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Realized variance option and options on quadratic variation normalized to unit expectation are analysed for the property of monotonicity in maturity for call options at a fixed strike. When this condition holds the risk-neutral densities are said to be increasing in the convex order. For Lévy processes, such prices decrease with maturity. A time series analysis of squared log returns on the S&P 500 index also reveals such a decrease. If options are priced to a slightly increasing level of acceptability, then the resulting risk-neutral densities can be increasing in the convex order. Calibrated stochastic volatility models yield possibilities in both directions. Finally, we consider modeling strategies guaranteeing an increase in convex order for the normalized quadratic variation. These strategies model instantaneous variance as a normalized exponential of a Lévy process. Simulation studies suggest that other transformations may also deliver an increase in the convex order.

Keywords: Equity options; Levy process; Mathematical finance; Stochastic volatility; Stochastic processes

1. Introduction

Financial markets now trade options on numerous underliers other than stocks and stock indices. Examples include options on the VIX index, realized variance on stocks and stock indices, cumulated losses from natural disasters, cumulated losses on defaults by a basket of firms, among other possibilities. The underlying outcomes on which these option contracts are written are not traded assets. As a consequence, the calendar spread inequality usually satisfied by call options on stocks need no longer hold. This property is often referred to as the condition for positive forward variance, reflecting the principle that total variance to the later maturity exceeds total variance to the earlier maturity.

Specifically, for stock options one may consider them as written on the price relative to the forward price for the appropriate maturity. Viewed this way, the underlier, now taken as the forward deflated stock price, has unit expectation for all maturities. If one now fixes a strike, at a pre-specified level of moneyness relative to the forward, it is well known by static arbitrage arguments that call prices for this strike are increasing in maturity. It then follows that all convex functions of the forward deflated stock price, delivered as promised payoffs, have a higher current market value for a longer maturity. Equivalently, one states that risk-neutral marginal densities for the forward deflated stock price are increasing in convex order as convex functions delivered later are worth more. We refer to Föllmer and Schied (2002), Carr and Madan (2005) and Davis and Hobson (2007) for the relationship between such convex orders and the existence of martingales meeting all the risk-neutral marginals. This same proposition allows one to define forward variance $v(K, T_1, T_2)$ at strike K over the interval $T_1 < T_2$ by the positive quantity $(\sigma^2(K, T_2)T_2 - \sigma^2(K, T_1)T_1)/(T_2 - T_1)$.

The arbitrage argument underlying this monotonicity in call prices relies quite critically on the ability to trade the underlying asset. When we have an underlying outcome that is not a traded asset price, it is no longer the case that risk-neutral marginal densities for outcomes deflated to a unit mean should be related in any way by the convex order for densities. Put another way, forward variances may be negative. The marginal densities may

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still be recovered from option prices in the usual way, as described, for example, by Breeden and Litzenberger (1978), but call option prices at fixed levels of moneyness relative to the mean may be increasing in maturity for some strikes and decreasing for others, or even lose monotonicity with respect to maturity at some strikes. The primary reason for such possibilities is that unlike an underlying traded asset, that refers at all times to the present value of some terminal cash flow, thereby constituting an underlying price process that is a martingale, for non-traded underliers the level of the underlier at different time points is more like two totally different stocks and then there is no reason for the volatility of one of them to be above or below another.

This paper considers the question of monotonicity in convex order of marginal densities, or the increase in price for calls with respect to maturity at a fixed strike, for options on realized variance normalized to a unit expectation. We shall consider both the physical and risk-neutral densities in this context or the monotonicity in maturity of the expected call payoff and its price. Though realized variance options are not yet exchange traded, there is a developing over the counter market in these contracts permitting the observation of some risk-neutral information. When working with data we shall take account of the necessary discretization of realized variance in terms of averaged squared daily log price relatives. At a theoretical level we study the behavior of the rate of realized quadratic variation, defined as the quadratic variation to time t deflated by the time to reflect the averaging implicit in the definition of the realized variance contract.

We begin with an analysis of some simple models. The classic model of geometric Brownian motion (Black and Scholes 1973, Merton 1973) is not a reasonable candidate for options on the rate of realized quadratic variation, as in this model this rate is a constant and not a random variable. A class of processes with independent increments, like Brownian motion, that has now successfully been employed for equity options is the class of infinite activity, pure jump Lévy processes with examples including the variance gamma model (Madan and Seneta 1990, Madan *et al.* 1998), the normal inverse Gaussian model (Barndorff-Nielsen 1998), the generalized hyperbolic model (Eberlein and Kellerer 1995, Eberlein 2001, Eberlein and Krause 2002) and the CGMY model (Carr *et al.* 2002). We show that the densities for the rate of realized quadratic variation in all these models are decreasing in the convex order. In fact, in these models the rate of realized quadratic variation is a backward martingale. A particularly simple example for the rate of realized quadratic variation is the rate of increase of the gamma process and we explicitly describe and graph its call option prices. For these models call options on realized quadratic variation display negative forward variance. The result may be intuitively understood on noting that, for reasons related to the law of large numbers, the variance of the rate of realized quadratic variation decreases as the reciprocal of maturity and the standard deviation falls as the reciprocal of the square root of maturity. Call prices on mean adjusted rates of realized quadratic variation should therefore fall with maturity. The issue is not connected with mean reversion in volatility as the normalization to unit expectation puts aside all matters of mean reversion, whether existent or not. The decline is a pure consequence of the effects of averaging sequences of independent centered variates. As a practical implication we note that if market data were to reveal an increase with respect to maturity for call prices at fixed strikes on realized quadratic variation normalized to unit expectation, then one would need to entertain models that keep the central limit theorem at bay. This is a modeling problem that has also been commented on by Eberlein and Madan (2009).

