially weighted average multiplied by $\sqrt{n}$, where the weighted average is given for $\rho > 1$ by

$$\frac{(\rho - 1)\sum_{k=1}^{n} \rho^{k-1} U_k}{(\rho^n - 1)}.$$

Defining $Z_k = \rho^{k-1} U_k$, as the scaled effect, then we are considering the limit of the sum S_n multiplied by

$$b_n = \frac{\sqrt{n}(\rho - 1)}{(\rho^n - 1)}.$$

The distribution of a random variable X is said to be self-decomposable (Sato 1999, page 90, Definition 15.1) if for any constant c, $0 < c < 1$ there exists an independent random variable say, $X^{(c)}$ such that

$$X \overset{\text{law}}{=} cX + X^{(c)}.$$

In other words, a random variable is self-decomposable if it has the same distribution as the sum of (cX) a scaled down version of itself and an independent residual random variable $(X^{(c)})$. Self-decomposable laws have the property that the associated densities are unimodal (Yamazato 1978; Sato 1999, p. 404).

The self-decomposable laws are an important sub-class of the class of infinitely divisible laws, as noted by Knight (2001) who studies in detail an interesting example. Lévy (1937) (see also Loève 1945) showed that self-decomposable laws are infinitely divisible with a special structure of their Lévy measure. Specifically, the characteristic function of these laws (Sato 1999, p. 95, Corollary 15.11) has the form

$$E\left[e^{iuX}\right] = \exp\left[ibu - \frac{1}{2}Au^2 + \int_{-\infty}^{\infty}(e^{iux} - 1 - iux\mathbf{1}_{|x|<1})\frac{h(x)}{|x|}\,dx\right],$$

where $A \geq 0$, b is a real constant, $h(x) \geq 0$, $\int_{-\infty}^{\infty}(|x|^2 \wedge 1)\frac{h(x)}{|x|}\,dx < \infty$, and $h(x)$ is increasing for negative x and decreasing for positive x. For explicit examples of convergence to self decomposable limit laws we refer the reader to Berkes (1991). Thus, an infinitely divisible law is self-decomposable if the corresponding Lévy density has the form $\frac{h(x)}{|x|}$ where $h(x)$ is increasing for negative x and decreasing for positive x. We call $h(x)$ the self-decomposability characteristic (SDC) of the random variable X. Note that it $X(t)$ is a Lévy process then $X(1)$ is self-decomposable if and only if $X(t)$ is self-decomposable for every $t > 0$. We then say that the process $X(t)$ enjoys the self-decomposability property. Note that this SDC representation holds for both processes of bounded and unbounded variation.

Sato (1999, p. 91, Theorem 15.3) shows that a random variable has a distribution of class L if and only if the law of the random variable is self-decomposable. Since it is desirable that a return distribution can be motivated as a limit law and that it be unimodal and infinitely divisible, we are led to consider self-decomposable laws as candidates for the unit period distribution of financial returns. Many of the older jump-diffusion models used in the option pricing literature have Gaussian or exponential jump sizes. These compound Poisson processes do not enjoy the self-decomposability property, as the Lévy densities do not assume the necessary form. In contrast, the recent Lévy models employed by Barndorff-Nielsen (1998), Eberlein, Keller, and Prause (1998), Madan, Carr, and Chang (1998), and Carr et al. (2002) all enjoy the self-decomposability property.

2.2. Processes Associated with Self-Decomposable Laws

The economic arguments which motivate option pricing formulas generally rely on dynamic trading in one or more assets whose prices follow stochastic processes in continuous time. Furthermore, financial contracting often involves the valuation of claims with payoffs depending on the time path of stock prices, as opposed to just the level of the price on a certain date. These considerations motivate modeling the stochastic process of prices instead of just the random variable associated with the price at some future date. Fortunately Sato (1991) established a connection between a self-decomposable law holding at a fixed time and a stochastic process reigning over the time interval.

First, note that a self-similar process is defined as a stochastic process ($Y(t)$, $t \geq 0$) with the property that for any $\lambda > 0$ and all t,

$$Y(\lambda t) \overset{\text{law}}{=} a(\lambda) Y(t). \tag{2.1}$$

It follows on considering $Y(\lambda\mu t)$ in two ways that:

$$Y(\lambda\mu t) \overset{\text{law}}{=} a(\lambda\mu) Y(t) \overset{\text{law}}{=} a(\lambda) Y(\mu t) \overset{\text{law}}{=} a(\lambda)a(\mu) Y(t)$$

and hence $a(t) = t^{\gamma}$ for some exponent γ. We then say that Y is γ-self-similar.

Sato (1991) defines additive processes as processes with inhomogeneous (in general) and independent increments. In the particular case when the increments are time homogeneous, the process is called a Lévy process. Sato (1991) showed that a law is self-decomposable if and only if it is the law at unit time of an additive process, that is also a self-similar process. As a result, we will refer to such processes as Sato processes.

To relate these concepts in a simple setting, suppose that a self-decomposable random variable X is the value at unit time of some pure jump Lévy process whose sample paths have bounded variation. We consider the case when the Lévy density integrates $|x|$ in the region $|x| < 1$ for which $b = \int_{|x|<1} x\frac{h(x)}{|x|}\, dx$. In this case the characteristic function of X has the form

$$E\big[e^{iuX}\big] = \exp\left[\int_{-\infty}^{\infty} (e^{iux} - 1)\frac{h(x)}{|x|}\, dx\right]. \tag{2.2}$$

Let $Y(t)$ be the value at time t of a self-similar additive process with paths of bounded variation. The characteristic function for $Y(t)$ may be written as

$$E\big[e^{iuY(t)}\big] = \exp\left[\int_0^t \int_{-\infty}^{\infty} (e^{iuy} - 1)\, g(y, s)\, dy\, ds\right], \tag{2.3}$$

for some *time-dependent Lévy system* $g(y, t)$. Suppose that we require that the law of the self-similar additive process at unit time be the self-decomposable law of the random variable X:

$$Y(1) \overset{\text{law}}{=} X. \tag{2.4}$$

Then the following theorem relates the *time-dependent Lévy system* to the *SDC*, $h(x)$ of the self-decomposable law.

THEOREM 1. *Given a self-decomposable law for the time one distribution (2.4) with a characteristic function satisfying (2.2), then there exists a self-similar process $Y(t)$ defined with respect to the increasing scaling function t^{γ} by (2.1) and which satisfies (2.3) when*

$$g(y, t) = \begin{cases} -\dfrac{h'\left(\frac{y}{t^{\gamma}}\right)\gamma}{t^{1+\gamma}}, & y > 0 \\[2ex] \dfrac{h'\left(\frac{y}{t^{\gamma}}\right)\gamma}{t^{1+\gamma}} & y < 0. \end{cases}$$

Proof. Combining equations (2.1), (2.4), and (2.2) we see that we must have

$$E\big[e^{iuY(t)}\big] = \exp\left[\int_{-\infty}^{\infty}(e^{iut^{\gamma}x}-1)\frac{h(x)}{|x|}\,dx\right] = \exp\left[\int_{-\infty}^{\infty}(e^{iuy}-1)\frac{h\left(\frac{y}{t^{\gamma}}\right)}{|y|}\,dy\right]. \tag{2.5}$$

Equating (2.5) to (2.3) separately for the positive and negative sides we get

$$\int_0^t\int_0^{\infty}(e^{iuy}-1)g(y,s)\,dy\,ds = \int_0^{\infty}(e^{iuy}-1)\frac{h\left(\frac{y}{t^{\gamma}}\right)}{y}\,dy \tag{2.6}$$

and on the negative side we have

$$\int_0^t\int_{-\infty}^{0}(e^{iuy}-1)g(y,s)\,dy\,ds = \int_{-\infty}^{0}(e^{iuy}-1)\frac{h\left(\frac{y}{t^{\gamma}}\right)}{|y|}\,dy. \tag{2.7}$$

Differentiating with respect to t in the equation (2.6) and substituting $-\lambda = iu$ we get that

$$\int_0^{\infty}(e^{-\lambda y}-1)g(y,t)\,dy = -\int_0^{\infty}(e^{-\lambda y}-1)\frac{h'\left(\frac{y}{t^{\gamma}}\right)\gamma}{t^{1+\gamma}}\,dy.$$

Now differentiate with respect to λ to get

$$-\int_0^{\infty}e^{-\lambda y}yg(y,t)\,dy = \int_0^{\infty}e^{-\lambda y}\frac{yh'\left(\frac{y}{t^{\gamma}}\right)\gamma}{t^{1+\gamma}}\,dy,$$

and it follows that

$$g(y,t) = -\frac{h'\left(\frac{y}{t^{\gamma}}\right)\gamma}{t^{1+\gamma}}, \quad y > 0. \tag{2.8}$$

