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A class of Lévy process models with almost exact calibration to both barrier and vanilla FX options

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Vanilla (standard European) options are actively traded on many underlying asset classes, such as equities, commodities and foreign exchange (FX). The market quotes for these options are typically used by exotic options traders to calibrate the parameters of the (risk-neutral) stochastic process for the underlying asset. Barrier options, of many different types, are also widely traded in all these markets but one important feature of the FX options markets is that barrier options, especially double-no-touch (DNT) options, are now so actively traded that they are no longer considered, in any way, exotic options. Instead, traders would, in principle, like to use them as instruments to which they can calibrate their model. The desirability of doing this has been highlighted by talks at practitioner conferences but, to our best knowledge (at least within the realm of the published literature), there have been no models which are specifically designed to cater for this. In this paper, we introduce such a model. It allows for calibration in a two-stage process. The first stage fits to DNT options (or other types of double barrier options). The second stage fits to vanilla options. The key to this is to assume that the dynamics of the spot FX rate are of one type before the first exit time from a 'corridor' region but are allowed to be of a different type after the first exit time. The model allows for jumps (either finite activity or infinite activity) and also for stochastic volatility. Hence, not only can it give a good fit to the market prices of options, it can also allow for realistic dynamics of the underlying FX rate and realistic future volatility smiles and skews. En route, we significantly extend existing results in the literature by providing closed-form (up to Laplace inversion) expressions for the prices of several types of barrier options as well as results related to the distribution of first passage times and of the 'overshoot'.

Keywords: Lévy processes; Option pricing; Barrier options; Continuous time finance; Credit models; Currency derivatives; Pricing of derivatives securities; Quantitative finance

1. Introduction

The option pricing framework of Black and Scholes (1973) and Merton (1973) was a major breakthrough and, when introduced, gained acceptance by market participants as the way to price and hedge options. However, it cannot account for the effect of volatility smiles and skews. More recently, models have been introduced which can account for or be calibrated to volatility smiles or skews observed in the market prices of vanilla (standard European) options. These include local volatility models (Derman and Kani 1994, Dupire 1994), stochastic volatility models (e.g. Heston 1993), models with Poisson jump processes (e.g. Merton 1976, Bates 1996, Kou 2002) and models based on general classes of (possibly time-changed) Lévy processes (Madan *et al*. 1998, Barndorff-Nielsen and Shephard 2001, Carr *et al*. 2002, 2003, Schoutens 2003, Carr and Wu 2007). These models can all be calibrated to the volatility smiles or skews observed in the vanilla options markets.

One of the largest options markets in the world is the market for foreign exchange (FX) options. This market is very active and liquid. One feature of this market is that, not only is there a very active market for vanilla options but there is also simultaneously an active market for barrier options. Many different types of barrier options are traded (e.g. partial, window, double knockout calls and puts, with or without rebates) but, by far, the most actively traded type of barrier option, in the

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inter-bank and inter-dealer broker markets, is the double-no-touch (henceforth DNT) option. This† is an exotic option which pays one unit of domestic currency if the spot FX rate (quoted as the number of units of domestic currency per unit of foreign currency), before maturity, never trades equal to or above an upper barrier and never trades equal to or below a lower barrier or it pays zero otherwise.

It is well known (Dupire 1994) that the prices of vanilla options only depend upon the terminal distribution of the spot FX rate. On the other hand, the prices of DNT options depend upon the full distribution of spot FX rates at all times up to and including maturity. Hence, market prices of DNT options contain finer information about future spot FX rates (in the risk-neutral pricing measure). In addition, since DNT options are so actively traded, traders would like to be able to calibrate a model to the prices of, not only, vanilla options, but also DNT options—simply in order to closely match market prices. The desirability of doing so has been highlighted at practitioner conferences (Afaf 2007, Crosby 2007, Kainth 2007) but it is easier said than done and, to our best knowledge, no existing models in the literature are specifically designed to facilitate this. This is the purpose of the present paper.

The reason why calibrating models to the prices of both DNT options and vanilla options is easier said than done is due to the following well-known (Dupire 1994, Schoutens *et al.* 2005) fact: It is possible to have more than one model which can be calibrated to the prices of vanilla options which will then give rise to different prices for exotic options. For example, one could calibrate (a) a Dupire (1994) local volatility model and (b) a Bates (1996) model (stochastic volatility and jumps, perhaps, making some parameters time-dependent) to the market prices of vanilla options. However, one would then get two different prices (perhaps very different (Schoutens *et al.* 2005)) for a DNT option—neither of which may coincide with the market price. Furthermore, the discrepancy between model and market prices might depend, say, on the maturity—with one model being better for short-dated DNT options and the other being better for longer-dated DNT options. The question is then, which model should one choose? This is by no means a rhetorical question. Indeed, practitioners note (Lipton and McGhee 2002) that local volatility models tend to under-price DNT options and stochastic volatility models tend to over-price DNT options, relative to the market prices, when the respective models are calibrated to the market prices of vanilla options.

