REAL OPTIONS AND THE TIMING OF IMPLEMENTATION OF EMISSION LIMITS UNDER ECOLOGICAL UNCERTAINTY

by

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ABSTRACT

Using real options, we analyze the timing of implementation of emission limits for a pollutant when its stock varies because of random environmental effects. Two types of irreversibility are present: ecological irreversibility, which results from long-term environmental damages, and investment irreversibility, which comes from sunk investments in pollution abatement. With reference to the deterministic case, we find that a small level of uncertainty may either delay or advance a reduction in pollutant emissions, depending on the cost of reducing emissions relative to the gain of reducing expected social costs. Thus, we cannot a-priori know the bias introduced by neglecting low levels of uncertainty in environmental problems. When environmental uncertainty is “high enough,” however, we find that pollutant emissions should be curbed immediately because of the prevalence of environmental damage. These results illustrate the importance of explicitly modeling uncertainty in environmental problems.

1. Introduction

The impact of uncertainty and irreversibility on environmental policy has long been a subject of debate among economists and policy makers, which has not yet been resolved. In the context of pollution, irreversibility has two basic sources. First, environmental damage may be so large that it becomes permanent or the amount of pollution may be so large that it leads to long-term damages. Second, investments to control or reduce a stock externality are often (at least partially) sunk. An example is the installation of scrubbers by a utility.

The extensive resources devoted to improving our understanding of environmental issues (e.g. global climate change, hazardous waste, or acid rain, to name a few) highlight the pervasive nature and importance of uncertainty in environmental problems. There are two aspects to uncertainty, which are not always distinguished: one is associated with risk aversion, and the other with the arrival of information over time. In this paper, we are only concerned with the latter.

A preoccupation with the combined effect of irreversibility and uncertainty led Weisbrod (1964) to introduce the concept of option value. He argues that if a decision has irreversible consequences, then the flexibility ("the option") to choose the timing of that decision should be included in a cost-benefit analysis. Weisbrod uses the case of a national park where total operating costs cannot be recovered. In this case, he argues, it could be socially detrimental to shut down the park and cut down its trees because many people would be willing to make a financial contribution in order to preserve the possibility for themselves or their children, to visit the park in the future.

Weisbrod's work has led to two competing interpretations of option value. The first, due to Cicchetti and Freeman (1971) and Schmalensee (1972), views option value as a risk premium paid by risk averse consumers to reduce the impact of uncertainty in the supply of an environmental good. The second interpretation, advanced independently by Arrow and Fisher (1974) and Henry (1974), stresses the intertemporal aspect of irreversibility and the arrival of new information over time. This has lead them to introduce the concept of "quasi option value," which is akin to a shadow tax on development to correct benefits from development in a traditional cost-benefit analysis. Although quite useful, this view has led to a debate not only on the magnitude of option value, but also on its sign.

Many papers in the quasi-option literature are based on two-period discrete time models. A standard result (e.g., see Arrow and Fisher, 1974, or Henry, 1974) is that, in the presence of environmental irreversibility, a standard cost-benefit analysis is biased against conservation. Freixas and Laffont (1984) generalize this result, and Conrad (1980) links option value to the expected value of information. More recently, Kolstad (1996), in a study of pollution stock effects and sunk emission control capital, shows that there is an irreversibility effect only when there is excess stock pollutant. However, Hanemann (1989) shows that conclusions drawn from this simple framework cannot be carried over when the benefit function is non-linear or when the decision-maker faces a continuum of irreversible development.

In this paper, we use another approach, which returns to Weisbrod's original intuition about option value. Since we are concerned with the value of flexibility in the presence of irreversibilites, we rely on the theory of real options from finance. A real option can be defined as the value of being able to choose some characteristics (e.g., the timing) of a decision with irreversible consequences, which affects a real asset (as opposed to a financial asset). This approach has been applied fruitfully in the growing real options literature.

One of the first papers on real options is by Brennan and Schwartz (1985); it analyzes the operation of a mine, which can be temporarily closed. Later, Paddock, Siegel, and Smith (1988) propose a model to value offshore petroleum leases as a function of the market price of oil. More recently, Clarke and Reed (1990) study the preservation of natural wilderness reserves. For a good introduction to investment under uncertainty using real options, see Dixit and Pindyck (1994). For examples of applications, see Trigeorgis (1996).

In this paper we revisit the problem of pollution reduction under environmental uncertainty in the presence of economic and environmental irreversibilities, which was recently considered by Kolstad (1996). However, instead of a discrete-time setting with two periods, we use a more general, continuous-time model. To obtain manageable closed-form results, we adopt a quadratic damage function and consider a class of square-root diffusions for the process followed by the stock of pollutant.

In Section 2, we introduce our simple continuous-time model, which features a single stock externality. The pollution control problem is formulated as a stopping problem where a risk-neutral social planner has to make a one-time decision on when and how much to reduce the emissions of a pollutant. In Section 3, we solve the corresponding deterministic problem to get a benchmark for the impact of randomness in the stock of pollutant. In Section 4, we tackle the stochastic case. Uncertainty is measured by the infinitesimal variance of the diffusion process of the stock of pollutant. Because it is difficult to analyze in general the impact of uncertainty, we look at vanishingly "small" and "large" uncertainty. A numerical application is provided in Section 5. The last section summarizes our conclusions.

2. Modeling Pollutant Stock Uncertainty

In this section, we consider a stylized model with one environmental pollutant, which decays at rate $\alpha \geq 0$. If $\alpha > 0$, we say that we have a decaying pollutant. A decaying pollutant is one for which the environment has some absorptive capacity (e.g., organic wastes discharged in oxygen rich waters, for example). On the other hand, if $\alpha = 0$ we say that we have a non-decaying pollutant. Examples of non-decaying pollutants include persistent synthetic chemicals such as PCBs.