Next we consider the behavior of realized variance for data on the S&P 500 index under the physical measure including the highly volatile period of the last quarter of 2008 in our study. Here we observe that the densities are slowly decreasing in the convex order. If we employ the operational concepts of acceptability introduced by Cherny and Madan (2009) and follow Madan (2009) to price options to levels of acceptability that are slightly increasing in maturity, with a view to reflecting a deteriorating confidence in the model used, we find the implied risk-neutral densities to be increasing in convex order. Hence there is a real possibility that these densities are increasing in convex order in the markets. A small sample of over the counter market prices is also suggestive of an increase in the convex order.

Numerically, we investigate the property of monotonicity in a wide class of stochastic volatility models, including the Heston (1993) model, and the stochastic volatility Lévy models of Carr *et al.* (2003) and Niccolato and Venardos (2003). We find that these models primarily deliver densities for the rate of realized quadratic variation that are both increasing and decreasing in convex order.

Finally, we explore modeling strategies that will deliver densities that are increasing in the convex order for the rate of realized quadratic variation. An increase is guaranteed when we model instantaneous volatility as a normalized exponential of a Lévy process. Simulation studies suggest that other functional transformations may also work.

The outline of the paper is as follows. Section 2 presents the results for Lévy processes and the example of the gamma process. In section 3 we describe the analysis of densities for the rate of realized quadratic variation on the S&P 500 index under the physical measure, and the risk-neutral measure as implied by pricing to acceptability and observing a small sample of over the counter prices. Section 4 takes up the stochastic volatility models followed by strategies for densities convex in the increasing order in section 5. Section 6 concludes.

2. Lévy process results

Suppose the stock price process $S=(S(t),\ t\geq 0)$ follows an exponential Lévy model with a driving Lévy process $X=(X(t),\ t\geq 0)$ with no Gaussian component, and

$$S(t)=S(0)\exp(rt+X(t)+\omega t),$$

where

$$\omega = -\log(E(X(1)).$$

Well-known examples of such Lévy processes employed in the finance literature were cited earlier in the introduction. The quadratic variation to time t, $Q(t)$, for such a process is given by

$$Q(t) = \sum_{s \le t} (\Delta X(s))^2,$$

and it was observed by Carr *et al.* (2005) that the process $Q(t)$ is itself a Lévy process with Lévy density $q(y)$, defined in terms of the Lévy density $k(x)$ for the process X by

$$q(y) = \frac{k(\sqrt{y})}{2\sqrt{y}} + \frac{k(-\sqrt{y})}{2\sqrt{y}}, \quad y > 0.$$

Now for any Lévy process $Z = (Z(t), t \ge 0)$ with $E[|Z(t)|] < \infty$ we have

$$\frac{Z(t)}{t} \underset{t \to \infty}{\longrightarrow} E[Z(1)],$$

and

$$\left(\frac{Z(t)}{t}, t > 0\right)$$

is a backwards martingale (Jacod and Protter 1988), i.e. if

$$\mathcal{F}_t^+ = \sigma\{Z(s), s \ge t\},$$

then

$$E\left[\frac{Z(s)}{s} \,\middle|\, \mathcal{F}_t^+\right] = \frac{Z(t)}{t}, \quad s < t. \tag{1}$$

Now from equation (1), one easily deduces that for every convex function $\psi(x)$

$$E\left[\psi\left(\frac{Z(t)}{t}\right)\right] \le E\left[\psi\left(\frac{Z(s)}{s}\right)\right].$$

It follows that the marginal densities for the rate of realized quadratic variation $Q(t)/t$ are decreasing in the convex order. A particular example is provided by the variance gamma model for which the quadratic variation is given by a gamma process $\gamma = (\gamma(t), t \ge 0)$ in the case of unit volatility or $\nu = 1$. In this case the backward martingale is particularly simple using the beta gamma algebra. Let $B(\alpha, \beta)$ be a beta random variable with parameters α, β and note that for $a < b$, γ_a/γ_b is distributed as $B(a, b-a)$ and is independent of γ_b. It follows that for $s < t$, and $\mathcal{F}_t^+ = \sigma\{\gamma_u \mid u \ge t\}$,

$$\begin{aligned} E\left[\frac{\gamma_s}{s} \,\middle|\, \mathcal{F}_t^+\right] &= E\left[\frac{\gamma_s}{\gamma_t}\frac{\gamma_t}{s} \,\middle|\, \mathcal{F}_t^+\right] \\ &= E\left[B(s, t-s)\frac{\gamma_t}{s} \,\middle|\, \mathcal{F}_t^+\right] \\ &= \frac{\gamma_t}{t}. \end{aligned}$$

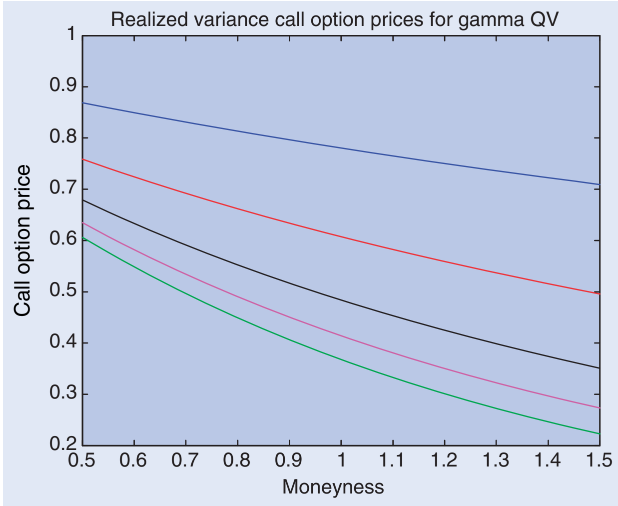

Figure 1. Graph showing prices of call options for gamma process quadratic variation as a function of the strike for maturities of 1, 3, 6, 9, and 12 months in blue, red, black, magenta and green, respectively.