We should mention that Lipton (2002) introduced into the option-pricing literature an analytical result (up to Laplace inversion) for the price of a type of barrier option under a jump-diffusion process with exponentially distributed jumps. Kou and Wang (2003, 2004) and Sepp (2004) price various types of barrier options within the Kou (2002) double exponential jump-diffusion (henceforth DEJD) model. Asmussen *et al.* (2007) price a type of barrier option within the CGMY model of Carr *et al.* (2002), by approximating the CGMY process with a jump-diffusion process with a large, finite, but otherwise essentially arbitrary number (in the limit that this number tends to infinity, the approximation becomes exact in the sense that one gets convergence in distribution) of sums of double exponential processes (i.e. by hyperexponential processes). Di Graziano and Rogers (2006) (see also Jobert and Rogers 2006) price barrier options within their regime-switching MMGBM model. Pricing barrier options for arbitrary Lévy processes is far from trivial. There are, in principle, some results (see Schoutens 2003 and the references therein) based on Weiner–Hopf analysis, although they involve inversion of triple Laplace transforms and it is open to debate as to whether this could be done efficiently enough for use in a trading environment. Some simplification occurs (Rogers 2000) if the Lévy process is spectrally one-sided (i.e. jumps are either always up or always down). This might, possibly, be appropriate for currencies in emerging markets but it is an unrealistic assumption for modelling major currency pairs. A simplification also occurs if the jumps are of phase-type (Asmussen *et al.* 2004, Pistorius 2004) which includes double exponential jumps as a special case.

We will also present results (in appendix A2) related to the distribution of first passage times and of the 'overshoot'. We thank a referee for drawing our attention to Jiang and Pistorius (2008) which contains some related results (although our method of proof is very different to theirs).

The rest of this paper is organized as follows. In section 2 we introduce the model. In section 3 we explain how we can calibrate our model to the market prices of DNT options. In section 4 we show how we can price vanilla options and hence, also, calibrate our model to the market prices of vanilla options. In section 5 we summarize our model, highlight its flexibility and illustrate its ability to calibrate to both DNT options and vanilla options by performing such calibrations on the market prices of options on cable (USD/STG). Section 6 is a short conclusion. All major proofs are relegated to the appendix.

2. The model

We define today (the initial time) to be time $t_0 \equiv 0$. We denote calendar time by t, with $t \geq t_0$.

We assume frictionless markets and the absence of arbitrage. The latter guarantees the existence of a risk-neutral equivalent martingale measure. However, because our model is incomplete, it is well known that the risk-neutral equivalent martingale measure is not unique. We shall calibrate our model to the market prices

†For all the barrier options we consider in this paper, we will assume that the barriers are monitored continuously to see if the relevant option has been knocked-in or out. This is, by far, the most common situation in the FX options markets. Barrier options with discretely monitored barriers are traded occasionally but are much less liquid.

of options and assume that the risk-neutral measure, which we shall denote by Q, is fixed through these market prices. This is a standard concept for incomplete markets. In this paper, we shall only be concerned with the dynamics of the spot FX rate in the risk-neutral measure Q. We fix a probability space $(\Omega, \Im, Q)$ and an information filtration $(\Im_t)_{t \geq t_0}$ which we assume satisfies the usual conditions. We denote by $E_t^Q[\bullet]$ the expectation operator, under Q, at time t.

We denote the spot FX rate, at time t, by $S(t)$. It is quoted as the number of units of domestic currency per unit of foreign currency. We denote $\log(S(t)/S(t_0)) \equiv X(t)$.

We assume that interest rates are constant and we denote the continuously compounded domestic (respectively, foreign) interest rate by r_d (respectively, r_f). We denote the price, at time t_0, of a zero coupon bond dominated in domestic currency, maturing at time T, by $P_d(t_0, T)$. Hence, $P_d(t_0, T) = \exp(-r_d(T - t_0))$. It is possible to allow for term structures of interest rates (and we briefly outline how in the appendix), but assuming that interest rates are constant simplifies the exposition, whilst aiding the understanding of the intuition behind our modelling framework.

We introduce lower and upper barriers, denoted by L and U respectively, which correspond to the barrier levels of DNT options† to which we wish to calibrate our model. We assume that $0 < L < S(t_0) < U < \infty$. We will call the region (L,U) the 'corridor'. We denote the first exit time of the spot FX rate from the corridor by τ, i.e. we define

$$\tau \equiv \inf\{t : S(t) \leq L \text{ or } S(t) \geq U\}.$$

If the spot FX rate $S(t)$, $t \geq t_0$, has always been strictly between L and U, i.e. the spot FX rate has never exited from the corridor, then the convention is that we set $\tau = \infty$.

We will now proceed to specify the dynamics of the spot FX rate $S(t)$ by specifying two auxiliary stochastic processes $S_1(t)$ and $S_2(t)$ whose dynamics are linked with two key assumptions.

The first key assumption is that, at time t_0, and until the spot FX rate first exits from the corridor, the dynamics of the spot FX rate are such that we can compute certain key quantities of interest. These key quantities of interest are, essentially, the probability density function of $X(\tau)$ conditional on $X(\tau) > \log(U/S(t_0))$ or on $X(\tau) < \log(L/S(t_0))$ (see lemma 4.1), and the Laplace Transforms of the joint probability distribution function of τ and $X(\tau)$ (see equation (4.6)).

Remark 2.1: The key quantities of interest that we need in our modelling framework can be computed in closed form in the DEJD model of Kou (2002) and Kou and Wang (2003). They can also be calculated in a jump-diffusion model with an arbitrary number of sums of double exponential jump processes. We also mention in passing that a possible alternative for the dynamics might be to assume that they follow the dynamics of the regime-switching MMGBM model of Di Graziano and Rogers (2006) (see also Jobert and Rogers 2006). We conjecture that the key quantities of interest that we need can be computed in this latter model, although we will not pursue this possibility here. In this paper, we assume that the dynamics follow a jump-diffusion model with an arbitrary number of sums of double exponential jumps. Furthermore, we also allow the diffusion volatility and the intensity rates of the jump processes to be functions of a continuous-time Markov chain with two states. Hence, we allow for stochastic volatility (or stochastic time-changes) as well as jumps. We will see in section 5 that allowing for a stochastic time-change improves the model fit when calibrated to market data. We compute the key quantities of interest that we need in lemma 4.1 and in the appendix. We christen these dynamics the Chain Extended Exponential Double jump process (henceforth the CEE2 process for brevity). This leads us immediately to assumption 2.2.