Let X denote the stock of this pollutant and E_1 its rate of emission. To focus solely on the impact of the variability of X, we assume that E_1 is constant. Because of the randomness of the physical and chemical processes that lead to changes in pollutant stock, we assume that X follows the diffusion process:

$$dX = (E_1 - \alpha X)\,dt + \sqrt{vX}\,dz = \alpha\,(x_{1c} - X)\,dt + \sqrt{vX}\,dz \qquad (1)$$

The parameter $v \geq 0$ characterizes the volatility of the process followed by the stock of pollutant and dz is an increment of a standard Wiener process. Thus, the infinitesimal variance of the process, which equals vX here, increases linearly with X. This specification seems a good compromise between a model with a fixed variance (which does not seem realistic), and a model where the infinitesimal variance of X varies with X^2, as for the geometric Brownian motion. Since diffusions have continuous trajectories, X remains non-negative and tends to revert to $x_{1c} \equiv E_1 / \alpha$, so the decay rate, α, also characterizes the speed of reversion. The initial stock of pollution, which is assumed known, is denoted X(0).

We further assume that the flow of social costs resulting from pollution damage, noted C(X), is the quadratic form:

$$C(X) = -cX^2 \tag{2}$$

where $c > 0$ is a valuation parameter. We adopt the convention that costs are negative.

The emission of pollution can be decreased from E_1 to a constant $E_2 < E_1$, at a cost K, which may depend on E_1 and the change $E_1 - E_2$. We suppose that K is completely sunk, which is reasonable for many pollution control measures (e.g. the installation of scrubbers by utilities). After the emission reduction investment is made, X follows the new process:

$$dX = (E_2 - \alpha X)\,dt + \sqrt{vX}\,dz = \alpha\,(x_{2c} - X)\,dt + \sqrt{vX}\,dz \tag{3}$$

where $x_{2c} \equiv E_2 / \alpha$.

To eliminate risk aversion from the problem, we consider a risk-neutral social planner. Her goal is to simultaneously select E_2 ($0 \le E_2 \le E_1$), the rate to which pollutant emission should be reduced, and T, the socially optimal time of doing so, in order to maximize (with respect to T and E_2) the present value function:

$$J(T, E_2) = \mathcal{E}_0 \int_0^{\infty} -cX^2 e^{-rt} dt - e^{-rT} K(E_2, E_1 - E_2) \tag{4}$$

subject to Equation (1) for $0 \le t \le T$ and to Equation (3) for $t > T$, given X(0). Here, $\mathcal{E}_0$ is the expectation operator based on information available at time $t = 0$ and r is the social discount (risk-free) rate.

This optimization problem can be solved in two steps. First, for an arbitrary value of E_2, such that $0 \le E_2 < E_1$, we calculate the critical (threshold) stock of pollutant, denoted x^*, at which the rate of pollution emission should be reduced from E_1 to E_2. Once x^* is known, we can find $\mathcal{E}_0 T(x^*; E_2)$, the expected time at which the stock of pollutant reaches x^* for the first time, given an initial stock of X(0). When $v > 0$, X is a random variable and $T(x^*; E_2)$ is a (first-passage) stopping time. For E_2 fixed, $X < x^*$ defines the so-called "continuation region," or region 1, where the optimal decision is to wait. As soon as $X \geq x^*$, which defines the so-called "stopping region," or region 2, the rate of pollutant emissions should be reduced to E_2. The second step consists in finding the value of E_2 that will maximize the objective function J.

In this paper we want to analyze the impact of uncertainty on the timing of the decision to make a sunk investment K to reduce pollution emissions, for arbitrary functional forms of K. Hence, we focus on the determination of x^* as a function of v for a fixed E_2, and we ignore the determination of the optimal value of T which comes into play in the determination of E_2.

We thus have a standard stopping problem that bears similarities with an optimal investment problem. We solve the problem by stochastic dynamic programming. Let $V_{i,m}(x)$ be the value function in region "i". The corresponding Hamilton-Jacobi-Bellman equation is:

$$rV_i = -cx^2 + (-\alpha x + E_i)\frac{dV_i}{dx} + \frac{vx}{2}\frac{d^2V_i}{dx^2}, \ i = 1,2 \tag{5}$$

The left side of Equation (5) can be interpreted as a return; the first term on the right side is the flow of social pollution costs; and the other terms, which result from Ito's lemma (see Karlin and Taylor, 1975), represent capital gains.

Equation (5) is a linear second-order equation. Its solution is thus the sum of a particular solution, denoted $P_i(x)$, and the general solution of the associated homogeneous equation, denoted $\varphi(x)$. We pick $P_i(x)$ to represent the expected social costs from continuing to emit the pollutant at rate E_i forever, given a current stock of pollutant of x. $\varphi(x)$ is the value of the option to choose the timing for reducing emissions. By construction, it has non-negative value. In this context, waiting reduces the present value of the cost of cutting pollution emissions while reducing emissions earlier decreases the present value of pollution damages. In financial terms, φ(x) is a perpetual American option.

When we consider a one-time reduction in pollutant emissions, there is no option term after pollutant emissions have been cut to E_2, so the solutions of Equation (5) in regions 1 and 2, respectively, are:

$$V_1(x) = \varphi(x) + P_1(x),\;\; V_2(x) = P_2(x) \tag{6}$$

To find x*, we need two additional conditions (for a heuristic proof, see Dixit and Pindyck, 1994; for a more rigorous treatment see Brekke and Oksendal, 1991). First, at x*, the value of the option plus the social cost of polluting forever at rate E_1 should equal the social cost of polluting forever at rate E_2 plus the cost of reducing emissions from E_1 to E_2.. This gives the continuity condition:

$$\varphi_1(x^*) + P_1(x^*) = P_2(x^*) - K \tag{7}$$

The second condition, called "smooth-pasting," says that when the option to reduce emissions should be exercised, a marginal change in the value of the option equals the marginal change in the difference of social pollution costs:

$$\frac{d\varphi(x^*)}{dx} = \frac{dP_2(x^*)}{dx} - \frac{dP_1(x^*)}{dx} \tag{8}$$

By combining these two conditions, we obtain a "stopping rule": for this problem, it is an equation whose smallest non-negative root defines the critical stock of pollutant at which pollution emissions should be reduced from E_1 to E_2.