The price of a call option $c(a, t)$ on the rate of realized quadratic variation with strike a and maturity t, for an interest rate of r, is

$$\begin{aligned} c(a,t) &= \mathrm{e}^{-rt} E\left[\left(\frac{\gamma_t}{t} - a\right)^+\right] \\ &= \mathrm{e}^{-rt} \int_{at}^{\infty} \left(\frac{x}{t} - a\right) \frac{x^{t-1}\,\mathrm{e}^{-x}}{\Gamma(t)}\,\mathrm{d}x \\ &= \mathrm{e}^{-rt}\left[\int_{at}^{\infty} \frac{x^t\,\mathrm{e}^{-x}}{\Gamma(t+1)}\,\mathrm{d}x - a\int_{at}^{\infty} \frac{x^{t-1}\mathrm{e}^{-x}}{\Gamma(t)}\,\mathrm{d}x\right]. \end{aligned}$$

The result is easily computed using the incomplete gamma function and figure 1 presents a graph of call prices for strikes relative to the mean ranging from 0.5 to 1.5 for the maturities of one month, and 3, 6, 9 and 12 months. The decrease in convex order is quite evident at this unit volatility for the gamma process.

3. Analysis of S&P 500 data

We analyse in this section the physical densities for the rate of realized quadratic variation on the S&P 500 index. For this purpose we took daily data on the level of the index, S_t, from 2 January 1990 to 17 December 2008 and we constructed the time series for daily squared log price relatives by

$$v_t = \left(\log\left(\frac{S_t}{S_{t-1}}\right)\right)^2.$$

In order to construct the densities for realized variance under the physical measure, and to investigate their monotonicity in convex order, it suffices to construct the expectation under the physical measure of the payout to call options on realized variance options. For this purpose we need to model the physical measure and to simulate paths for v_t. It is well known that v_t is highly autocorrelated. The property we refer to is also called long memory as reflected in an autocorrelation function that sums to infinity across the lags. Long memory is an interesting property from a financial viewpoint as it will keep monotonicity in maturity for call prices written on the rate of the realized quadratic variation. These considerations suggest a regression model for v_t based on many lagged values for v_t. However, such a model would not give positive values for v_t when simulated forward. For this reason we consider a regression model on $y_t = \log(v_t)$. We then exponentiate simulated paths for y_t to build the paths for v_t.

The model for y_t regressed y_t on its lagged values using a robust regression procedure, given the length of the data period and the presence of some fairly volatile periods in the data set. The specific model used is

$$y_t = a + \sum_{i=1}^{20} b_i y_{t-i} + u_t.$$

The results of the robust regression are presented in table 1. We observe the pattern of possible long range dependence in the significance of t-statistics lagged up to 20 days.

For the simulation we draw from the empirical density of the residuals. We present in figure 2 the density for the residual employed in the simulation.

We simulate forward from the end of the data set on 17 December 2008 for 252 days 10,000 paths for v_t in this model. We then compute the realized variance at maturities of 1, 3, 6, 9 and 12 months for each of the 10,000 paths and divide by the mean value for each maturity. This gives us 10,000 readings for realized variance normalized to a unit expectation for our five maturities and we evaluate the price of call option payoffs under this physical measure for a range of strike ranging from 0.5 to 1.5. We present in figure 3 the prices of these call options for all the five maturities, and we present in figure 4 a graph of the densities for realized variance normalized to a unit mean.

We observe clearly that these densities are slightly decreasing in the convex order. We have explored this construction over varied time sub-intervals with similar results. The physical densities reflect the force of averaging in generating densities that are decreasing in the convex order.

The question remains as to what one may expect of risk neutrally. For a potential perspective on this we follow Madan (2009) and consider pricing to pre-specified levels of acceptability the residual cash flow held on selling the realized variance option for an ask price. The levels of acceptability of residual cash flows were axiomatized by Cherny and Madan (2009). For each level γ of acceptability for a residual cash flow X, there is a convex

Table 1. Regression results for log squared returns.

	Coefficient	t-Stat
Constant	−2.4798	−7.465
Lag 1	−0.0007	−0.539
Lag 2	0.0285	2.0734
Lag 3	0.0506	3.6805
Lag 4	0.0524	3.8111
Lag 5	0.0753	5.4666
Lag 6	0.0573	4.1504
Lag 7	0.0304	2.1985
Lag 8	0.0457	3.3062
Lag 9	0.0299	2.1605
Lag 10	0.0540	3.9111
Lag 11	0.0444	3.2115
Lag 12	0.0393	2.8418
Lag 13	0.0296	2.1422
Lag 14	0.0281	2.0313
Lag 15	0.0292	2.1138
Lag 16	0.0218	1.5849
Lag 17	0.0233	1.6955
Lag 18	0.0529	3.8441
Lag 19	0.0294	2.1347
Lag 20	0.0315	2.2864
R^2	11.01%	

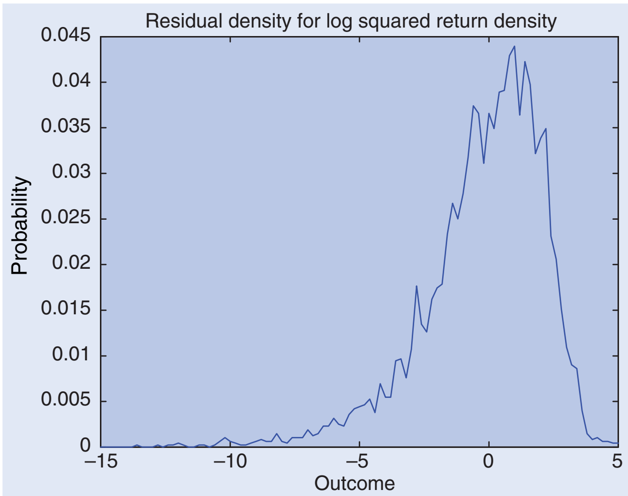

Figure 2. Density of residuals in the log squared return regression.