For the negative side we rewrite equation (2.7) with $\lambda = iu$ as

$$\int_0^t\int_0^{\infty}(e^{-\lambda y}-1)g(-y,s)\,dy\,ds = \int_0^{\infty}(e^{-\lambda y}-1)\frac{h\left(-\frac{y}{t^{\gamma}}\right)}{y}\,dy.$$

Differentiation with respect to t yields

$$\int_0^{\infty}(e^{-\lambda y}-1)g(-y,t)\,dy = \int_0^{\infty}(e^{-\lambda y}-1)\frac{h'\left(-\frac{y}{t^{\gamma}}\right)\gamma}{t^{1+\gamma}}\,dy.$$

Differentiation with respect to λ yields

$$-\int_0^{\infty}e^{-\lambda y}yg(-y,t)\,dw = -\int_0^{\infty}e^{-\lambda y}\frac{yh'\left(-\frac{y}{t^{\gamma}}\right)\gamma}{t^{1+\gamma}}\,dw,$$

and it follows that

$$g(y,t) = \frac{h'\left(\frac{y}{t^{\gamma}}\right)\gamma}{t^{1+\gamma}}, \quad y < 0. \tag{2.9}$$

□

Observe that it is precisely the property of h that it be increasing on the left and decreasing on the right that yields g as a positive inhomogeneous Lévy density. Different choices of γ or exponents lead to unique representations of additive processes or processes with independent and inhomogeneous increments with the self-decomposable law as the unit time distribution. In the terminology for a fixed γ, the process is called self-similar with exponent γ. The proof of Theorem 1 has been presented for the bounded variation case but a similar argument extends this result to pure jump processes of infinite variation.

2.2.1. Some Other Processes Associated with Self-Similar Processes. It is shown in Lamperti (1962), (see also Embrechts and Maejima 2002) that one may associate with any γ-self-*similar* process $Y(t)$ a stationary process Z_t defined by

$$Z_u = e^{-\gamma u} Y(e^u)$$

$$Y(t) = t^\gamma Z(\log(t))$$

and so we observe that our scaled self-decomposable process $Y(t)$ is also a scaled and time changed stationary process.

It is further shown in Jeanblanc, Pitman, and Yor (2001) that the stationary process Z_u, $u \geq 0$ is the solution to the Ornstein–Uhlenbeck equation associated with a Background Driving Lévy Process (Barndorff-Nielsen and Shephard 2001) $U(t)$

$$dZ = -\gamma Z\, dt + dU$$

with initial condition $Z(0) = X$.

The Lévy process may itself be constructed from the γ-self-*similar* process $Y(t)$ in accordance with

$$U(t) = \int_1^{e^t} \frac{1}{s^\gamma}\, dY(s).$$

2.2.2. Financial Relevance of Self-Similarity. The law of any infinitely divisible random variable may be used to construct a Lévy process. This construction is employed in Madan, Carr, and Chang (1998), Barndorff-Nielsen (1998), and Eberlein, Keller, and Prause (1998) to develop several Lévy processes. Konikov and Madan (2002), noted that the term t skewness for such processes falls like $1/\sqrt{t}$ while the excess kurtosis falls like $1/t$. They also empirically determined the term structures of these moments from market option prices and found that these moments may be rising somewhat or be constant, but they are not falling. Self-similar processes have the property that these higher moments are constant over the term by construction and hence in this respect they are consistent with our observations. This feature of the resulting process provides further encouragement in evaluating the potential relevance of these processes as models for risk-neutral returns over differing horizons.

3. SOME SPECIFIC SELF-DECOMPOSABLE RISK NEUTRAL PROCESSES

In this section, we consider six examples of self-decomposable laws with which we shall associate a γ-self-similar additive process $Y(t)$. We then define the risk-neutral process for the stock price process $S(t)$ in terms of $Y(t)$ by

$$S(t) = S(0)e^{rt}\frac{e^{Y(t)}}{E[e^{Y(t)}]}. \tag{3.1}$$

The stock price process defined by (3.1) is a Markov process whose proportional drift is the interest rate r. The discounted price process $S(t)e^{-rt}$ is a martingale, since $Y(t)$ is a process of independent increments. The characteristic function for $\ln(S(t))$ is easily written in terms of the characteristic function for $Y(t)$,

$$\phi_{Y(t)}(u) = E\left[e^{iuY(t)}\right]$$

and specifically we have that

$$E[e^{iu\ln(S(t))}] = \exp(iu(\ln(S(0)) + rt - \ln(\phi_{Y(t)}(-i)))\phi_{Y(t)}(u). \tag{3.2}$$

The characteristic function (3.2) is employed to estimate the parameters of the risk-neutral process using market closing prices of European options. Model option prices are obtained using the *FFT* methodology described in Carr and Madan (1998).

We note that in all the cases considered the stock price process is a pure jump process with a mean rate of return equal to the interest rate. Furthermore, we consider special semi-martingales where one may employ the identity function as a truncation function and hence we may write the infinitesimal generator of the Markov process for the stock price, in terms of the Lévy system $g(y, t)$ identified in Theorem 1, for a test function $f(S)$ as

$$\begin{aligned}\mathcal{I}(f) = &\left(r - \int_{-\infty}^{\infty}(e^y - 1)g(y, t)\,dy\right)S\frac{\partial}{\partial S}f \\ &+ \int_{-\infty}^{\infty}\left(f(Se^y) - f(S) - S(e^y - 1)\frac{\partial}{\partial S}f\right)g(y, t)\,dy.\end{aligned}$$

We now develop the explicit form for the characteristic functions employed in our study. The first three self-decomposable laws are those for the unit time variance gamma (VG) model, of Madan, Carr, and Chang (1998), the normal inverse Gaussian (NIG) model of Barndorff-Nielsen (1998), and the Meixner process (MXNR) developed by Grigelionis (1999) and Schoutens (2001). In addition, we develop three new processes based on laws related to the hyperbolic functions and studied by Pitman and Yor (2000): the three processes involved employ the hyperbolic cosine, sine, and tangent functions in their analytical structure. These processes time change Brownian motion using processes related to the hyperbolic functions. Skewness is then added using an Esscher transform which is equivalent to risk shifting using a power utility function defined over the final stock price. We term the three new processes (VC), (VS), and (VT) as the variance conditional on the time change is related to processes employing the cosh, sinh, or tanh functions. The details for the VG, NIG, and MXNR processes are presented in separate subsections while the three hyperbolic processes are treated collectively in a fourth subsection. The six processes are denoted VGSSD, NIGSSD, MXNRSSD, VCSSD, VSSSD, and VTSSD where the addition of the extension SSD signifies that the risk-neutral density varies with maturity by a scaled self-decomposable law. We shall see that all six processes are described by exactly four parameters. The first three parameters provide respective control over the variance, skewness and excess kurtosis, while the fourth parameter provides control over the effect of the horizon length on the risk-neutral distribution.

3.1. VGSSD

The VG process is defined by time changing an arithmetic Brownian motion with drift θ and volatility σ by an independent gamma process with unit mean rate and variance rate ν. Let $G(t;\nu)$ be the gamma process, then the variance gamma process may be written as

$$X_{\mathrm{VG}}(t;\sigma,\nu,\theta) = \theta G(t;\nu) + \sigma W(G(t;\nu)),$$

where $W(t)$ is an independent standard Brownian motion. Madan, Carr, and Chang (1998) show that the VG process can also be expressed as the difference of two independent gamma processes. Carr et al. (2002) show that the VG process is a Lévy process whose Lévy density has the form

$$k_{\mathrm{VG}}(x) = \begin{cases} C\dfrac{\exp(Gx)}{|x|} & x < 0 \\ C\dfrac{\exp(-Mx)}{x} & x > 0, \end{cases}$$

where the parameters C, G, M are explicitly related to the original parameters by

$$\begin{aligned} C &= \frac{1}{\nu} \\ G &= \left(\sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} - \frac{\theta\nu}{2}\right)^{-1} \\ M &= \left(\sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} + \frac{\theta\nu}{2}\right)^{-1}. \end{aligned}$$

We observe that the SDC for the VG process is

$$h_{\mathrm{VG}}(x) = \begin{cases} C\exp(Gx) & x < 0 \\ C\exp(-Mx) & x > 0. \end{cases}$$

The exponential and negative exponential are classic examples of functions, which are increasing and decreasing, when the domains are restricted to the negative and positive axis, respectively. Therefore the unit time VG law is a self-decomposable law.

The characteristic function for the VG process may be computed by conditioning on the gamma time change and recognizing that the conditional characteristic function is Gaussian. The resulting integral is easily calculated by recognizing it as a Laplace transform. We thus obtain that

$$E\left[e^{iuX(1)}\right] = \left(\frac{1}{1 - iu\theta\nu + \dfrac{\sigma^2\nu}{2}u^2}\right)^{\frac{1}{\nu}}.$$

By the scaling property we want the law of $Y(t)$ to be that of $t^{\gamma}X(1)$ and hence it follows that

$$\phi_{\mathrm{VGSSD}}(u,t) = \left(\frac{1}{1 - iu\theta\nu t^{\gamma} + \dfrac{\sigma^2\nu}{2}u^2 t^{2\gamma}}\right)^{\frac{1}{\nu}}. \tag{3.3}$$

Substituting (3.3) into (3.2) completes the specification of the log characteristic function for the stock price at an arbitrary maturity.