3. Solution of the Deterministic Case

We implement the above formulation in a deterministic setting (v = 0) to illustrate the concept of real option. In this context, the real option term gives the value of being able to choose the timing and magnitude of a one-time investment to reduce pollution emissions. In this case, there is no arrival of information over time so the option term measures the value of the flexibility of a decision with irreversible consequences. Equations (1), (3), and (5) degenerate to first order linear differential equations. Integrating Equation (1) gives:

$$X(t) = \frac{E_1}{\alpha} + (X(0) - \frac{E_1}{\alpha})e^{-\alpha t} \tag{9}$$

This shows that, if pollutant emission rates are unchanged, X converges asymptotically towards $x_{1c} \equiv E_1 / \alpha$, from above if $X(0) > x_{1c}$, and from below otherwise. In this case, x_{1c} is a barrier. Next, we calculate the present value of social pollution costs.

LEMMA 1: If the rate of pollutant emission is fixed at E_i and the initial stock of pollutant is x, the present value of social pollution costs is:

$$P_i(x) = -\frac{cx^2}{(r+2\alpha)} - \frac{2cE_i x}{(r+\alpha)(r+2\alpha)} - \frac{2cE_i^2}{r(r+\alpha)(r+2\alpha)} \tag{10}$$

To prove Lemma 1, simply calculate $\int_0^{+\infty} -cX^2 e^{-rt} dt$ with $X(t) = (x - \frac{E_i}{\alpha})e^{-\alpha t} + \frac{E_i}{\alpha}$.

The option term for this case is:

$$\varphi(X) = \text{Max}(\,\tilde{\varphi}(X),\ 0\,) \tag{11}$$

where $\tilde{\varphi}(x)$ is the solution of the homogeneous equation associated to Equation (5):

$$\tilde{\varphi}(x) = \begin{cases} A_0 \left|\alpha x - E_1\right|^{\frac{-r}{\alpha}}, \text{if } \alpha > 0 \\ \tilde{\varphi}(x) = A_0\, e^{\frac{r}{E_1}x}, \text{if } \alpha = 0 \end{cases} \tag{12}$$

In Equation (12), A_0 is a non-negative constant to be determined jointly with x_0^*, the critical stock of pollutant at which it is optimal to reduce pollution emissions from E_1 to E_2.

To obtain x_0^*, we substitute Equations (10), (11) and (12) into the continuity and smooth-pasting conditions and take their ratio to get rid of unknown parameter A_0. We find:

$$x_0^* = \frac{r(2\alpha + r)K}{2c(E_1 - E_2)} - \frac{E_2}{\alpha + r} \tag{13}$$

As expected, x_0^* increases with K, α, and r, and decreases when c or E_1 increase. Also note that x_0^* can be negative if the cost of adoption is "low," in which case the emission reduction policy should be adopted immediately.

However, if $x_0^* > x_{1c}$, the smooth-pasting condition cannot be met and Equation (13) is not valid. Indeed, $\frac{d\tilde{\varphi}(x)}{dx} = \frac{r}{E_1 - \alpha x}\tilde{\varphi}(x)$ so $\frac{d\tilde{\varphi}(x)}{dx}$, the left side of the smooth-pasting condition, is positive if $x<E_1/\alpha$ and negative otherwise. From Equation (10), $\forall x \geq 0,\ \frac{dP_{2,m}(x)}{dx} - \frac{dP_{1,m}(x)}{dx} > 0$, which is the right-hand side of the smooth pasting condition. Hence, when $x_0^* > E_1/\alpha$, the option value is zero, so making a sunk investment to reduce pollution emissions is a "now or never" proposition: we should invest now in pollution reduction if $J(0,E_2)>J(+\infty,E_2)$, and never otherwise. Using Equations (4) and (10), we find that this condition is equivalent to the rule: when $x_0^* > E_1/\alpha$, invest now if $x > \frac{(r+\alpha)x_0^* - E_1}{r}$, where x is the current stock of pollutant, and never otherwise.

Once $x_0^* < x_{1c}$ is known, we can calculate the first-passage time T* by inverting Equation (9). We find:

$$T^* = \begin{cases} \frac{1}{\alpha} \mathrm{Ln}\left(\frac{E_1 - \alpha x_0}{E_1 - \alpha x_0^*}\right), \text{if } \alpha > 0 \\ (x_0^* - x_0)/E_1, \quad \text{if } \alpha = 0 \end{cases} \tag{14}$$

Finally, we can derive explicitly the option term:

$$\tilde{\varphi}(x) = \begin{cases} \frac{2c(E_1 - E_2)}{r(\alpha + r)(2\alpha + r)}\left(E_1 - \alpha x_0^*\right)\left|\frac{E_1 - \alpha x_0^*}{E_1 - \alpha x}\right|^{\frac{r}{\alpha}}, \text{if } \alpha > 0 \\ \frac{2c(E_1 - E_2)E_1}{r^3} e^{\frac{r}{E_1}(x - x_0^*)}, \text{if } \alpha = 0 \end{cases} \tag{15}$$

Once x_0^* is known for all values of E_2 of interest, the optimal level of pollution reduction E_2^* can be calculated by a simple optimization.

The expressions of T* and x_0^* can also be obtained by solving the social planner's problem directly. In that case, the second order condition for a maximum confirms that when $x_0^* > E_1/\alpha$, we should invest now in pollution reduction if $J(0,E_2)>J(+\infty,E_2)$, and never otherwise.

4. The Stochastic Case

In this section, we assume that the variance parameter of the stock of pollutant, v, is greater than 0.