Assumption 2.2: *The dynamics, under the risk-neutral measure Q, of the spot FX rate $S(t)$, at time t, for $t \in (t_0, \tau)$, are constructed in terms of the auxiliary stochastic process $S_1(t)$. Note that $S_1(t)$ is defined for all $t \in [t_0, \infty)$ and it is a CEE2 process constructed as follows.*

We introduce M Poisson (counting) *processes, denoted by $N_i(t)$, $i = 1, \ldots, M$, with $N_i(t_0) \equiv 0$. For notational simplicity, we assume M is an even number. The Poisson processes $N_i(t)$, for $1 \leq i \leq M/2$, are associated with up jumps and the Poisson processes $N_i(t)$, for $1 + M/2 \leq i \leq M$, are associated with down jumps. Associated with each Poisson process $N_i(t)$, $i = 1, \ldots, M$, is an exponentially distributed random variable γ_i with mean $1/b_i$ (under Q). We assume $1 < b_i < \infty$, for $1 \leq i \leq M$. We introduce a standard Brownian motion, denoted by $z(t)$, with $z(t_0) \equiv 0$. We introduce a continuous-time Markov chain $\Psi(t)$ with two states, labelled 1 and 2. Transitions take place between state j and state k of the Markov chain, $j = 1, 2$, $k = 1, 2$, $j \neq k$ with constant instantaneous jump rates ε_{jk} (under Q), where $\varepsilon_{jk} > 0$. We‡ write $\Psi(t_0) = j_0$, where j_0 is either 1 or 2. For each $i = 1, \ldots, M$, the Poisson process $N_i(t)$ has (under Q)*

†This statement provides the intuition behind the role of L and U. However, as we will see later, we can, in fact calibrate our model, not only to DNT options with barrier levels L and U, but also to other DNT options and double barrier knockout options of other types provided that they have lower barrier levels which are greater than or equal to L and upper barrier levels which are less than or equal to U.

‡We thank a referee for commenting that, in theory, j_0 is observable in the sense that (and we quote) "if the values of $\sigma(\Psi(t))$ are different for $j_0 = 1, 2$, then after a continuous observation [of the spot FX rate] of an infinitesimal length, calculation of the quadratic variation reveals the state j_0". When observations are not continuous, one can only estimate j_0 with error. Hence, we will take a more pragmatic view and regard j_0 as latent and unobservable. We will leave the relative orderings of $\sigma(1)$ and $\sigma(2)$ and of $a_i(1)$ and $a_i(2)$, for each i, unspecified at this point. When we calibrate our model to market data, we can (without loss of generality) choose $j_0 = 1$, say, and then our calibration to market data will implicitly choose the relative orderings.

intensity rate $a_i(\Psi(t))$, *with* $0 < a_i(\Psi(t)) < \infty$. *The Brownian motion has an associated volatility term* $\sigma(\Psi(t))$, *with* $0 < \sigma(\Psi(t)) < \infty$.

We assume that the dynamics, under the risk-neutral measure Q, *of the spot FX rate* $S(t)$, *at time* t, *for* $t \in (t_0, \tau)$, *are*: $S(t) = S_1(t)$, *where* $S_1(t)$, *for all* $t \in [t_0, \infty)$, *follows*:

$$S_1(t) = S(t_0)\exp((r_d - r_f)(t - t_0))\exp\left(-\frac{1}{2}\int_{t_0}^{t} \sigma(\Psi(s))^2 \mathrm{d}s + \int_{t_0}^{t} \sigma(\Psi(s))\mathrm{d}z(s) + \sum_{i=1}^{M} \rho_i \sum_{n=1}^{N_i(t)} \gamma_i^{(n)} - \int_{t_0}^{t} \sum_{i=1}^{M} \left(a_i(\Psi(s)) b_i \left\{ \frac{1}{b_i - \rho_i} - \frac{1}{b_i} \right\} \right) \mathrm{d}s \right) \tag{2.1}$$

where $\rho_i = 1$ *if* $1 \le i \le M/2$ *and* $\rho_i = -1$ *if* $1 + M/2 \le i \le M$ (*the corresponding processes produce up and down jumps respectively*) *and where, for each* $i = 1, \ldots, M$, $\gamma_i^{(n)}$ *is the realized outcome of the random variable* γ_i *for the nth jump of the ith Poisson process. Note that* $S(t_0)$ *and* $S_1(t_0)$ *are equal and known at time* t_0 *and that, for all* $t \in [t_0, \tau)$, *the spot FX rate satisfies* $E_{t_0}^{Q}[S(t)] = E_{t_0}^{Q}[S_1(t)] = S(t_0)\exp((r_d - r_f)(t - t_0))$.