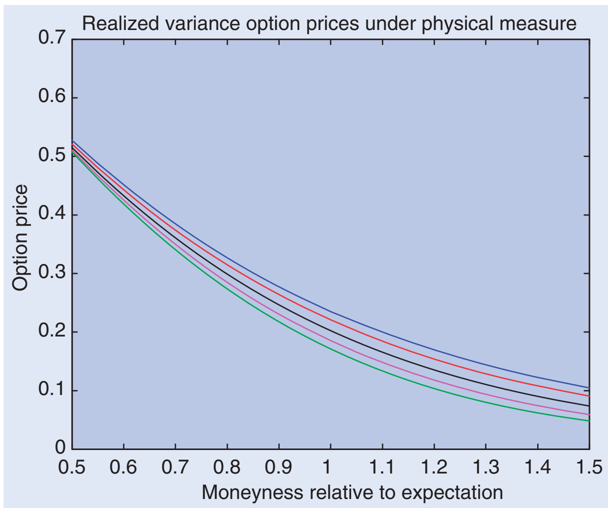

Figure 3. Prices of call options on normalized realized variance under the physical measure for maturities of 1, 3, 6, 9 and 12 months in blue, red, black, magenta and green, respectively.

set of measures $\mathcal{D}_\gamma$ supporting such acceptability with the requirement that $E^Q[X] \geq 0$, for all $Q \in \mathcal{D}_\gamma$. The higher the level of acceptability the larger is the set of supporting measures with $\mathcal{D}_\gamma \subseteq \mathcal{D}_{\gamma'}$ for $\gamma < \gamma'$. The set of cash flows acceptable at level γ, $\mathcal{A}_\gamma$, forms a convex cone of random variables that contains all the non-negative cash flows. When the acceptability of a cash flow is just a function of its probability law, one may define acceptability using a concave distortion. In this case, one associates with each level γ a concave distribution function Ψ^γ defined on the unit interval and X is acceptable at level γ just if

$$\int_{-\infty}^{\infty} x \, \mathrm{d}\Psi^\gamma(F_X(x)) \geq 0,$$

where F_X is the distribution function of the random variable X. The set of supporting measures related to a particular distortion are defined by Cherny and Madan (2009).

The ask price for a cash flow X attaining the acceptability level γ is the smallest constant a one may add to the cash flow to make $a + X$ acceptable at level γ. It is shown by Madan (2009) that this ask price is the negative of the expectation under concave distortion at level γ of the distribution function for negative of the cash flow. We employ here just a slight increase in the level of acceptability for longer maturities, reflecting a decreased confidence in the underlying model employed. We used an initial acceptability level of 0.025, that increases monthly by 0.025, for the distortion MINMAXVAR. For this distortion,

$$\Psi^\gamma(u) = 1 - (1 - u^{1/(1+\gamma)})^{1+\gamma}.$$

Figure 5 presents a graph for the resulting call prices across a range of strikes for our five maturities. We observe that these prices are increasing in the convex order. Hence we conclude that it is a real possibility that financial markets may well display marginals for normalized realized variance options that are increasing in the convex order.

4. Prices in markets

We obtained data for three at-the-money straddle prices for options on realized variance on the SPX. There were two at-the-money straddle prices on 4 February 2009 maturing December 2009 and December 2010 with bid and ask at 14.7/16.0 and 13.85/15.5, respectively, with the variance swap reference price at 41.5 and 39.5. We also have an at-the-money straddle quoted on 15 January 2009 for a 9 June maturity with a bid and ask at 16.25/18.25 at a variance swap reference of 48.5. The maturities for the first two straddles are 0.8685 and 1.8675, while for the third straddle it is 0.4247.

For the monotonicity in convex order we are interested in the prices of the options written on random variables of unit expectation and so we relativize the strikes and option prices to the level of the variance swap rate or the level of the risk-neutral expectation of realized variance. The dollar mid-quote price of the first two relativized unit strike straddles are 0.7397 and 0.7430. The relativized dollar mid-quote price of the third straddle is 0.7113. Since the longer maturities have the higher relativized price these observations support the hypothesis that in the market we possibly have a slight increase in the convex order.

We also obtained two other prices, a 4 February 2009 quote for a 60 strike call of 3.1 with a variance swap reference of 42, and a 23 January 2009 quote for a 9 March at-the-money put at 9.0 for a variance swap reference of 50.5.

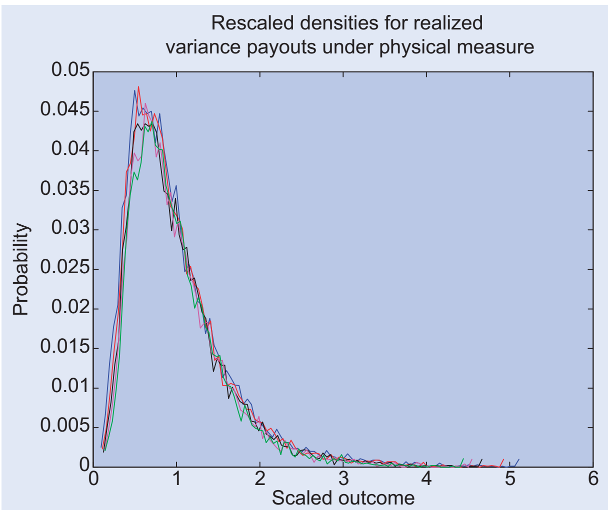

Figure 4. Densities for realized variance normalized to unit expectation under the physical measure for 1, 3, 6, 9 and 12 months in blue, red, black, magenta and green, respectively.