LEMMA 2: The expected social costs from continuing to pollute at rate E_i forever, given an initial pollutant stock x, are:

$$P_i(x) = -\frac{cx^2}{(r+2\alpha)} - \frac{c(2E_i+v)x}{(r+\alpha)(r+2\alpha)} - \frac{cE_i(2E_i+v)}{r(r+\alpha)(r+2\alpha)} \tag{16}$$

To prove this result, we need to calculate the moment generating function of the process followed by X. Calculations are shown in Appendix A. From this expression for $P_i(x)$, it is clear that an increase in v augments the expected social costs of pollution. As before, P_i increases when r (or α) decrease, and it increases when E_i increases. Moreover, since $P_1(x) < P_2(x)$ (recall that $E_1 > E_2$), the smooth-pasting condition requires the option term to be increasing in x. Because the expression of the option term differs between the cases $\alpha = 0$ and $\alpha > 0$, we treat decaying and non-decaying pollutants separately.

4.1 The Case of a Decaying Pollutant

In this case ($\alpha > 0$), we have:

LEMMA 3: For a decaying pollutant, the stochastic option term is given by Equation (11) with:

$$\tilde{\varphi}(x) = B_0\Phi\left(\frac{r}{\alpha}, \frac{2E_1}{v}; \frac{2\alpha}{v}x\right) \tag{17}$$

See Appendix B for a proof. In the above, B_0 is a constant to be determined jointly with x*, and $\Phi(a, b; y)$ is the confluent hypergeometric function with argument y and parameters a and b (see Lebedev, 1972).

Substituting Equations (16) and (17) into the continuity and smooth-pasting conditions (Equations (7) and (8)), we find that x* is the smallest non-negative real value which verifies:

$$\frac{\Phi\left(\frac{r}{\alpha},\frac{2E_1}{v};\frac{2\alpha}{v}x^*\right)}{\frac{r}{E_1}\Phi\left(\frac{r}{\alpha}+1,\frac{2E_1}{v}+1;\frac{2\alpha}{v}x^*\right)}=x+\frac{E_1-(r+\alpha)x_0^*}{r}+\frac{v}{2r} \tag{18}$$

Since $\frac{\left[P_2(x^*)-P_1(x^*)-K\right]}{\left[P_2'(x^*)-P_1'(x^*)\right]}=x+\frac{E_1-(r+\alpha)x_0^*}{r}+\frac{v}{2r}$, Equation (18) says that, at x*, the net savings from reducing emissions from E_1 to E_2 divided by the corresponding marginal savings equals the ratio of the option term by its marginal value. This relationship defines x* as a function of v.

In general, it is not possible to find an explicit expression for x*, and it is quite difficult to examine how x* changes with v because of the complexity of the derivative of $\Phi\left(\frac{r}{\alpha},\frac{2E_1}{v};\frac{2\alpha}{v}x\right)$ with respect to v. We thus examine how x* changes for large values of v and for v close to zero. Considering first "large" values of v, we have:

PROPOSITION 1. For "large enough" values of the pollutant stock volatility v, x decreases to zero. If* x_0^* *> 0, then the value of v for which x*=0 is* $\tilde{v}$ *such that:*

$$\tilde{v}=2(r+\alpha)x_0^* \tag{19}$$

If $x_0^*\leq 0$, *pollution emissions should be reduced to* E_2 *right away.*

This result is intuitive because an increase in v augments expected social costs, as we remarked above, but leaves investment K unchanged. For large enough values of v, environmental irreversibility prevails over sunk pollution control costs and it becomes optimal to act immediately.

Proof of PROPOSITION 1. Equation (19) is obtained from Equation (18) by setting x* to zero and simplifying. If $x_0^* \leq 0$, we have seen above that the sunk investment needed for curbing emissions down to E_2 is so small relative to the expected social gains from a cleaner environment that pollution emissions should be curbed right away.

We next investigate how x* changes when v goes to zero.

PROPOSITION 2. Let $\tilde{x}_0^* = \lim_{v \to 0+} x^*(v)$, *where x*(v) is the solution of Equation (18). When* $x_0^* < \frac{E_1}{\alpha}$, $\tilde{x}_0^* = x_0^*$. *However, when* $x_0^* > \frac{E_1}{\alpha}$, $\tilde{x}_0^*$ *differs from* x_0^* *given by Equation (13) and just as in the deterministic case, we find:*

$$\tilde{x}_0^* = \frac{(r+\alpha)x_0^* - E_1}{r} > x_0^* \tag{20}$$

The limit of the stochastic case, when the volatility of the stock of pollutant goes to zero, thus gives the deterministic results.

Proof of PROPOSITION 2. When $\tilde{x}_0^* < \frac{E_1}{\alpha}$, it can be shown (see Appendix C) that $\lim_{v \to 0} \Phi(\frac{r}{\alpha} + k, \frac{2E_1}{v} + k; \frac{2\alpha x^*}{v}) = \left(1 - \frac{\alpha \tilde{x}_0^*}{E_1}\right)^{-\frac{r}{\alpha} - k}$, with k = 0 or 1. Inserting these results into Equation (18) yields Equation (13).

However, when $\tilde{x}_0^* > \frac{E_1}{\alpha}$, $\lim_{v \to 0} \Phi\left(\frac{r}{\alpha}, \frac{2E_1}{v}; \frac{2\alpha}{v} x^*\right) \Big/ \Phi\left(\frac{r}{\alpha} + 1, \frac{2E_1}{v} + 1; \frac{2\alpha}{v} x^*\right) = 0$, which leads to Equation (20).

Let us now examine how x*(v) changes when v increases from 0.

PROPOSITION 3. For small values of v, the variance parameter of the process followed by the stock of pollutant, x may be larger or smaller than* $\tilde{x}_0^*$, *depending on the cost of reducing emissions relative to the expected social gains from reducing the emissions of pollution. If we define:* $x^*(v) = \tilde{x}_0^* + Dv + o(v)$, *then:*

$$D = \begin{cases} \dfrac{(2\alpha + r)x_0^* - E_1}{2(\alpha + r)(E_1 - \alpha x_0^*)}, \text{ if } \tilde{x}_0^* < \dfrac{E_1}{\alpha} \\ \dfrac{(\alpha - r)\tilde{x}_0^* - E_1}{2r(E_1 - \alpha \tilde{x}_0^*)}, \text{ if } \tilde{x}_0^* > \dfrac{E_1}{\alpha} \end{cases} \tag{21}$$

Proof of PROPOSITION 3. To derive this result, we substitute the first order expansion of x*(v) as a function of v into the expression of the stopping frontier, given by Equation (18). Details are provided in Appendix C.