Remark 2.3: We know from Asmussen *et al.* (2007) that we can approximate a CGMY process (in distribution) by a jump-diffusion process with a large number of sums of double-exponential processes. In fact, any Lévy process whose Lévy density is (completely) monotonic as one moves away from the origin can be approximated, arbitrarily closely, in this way. This includes† not only the CGMY process, but, also, other Lévy processes such as the Generalized Hyperbolic process and the NIG process. Hence, if we were to assume that

$$\frac{\sigma^2(1)}{\sigma^2(2)} = \frac{a_1(1)}{a_1(2)} = \cdots = \frac{a_M(1)}{a_M(2)},$$

then we can approximate a time-changed CGMY process, or time-changed versions of these other Lévy processes, where the stochastic time-change is driven by a two-state Markov chain. However, clearly our CEE2 process can be more flexible than that. Alternatively, we could, for example, assume $a_i(1) = a_i(2)$, for each $i = 1, \ldots, M$, and hence just have a stochastic diffusion volatility term. The DEJD model of Kou (2002) is a special case of our CEE2 process when $M = 2$ and the intensity rates and the diffusion volatility are constants.

The first key assumption defined the dynamics of the spot FX rate for $t \in (t_0, \tau)$. The second key assumption is that the dynamics of the spot FX rate can change at the first exit time from the corridor.

Remark 2.4: At the instant $t = \tau$, at which the spot FX rate first exits from the corridor, i.e. at the first exit time when the spot FX rate is equal to or is strictly outside the barriers, we assume that the dynamics of the spot FX rate, under the risk-neutral measure Q, can change to a different arbitrage-free stochastic process. The only requirement from a practical modelling viewpoint is that we will need to know, in closed form, the Laplace transform (with respect to time) of the Characteristic Function of (the log of) this process. Hence, we have the flexibility to choose from a general class of (possibly time-changed) Lévy processes (either finite or infinite activity).

Assumption 2.5: *The dynamics of the spot FX rate, for all* $t \in [t_0, \infty)$, *are constructed as follows. We have already defined the stochastic process* $S_1(t)$, *for all* $t \in [t_0, \infty)$, *via equation* (2.1). *In particular, for all* $t \in [t_0, \infty)$, $S_1(t)$ *satisfies*

$$E_{t_0}^{Q}[S_1(t)] = S_1(t_0)\exp((r_d - r_f)(t - t_0)) = S(t_0)\exp((r_d - r_f)(t - t_0)). \tag{2.2}$$

Now we define $S_2(t)$ *to be an arbitrage-free stochastic process which is such that vanilla option prices are linear homogenous in* $S_2(t)$ *and strike and for which the Laplace transform (with respect to time) of the Characteristic Function of* $log(S_2(t))$ *is known. In particular, for all* $t \ge \tau$, $S_2(t)$ *satisfies*

$$E_{\tau}^{Q}[S_2(t)] = S_2(\tau)\exp((r_d - r_f)(t - \tau)). \tag{2.3}$$

Furthermore, we require

$$S_2(\tau) = S_1(\tau). \tag{2.4}$$

We assume that the dynamics, under the risk-neutral measure Q, *of the spot* FX *rate* $S(t)$, *at time* t, *for* $t \in [\tau, \infty)$, *are* $S(t) = S_2(t)$.

Hence, it is clear that the dynamics of the spot FX rate $S(t)$, *for all* $t \in [t_0, \infty)$, *are of the form*

$$S(t) = S_1(t)I(t < \tau) + S_2(t)I(t \ge \tau), \tag{2.5}$$

where $I(\bullet)$ *denotes the indicator function. We remark that the dynamics in our modelling framework are those of a mixture model but with random weights governed by the stopping time* τ. *Note, it can easily be verified using the optional stopping theorem and equation* (2.4) *that, for all* $t \in [t_0, \infty)$,

$$E_{t_0}^{Q}[S(t)] = S(t_0)\exp((r_d - r_f)(t - t_0)),$$

which is required in the absence of arbitrage.

2.1. *Financial motivation for assumption 2.5*

Whilst we would concede that the second key assumption (assumption 2.5) is mostly made for mathematical convenience, there is a financial motivation for it. If a large volume of barrier options have been traded with a particular barrier level and that particular barrier level is hit, it is often noted that there are significant changes in the market as traders unwind or execute hedges, both in the spot market and in the vanilla FX options market. The former market often sees large jumps (which, whilst observed in the real-world physical measure P, may also

†To be precise, the CGMY process only has a monotonic Lévy density when the parameter $Y \ge -1$.

be present in the risk-neutral measure Q). This is often† attributed to large numbers of market-makers, who may not have perfectly delta-hedged these barrier options (they sometimes have large gammas and so traders can incur significant transactions costs if they very frequently rebalance their delta-hedges), executing stop-loss orders immediately after a barrier level is hit. The latter market often sees changes in the magnitude, and even the sign, of risk-reversals. Traders (Afaf 2007) note that if an upper (respectively, lower) barrier level is hit, risk-reversals often become positive (or more positive) (respectively, negative (or more negative)). This is an example of the more general behaviour, detailed by Carr and Wu (2007), that risk-reversals are stochastic and tend to be positively correlated with the spot FX rate. In addition, the risk-neutral dynamics of the spot FX rate could change at the first exit time from the corridor because $\mathrm{d}Q/\mathrm{d}P$ changes. This might occur for two reasons. Firstly, barrier levels are often chosen because they represent psychologically important levels. The breaching of these levels might lead to speculators or liquidity-providers displaying a different degree of risk-aversion. Secondly, since our market is incomplete, barrier options may have an important role to play in hedging other contingent claims. The fact that, for example, DNT options are knocked-out means these options no longer exist. Hence, the universe of market-traded options available for hedging other contingent claims has diminished (or more formally, we have changed the set of securities available for (partial) spanning of the Arrow–Debreu state-space). This can lead to changes in Q without any changes in P. We will have more to say about assumption 2.5 later.