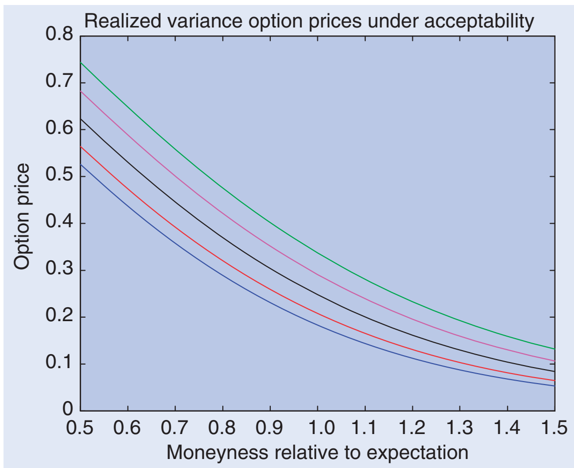

Figure 5. Call prices under acceptability pricing with acceptability levels slowly rising with maturity. The maturities are 1, 3, 6, 9 and 12 months in blue, red, black magenta and green, respectively.

5. Stochastic volatility models

There are two important classes of stochastic volatility models in the literature. These are the Heston (1993) model and its extensions to underlying Lévy processes by Carr *et al.* (2003) and the OU model driven background Lévy processes with only positive jumps entertained by Barndorff-Nielsen and Shepard (2001) and Nicolato and Venardos (2003). We investigate in this section the behavior in convex order of the marginal densities for the rate of realized quadratic variation normalized to a unit expectation in the Heston (1993) model (HSV), the CGMYSA model and the model CGMYSG, which were developed by Carr *et al.* (2003). Given the relevance of stationary solutions to the OU equations employed and the resulting impact of ergodic theorems on the behavior of averages, we anticipate that although one may have an initial increase in the convex order, these models will primarily be characterized by an eventual decrease in convex order for the relevant marginals. The task of creating risk-neutral models generically reflecting an increase in the convex order in then taken up in the final section of the paper.

We begin with the HSV model. In this model, realized quadratic variation to time t takes the form

$$Q(t) = \int_0^t y(u)\mathrm{d}u.$$

The characteristic function for $Q(t)$ is readily available from the cited papers and may be used to build the Laplace transform of the rate of realized quadratic variation normalized to unit expectation or $Q(t)/E[Q(t)]$. We then numerically price options on this variable for all the models using an extension of the Carr and Madan (1999) method to Laplace transforms that was also employed by Carr *et al.* (2005).

More specifically we define

$$\phi(\lambda, t) = E[\exp(-\lambda Q(t))].$$

We may obtain by differentiation that

$$E[Q(t)] = -\phi_\lambda(0, t).$$

The Laplace transform of the normalized quadratic variation is then

$$\eta(\lambda, t) = \phi\left(\frac{\lambda}{-\phi_\lambda(0, t)}, t\right).$$

The expectation of the normalized random variable is unity and hence following Carr *et al.* (2005) the Laplace transform in the strike a of the option prices,

$$w(a, t) = \mathrm{e}^{-rt} E\left[\left(\frac{Q(t)}{E[Q(t)]} - a\right)^+\right],$$

is given by

$$\zeta(\lambda, t) = \mathrm{e}^{-rt}\left[\frac{\eta(\lambda, t) - 1}{\lambda^2} + \frac{1}{\lambda}\right],$$

where

$$\zeta(\lambda, t) = \int_0^\infty \mathrm{e}^{-\lambda a} w(a, t)\mathrm{d}a.$$

The option prices follow on Laplace inversion.

For the CGMYSA model the quadratic variation to time t is the quadratic variation of the CGMY process up to the random time given by the integral of the square root process. The Laplace transform of the quadratic variation of the CGMY process to time t, $Q_{\mathrm{CGMY}}(t)$, was derived by Carr *et al.* (2005) and we have

$$\begin{aligned} E[\exp(-\lambda Q_{\mathrm{CGMY}}(t))] &= \Phi(\lambda, t) \\ &= \exp(-t\Psi(\lambda)). \end{aligned}$$

We are now interested in the expectation of

$$\begin{aligned} &E\left[\exp\left(-\lambda Q_{\mathrm{CGMY}}\left(\int_0^t y(u)\mathrm{d}u\right)\right)\right] \\ &\quad = E\left[\exp\left(\int_0^t y(u)\mathrm{d}u\Psi(\lambda)\right)\right] \\ &\quad = \phi(\Psi(\lambda), t). \end{aligned}$$

A similar construction is made for the CGMYSG model. For the details on the two functions $\phi(\lambda)$ and $\Psi(\lambda)$ we refer, respectively, to Carr *et al.* (2003, 2005). For the numerical inversion of Laplace transforms we follow Abate and Whitt (1995) and Rogers (2000).

Before proceeding with this investigation we comment on the consequences for the Sato process introduced by Carr *et al.* (2007) and studied further with respect to options on variance by Eberlein and Madan (2009). The Sato process is an additive process with independent but inhomogeneous increments. It is constructed from a self-decomposable random variable X at unit time by scaling and defining the probably law of $X(t)$ at time t as that of $t^\gamma X$. Sato (1999) shows that there exists an additive process $X(t)$ with these marginal laws for each time t. The Lévy system for this process may be explicitly derived from the Lévy measure of X at unit time and is given by Carr *et al.* (2005). It was demonstrated by Eberlein and Madan (2009) that, for the Sato process, options on realized variance remain a random variable and do not lose variance with maturity provided the scaling coefficient is equal to or above 1/2.