From Equation (21), we see that, if $x_0^* < \dfrac{E_1}{2\alpha + r}$ then D < 0, while $\dfrac{E_1}{2\alpha + r} < x_0^* < \dfrac{E_1}{\alpha}$ implies D > 0. If we recall the expression of x_0^*, the first case corresponds to relatively small values of K, so the environmental irreversibility dominates. An increase in v augments the expected social costs, so action is required earlier than in the deterministic case. If K (and thus x_0^* are larger), sunk investment costs dominate. When $x_0^* > \dfrac{E_1}{\alpha}$, the sign of D depend on the value of r (the social discount rate) compared to that of α (the pollutant rate of decay). A little algebra shows that if α < r (i.e. the pollutant decays faster than the social discount rate), then D > 0. If α > r, however, D is positive for $\dfrac{E_1}{\alpha} < \tilde{x}_0^* < \dfrac{E_1}{\alpha - r}$ and negative for $\tilde{x}_0^* > \dfrac{E_1}{\alpha - r}$.

4.2 The Case of a Non-decaying Pollutant

For completeness, we now deal with the case of a non-decaying pollutant (α = 0). Proceeding as above, we obtain qualitatively the same results as for a decaying pollutant in the region $x_0^* < x_{1c}$ (when $\alpha = 0$, $x_{1c} = +\infty$). More specifically, we find that:

LEMMA 4: For a non-decaying pollutant (α = 0), the stochastic option term is given by Equation (11) with:

$$\tilde{\varphi}(x) = D_0 \Theta\left(\frac{2E_1}{v}; \frac{2r}{v} x\right) \tag{22}$$

(See Appendix D for a proof.) As before, D_0 is a constant which has to be evaluated jointly with x*, and Θ is given by:

$$\Theta(a; y) = \sum_{n=0}^{+\infty} \frac{1}{(a)_n} \frac{y^n}{n!} = \Gamma(a) y^{\frac{1-a}{2}} I_{a-1}(2\sqrt{y}) \tag{23}$$

$I_\nu(z)$ is the modified Bessel function of order ν. Equation (16), which gives $P_i(x)$, is still valid after setting α to zero. Substituting Equations (22) and (16) (with α = 0) into the continuity and smooth-pasting conditions (Equations (7) and (8)), we find that x* is the smallest non-negative root of:

$$\frac{\Theta\left(\frac{2E_1}{v}; \frac{2r}{v} x\right)}{\frac{r}{E_1} \Theta\left(\frac{2E_1}{v} + 1; \frac{2r}{v} x\right)} = x + \frac{E_1 - r x_0^*}{r} + \frac{v}{2r} \tag{24}$$

This equation has the same interpretation as Equation (18). Proposition 1, the first part of Proposition 2 (here $\alpha = 0$ so $x_{1c} = \infty$) and Proposition 3 remain valid for this case after setting α to zero. For small values of v, we find that x*(v) decreases when v increases if x_0^* is "small", which happens when environmental irreversibility dominates. Conversely, x*(v) increases with v when x_0^* is "large," which happens when sunk costs from pollution control are larger.

5. A Numerical Application

A numerical illustration is presented in Figure 1, panels A and B, and in Table 1. Figure 1, panels A and B, shows how x^*/x_0^* (the critical stock of pollutant in the stochastic case, also noted x*(v), normalized by its deterministic value) changes as a function of $\sqrt{v}$ over a range of parameter values for α and r, respectively. Table 1 gives the value of the critical stock of pollutant, x*(v), and option value at x*(v) per unit cost K (noted φ^*/K), for small values of v and for a wide range of values of α and r. Results in Table 1 are for $E_1 = 1$ and $E_2 = 0.7$, which corresponds to a 30% reduction in pollutant emissions.

From the figures, observe that x*(v) goes to zero for large enough values of $\sqrt{v}$, as shown in Proposition 1. In that case, expected environmental damages, which increase with v, dominate sunk pollution control investments. Moreover, α and r appear to have symmetric effects on the determination of x*(v). This is expected since a decrease in α increases future levels of the stock of pollutant, everything else being the same, and thus increases expected social pollution costs. A decrease in r has the same effect, although more directly. For small values of v, x*(v) can either increase or decrease with v, as shown in Proposition 3. However, for relatively large values of α and r, the expected social costs of pollution are given less weight and irreversibility linked to pollution control dominates.

The evolution of $x^*(v)$ when v is small is examined in more detail in Table 1, which provides values for the relative magnitude of the option value at the critical stock of pollutant $x^*(v)$. First, we see that option value can be larger than K. Ignoring it (as would be done in a conventional cost-benefit analysis) is likely to yield sub-optimal decisions. We also observe that option value increases with v, which makes sense because an increase in v augments expected environmental damages while leaving K unchanged. The flexibility to reduce expected social costs then becomes more valuable. As expected, for $\tilde{x}_0^* > E_1 / \alpha$, we see that option value is close to zero when v is small since there is no option value when $x_0^* > E_1 / \alpha$ in the deterministic case. Finally, Table 1 illustrates the large sensitivity of x^* to α and r.

6. Conclusions

In this paper, we use the theory of real options to examine the tension between ecological irreversibility and investment irreversibility (sunk pollution control costs), in the presence of ecological uncertainty. This uncertainty is represented by the stochastic variations of a stock of pollutant, which result from natural fluctuations in its absorption or production by ecosystems (as in the case of some greenhouse gases, for example). After developing a simple deterministic model to obtain a benchmark for the impact of uncertainty, we analyze a stochastic model where the infinitesimal variance of the pollutant stock, denoted vX, varies linearly with X. When uncertainty is low, we show that a small level of uncertainty may either delay or advance a reduction in pollutant emissions, depending on the cost of reducing pollution relative to the expected social gains from reducing pollution. This is because at low levels of uncertainty, either ecological (through natural damage) or investment (through sunk investments needed to reduce emissions) irreversibility may prevail. Thus, we cannot know a-priori the bias introduced by neglecting low levels of uncertainty in environmental problems. Moreover, we show that when uncertainty is high enough, expected environmental damages become dominant and pollutant emissions should be reduced immediately.