3.2. NIGSSD

The NIG process also has a characteristic function defined by three parameters (see Barndorff-Nielsen 1998). To obtain the characteristic function, we follow the presentation in Carr et al. (2003). From this perspective, we first define inverse Gaussian time I_t^ν as the time it takes an independent Brownian motion with drift ν to reach the level t. It is well known that the Laplace transform of this random time is

$$E\big[\exp\big(-\lambda I_t^\nu\big)\big] = \exp\big(-t(\sqrt{2\lambda+\nu^2}-\nu)\big). \tag{3.4}$$

The process is well defined for $\nu > 0$, while for $\nu < 0$ it gets infinite almost surely; more precisely, $P(I_t^\nu < \infty) = \exp(2t\nu)$. Next, we evaluate an independent arithmetic Brownian motion with drift θ and volatility σ at this inverse Gaussian time:

$$X_{\text{NIG}}(t;\sigma,\nu,\theta) = \theta I_t^\nu + \sigma W\big(I_t^\nu\big) \tag{3.5}$$

on the set $I_t^\nu < \infty$. The characteristic function of the resulting process is evaluated in Carr et al. (2003) as

$$E\left[e^{iuX_{\text{NIG}}(t)}\right] = \exp\left(-t\sigma\left(\sqrt{\frac{\nu^2}{\sigma^2}+\frac{\theta^2}{\sigma^4}-\left(\frac{\theta}{\sigma^2}+iu\right)^2}-\frac{\nu}{\sigma^2}\right)\right).$$

To obtain the NIG Lévy density, we condition on a jump of magnitude g in the time change. The conditional move is then normally distributed with mean θg and variance $\sigma^2 g$. The arrival rate for the jumps is given by the following Lévy density for inverse Gaussian time:

$$k(g) = \frac{\exp\left(-\frac{\nu^2}{2}g\right)}{g^{3/2}}.$$

It follows, on applying Sato's (1999) theorem 30.1, that the Lévy density for NIG is

$$\begin{aligned}
&\int_0^\infty \frac{1}{\sigma\sqrt{2\pi g}}\exp\left(-\frac{(x-\theta g)^2}{2\sigma^2 g}\right)\frac{1}{g^{3/2}}\exp\left(-\frac{\nu^2}{2}g\right)dg\\
&= \frac{1}{\sigma}\int_0^\infty \frac{1}{\sqrt{2\pi}}g^{-2}\exp\left(-\frac{x^2}{2\sigma^2 g}-\frac{\nu^2}{2}g+\frac{\theta}{\sigma^2}x-\frac{\theta^2}{2\sigma^2}g\right)dg\\
&= \frac{e^{\frac{\theta}{\sigma^2}x}}{\sigma}\int_0^\infty \frac{1}{\sqrt{2\pi}}t^{-2}\exp\left(-\frac{\nu^2+\frac{\theta^2}{\sigma^2}}{2}t-\frac{x^2}{2\sigma^2 t}\right)dt\\
&= \frac{e^{\frac{\theta}{\sigma^2}x}}{\sigma}\int_0^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-s-\frac{x^2\alpha^2}{4s}\right)s^{-2}\frac{\sigma^2\alpha^2}{2}\,ds,
\end{aligned}$$

where

$$\alpha = \sqrt{\frac{\nu^2}{\sigma^2}+\frac{\theta^2}{\sigma^4}}.$$

We now recall the integral representation of the McDonald function

$$K_a(x) = \frac{1}{2}\left(\frac{x}{2}\right)^a \int_0^\infty \exp\left(-\left(t + \frac{x^2}{4t}\right)\right) t^{-a-1}\, dt.$$

Hence, we may write

$$\int_0^\infty \exp\left(-\left(t + \frac{x^2}{4t}\right)\right) t^{-a-1}\, dt = 2K_a(x)\left(\frac{2}{x}\right)^a$$

and so by Sato's (1999) theorem 30.1, the NIG Lévy density is given by

$$k_{\mathrm{NIG}}(x) = \sqrt{\frac{2}{\pi}}\sigma\alpha^2 \frac{e^{\frac{\theta}{\sigma^2}x} K_1(|x|)}{|x|}. \tag{3.6}$$

For an alternative derivation of the Lévy density of NIG, we refer the reader to Barndorff-Nielsen (1998). It follows that the SDC for NIG process is given by

$$h_{\mathrm{NIG}}(x) = \sqrt{\frac{2}{\pi}}\sigma\alpha^2 e^{\frac{\theta}{\sigma^2}x} K_1(|x|)$$

and hence the law is self-decomposable for θ/σ^2 sufficiently small.

The characteristic function for the NIGSSD process is easily obtained by evaluating the NIG characteristic function for $X_{\mathrm{NIG}}(1)$ at ut^γ. This result is

$$\phi_{\mathrm{NIGSSD}}(u,t) = \exp\left(-\sigma\left(\sqrt{\frac{\nu^2}{\sigma^2} + \frac{\theta^2}{\sigma^4} - \left(\frac{\theta}{\sigma^2} + iut^\gamma\right)^2} - \frac{\nu}{\sigma^2}\right)\right). \tag{3.7}$$

3.3. MXNRSSD

The Meixner process has recently been proposed by Grigelionis (1999) and Schoutens (2001). The characteristic function for zero drift is

$$E\left[e^{iuX_{\mathrm{MXNR}}(t)}; a, b, d\right] = \left(\frac{\cos\left(\frac{b}{2}\right)}{\cosh\left(\frac{au - ib}{2}\right)}\right)^{2dt}.$$

We note here that $X_{\mathrm{MXNR}}(t; a, b, d) \overset{(d)}{=} aX_{\mathrm{MXNR}}(dt; 1, b, 1)$, and furthermore the process $(X_{\mathrm{MXNR}}(u; 1, b, 1), u \geq 0)$ is obtained by an Esscher transform applied to the case $b = 0$. When $b = 0$ we observe that, for fixed t,

$$X_{\mathrm{MXNR}}(t; 1, 0, 1) \overset{(d)}{=} \beta\left(\int_0^1 R_{4t}^2(s)\, ds\right),$$

where β is a Brownian motion and R_{4t} is an independent Bessel process of dimension $4t$. The Esscher transform is comparable to the procedure described below in equation (3.11) in the context of the hyperbolic processes.

The probability density of the Meixner distribution is given on Fourier inversion of the characteristic function by

$$f(x; a, b, d) = \frac{\left(2\cos\left(\frac{b}{2}\right)\right)^{2d}}{2a\pi\Gamma(2d)} \exp\left(\frac{b}{a}x\right)\left|\Gamma\left(d + i\frac{x}{a}\right)\right|^2,$$

where $\Gamma(z)$ is the gamma function with complex argument z.

The Lévy density is given by

$$k_{\text{MXNR}}(x) = d\frac{\exp\left(\frac{b}{a}x\right)}{x\sinh\left(\frac{\pi x}{a}\right)}.$$

Hence, the SDC of the Meixner process is given by

$$h_{\text{MXNR}}(x) = d\frac{\exp\left(\frac{b}{a}x\right)}{\left|\sinh\left(\frac{\pi x}{a}\right)\right|}.$$

This function also satisfies the self-decomposability condition for small enough values of b/a.

The characteristic function MXNRSSD is obtained as usual as

$$\phi_{\text{MXNRSSD}}(u, t) = \left(\frac{\cos\left(\frac{b}{2}\right)}{\cosh\left(\frac{aut^{\gamma} - ib}{2}\right)}\right)^{2d}.$$

3.4. The Hyperbolic Processes VCSSD, VSSSD, and VTSSD

We define two increasing additive processes denoted by C_t, S_t by their Laplace transforms:

$$E[e^{-\lambda C_t}] = \left(\frac{1}{\cosh(\sqrt{2\lambda t})}\right)$$

$$E[e^{-\lambda S_t}] = \left(\frac{\sqrt{2\lambda t}}{\sinh(\sqrt{2\lambda t})}\right).$$

These processes may be described by

$$C_t = \inf\{s : |B_s| = t\}$$

$$S_t = \inf\{s : \text{BES}(3, s) = t\},$$

where B_s is a standard Brownian motion and BES$(3, s)$ is the Bessel process of dimension 3, i.e., the norm of a three-dimensional standard Brownian motion.

Using Lévy's theorem for C_t and the results of Pitman (1975) on three-dimensional Brownian motion for S_t, we write alternative characterizations for these processes as

$$C_t \overset{(d)}{=} \inf\{s : M_s - B_s = t\}$$

$$S_t \overset{(d)}{=} \inf\{s : 2M_s - B_s = t\},$$

where $M_t = \sup_{s \le t} B_s$.