Furthermore, it is shown by Carr *et al.* (2005, theorem 5) that the quadratic variation of a Sato process with scaling coefficient γ is itself a Sato process with scaling coefficient 2γ. One may explicitly derive the Lévy system of quadratic variation as an additive process in its own right. The characteristic exponent at unit time is then an integral of $(\mathrm{e}^{iux} - 1)$ against this Lévy system that is then observed to be of the form required for a self-decomposable law. One then shows that the Lévy system of this self-decomposable law when scaled at 2γ coincides with the Lévy system for the quadratic variation of the original process. Hence for a Sato process with scaling coefficient γ, its quadratic variation satisfies

$$Q(t) \stackrel{(d)}{=} t^{2\gamma} Q(1).$$

It follows that

$$E[Q(t)] = t^{2\gamma} E[Q(1)],$$

and therefore

$$\frac{Q(t)}{E[Q(t)]} \stackrel{(d)}{=} \frac{Q(1)}{E[Q(1)]},$$

whereby we have that the distribution of the realized quadratic variation normalized to a unit expectation is constant in convex order. The property of an increase in convex order will therefore not be delivered by the Sato process, even if it does give some reasonable value to options on realized variance as argued by Eberlein and Madan (2009).

We estimate on data for 130 SPX options on 4 February 2009 three stochastic volatility models. These are the Heston stochastic volatility model, the model CGMYSA (Carr *et al.* 2003), both of which have instantaneous volatility modeled by a square root process, along with the model CGMYSG, also studied by Carr *et al.* (2003), that takes the instantaneous volatility to be given by an OU equation driven by a process that only jumps upwards with a finite jump arrival rate and exponential jump size distribution.

We present first in table 2 the fit statistics and in table 3 the parameter estimates. Graphs of the fit of the model to market prices are also presented in figures 6–8.

For each of these models we have the Laplace transform in strike of the option price on the normalized

Table 2. Fit statistics for SPX 4 February 2009.

	Model		
	HSV	CGMYSA	CGMYSG
RMSE	1.0036	1.1294	0.8172
AAE	0.8340	0.9134	0.6657
APE	0.0206	0.0226	0.0165

Table 3. Parameter values.

HSV		CGMYSA		CGMYSG	
v_0	0.4029	C	0.8065	C	0.6210
η	0.4316	G	4.7142	G	2.4159
κ	1.7358	M	15.3727	M	44.5588
λ	1.0182	Y	0.9187	Y	0.9390
ρ	−0.7961	κ	4.2242	κ	2.8320
		η	0.3503	λ	1.0503
		λ	2.7448	ζ	0.3423

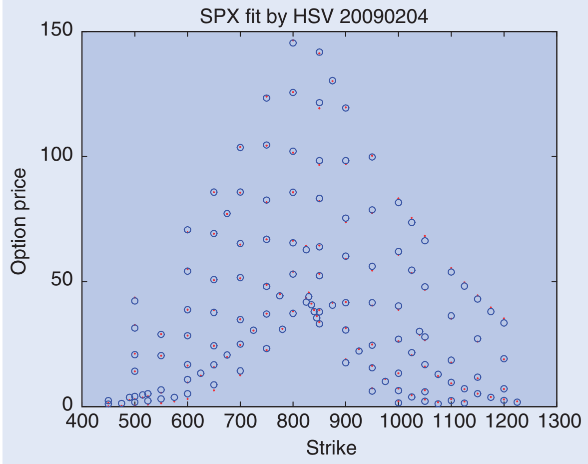

Figure 6. Model (red dots) and market prices (blue circles) for HSV. SPX 4 February 2009.

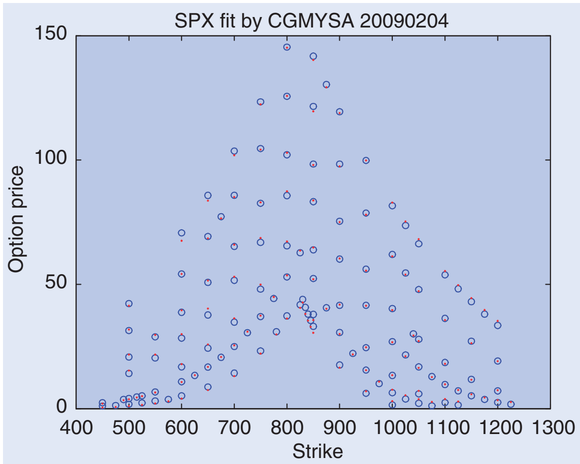

Figure 7. Model (red dots) and market prices (blue circles) for CGMYSA. SPX 4 February 2009.

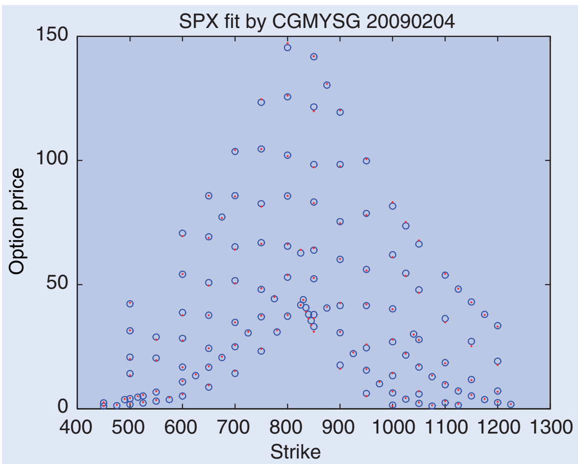

Figure 8. Model (red dots) and market prices (blue circles) for CGMYSG. SPX 4 February 2009.