3. Calibration to the market prices of DNT options

We will want to calibrate our model to the market prices of DNT options and vanilla options. In section 4, we will explain how we calibrate our model to vanilla options. In this section, we explain how to calibrate our model to DNT options.

Kou and Wang (2003) and Sepp (2004) (see also Asmussen *et al.* (2007)) explain how to price DNT options within the Kou (2002) DEJD model. We extend these results to the case when the dynamics of the underlying spot FX rate are those of our CEE2 process (equation (2.1)) in the appendix. It is now obvious that these results are all we need to price DNT options whose barrier levels are L and U. The change of dynamics in the spot FX rate after the first exit from the corridor is irrelevant since, in this case, the DNT option expires worthless.

Hence, we can immediately calibrate the model parameters in equation (2.1) to the market prices of DNT options independently of any assumption about the dynamics of the spot FX rate after the first exit time from the corridor.

As already briefly alluded to, we can also, if we wish, easily include, within our calibration, DNT options whose barrier levels are inside the corridor (L,U) because the change of dynamics in the spot FX rate at the first exit time from the corridor cannot affect their value. We cannot use, within our calibration, DNT options which have a barrier level (or both barrier levels) outside the closed interval $[L,U]$. To put this another way, all the DNT options used within our calibration will have been knocked out by the time that the dynamics of the spot FX rate are allowed to change.

4. Calibration to the market prices of vanilla options

We would now like to calibrate our model to the market prices of vanilla options. To do this, we need a way to evaluate the price of a vanilla option. The first step is to note that by 'in-out' parity, the price of a vanilla option is equal to the price of a knockout option plus the price of a knock-in option, where these barrier options have the same strikes and maturities as the vanilla option.

Kou and Wang (2003) and Sepp (2004) explain how to price double barrier knockout options within the Kou (2002) DEJD model. Again (see the appendix), we extend these results to the case when the dynamics of the underlying spot FX rate are those of our CEE2 process (equation (2.1)). As with DNT options, the change of dynamics in the spot FX rate after the first exit time from the corridor is irrelevant and, hence, we can immediately price a double barrier knockout option, with barrier levels L and U, with the same strike and maturity as the vanilla option in question. Hence, by 'in-out' parity, all we need to do to establish a pricing formula for vanilla options is to price a knock-in option. To be more explicit, we need a pricing formula for a double barrier knock-in option, whose payoff is the same as that of a vanilla option if either barrier is touched or breached before maturity and whose payoff is zero otherwise. We denote by $C(S(t), K, L, U, T-t, \Psi(t))$ the price, at time t, of such a double barrier knock-in option, with strike K and lower and upper barrier levels L and U respectively, and with remaining time to maturity equal to $T-t$. We will consider how to evaluate $C(S(t_0), K, L, U, T-t_0, \Psi(t_0))$ in this next section.

Recall $\log(S(t)/S(t_0)) \equiv X(t)$ and define

$$u \equiv \log(U/S(t_0)), \; l \equiv \log(L/S(t_0)). \tag{4.1}$$

It is important to understand the nature of the process at the first exit time from the corridor. Specifically, the spot FX rate can first exit in one of four possible ways.

†We mention in passing that an alternative situation might be when a central bank targets the FX rate to stay within a semi-fixed band. If the FX rate exits the band, there may be large jumps (both in the spot and in implied volatilities) as happened with Sterling when it left the ERM in 1992. However, we only mention this situation for illustration as it is not the intended application of our model.

- Either (1): The spot FX rate diffuses through the upper barrier, in which case $S(\tau)=U$.
- Or (2): The spot FX rate jumps through (i.e. overshoots) the upper barrier, in which case $S(\tau)$ is strictly greater than U. We write $S(\tau)=S(t_0)\exp(u+x)$, for some $x>0$. We know that, in this case, a jump must have occurred in one (and, with probability one, only one) of the Poisson processes $N_i(t)$, with $1\le i\le M/2$, i.e. for one, and only one, of the i, $1\le i\le M/2$, $\Delta N_i(\tau)\equiv N_i(\tau)-N_i(\tau-)=1$.
- Or (3): The spot FX rate jumps through (i.e. overshoots) the lower barrier, in which case $S(\tau)$ is strictly less than L. We write $S(\tau)=S(t_0)\exp(l+x)$, for some $x<0$. We know that, in this case, a jump must have occurred in one (and, with probability one, only one) of the Poisson processes $N_i(t)$, with $1+M/2\le i\le M$, i.e. for one, and only one, of the i, $1+M/2\le i\le M$, $\Delta N_i(\tau)\equiv N_i(\tau)-N_i(\tau-)=1$.
- Or (4): The spot FX rate diffuses through the lower barrier, in which case $S(\tau)=L$.

Let us denote by $P_d(t,T)V(S(t),K,T-t)$ the price, at time t, when the spot FX rate is $S(t)$, of a vanilla option with strike K and with remaining time to maturity equal to $T-t$. In other words, $V(S(t),K,T-t)$ is the *undiscounted* price of the vanilla option.