We now allow for drift in the Brownian motion. Hence, let

$$B_t^{(\nu)} = \nu t + B_t$$

and define

$$C_t^{(\nu)} = \inf\left\{s : M_s^{(\nu)} - B_s^{(\nu)} = t\right\}$$
$$S_t^{(\nu)} = \inf\left\{s : 2M_s^{(\nu)} - B_s^{(\nu)} = t\right\},$$

where $M_t^{(\nu)} = \sup_{s \le t} B_s^{(\nu)}$.

We also consider a one dimensional diffusion $Z_t^{(\nu)}$ with infinitesimal generator

$$\frac{1}{2}\frac{\partial^2}{\partial x^2} + \nu \tanh(\nu x)\frac{\partial}{\partial x}$$

and define

$$T_t^{(\nu)} = \inf\left\{s : \left|Z_s^{(\nu)}\right| = t\right\}.$$

We note that $(|Z_s^{(\nu)}|, s \ge 0) \stackrel{(d)}{=} (|B_s^{(\nu)}|, s \ge 0)$ when both start at zero.

On identifying the infinitesimal generators of the Markov processes $M_t^{(\nu)} - B_t^{(\nu)}$ (Fitzsimmons 1987; Cherny and Shiryaev 1999), $2M_t^{(\nu)} - B_t^{(\nu)}$ (Pitman and Rogers 1981) and given the infinitesimal generator of $Z_t^{(\nu)}$ we apply Girsanov's theorem to compute expectations working with the zero drift measure and employ the appropriate Radon–Nikodym measure change density to determine the Laplace transforms of these times. Alternative derivations may also be found in Williams (1976) and Taylor (1975). The resulting transforms are

$$E\left[e^{-\lambda C_t^{(\nu)}}\right] = \frac{\exp(-\nu t)\sqrt{\nu^2 + 2\lambda}}{\sqrt{\nu^2 + 2\lambda}\cosh(t\sqrt{\nu^2 + 2\lambda}) - \nu \sinh(t\sqrt{\nu^2 + 2\lambda})}$$
$$E\left[e^{-\lambda S_t^{(\nu)}}\right] = \frac{\sinh(\nu t)}{\nu}\frac{\sqrt{\nu^2 + 2\lambda}}{\sinh(t\sqrt{\nu^2 + 2\lambda})}$$
$$E\left[e^{-\lambda T_t^{(\nu)}}\right] = \frac{\cosh(\nu t)}{\cosh(t\sqrt{\nu^2 + 2\lambda})}.$$

These three processes are additive processes associated with the hyperbolic functions, and are different from and not to be confused with the three Lévy processes in Pitman and Yor (2003) also denoted by C, S, and T. The processes VC, VS, and VT are constructed by first evaluating an independent Brownian with volatility σ at the times $C_t^{(\nu)}$, $S_t^{(\nu)}$, and $T_t^{(\nu)}$, respectively, to obtain the characteristic functions:

$$E\left[e^{iu\sigma B(C_t^{(\nu)})}\right] = \frac{\exp(-\nu t)\sqrt{\nu^2 + \sigma^2 u^2}}{\sqrt{\nu^2 + \sigma^2 u^2}\cosh(t\sqrt{\nu^2 + \sigma^2 u^2}) - \nu \sinh(t\sqrt{\nu^2 + \sigma^2 u^2})} \tag{3.8}$$

$$E\left[e^{iu\sigma B(S_t^{(\nu)})}\right] = \frac{\sinh(\nu t)}{\nu}\frac{\sqrt{\nu^2 + \sigma^2 u^2}}{\sinh(t\sqrt{\nu^2 + \sigma^2 u^2})} \tag{3.9}$$

$$E\left[e^{iu\sigma B(T_t^{(\nu)})}\right] = \frac{\cosh(\nu t)}{\cosh(t\sqrt{\nu^2 + \sigma^2 u^2})}. \tag{3.10}$$

As the resulting processes are symmetric, they cannot match market skews. To add asymmetry, one may use Esscher transforms or one may consider evaluating Brownian motion with drift at the time changes given by $C_t^{(\nu)}$, $S_t^{(\nu)}$, and $T_t^{(\nu)}$. For the VG process, a comparison of the results of Madan, Carr, and Chang (1998) with those of Madan and Milne (1991) implies that the two methods lead to identical processes. In general this is not the case. Here, we considered subordinating Brownian motion with drift to the proposed time changes and found that the models did not calibrate well to the options data. The Esscher transforms performed much better and so we only report on them.

For a transform parameter θ we define for $H_t \in \{C_t^{(\nu)}, S_t^{(\nu)}, T_t^{(\nu)}\}$

$$
\begin{aligned}
E^{(\theta)}\left[e^{iu\sigma B(H_t)}\right] &= \frac{E\left[e^{iu\sigma B(H_t)}e^{\theta\sigma B(H_t)}\right]}{E\left[e^{\theta\sigma B(H_t)}\right]} \\
&= \frac{E\left[e^{i(u-i\theta)\sigma B(H_t)}\right]}{E\left[e^{i(-i\theta)\sigma B(H_t)}\right]}.
\end{aligned}
\tag{3.11}
$$

The characteristic functions for $X_{\mathrm{VC}}(t)$, $X_{\mathrm{VS}}(t)$, $X_{\mathrm{VT}}(t)$ may then be obtained by substituting (3.8), (3.9), and (3.10) into (3.11):

$$
\phi_{\mathrm{VHSSD}}(u, t) = E\left[e^{iut^{\gamma}X_{VH}(1)}\right]
$$

for $H \in \{C_t^{(\nu)}, S_t^{(\nu)}, T_t^{(\nu)}\}$.

4. SUMMARY OF DATA AND DESIGN OF STUDY

The ultimate objective in options modeling is to capture the variation in option prices across strike, maturity, and calendar time using a parsimonious model whose parameters are stable over wide ranges of the these three variables. To our knowledge, no such option pricing model presently exists. In fact, no model with fewer than five parameters appears able to accurately capture the variation in just two of these variables. Rather than cast about for this elusive model, our present interest is in obtaining a parsimonious model which accurately captures the variation in option prices across all strikes and maturities, and whose parameters are stable over wide ranges of these two variables. The main interest in such a model is that for short periods of time over which the parameters are stable, it can be relied upon for quoting option prices required to enter new positions and for capturing the market value of existing positions. Over longer periods of time, models of this type must be recalibrated in order to retain their accuracy. We recognize that models which require frequent recalibration are internally inconsistent, and are generally unreliable for hedging and investment decisions. However, for S&P 500 index options, the modeller faces the daunting task of explaining over 150 closing market prices, over periods that may last more than several years. Given the current state of practice, we regard it as a significant advance if a parsimonious model is forwarded which can capture just the variation in option prices across strike and maturity at a fixed point in time using risk-neutral valuation by a measure on the paths of the stock price process, thus avoiding static arbitrages. Just as stochastic volatility models were originally proposed as a remedy for deficiencies in the constant volatility model, future work can focus on specifying the stochastic evolution of the parameters of the relatively parsimonious models examined here.

At the present stage of this research program, we are therefore engaged in effectively synthesizing the information content of the surface of option prices in a parsimonious way. For this purpose, we evaluate the average absolute error as a percentage of the average option price. Based on market practice, we regard the particular model for a particular name on a particular day as having failed if this average percentage error (APE) is over 5%.

In developing summary statistics for model parameters, we exclude all failures defined by the 5% cut-off for the APE. If this is not done, then outliers arising from a variety of error sources unduly influence the summary parameter set making it an inappropriate representation of the results. Needless to say if the APE target is infrequently attained then the particular modeling exercise has been a failure. As mentioned earlier, the hyperbolic process time changes when used to evaluate Brownian motion with drift are an example of such failed models, and hence we do not present the results.

Given the focus described above, we obtained data for equity options on 21 different underlyings, including both single names and market indices, and with varying strikes and maturities. For each of the 14 months from September 2000 to October 2001, we used closing market prices for the second Wednesday of the month. In September 2001, we took the third Wednesday as markets were closed on the second Wednesday for obvious reasons. The names employed were, amzn, ba, bkx, c, csco, ge, hwp, ibm, intc, jnj, ko, mcd, mrk, msft, orcl, pfe, spx, sunw, wmt, xom, and yhoo. The specific dates employed, reported in the format *YYYYMMDD*, were, 20000913, 20001011, 20001108, 20001213, 20010110, 20010214, 20010314, 20010411, 20010509, 20010613, 20010711, 20010808, 20010919, and 20011010.

For each name and each day we used closing prices of out-of-the-money (spot) options, to reduce any bias due to the possibility of early exercise. We also excluded options expiring in the current month or maturing after 15 months. Strikes used were within 35% of the spot on both sides and we excluded options with a price below .00075 times the spot price. In all, we had 11,988 option prices and we performed 1,764 estimations covering $294 = (21 \times 14)$ estimations for each of the six models.