These results illustrate the need to model uncertainty explicitly in environmental problems as uncertainty may have a key impact both on the timing and intensity of pollution reduction measures. In particular, our results may be useful in the debate on global warming where it is often argued that action on curbing emission of greenhouse gases should be delayed to wait for the arrival of new information on the effect of global warming.

Finally, this paper shows that the theory of real options provides a natural framework to analyze environmental policy because irreversibility and uncertainty are dominant features of many environmental problems. Indeed, while the theory of real options embodies Weisbrod's original intuition on option value, it does not suffer from some of the conceptual problems that plague the concept of option-value in the environmental economics literature.

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Appendix A

Let $M(\theta,t)$ be the moment generating function of $X(t)$ with $\alpha > 0$:

$$M(\theta,t) = \mathcal{E}(e^{-\theta x}) = \int_{-\infty}^{+\infty} \phi(x_0,t_0;x,t)e^{-\theta x}dx \tag{A1}$$

where $\phi(x_0,t_0;x,t)$ is the probability density function of x at t, given that $x(t_0) = x_0$. Then

$$\frac{\partial M}{\partial t} = \int_{-\infty}^{+\infty} \frac{\partial \phi}{\partial t} e^{-\theta x}dx \tag{A2}$$

The Kolmogorov forward equation for this process is (see Cox and Miller, 1965):

$$\frac{\partial \phi}{\partial t} = \frac{vx}{2}\frac{\partial^2 \phi}{\partial x^2} + (v + \alpha x - E_1)\frac{\partial \phi}{\partial x} + \alpha\phi \tag{A3}$$

We substitute (A3) into (A2) and integrate to obtain:

$$\frac{\partial M}{\partial t} = -\theta(\alpha + \frac{v}{2}\theta)\frac{\partial M}{\partial \theta} - E_1\theta M \tag{A4}$$

This partial differential equation must be solved subject to the boundary conditions:

$$M(0,t) = 1, \quad \frac{\partial M(0,0)}{\partial \theta} = -x_0, \quad \frac{\partial^2 M(0,0)}{\partial \theta^2} = x_0^2 \tag{A5}$$

We find:

$$M(\theta,t) = \left(1 + \frac{v\theta}{2\alpha}\right)^{-\frac{2E}{v}} \left[1 + C_1 \frac{2\theta e^{-\alpha t}}{2\alpha + \theta v} + C_2\left(\frac{2\theta e^{-\alpha t}}{2\alpha + \theta v}\right)^2\right], \text{ with} \tag{A6}$$

$$C_1 = E - \alpha x_0, \quad C_2 = \frac{1}{2}\left(\left(E - \alpha x_0 + \frac{v}{2}\right)^2 - \frac{v}{2}\left(E + \frac{v}{2}\right)\right) \tag{A7}$$

We then use the relationship:

$$\frac{\partial^n M(0,t)}{\partial \theta^n} = (-1)^n \mathcal{E}(x_t^n) \tag{A8}$$

and integrate with respect to time to obtain Equation (16).

Appendix B

In this Appendix, we obtain the option term of the stochastic problem. It must be defined at X = 0, non-negative, and increasing with X. Let $Y = \frac{2\alpha X}{v}$, and define W(Y)≡V(X). With this change of variables, the homogeneous equation corresponding to Equation (5) becomes:

$$Y\frac{d^2W}{dY^2} + (\frac{2E_1}{v} - Y)\frac{dW}{dY} - \frac{r}{\alpha}W = 0 \tag{A9}$$

This is Kummer's Equation (see Lebedev, 1972), a second order ordinary differential equation. A general solution of Equation (A9) can be written:

$$W(Y) = B_0\ \Phi(\frac{r}{\alpha}, \frac{2E_1}{v}; Y) + B_1\ z^{1\text{-}b}\ \Phi(1 + \frac{r}{\alpha} - \frac{2E_1}{v}, 2 - \frac{2E_1}{v}; Y), \tag{A10}$$

Φ is the confluent hypergeometric function of the first kind, and B_0 and B_1 are two unknown constants. Φ has the series representation:

$$\Phi(a,b;z) = \sum_{k=0}^{+\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!} \tag{A11}$$

As z→0, the derivative of the second solution tends to +∞, and for v small enough, the second solution itself tends to +∞. We thus retain only the term in $B_0\Phi$ as our solution. In the above:

$$(b)_0 = 1, (b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = b\,(b+1)\ldots(b+k-1) \text{ for } k \geq 1 \tag{A12}$$

and Γ is the Gamma function. To find the stopping frontier, we need the derivative of Φ. From its series expansion, it is easy to see that:

$$\frac{d\Phi(a,b;z)}{dz} = \frac{a}{b}\Phi(a+1,b+1;z) \tag{A13}$$

Appendix C

In this Appendix we derive a first-order approximation to x*(v) when v is close to zero. When $v \to 0$, we have the formal convergence:

$$\Phi(\frac{r}{\alpha}, \frac{2E_1}{v}; \frac{2\alpha}{v} x^*(v)) \to S(\tilde{x}_0^*) \equiv \sum_{n=0}^{+\infty} \left(\frac{r}{\alpha}\right)_n \frac{1}{n!} \left(\frac{\alpha \tilde{x}_0^*}{E_1}\right)^n \tag{A14}$$

This series converges provided $\frac{\alpha \tilde{x}_0^*}{E_1} < 1$, or $\tilde{x}_0^* < \frac{E_1}{\alpha} \equiv x_{1c}$, where $\tilde{x}_0^* = \lim_{v \to 0} x^*(v)$. When this condition does not hold, $S(\tilde{x}_0^*) = +\infty$. We thus distinguish between two cases:

Case 1: $\tilde{x}_0^* < x_{1c}$

In this case, we can use Equation (4.3.7) in Slater (1960):

$$\Phi(a, b; by) = (1-y)^{-a}\left\{1 - \frac{a(a+1)}{2b}\left(\frac{y}{1-y}\right)^2 + O(|b|^{-2})\right\} \tag{A15}$$

This expression is valid for a and y bounded complex variables and b real and “large”. Then:

$$\Phi(a+1, b+1; by) = (1-ky)^{-a-1}\left\{1 - \frac{(a+1)(a+2)}{2(b+1)}\left(\frac{ky}{1-ky}\right)^2 + O(|b|^{-2})\right\} \tag{A16}$$

with $k \equiv b / (b + 1)$. Here, $a = \frac{r}{\alpha}$, $b = \frac{2E_1}{v}$, $z = \frac{2\alpha x^*}{v}$, so $y = \frac{\alpha x^*}{E_1}$ and $k = \frac{2E_1}{2E_1 + v}$. Using repeatedly $(1+\varepsilon)^r = 1 + r\varepsilon + o(\varepsilon)$ where ε is small, we find:

$$\text{As } v \to 0+, \frac{\Phi(\frac{r}{\alpha}, \frac{2E_1}{v}; \frac{2\alpha x^*}{v})}{\Phi(\frac{r}{\alpha}+1, \frac{2E_1}{v}+1; \frac{2\alpha x^*}{v})} = (1 - \frac{\alpha x^*}{E_1})(1 + \frac{\alpha + r}{2} \frac{x^* v}{(E_1 - \alpha x^*)^2} + o(v)) \quad \text{(A17)}$$

Case 2: $\tilde{x}_0^* > x_{1c}$

In this case, we extend (9.12.8) in Lebedev (1972) to obtain:

$$\Phi(a; b; by) = \frac{\Gamma(b)}{\Gamma(a)} e^{by} (by)^{a-b} \left\{ \left(1 - \frac{1}{y}\right)^{a-1} + o(1) \right\} \quad \text{(A18)}$$

so, after substituting and simplifying:

$$\text{As } v \to 0^+, \frac{\Phi(\frac{r}{\alpha}, \frac{2E_1}{v}; \frac{2\alpha}{v} x^*)}{\Phi(\frac{r}{\alpha}+1, \frac{2E_1}{v}+1; \frac{2\alpha}{v} x^*)} = \frac{r x^* v}{2E_1(\alpha x^* - E_1)} + o(v) \quad \text{(A19)}$$

Using this result into Equation (18) leads to Equation(20).

To derive the results of Proposition 3, we start by substituting the expression:

$$x^*(v) = \tilde{x}_0^* + Dv + o(v) \quad \text{(A20)}$$

into the right-hand side (noted RHS) of Equation (18). We obtain:

$$\text{RHS} = 1 + \frac{r\tilde{x}_0^*}{E_1} - \frac{r+\alpha}{E_1} x_0^* + (\frac{rD}{E_1} + \frac{1}{2E_1})v + o(v) \quad \text{(A21)}$$

Combining this result with Equations (A17) and (A19) yields Equation (21).

Appendix D

To find the option term when $\alpha = 0$, we need to find a solution to the following equation (which is positive, increasing, and well defined for x non-negative):

$$rV = E\frac{dV}{dx} + \frac{vx}{2}\frac{d^2V}{dx^2} \tag{A22}$$

Trying a series solution we find:

$$V(x) = D_0\Theta\left(\frac{2E_1}{v};\frac{2rx}{v}\right) \equiv D_0\sum_{0}^{+\infty}\frac{1}{(2E_1/v)_n}\frac{1}{n!}\left(\frac{2rx}{v}\right)^n \tag{A23}$$

where D_0 is a constant. Equation (A22) is a second-order ordinary differential equation so we need to find another independent solution. A simple calculation shows that

$$\Omega(x) \equiv x^{1-\frac{2E_1}{v}}\,\Theta\left(\frac{2E_1}{v};\frac{2rx}{v}\right) = x^{1-\frac{2E_1}{v}}\sum_{0}^{+\infty}\frac{1}{(2E_1/v)_n}\frac{1}{n!}\left(\frac{2rx}{v}\right)^n \tag{A24}$$

is also a solution of Equation (A22). We discard this solution, however, because its derivative is infinite at 0 (and it may not be well defined at x = 0 when v is small).

Table 1: Critical pollutant stock (x*) and standardized option value at x*.

		r=0.02		r=0.03		r=0.04		r=0.05	
α	$\sqrt{v}$	x*	φ*/K	x*	φ*/K	x*	φ*/K	x*	φ*/K
K/c = 20000									
0.00	0.00	< 0	- -	6.67	111.1%	35.83	46.9%	69.33	24.0%
	0.05	< 0	- -	6.63	111.1%	35.85	47.0%	69.39	24.1%
	0.10	< 0	- -	6.53	111.2%	35.89	47.2%	69.57	24.4%
0.01	0.00	3.33	120.8%	32.50	33.8%	66.00	8.5%	106.00‡	0.0%
	0.05	3.30	120.9%	32.53	33.9%	66.21	8.7%	107.09	0.8%
	0.10	3.18	121.1%	32.61	34.2%	66.76	9.4%	108.55	1.9%
K/c = 6000									
0.02	0.00	< 0	- -	7.00	81.9%	20.33	30.9%	35.00	9.5%
	0.05	< 0	- -	6.99	82.0%	20.36	31.0%	35.12	9.8%
	0.10	< 0	- -	6.94	82.2%	20.42	31.3%	35.45	10.4%
0.03	0.00	2.00	117.5%	15.33	33.3%	30.00	3.6%	54.00‡	0.0%
	0.05	1.98	117.6%	15.35	33.4%	30.29	4.0%	54.08	0.1%
	0.10	1.91	117.9%	15.39	33.7%	30.88	5.0%	54.32	0.5%
0.04	0.00	8.33	55.6%	23.00	3.5%	53.50‡	0.0%	83.00‡	0.0%
	0.05	8.33	55.7%	23.27	3.9%	53.53	0.1%	83.02	0.0%
	0.10	8.31	55.9%	23.81	4.7%	53.61	0.2%	83.08	0.2%