A double barrier knock-in option can be viewed as an option which pays the holder a vanilla option at the time it is knocked-in. Hence, by the law of total probability, we can write

$$C(S(t_0),K,L,U,T-t_0,\Psi(t_0))=P_d(t_0,T)\int_{t_0}^{T}g(s)\,\mathrm{d}s,$$

where

$$\begin{aligned}
g(s)\equiv{}&\Pr(X(\tau)=u\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)V(S(t_0)\exp(u),K,T-\tau)\\
&+\sum_{i=1}^{M/2}\int_{0+}^{\infty}\Pr(\Delta N_i(\tau)=1\,\&\,X(\tau)=u+x\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)\\
&\times V(S(t_0)\exp(u+x),K,T-\tau)\,\mathrm{d}x\\
&+\sum_{i=1+M/2}^{M}\int_{-\infty}^{0-}\Pr(\Delta N_i(\tau)=1\,\&\,X(\tau)=l+x\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)\\
&\times V(S(t_0)\exp(l+x),K,T-\tau)\,\mathrm{d}x\\
&+\Pr(X(\tau)=l\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)V(S(t_0)\exp(l),K,T-\tau).
\end{aligned}\tag{4.2}$$

In equation (4.2), we have used the notation $0+$ and $0-$ in the limits of the integrals on the second and third lines to indicate that the barrier has been overshot. Hence in the second line, x takes on only strictly positive values, and in the third line, x takes on only strictly negative values.

The key to making further progress in evaluating equation (4.2) is the following lemma (a conceptually similar result can be found in section 2 of Kou and Wang (2003), for the special case of a single up barrier and a Kou (2002) DEJD process).

Lemma 4.1: *For* $1\le i\le M/2$ *(and recalling* $x>0$ *and* $\rho_i=1$*),*

$$\begin{aligned}
&\Pr(\Delta N_i(\tau)=1\,\&\,X(\tau)=u+x\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)\\
&=\Pr(\Delta N_i(\tau)=1\,\&\,X(\tau)>u\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)b_i\exp(-\rho_i b_i x),
\end{aligned}$$

and similarly for $1+M/2\le i\le M$ *(and recalling* $x<0$ *and* $\rho_i=-1$*),*

$$\begin{aligned}
&\Pr(\Delta N_i(\tau)=1\,\&\,X(\tau)=l+x\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)\\
&=\Pr(\Delta N_i(\tau)=1\,\&\,X(\tau)<l\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)b_i\exp(-\rho_i b_i x).
\end{aligned}\tag{4.3}$$

Proof: See the appendix.

Hence, using lemma 4.1, we have

$$\begin{aligned}
g(s)={}&\Pr(X(\tau)=u\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)V(S(t_0)\exp(u),K,T-\tau)\\
&+\sum_{i=1}^{M/2}\int_{0+}^{\infty}\{\Pr(\Delta N_i(\tau)=1\,\&\,X(\tau)>u\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)\\
&\times b_i\exp(-\rho_i b_i x)V(S(t_0)\exp(u+x),K,T-\tau)\}\,\mathrm{d}x\\
&+\sum_{i=1+M/2}^{M}\int_{-\infty}^{0-}\{\Pr(\Delta N_i(\tau)=1\,\&\,X(\tau)<l\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)\\
&\times b_i\exp(-\rho_i b_i x)V(S(t_0)\exp(l+x),K,T-\tau)\}\,\mathrm{d}x\\
&+\Pr(X(\tau)=l\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)V(S(t_0)\exp(l),K,T-\tau).
\end{aligned}\tag{4.4}$$

Observing that the probability terms are now independent of x, so that we can take them outside the integrals, and using the linear homogeneity property of vanilla option prices (which is certainly valid by assumption 2.5), and substituting from equation (4.1), we have

$$\begin{aligned}
g(s)={}&\Pr(X(\tau)=u\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)V(U,K,T-\tau)\\
&+\sum_{i=1}^{M/2}\begin{pmatrix}\Pr(\Delta N_i(\tau)=1\,\&\,X(\tau)>u\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)\\ \int_{0+}^{\infty}b_i\exp((1-\rho_i b_i)x)V(U,K\exp(-x),T-\tau)\,\mathrm{d}x\end{pmatrix}\\
&+\sum_{i=1+M/2}^{M}\begin{pmatrix}\Pr(\Delta N_i(\tau)=1\,\&\,X(\tau)<l\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)\\ \int_{-\infty}^{0-}b_i\exp((1-\rho_i b_i)x)V(L,K\exp(-x),T-\tau)\,\mathrm{d}x\end{pmatrix}\\
&+\Pr(X(\tau)=l\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)V(L,K,T-\tau).
\end{aligned}\tag{4.5}$$

Now we can make progress in evaluating

$$C(S(t_0),K,L,U,T-t_0,\Psi(t_0))/P_d(t_0,T)=\int_{t_0}^{T}g(s)\,\mathrm{d}s$$

by noticing that each of the $M+2$ terms coming from equation (4.5) is the form of a convolution. We denote the Laplace transform operator by $L[\bullet]$, i.e. for any function $f(\tau)$ and for $\alpha>0$, $L[f(\tau)]=\int_0^{\infty}\exp(-\alpha\tau)f(\tau)\,\mathrm{d}\tau$.