5. RESULTS

The results of the study are presented in the following format. First, we cover the aggregate performance levels as measured by APE across all days and names for each of the six models. Second, we present the results for the names averaged across the days, and for the days averaged across the names. Third, we report the average and standard deviation of the four parameters for each of the six models for all 21 names and across all 14 days.

5.1. Aggregate Results

Table 5.1 indicates the proportion of times that the six models achieved the 5% APE target. It also reports the means and standard deviations of the APE across the entire data set of 294 estimations for each model.

TABLE 5.1
Average Percentage Errors across Names and Days

Model	Proportion below .05	Mean	*SD*
VG	0.9320	0.02488	0.008937
NIG	0.9456	0.02492	0.009001
MXNR	0.9456	0.02487	0.008966
VC	0.9149	0.02539	0.008741
VS	0.9014	0.02600	0.008728
VT	0.9150	0.02548	0.008853

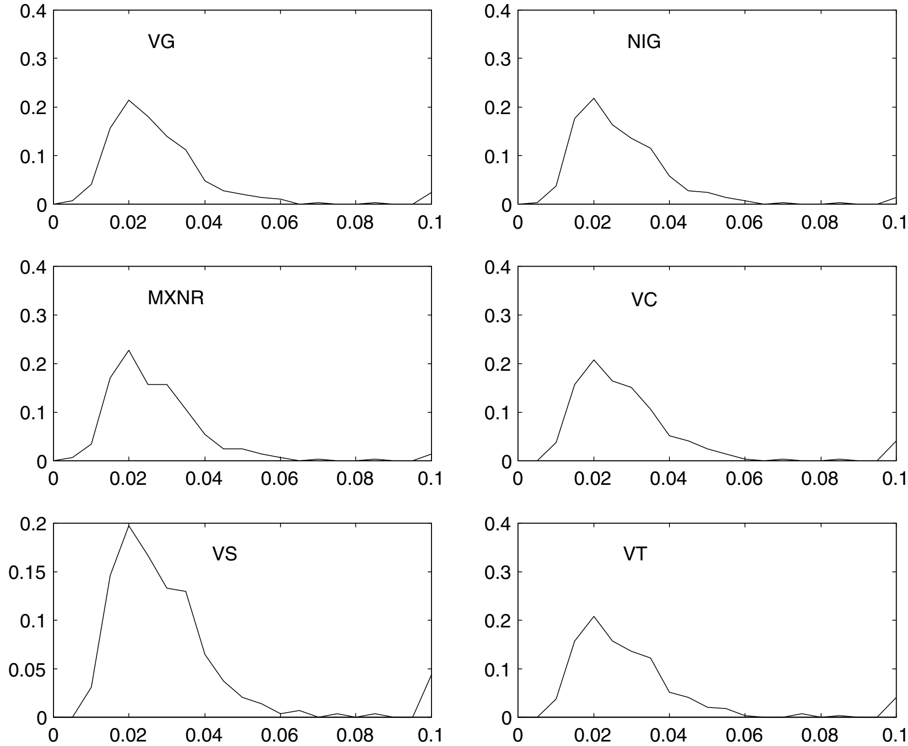

FIGURE 5.1. Graphs of densities of average pricing errors for the six models across 21 names for the 14 days.

TABLE 5.2
Average Rank of Model

VG	NIG	MXNR	VC	VS	VT
1.97	3.25	3.68	4.21	4.29	3.58

We observe from Table 5.1 that all six models are acceptable, with errors that are generally around half of the 5% mark. To get a better picture of the distribution of pricing errors, Figure 5.1 graphs the six densities of average pricing errors constructed across all 294 estimations for each model. More specifically the figure graphs the histogram of the average percentage pricing error across all the options for each model in the 294 cases generated by estimating the model for 21 names on each of 14 days.

We observe that the models are fairly comparable in their estimation performance. For all of the days and names, we ranked the models by APE for each of the 294 cases and computed the average rank. As indicated in Table 5.2, the VG models has the highest rank followed by NIG, VT, MXNR, VC, and VS models.

5.2. Results Sorted by Name and Day

In this section, we first consider the performance of the models for each of the 21 names averaged across the days. Table 5.3 reports the APE for each model across all days, for each name in the sample.

TABLE 5.3
Average Pricing Errors across Days for Each Name and $\bar{M}$ the Average Number of Options Used

Name	VG	NIG	MXNR	VC	VS	VT	$\bar{M}$
amzn	0.0284	0.02937	0.02947	0.02948	0.0335	0.0305	18
ba	0.0268	0.0269	0.0269	0.0269	0.0295	0.0276	27
bkx	0.0224	0.0219	0.0218	0.0218	0.0234	0.0219	159
c	0.0298	0.0294	0.0292	0.0295	0.0306	0.0295	48
csco	0.0197	0.0200	0.1986	0.0214	0.0212	0.0208	25
ge	0.0336	0.0344	0.0340	0.0344	0.0360	0.0345	40
hwp	0.0267	0.0267	0.0268	0.0277	0.0302	0.0274	24
ibm	0.0209	0.0200	0.0203	0.0204	0.0221	0.0204	41
intc	0.0160	0.0162	0.0160	0.0166	0.0182	0.0165	30
jnj	0.0276	0.0279	0.0277	0.0280	0.0282	0.0289	32
ko	0.0261	0.0265	0.0265	0.0278	0.0289	0.0266	24
mcd	0.0322	0.0316	0.0319	0.0321	0.0326	0.0323	22
mrk	0.0174	0.0175	0.0175	0.0175	0.0175	0.0202	27
msft	0.0207	0.0204	0.0201	0.0213	0.0219	0.0197	30
orcl	0.0195	0.0197	0.0197	0.0207	0.0231	0.0201	30
pfe	0.0274	0.0275	0.0274	0.0280	0.0283	0.0277	34
spx	0.0319	0.0302	0.0305	0.0306	0.0302	0.0301	135
sunw	0.0207	0.0209	0.0209	0.0225	0.0256	0.0216	30
wmt	0.0224	0.0224	0.0223	0.0234	0.0281	0.0226	34
xom	0.0299	0.3030	0.0301	0.0303	0.0320	0.0303	25
yhoo	0.0247	0.0248	0.0248	0.0251	0.0287	0.0284	21

TABLE 5.4
Average Pricing Errors across Names for Each Day

Day	VG	NIG	MXNR	VC	VS	VT
September 2000	0.0260	0.0263	0.0262	0.0256	0.0292	0.0268
October 2000	0.0245	0.0245	0.0243	0.0253	0.0278	0.0248
November 2000	0.0254	0.0261	0.0257	0.0264	0.0270	0.0270
December 2000	0.0252	0.0250	0.0250	0.0259	0.0289	0.0258
January 2001	0.0252	0.0282	0.0228	0.0237	0.0256	0.0232
February 2001	0.0254	0.0252	0.0252	0.0255	0.0272	0.0254
March 2001	0.0274	0.0273	0.0273	0.0267	0.0276	0.0264
April 2001	0.0248	0.0243	0.0243	0.0253	0.0280	0.0248
May 2001	0.0227	0.0242	0.0241	0.0247	0.0244	0.0245
June 2001	0.0236	0.0236	0.0235	0.0243	0.0268	0.0251
July 2001	0.0240	0.0239	0.0237	0.0240	0.0260	0.0239
August 2001	0.0301	0.0302	0.0300	0.0294	0.0308	0.0305
September 2001	0.0267	0.0262	0.0264	0.0268	0.0281	0.0270
October 2001	0.0230	0.0222	0.0225	0.0239	0.0238	0.0234

TABLE 5.5
Average Model Rankings

	Average rank across names	Average rank across days
VG	2.0	2.36
NIG	2.62	2.79
MXNR	2.57	1.43
VC	4.76	4.86
VS	4.24	4.64
VT	4.81	4.93

When we instead averaged across names for each day, the average number of options each day was around 50. Table 5.4 reports the model performances by day, averaged across the 21 names.

We rank the models on the basis of the average rank across days, for the 21 names. On the basis of the average rank across names for the 14 days we get the average rankings displayed in Table 5.5 in columns 1 and 2, respectively.

From Table 5.5 we see that the VG model is best for most names, based on the average absolute percentage error across the days of the year, while the Meixner model is best on most days based on the average absolute percentage error across the 21 names estimated on that day. The differences in rankings between VG, NIG, and MXNR are small, as are the differences between VC, VS, and VT models.

6. PARAMETER ESTIMATES BY NAME FOR ALL MODELS

In Tables 6.1–6.6, we report the mean and standard error of the four parameter estimates, across all days for which the 5% APE cutoff was met. We observe that the parameter estimates for the scaling parameter γ are very stable overtime, across both models and names. This is reflected in the low standard deviation for this parameter across the 14 days. The values of γ are also observed to be close to and just under the value of $1/2$. We note that at $\gamma = 1/2$ variances scale linearly with time, a property consistent with the process being one of independent identically distributed increments. However, excepting Brownian motion, the validity of the scaling refutes the independent identically distributed increment property. We also note that NIG, VC, VS, and VT have considerably large standard deviations for the drift in the Brownian motion used for the time change, as well as for the skewness parameter. In contrast the parameters of MXNR and VG are relatively stable over the time dimension. Given the high rankings of these models in the ranking performance on the names and the days we conclude that these two models are probably better suited for further investigation.