Note: a "‡" indicates that $\tilde{x}_0^*$ differs from x_0^* given by Equation (13). x is the critical stock of pollutant; α is the pollutant rate of decay; r is the social discount rate; v is the volatility of X; K is the sunk investment needed to reduce pollution emissions from E_1 to E_2; c is the coefficient of valuation of pollution; and φ*/K is the option value at x* normalized by K.*

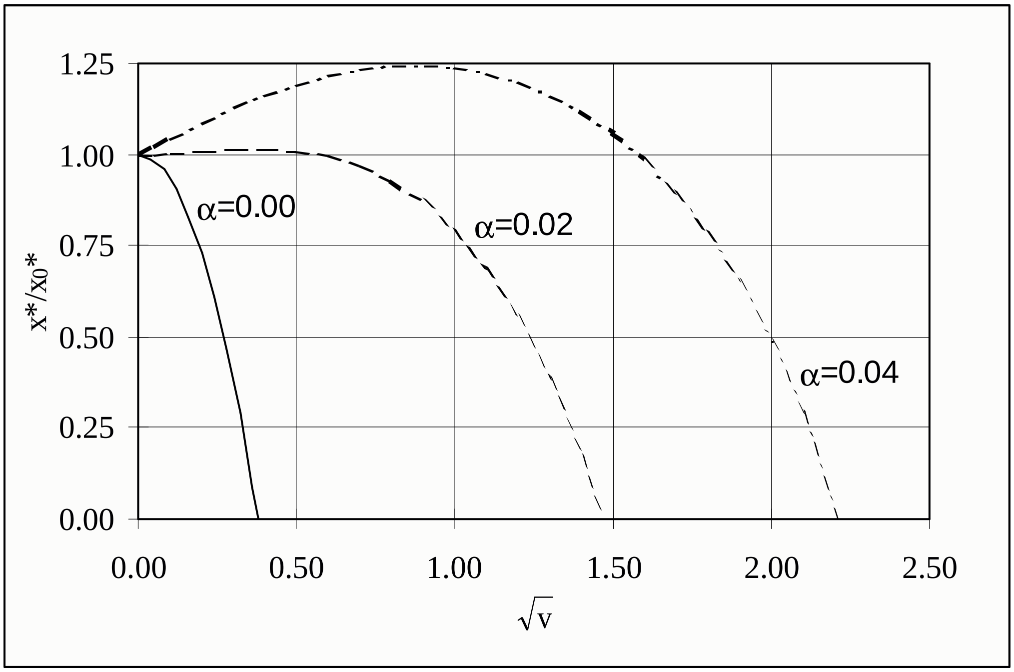

Figure 1.A: Variations of x^*/x_0^* with $\sqrt{v}$ and α for r = 0.06, E_1 = 1, E_2 = 0.75, & K/c=1900.

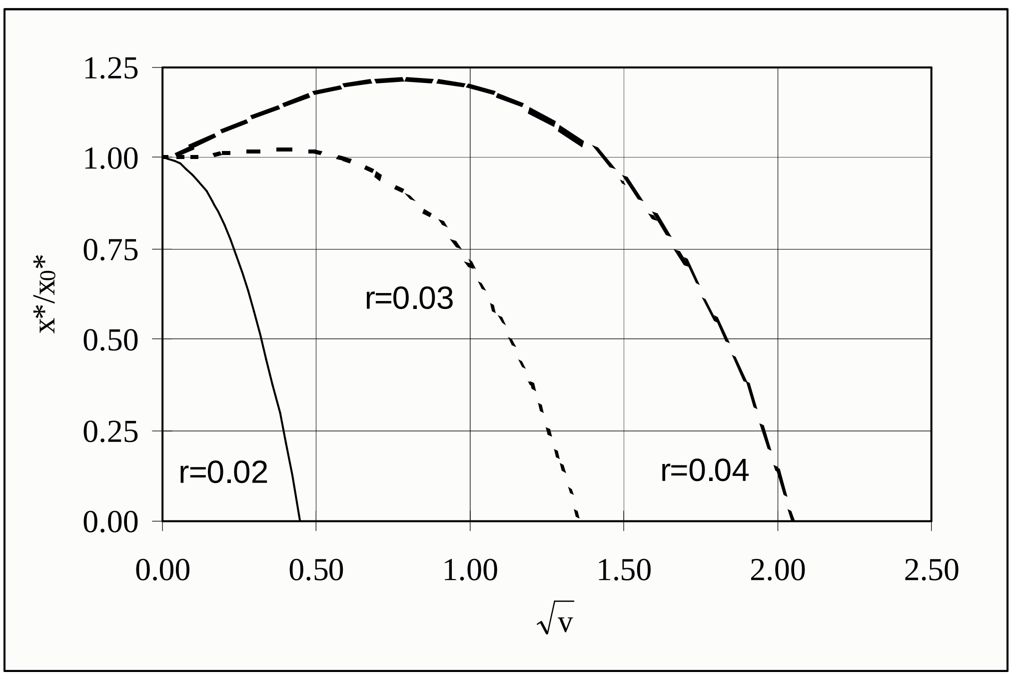

Figure 1.B: Variations of x^*/x_0^* with $\sqrt{v}$ and r for α = 0.03, E_1 = 1, E_2 = 0.7, & K/c = 6000.

Note: x^* is the critical stock of pollutant; α is the pollutant rate of decay; r is the social discount rate; v is the volatility of X; K is the sunk investment needed to reduce pollution emissions from E_1 to E_2; and c is the coefficient of valuation of pollution.