Then, we have that the Laplace transform of $C(S(t_0), K, L, U, T-t_0, \Psi(t_0))/P_d(t_0, T)$ is

$$
\begin{aligned}
&L[C(S(t_0), K, L, U, T-t_0, \Psi(t_0))/P_d(t_0, T)] \\
&= L[\Pr(X(\tau)=u\,\&\,\tau\in \mathrm{d}s|\Psi(t_0)=j_0)]L[V(U,K,\tau)] \\
&+\sum_{i=1}^{M/2}\begin{pmatrix} L[\Pr(\Delta N_i(\tau)=1\,\&\,X(\tau)>u\,\&\,\tau\in \mathrm{d}s|\Psi(t_0)=j_0)] \\ L\left[\int_{0+}^{\infty} b_i \exp((1-\rho_i b_i)x)V(U,K\exp(-x),\tau)\mathrm{d}x\right]\end{pmatrix} \\
&+\sum_{i=1+M/2}^{M}\begin{pmatrix} L[\Pr(\Delta N_i(\tau)=1\,\&\,X(\tau)<l\,\&\,\tau\in \mathrm{d}s|\Psi(t_0)=j_0)] \\ L\begin{bmatrix}\int_{-\infty}^{0-} b_i \exp((1-\rho_i b_i)x) \\ V(L,K\exp(-x),\tau)\mathrm{d}x\end{bmatrix}\end{pmatrix} \\
&+L[\Pr(X(\tau)=l\,\&\,\tau\in \mathrm{d}s|\Psi(t_0)=j_0)]L[V(L,K,\tau)].
\end{aligned} \tag{4.6}
$$

The RHS of equation (4.6) involves a total of $2(M+2)$ Laplace transforms. We show in the appendix how it is possible to evaluate all of them in a form suitable for rapid computation. Hence, it is possible to evaluate $L[C(S(t_0), K, L, U, T-t_0, \Psi(t_0))/P_d(t_0, T)]$.

We can compute $C(S(t_0), K, L, U, T-t_0, \Psi(t_0))/P_d(t_0, T)$ by inverting the Laplace transform and hence obtain $C(S(t_0), K, L, U, T-t_0, \Psi(t_0))$. This gives us the price of a double barrier knock-in option. As we explained at the start of this section, following Kou and Wang (2003) and Sepp (2004), we can price double barrier knockout options (see the appendix for details). Hence, we can price vanilla options by 'in-out' parity. This is what we set out to achieve.

5. Model summary, choice of dynamics and illustrative calibrations

It should be clear that, what we now have is a very flexible framework within which we can price both DNT options (and, in fact, other types of barrier options) and vanilla options. The dynamics of the spot FX rate after the first exit time from the corridor can be any stochastic process for which we know the Laplace transform of the characteristic function. One of the simplest specifications would be to assume that the dynamics of the spot FX rate after the first exit time from the corridor are those of a CEE2 process but with different parameters compared to the process before the spot FX rate first exits from the corridor (and possibly a different value of M). Alternative specifications include Lévy processes such as variance gamma (Madan *et al.* 1998), CGMY (Carr *et al.* 2002), Generalized Hyperbolic, NIG and Meixner (Schoutens 2003) processes, which may, possibly, be time-changed (Barndorff-Nielsen and Shephard 2001, Carr *et al.* 2003, Carr and Wu 2007).

It is straightforward to see that the 'recipe' for calibrating our model is as follows. Calibration is a two-stage process.

- First stage: Before the first exit time from the corridor, we assume that the spot FX rate follows our CEE2 process (equation (2.1)) with some chosen value of M. We can calibrate the parameters of this model to the market prices of DNT options (or, possibly, other types of double barrier options). We then take these parameters as given. We price double barrier knockout options (with the same strikes and maturities as the vanilla options to which we will calibrate in the second stage) using these estimated parameters. Subtracting these double barrier knockout option prices from the market prices of the corresponding vanilla options gives us the prices of double barrier knock-in options which we will use in the second stage of the calibration.
- Second stage: We choose a specific stochastic process (out of the wide class of possible Lévy processes (with or without a stochastic time-change)). We then, taking the parameters of the first stage as given, calibrate the model parameters of the chosen stochastic process to the market prices of vanilla options using the results we have derived in section 4 and the prices of double barrier knock-in options obtained in the first stage.

By separating the two stages of the calibration procedure, we reduce the dimensionality of the optimization problem of finding the model parameters. In addition, our two-stage calibration procedure enables, by design, a good calibration to the market prices of both DNT options and to vanilla options.

In order to illustrate our modelling approach, we obtained the market prices of DNT options and vanilla options on cable (USD/STG) as of 31 May 2007 and as of 6 July 2007. We calibrated our model to these prices, as just described, for various specifications of the dynamics of the spot FX rate after the first exit time from the corridor and we report the results in sections 5.1 and 5.2. We should stress that our calibrations are designed to be illustrative rather than exhaustive.

Before we proceed to discuss the calibrations, we should briefly mention the numerical implementation of the algorithms we used. For the sake of brevity, we have put all the details into an on-line supplement (Ambrose *et al.* 2008) to this paper. As well as describing key aspects of the numerical implementation, it also provides the market data used for the calibration as of 6 July 2007 and some intermediate steps used in some of the calibrations. It also describes tests that we did (such as comparing prices obtained against Monte Carlo simulation) to benchmark the accuracy of our numerical implementation. For Laplace transform inversion, we used the Gaver-Stehfest algorithm (see, for example, Kou and Wang 2003 and Sepp 2004). It is well known that the use of high-precision arithmetic is very desirable when using this algorithm. Unless otherwise stated, our calculations used 'quad double' precision (that is to say, using 60–64 significant figures of accuracy instead of the usual double precision which offers 15–16 significant figures of accuracy). The code to do this 'quad double' precision arithmetic was written for us by Alan Ambrose to whom we, again, express our sincere thanks. His code was in turn based on work by Bailey (1990) and Chatterjee (1998).

5.1. Calibration to market data as of 31 May 2007

In this sub-section, we calibrate our modelling approach to the market prices of DNT options and vanilla options as of 31 May 2007. We used three different specifications of the dynamics of the spot FX rate after the first exit time from the corridor. These three specifications were as follows.