TABLE 6.1
Average Parameter Values and (*SD*) for VG by Name

Name	σ	ν	θ	γ	NDays
amzn	0.7721	0.7077	−1.1354	0.4465	12
	(0.1641)	(0.1963)	(0.4983)	(0.0202)	
ba	0.3304	0.2857	−0.3390	0.4073	12
	(0.0115)	(0.0019)	(0.0341)	(0.0167)	
bkx	0.2892	0.6003	−0.1463	0.4713	12
	(0.0079)	(0.0443)	(0.0022)	(0.0193)	
c	0.3433	0.4477	−0.3077	0.4714	14
	(0.0099)	(0.0489)	(0.0122)	(0.0184)	
csco	0.5479	0.4244	−0.5871	0.4197	14
	(0.0366)	(0.0392)	(0.0733)	(0.0153)	
ge	0.2866	0.4764	−0.3392	0.4257	14
	(0.0106)	(0.0448)	(0.0307)	(0.0165)	
hwp	0.4843	0.2863	−0.5974	0.4083	13
	(0.0206)	(0.0420)	(0.2956)	(0.0160)	
ibm	0.3661	0.5064	−0.2897	0.4178	13
	(0.0123)	(0.0492)	(0.0085)	(0.0166)	
intc	0.4774	0.2986	−0.5181	0.4172	14
	(0.0182)	(0.0268)	(0.0594)	(0.0138)	
jnj	0.2540	0.4243	−0.2135	0.4620	13
	(0.0054)	(0.0175)	(0.0047)	(0.0180)	
ko	0.2782	0.2082	−0.3589	0.4586	12
	(0.0068)	(0.0096)	(0.0411)	(0.0201)	
mcd	0.3083	0.4623	−0.1603	0.4642	13
	(0.0083)	(0.0516)	(0.0048)	(0.0183)	
mrk	0.2820	0.2534	−0.2916	0.4660	14
	(0.0061)	(0.0068)	(0.0079)	(0.0167)	
msft	0.4107	0.6185	−0.2606	0.4339	13
	(0.0152)	(0.0545)	(0.0089)	(0.0172)	
orcl	0.5235	0.2223	−0.9821	0.4086	14
	(0.0332)	(0.0112)	(0.3047)	(0.0140)	
pfe	0.3224	0.3746	−0.2667	0.4736	14
	(0.0080)	(0.0267)	(0.0104)	(0.0181)	
spx	0.1783	0.5848	−0.1914	0.4677	12
	(0.0033)	(0.0347)	(0.0049)	(0.0201)	
sunw	0.4690	0.2156	−1.2908	0.4100	12
	(0.0435)	(0.0186)	(0.5704)	(0.0174)	
wmt	0.3584	0.2644	−0.3769	0.4694	14
	(0.1093)	(0.0328)	(0.0437)	(0.0184)	
xom	0.2191	0.2144	−0.3011	0.4644	12
	(0.0061)	(0.0198)	(0.0811)	(0.0208)	
yhoo	0.6456	0.2062	−1.4052	0.3771	13
	(0.0650)	(0.0265)	1.4860	(0.0142)	

TABLE 6.2
Average Parameter Values and (*SD*) for NIG by Name

Name	σ	ν	θ	γ	NDays
amzn	0.8847	2.1919	−4.5988	0.4538	14
	(0.0194)	(2.4009)	(21.596)	(0.0183)	
ba	0.6154	4.7638	−3.5501	0.4078	12
	(0.0573)	(15.3077)	(37.722)	(0.0167)	
bkx	0.3463	1.4036	−0.2597	0.4725	12
	(0.0121)	(0.2763)	(0.0142)	(0.0194)	
c	0.5665	3.5821	−2.3455	0.4724	14
	(0.0552)	(15.4488)	(22.141)	(0.0185)	
csco	0.8748	3.3396	−3.4704	0.4214	14
	(0.1930)	(5.2986)	(12.983)	(0.0154)	
ge	0.4046	3.0785	−1.9875	0.4269	14
	(0.0333)	(3.3546)	(5.911)	(0.0166)	
hwp	1.1130	7.5444	−10.4969	0.4089	13
	(0.4717)	(131.9591)	(572.164)	(0.0161)	
ibm	0.5146	2.2696	−1.0283	0.4198	13
	(0.0377)	(3.6108)	(1.8754)	(0.0166)	
intc	0.9912	4.8189	−4.2061	0.4181	14
	(0.1797)	(12.8605)	(29.669)	(0.0139)	
jnj	0.3931	2.4982	−0.7561	0.4629	13
	(0.0136)	(0.7155)	(0.1237)	(0.0181)	
ko	0.6808	6.8884	−4.8917	0.4619	13
	(0.0693)	(27.2538)	(42.628)	(0.0189)	
mcd	0.4308	2.0235	−0.3958	0.4652	13
	(0.0300)	(2.4329)	(0.1814)	(0.0184)	
mrk	0.5897	4.6321	−2.0588	0.4665	14
	(0.0305)	(4.2441)	(2.3629)	(0.0167)	
msft	0.4840	1.3941	−0.4511	0.4390	14
	(0.0202)	(0.3605)	(0.0485)	(0.0163)	
orcl	1.1195	6.9661	−12.5825	0.4093	14
	(0.2063)	(13.9692)	(138.21)	(0.0141)	
pfe	0.5560	3.3970	−1.7111	0.4744	14
	(0.0459)	(5.7138)	(5.7787)	(0.0182)	
spx	0.2299	1.7070	−4.2667	0.4700	12
	(0.0047)	(0.309)	(0.0239)	(0.0202)	
sunw	1.0197	9.2134	−23.8656	0.4105	12
	(0.2884)	(48.796)	(820.007)	(0.0174)	
wmt	0.8340	6.6603	−6.0660	0.4701	14
	(0.1599)	(39.991)	(96.388)	(0.0184)	
xom	0.5375	1.0590	−11.8701	0.4646	12
	(0.0786)	(208.40)	(646.61)	(0.0208)	
yhoo	1.7093	1.1686	−37.0183	0.3783	13
	(1.0657)	(202.12)	(3570.7)	(0.0143)	

TABLE 6.3
Average Parameter Values and (*SD*) for MXNR by Name

Name	a	b	d	γ	NDays
amzn	1.4021	−1.8664	0.0643	0.4506	14
	(1.0201)	(0.5696)	(0.1359)	(0.0184)	
ba	0.3925	−1.1532	1.3478	0.4076	12
	(0.0310)	(0.2508)	(0.6479)	(0.0167)	
bkx	0.5341	−0.9343	0.5041	0.4717	12
	(0.0333)	(0.0761)	(0.0349)	(0.0194)	
c	0.4738	−1.2992	0.9891	0.4717	14
	(0.0372)	(0.1568)	(0.5665)	(0.0184)	
csco	0.7403	−1.4104	0.9559	0.4205	14
	(0.0995)	(0.2629)	(0.3136)	(0.0154)	
ge	0.3834	−1.6021	0.8574	0.4261	14
	(0.0320)	(0.3973)	(0.1898)	(0.0165)	
hwp	0.6426	−0.9846	1.3666	0.4090	13
	(0.0842)	(0.0942)	(1.2189)	(0.0160)	
ibm	0.5851	−1.2506	0.7238	0.4189	13
	(0.0611)	(0.1444)	(0.2101)	(0.0166)	
intc	0.5567	−1.1976	1.3588	0.4177	14
	(0.0372)	(0.0139)	(0.6237)	(0.0138)	
jnj	0.3514	−1.2486	0.7994	0.4622	13
	(0.0121)	(0.0156)	(0.0690)	(0.0181)	
ko	0.2839	−1.1442	1.7778	0.4617	13
	(0.0102)	(0.0158)	(0.8890)	(0.0188)	
mcd	0.5249	−0.7972	0.7431	0.4646	13
	(0.0396)	(0.0999)	(0.2843)	(0.0184)	
mrk	0.3079	−1.1801	1.3721	0.4662	14
	(0.0094)	(0.1010)	(0.2436)	(0.0167)	
msft	0.7497	−1.1732	0.5079	0.4379	14
	(0.0615)	(0.1647)	(0.0423)	(0.0162)	
orcl	0.5778	−1.4515	1.5222	0.4092	14
	(0.0602)	(0.2662)	(0.3909)	(0.0141)	
pfe	0.4327	−1.1265	0.9975	0.4740	14
	(0.0223)	(0.1314)	(0.2797)	(0.0181)	
spx	0.2859	−1.5852	0.5473	0.4689	12
	(0.0081)	(0.2143)	(0.0286)	(0.0201)	
sunw	0.5274	−1.5945	1.8312	0.4105	12
	(0.0729)	(0.3673)	(1.2941)	(0.0174)	
wmt	0.4023	−1.0651	1.6047	0.4698	14
	(0.0265)	(0.1383)	(0.9919)	(0.0184)	
xom	0.2399	−0.9904	2.1091	0.4646	12
	(0.0128)	(0.2064)	(2.8735)	(0.0208)	
yhoo	0.8216	−1.0675	1.8136	0.3778	13
	(0.1674)	(0.1572)	(1.8085)	(0.0143)	