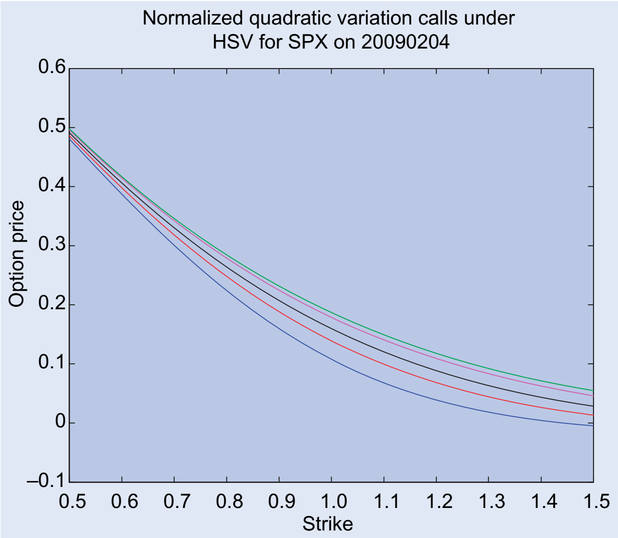

Figure 9. Call prices on normalized quadratic variation in the Heston model for an increasing set of maturities in blue, red, black, magenta and green.

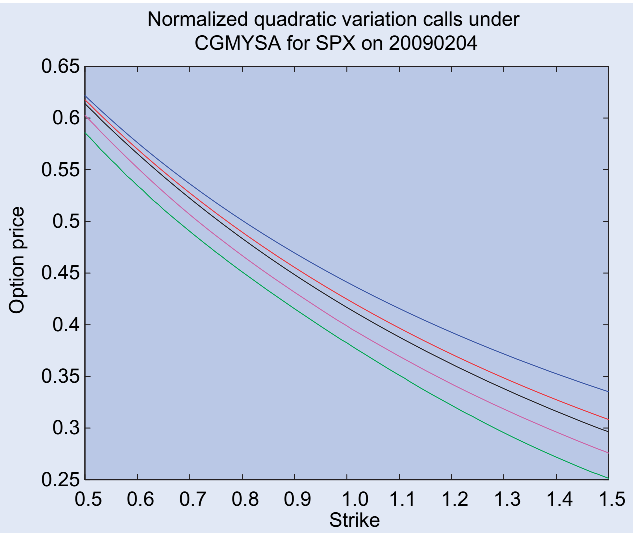

Figure 10. Call prices on normalized quadratic variation in the CGMYSA model for an increasing set of maturities in blue, red, black, magenta and green.

quadratic variation and we present in figures 9–11 the graphs of these call prices for the five maturities, 0.15, 0.27, 0.40, 0.65 and 0.90, that match the option maturities to which the models were calibrated. We observe an increase in convex order for HSV, and a decrease in convex order for CGMYSA and CGMYSG.

6. Exponential Lévy models for instantaneous quadratic variation

It may well turn out that, in markets, call prices on realized variance options characteristically display an increase with respect to maturity for a fixed strike. Model calibrations may however still be done by fitting prices of

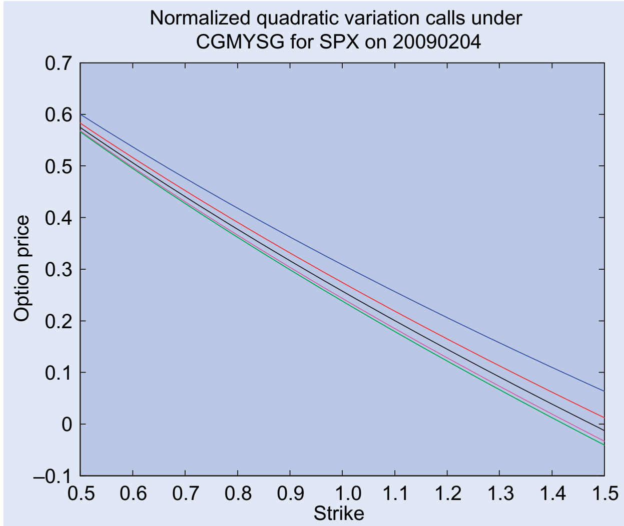

Figure 11. Call prices on normalized quadratic variation in the CGMYSG model for an increasing set of maturities in blue, red, black, magenta and green.

options on the index or underlying asset. It is then of interest to know when we have a structure for the asset dynamics that guarantees an increase in convex order for the density of the rate of realized quadratic variation. We are then led to consider modeling strategies guaranteeing an increase in convex order for normalized quadratic variation. We do not wish to rely on chance calibrations delivering this property, but must organize it up front.

We begin by following Carr *et al.* (2008) and Baker and Yor (2008) by taking the instantaneous variance of the stock to be modeled by a geometric Brownian motion. The absence of mean reversion in drift is not an issue as our focus is on the law of normalized quadratic variation and the drift will be put aside in any case by the normalization. Hence we take the stock price $(S(t),\ t \geq 0)$ to be driven by a Brownian motion $(W_S(t),\ t \geq 0)$ with an instantaneous variance process $(v(t),\ t \geq 0)$ driven by an independent Brownian motion $(W_V(t),\ t \geq 0)$ satisfying

$$\begin{aligned} \mathrm{d}S(t) &= rS(t)\mathrm{d}t + \sqrt{v(t)}S(t)\mathrm{d}W_S(t), \\ \mathrm{d}v(t) &= \lambda v(t)\mathrm{d}W_v(t). \end{aligned}$$

The normalized quadratic variation to time t, $U(t)$, is then

$$U(t) = \frac{1}{t}\int_0^t \mathrm{e}^{\lambda W_v(u) - (\lambda^2/2)u}\,\mathrm{d}u.$$

Carr *et al.* (2008) provide a proof that the process $U(t)$ is increasing in convex order and Baker and Yor (2008) provide a short proof of this result. It is well known (Strassen 1965, Doob 1968, Kellerer 1972) that a sequence of marginal densities are increasing in the convex order just if there exists a martingale on possibly another probability space with the same marginal densities. Baker and Yor (2008) show explicitly the martingales supporting the increasing convex order of the densities $U(t)$.