TABLE 6.4
Average Parameter Values and (*SD*) for VC by Name

Name	σ	ν	θ	γ	NDays
amzn	1.0742	−7.3852	−6.2111	0.4381	12
	(0.1531)	(7.0528)	(4.7875)	(0.0187)	
ba	0.6706	−5.8178	−5.0463	0.4076	12
	(0.0615)	(28.658)	(33.851)	(0.0168)	
bkx	0.2820	−11.258	−1.6672	0.4718	12
	(0.0121)	(0.6763)	(0.4167)	(0.0194)	
c	0.5633	−4.3202	−4.3973	0.4718	14
	(0.0731)	(38.356)	(17.410)	(0.0185)	
csco	0.9323	−3.6493	−5.0193	0.4136	13
	(0.1351)	(40.581)	(4.3293)	(0.0157)	
ge	0.4319	−3.5052	−6.1607	0.4250	13
	(0.0308)	(13.400)	(36.391)	(0.0177)	
hwp	1.3586	−10.421	−4.0699	0.4062	13
	(0.2138)	(15.570)	(1.5958)	(0.0158)	
ibm	0.4645	−1.4936	−2.3992	0.4180	12
	(0.0421)	(3.177)	(1.2864)	(0.0179)	
intc	1.1205	−7.6666	−3.8770	0.4171	14
	(0.2467)	(18.874)	(2.2711)	(0.0138)	
jnj	0.3903	−2.4680	−4.1707	0.4627	13
	(0.0146)	(1.3883)	(3.3296)	(0.0181)	
ko	0.6689	−6.9960	−5.1770	0.4628	12
	(0.0659)	(35.912)	(14.113)	(0.0203)	
mcd	0.4079	−1.3050	−1.6087	0.4647	13
	(0.0427)	(5.2264)	(1.0510)	(0.0184)	
mrk	0.5997	−5.2893	−4.7985	0.4665	14
	(0.0319)	(8.2719)	(6.0315)	(0.0167)	
msft	0.4234	−0.7815	−2.0143	0.4397	13
	(0.0277)	(2.8996)	(2.3860)	(0.0173)	
orcl	1.1193	−4.2016	−4.7485	0.4104	14
	(0.1922)	(48.010)	(2.0724)	(0.0141)	
pfe	0.5612	−3.2062	−3.0348	0.4742	14
	(0.0603)	(6.5781)	(1.8581)	(0.0182)	
spx	0.2016	−1.1092	−6.0776	0.4664	11
	(0.0041)	(0.4466)	(4.6708)	(0.0218)	
sunw	1.1191	−4.0469	−4.4024	0.4120	12
	(0.2785)	(45.520)	(2.0061)	(0.0175)	
wmt	0.9104	−12.725	−6.7275	0.4694	14
	(0.2163)	(455.95)	(107.85)	(0.0184)	
xom	0.5735	−9.9126	−7.9885	0.4647	12
	(0.0759)	(160.20)	(137.12)	(0.0208)	
yhoo	2.0048	−11.583	−4.1807	0.3810	12
	(0.7647)	(113.97)	(2.1309)	(0.0152)	

TABLE 6.5
Average Parameter Values and (*SD*) for VS by Name

Name	σ	ν	θ	γ	NDays
amzn	0.6762	−0.0033	−4.7136	0.4295	8
	(0.1274)	(0.0010)	(8.0115)	(0.0259)	
ba	0.5482	0.4109	−3.5142	0.4103	12
	(0.0337)	(1.9401)	(14.583)	(0.0170)	
bkx	0.4362	0.0039	−2.6679	0.4698	12
	(0.0187)	(0.0004)	(0.8780)	(0.0192)	
c	0.5187	0.0008	−3.2430	0.4733	14
	(0.0264)	(0.0008)	(8.4956)	(0.0186)	
csco	0.7537	−0.0077	−4.5348	0.4202	13
	(0.0891)	(0.00005)	(13.115)	(0.0164)	
ge	0.4412	0.0074	−4.1959	0.4279	14
	(0.0174)	(0.0004)	(10.797)	(0.0164)	
hwp	0.6877	−0.0003	−4.6375	0.4134	13
	(0.0644)	(0.0001)	(16.639)	(0.0166)	
ibm	0.4878	−0.6834	−7.3176	0.4193	13
	(0.0415)	(7.9552)	(26.632)	(0.0166)	
intc	0.7587	−0.0556	−2.4846	0.4205	14
	(0.0581)	(0.0343)	(7.1605)	(0.0139)	
jnj	0.4028	−0.2184	−3.3331	0.4630	13
	(0.0137)	(0.6085)	(1.7648)	(0.0181)	
ko	0.5406	0.9748	−2.1484	0.4648	13
	(0.0340)	(4.3453)	(0.6295)	(0.0190)	
mcd	0.5199	0.4645	−1.9539	0.4578	12
	(0.0253)	(2.4434)	(1.4556)	(0.0192)	
mrk	0.5375	1.7864	−3.0694	0.4682	14
	(0.0328)	(10.120)	(3.1822)	(0.0169)	
msft	0.5402	0.0038	−5.0184	0.4362	14
	(0.0329)	(0.0009)	(23.191)	(0.0161)	
orcl	0.8008	−0.0162	−4.9636	0.4131	14
	(0.0936)	(0.0013)	(14.183)	(0.0145)	
pfe	0.5363	1.0698	−3.0744	0.4736	13
	(0.0300)	(14.529)	(10.429)	(0.0196)	
spx	0.2454	−0.0063	−6.9053	0.4687	12
	(0.0054)	(0.00002)	(4.4521)	(0.0202)	
sunw	0.8940	−0.0176	−2.9397	0.4121	10
	(0.1150)	(0.00012)	(7.7327)	(0.0211)	
wmt	0.4737	0.0023	−7.2048	0.4742	13
	(0.0026)	(0.00011)	(23.744)	(0.0200)	
xom	0.5211	3.7511	−6.1284	0.4654	12
	(0.0063)	(165.26)	(128.20)	(0.0208)	
yhoo	1.0503	−0.0186	−3.0005	0.3836	12
	(0.1750)	(0.00006)	(7.8931)	(0.0152)	

TABLE 6.6
Average Parameter Values and (*SD*) for VT by Name

Name	σ	ν	θ	γ	NDays
amzn	0.9807	−0.6425	−5.6473	0.4483	11
	(0.1113)	(9.3187)	(3.3910)	(0.0226)	
ba	0.6864	−0.6570	−5.1889	0.4059	11
	(0.0694)	(25.406)	(30.705)	(0.0182)	
bkx	0.3037	−0.6830	−1.7716	0.4716	12
	(0.0102)	(0.4989)	(0.3735)	(0.0193)	
c	0.5633	−4.1628	−3.7947	0.4716	14
	(0.0705)	(17.039)	(5.2529)	(0.0185)	
csco	1.0567	−7.1514	−4.7568	0.4194	14
	(0.2363)	(14.482)	(2.7087)	(0.0152)	
ge	0.4303	−4.0828	−6.6586	0.4264	14
	(0.0259)	(10.959)	(34.789)	(0.0165)	
hwp	1.2715	−10.191	−4.1911	0.4070	13
	(0.3009)	(46.642)	(5.0878)	(0.0158)	
ibm	0.4766	−2.0124	−2.4433	0.4185	13
	(0.0348)	(3.3663)	(1.2919)	(0.0167)	
intc	1.0955	−7.7125	−3.7720	0.4171	14
	(0.2349)	(17.249)	(2.1290)	(0.0138)	
jnj	0.3857	−2.9425	−4.1016	0.4582	12
	(0.0163)	(1.5215)	(3.0704)	(0.0192)	
ko	0.6712	−7.6452	−5.4547	0.4620	13
	(0.0638)	(35.943)	(14.775)	(0.0188)	
mcd	0.4079	−1.6715	−1.5911	0.4648	13
	(0.0321)	(3.4571)	(0.8343)	(0.0184)	
mrk	0.5871	−5.5057	−4.6984	0.4665	14
	(0.0302)	(7.3340)	(5.4601)	(0.0167)	
msft	0.4234	−0.8295	−1.5959	0.4396	13
	(0.0185)	(0.7021)	(0.3757)	(0.0173)	
orcl	1.3337	−9.4482	−4.8852	0.4093	14
	(0.2461)	(17.195)	(1.9136)	(0.0141)	
pfe	0.5429	−4.0225	−3.6854	0.4742	14
	(0.0501)	(13.918)	(12.897)	(0.0182)	
spx	0.2034	−1.6843	−6.1219	0.4690	12
	(0.0038)	(0.5312)	(4.2939)	(0.0202)	
sunw	1.2275	−7.5918	−4.7830	0.4084	11
	(0.3288)	(31.986)	(2.2890)	(0.1881)	
wmt	0.7648	−5.2872	−3.1186	0.4704	13
	(0.1061)	(8.5511)	(2.2215)	(0.0197)	
xom	0.5687	−10.080	−7.8559	0.4646	12
	(0.0741)	(146.79)	(132.07)	(0.0208)	
yhoo	1.7314	−10.928	−4.3397	0.3843	12
	(0.9138)	(81.454)	(1.7769)	(0.0150)	