Hirsch and Yor (2009a) take up a general approach to constructing processes increasing in the convex order and simultaneously exhibiting the martingales with the same marginal densities. We note in this context that

Roynette (2009) has recently demonstrated that, for any martingale $(M(t), t \geq 0)$ and an increasing continuous process $\alpha = (\alpha(t), t \geq 0)$, the marginal densities of the process

$$\frac{1}{\alpha(t)} \int_0^t M(u) \mathrm{d}\alpha(u)$$

are increasing in the convex order. It follows from here that, for any Lévy process $(X(t),\ t \geq 0)$ admitting exponential moments, the process

$$\frac{1}{t} \int_0^t \frac{\mathrm{e}^{\lambda X(u)}}{E[\mathrm{e}^{\lambda X(u)}]} \mathrm{d}u$$

has marginals increasing in the convex order. Hence instantaneous variance modeled as an exponential Lévy processes normalized to unit expectation delivers normalized quadratic variations increasing in the convex order. The task of explicitly exhibiting the martingales with these marginal densities is taken up by Hirsch and Yor (2009b).

We now consider other transformations that give results in both directions. We leave for future research the characterization question of what result to expect from each functional transformation. For an example of another potential transformation we first consider constructing normalized daily instantaneous variance for $N(x)$, the standard normal distribution function, as

$$v_t = \frac{N(X(t))}{E[N(X(t))]},$$

where we take for $X(t)$ the VG process with parameters $\sigma = 0.5$, $\nu = 0.15$, and $\theta = -0.1$. We simulated the VG process on 10,000 paths of length 252 and constructed 10,000 simulated paths for v_{ts}. We then constructed readings on realized variance as

$$R_{Ns} = \frac{1}{N} \sum_{t=1}^{N} v_{ts},$$

obtaining 10,000 observations for N corresponding to 1, 3, 6, 9 and 12 months. We graph in figure 12 the resulting option prices for a variety of strikes.

For the opposite result, consider the square of the VG process for v_t. In this case we obtain a decrease in the convex order as shown in figure 13.

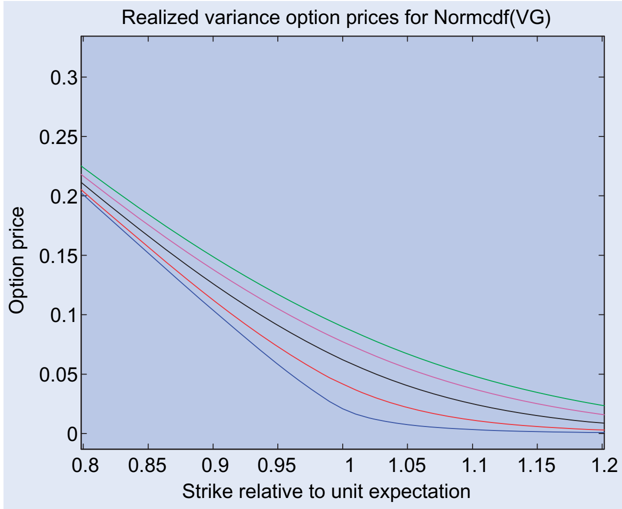

Figure 12. Realized variance option prices for instantaneous variance given by the cum norm function evaluated on the VG process for maturities of 1, 3, 6, 9 and 12 months in blue, red, black, magenta and green, respectively.

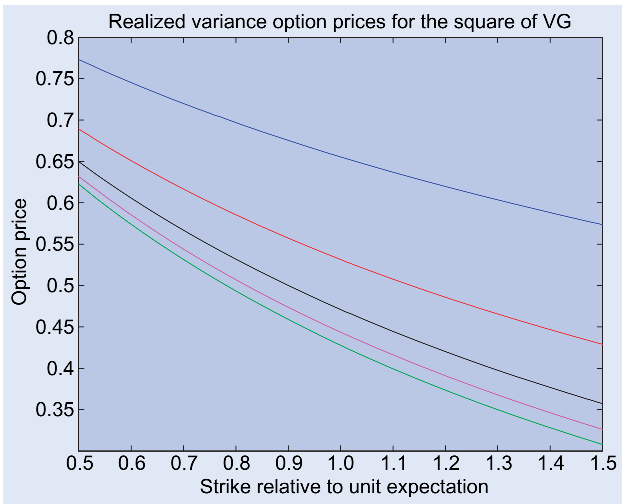

Figure 13. Realized variance option prices for instantaneous variance given by the cum norm function evaluated on the square of the VG process for maturities of 1, 3, 6, 9 and 12 months in blue, red, black, magenta and green, respectively.

7. Conclusion

Options on realized variance and quadratic variation normalized to a unit expectation more generally are investigated with respect to the property of monotonicity in convex order for their one-dimensional marginal distributions. It is observed that for Lévy processes these densities are decreasing in the convex order. A time series analysis of squared log returns on the S&P 500 index also reveals that the densities for realized variance are decreasing in the convex order under the physical measure. Hence we have the reverse situation for calendar spreads to that known to exist for stock options, with longer maturity calls declining in value for the same strike.

It is observed that if options are priced to a slightly increasing level of acceptability then the risk-neutral densities would be increasing in the convex order. Calibrated stochastic volatility models yield possibilities in both directions. Finally, we consider modeling strategies that guarantee an increase in convex order for the normalized quadratic variation based on modeling instantaneous volatility as an exponential of a Lévy process normalized to a unit expectation. Simulation studies suggest that transformations other than the exponential may also deliver an increase in the convex order. A more detailed investigation of such transformations is left as a topic for further research.

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