7. SUMMARY

Self-decomposable laws for the risk-neutral density at unit time were motivated as limit laws and were associated with self-similar additive processes using the results of Sato (1991). These processes were then employed to build option pricing models amenable to empirical evaluation using the Fast Fourier Transform method of Carr and Madan (1998). Six scaled, self-decomposable laws were formulated based on the variance gamma law, the normal inverse Gaussian law, the Meixner process, and processes related to the hyperbolic functions. Three additional models based on subordinating Brownian motion with drift to the processes related to the hyperbolic functions were ruled out of the study as they failed to meet acceptable error targets.

Empirical investigations were conducted using some 12,000 option prices on 21 names, for 14 days and for a wide range of strikes and maturities. It was observed that all models could explain option prices consistently across strike and maturity, with an average APE of around 2.5%. Based on a variety of performance rankings, there was a slight preference for the VGSSD, NIGSSD, and MXNRSSD models over the other three. Furthermore, an analysis of parameter stability over time gives a clear preference for the use of the VGSSD or the MXNRSSD models over the other models tested. Hence future research should focus on the use of such models in an effort to capture the variation of option prices across calendar time.

From the empirical adequacy of the calibrations based on the scaled self decomposable laws we may infer that the risk-neutral distributions indeed satisfy the scaling laws introduced in Section 2. The process implications of the validity of such scaling are an open question.

REFERENCES

APPLEBAUM, D. (2004): *Lévy Processes and Stochastic Calculus*. Cambridge: Cambridge University Press.

BAKSHI, G., C. CAO, and Z. CHEN (1997): Empirical Performance of Alternative Option Pricing Models, *J. Finance* 52, 2003–2049.

BARNDORFF-NIELSEN, O. E. (1998): Processes of Normal Inverse Gaussian Type, *Finance Stochast.* 2, 41–68.

BARNDORFF-NIELSEN, O. E., and N. SHEPHARD (2001): Non-Gaussian Ornstein-Uhlenbeck-Based Models and Some of Their Uses in Financial Economics, *J. Roy. Statist. Soc. B*, 63, 167–241.

BATES, D. (1991): The Crash of '87: Was It Expected? The Evidence from Options Markets, *J. Finance* 46, 1009–1044.

BATES, D. (1996): Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutschemark Options, *Rev. Financial Studies* 9, 69–108.

BATES, D. (2000): Post-'87 Crash Fears in S&P 500 Futures Options, *J. Econometrics* 94, 181–238.

BERKES, I. (1991): Non-Gaussian Limit Distributions of Lacunary Trigonometric Series, *Canadian J. Math.* 43, 948–959.

BLACK, F., and M. SCHOLES (1973): The Pricing of Corporate Liabilities, *J. Political Econ.* 81, 637–654.

CARR, P., H. GEMAN, D. MADAN, and M. YOR (2002): The Fine Structure of Asset Returns: An Empirical Investigation, *J. Business* 75(2), 305–332.

CARR, P., H. GEMAN, D. MADAN, and M. YOR (2003): Stochastic Volatility for Lévy Processes, *Math. Finance* 13, 345–382.

CARR, P., and D. MADAN (1998): Option Valuation Using the Fast Fourier Transform, *J. Comput. Finance* 2, 61–73.

CHERNY, A. S., and A. N. SHIRYAEV (1999): Some Distributional Properties of the Brownian Motion with Drift and an Extension of P. Lévy's Theorem, *Theory Probab. Appl.* 44(2), 466–472.

CONT, R., and P. TANKOV (2004): *Financial Modelling with Jump Processes*. Boca Raton, FL: Chapman and Hall, CRC.

DUFFIE, D., J. PAN, and K. SINGLETON (2000): Transform Analysis and Asset Pricing for Affine Jump-Diffusions, *Econometrica* 68, 1343–1376.

EBERLEIN, E. U. KELLER, and K. PRAUSE (1998): New Insights into Smile, Mispricing and Value at Risk, *J. Business* 71, 371–406.

EMBRECHTS, P., and M. MAEJIMA (2002): *Self-Similar Processes*. Princeton Series in Applied Mathematics. Princeton, NJ: Princeton University Press.

FITZSIMMONS, P. (1987): A Converse to a Theorem of P. Lévy, *Ann. Probab.* 15, 1515–1523.

GRIGELIONIS, B. (1999): Processes of Meixner Type, *Lithuanian Math. J.* 39, 33–41.

HESTON, S. (1993): A Closed-form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Rev. Financial Studies* 6, 327–343.

HULL, J. (2000): *Options, Futures and Other Derivative Securities*. Upper Saddle River, NJ: Prentice Hall.

JARROW, R. A., and S. TURNBULL (2000): *Derivative Securities*. Cincinnati, OH: South Western Publishing Company.

JEANBLANC, M., J. PITMAN, and M. YOR (2001): Self-Similar Processes with Independent Increments Associated with Lévy and Bessel Processes, *Stochast. Proc. Appl.* 100, 223–232.

JONES, E. P. (1984): Option Arbitrage and Strategy with Large Price Changes, *J. Financial Econ.* 13, 91–113.

KHINTCHINE, A. Y. (1938): Limit Laws of Sums of Independent Random Variables, *ONTI*, Moscow, Russia.

KNIGHT, F. B. (2001): On the Path of an Inert Object Impinged on One Side by a Brownian Particle, *Probab. Theory Related Fields* 121, 577–598.

KONIKOV, M., and D. MADAN (2002): Option Pricing using Variance Gamma Markov Chains, *Rev. Derivatives Res.* 5, 81–115.

LAMPERTI, J. (1962): Semi-stable Stochastic Processes, *Trans. Am. Math. Soc.* 104, 62–78.

LÉVY, P. (1937): *Théorie de l'Addition des Variables Aléatoires*. Paris: Gauthier-Villars.

LOÈVE, M. (1945): Nouvelles classes de Lois Limites, *Bull. Math. Soc. France* 73, 107–126.

MADAN, D., P. CARR, and E. CHANG (1998): The Variance Gamma Process and Option Pricing, *Eur. Finance Rev.* 2, 79–105.

MADAN, D., and F. MILNE (1991): Option Pricing with VG Martingale Components, *Math. Finance* 1, 39–55.

MERTON, R. C. (1973): Theory of Rational Option Pricing, *Bell J. Econ. Management* 4, 141–183.

MERTON, R. C. (1976): Option Pricing When Underlying Returns Are Discontinuous, *J. Financial Econ.* 3, 125–144.

NAIK, V., and M. LEE (1990): General Equilibrium Pricing of Options on the Market Portfolio with Discontinuous Returns, *Rev. Financial Studies* 3, 493–522.

PITMAN, J. (1975): One Dimensional Brownian Motion and the Three-Dimensional Bessel Process, *Adv. Appl. Probab.* 7, 511–526.

PITMAN, J., and L. C. G. ROGERS (1981): Markov Functions, *Ann. Probab.* 9, 573–582.

PITMAN, J., and M. YOR (2003): Infinitely Divisible Laws Associated with Hyperbolic Functions, *Canadian J. Math.* 55(2), 292–330.

SATO, K. (1991): Self-Similar Processes with Independent Increments, *Probab. Theory Related Fields* 89, 285–300.

SATO, K. (1999): *Lévy Processes and Infinitely Divisible Distributions*. Cambridge: Cambridge University Press.

SCHOUTENS, W. (2002): The Meixner Process in Finance, *Eurandom Report 2001–002*, Eurandom, Eindhoven.

SCHOUTENS, W. (2003): *Lévy Processes in Finance*. Hoboken, NJ: Wiley.

TAYLOR, H. M. (1975): A Stopped Brownian Motion Formula, *Ann. Probab.* 3, 234–246.

WILLIAMS, D. (1976): On a Stopped Brownian Motion Formula of H.M.Taylor, Seminaire de Probabilité X, *Lecture Notes in Mathematics* 511, Springer-Verlag, Berlin, 235–240.

YAMAZATO, M. (1978): Unimodality of Infinitely Divisible Distribution Functions of Class L, *Ann. Probab.* 6, 523–531.