(1) Our CEE2 process (equation (2.1)) with $M=2$ and the intensity rates and the diffusion volatility assumed to be constants which is the same as the Kou (2002) DEJD model.
(2) The CGMY model of Carr *et al.* (2002) with the addition of a Brownian motion component with constant volatility.
(3) As with the first two specifications, specification (3) also has a Brownian motion component with constant volatility. In addition, it has a time-changed Lévy process constructed as follows: Firstly, we introduce two independent continuous-time Markov chains, denoted by $\zeta_1(t)$ and $\zeta_2(t)$, each of which has two states, which we denote by N (the 'normal' state) and A (the 'abnormal' state). Our time-changed Lévy process used two independent tempered stable processes (see chapter 5 of Schoutens 2003), one of which produces only down jumps and the other produces only up jumps. Conditional upon the states of the two Markov chains, the Lévy densities of the two tempered stable processes are of the form
$$\frac{C_1(\zeta_1(t))\exp(-G(-x))}{(-x)^{1+Y}},$$
for $x<0$ (this process produces down jumps), and
$$\frac{C_2(\zeta_2(t))\exp(-Mx)}{x^{1+Y}},$$
for $x>0$ (this process produces up jumps). The parameters G, M and Y are all constants, with $G>0$, $M>0$ and $Y<2$. If the quantities $C_1(\zeta_1(t))$ and $C_2(\zeta_2(t))$ were (positive) constants, then the dynamics of the spot FX rate would be the same as for specification (2). However, they are not constant, but instead stochastic, which we know from Carr *et al.* (2003) is equivalent to stochastically time-changing the two tempered stable processes. For each $i=1,2$, $C_i(\zeta_i(t))$ is a strictly positive function of the respective Markov chain $\zeta_i(t)$, and it can take on one of two possible values which we denote by C_i^{N} and C_i^{A}, where the superscripts N and A refer to the 'normal state and the 'abnormal' state respectively. Transitions take place between state j and state k, for Markov chain $\zeta_i(t)$, for each $i=1,2$, $j=\mathrm{N},\mathrm{A}$, $k=\mathrm{N},\mathrm{A}$, $j\neq k$ with constant instantaneous jump rates λ_i^{jk}, where $\lambda_i^{jk}>0$. We assume, without loss of generality, that, for each $i=1,2, C_i^{\mathrm{A}}\geq C_i^{\mathrm{N}}$. To summarize specification (3), we have two independent tempered stable processes, one producing down jumps and the other producing up jumps, each of which is independently time-changed by a two-state Markov chain. Hence, from Carr and Wu (2007), we know this specification generates, not only stochastic volatility, but also stochastic skew. We would like our specification to be able to model the empirically observed behaviour (see the end of section 2 and Afaf 2007) that if an upper barrier level is hit, risk-reversals (often) become (more) positive, whilst conversely, if a lower barrier is hit, risk-reversals (often) become (more) negative. We can achieve this by specifying that if the spot FX rate first exits from the corridor through the lower barrier, i.e. if $S(\tau)\leq L$, then $\zeta_1(\tau)=\mathrm{A}$ and $\zeta_2(\tau)=\mathrm{N}$, whereas if the spot FX rate first exits from the corridor through the upper barrier, i.e. if $S(\tau)\geq U$, then $\zeta_1(\tau)=\mathrm{N}$ and $\zeta_2(\tau)=\mathrm{A}$. Hence this specification is able to capture the aforementioned empirically observed behaviour as well as to capture stochastic skew, i.e. capture the fact that the sign and magnitude of risk-reversals can vary stochastically through time. Hence, specification (3) is a very rich specification. Note that this specification requires an obvious extension to assumption 2.5 in that we now have a different process if the upper barrier U is exited first compared to if the lower barrier L is exited first. In order to make for a more parsimonious calibration, we assumed that $C_1^{\mathrm{A}}=C_2^{\mathrm{A}}$, $C_1^{\mathrm{N}}=C_2^{\mathrm{N}}$, $\lambda_1^{\mathrm{NA}}=\lambda_2^{\mathrm{NA}}$, $\lambda_1^{\mathrm{AN}}=\lambda_2^{\mathrm{AN}}$. Hence specification (3) has eight parameters (namely the constant volatility of the Brownian motion component as well as G, M, Y, $C_1^{\mathrm{A}}\equiv C_2^{\mathrm{A}}$, $C_1^{\mathrm{N}}\equiv C_2^{\mathrm{N}}$, $\lambda_1^{\mathrm{NA}}\equiv\lambda_2^{\mathrm{NA}}, \lambda_1^{\mathrm{AN}}\equiv\lambda_2^{\mathrm{AN}}$).

Specifications (1) and (2) both have five parameters.

The calibration to DNT options is the same for all three specifications. The dynamics of the spot FX rate before the first exit time from the corridor are assumed to be those of our CEE2 process (equation (2.1)) with $M=2$ and the intensity rates and the diffusion volatility assumed to be constants, i.e. the same as the Kou (2002) DEJD model. Hence, there are five parameters to fit and we calibrated them to the mid-market prices of six DNT options. The spot FX rate (mid-market) was 1.97575. The barrier levels of five of the DNT options were 1.9200 and 2.0200. These correspond to the levels L and U at which the dynamics of the spot FX rate change. The maturities of these five DNT options were one month (1 m), six weeks (6 w), three months (3 m), six months (6 m) and nine months (9 m). In addition, we used one DNT option with a maturity of one month with barrier levels at 1.9500 and 2.0000. The reason for using this option was simply to provide some additional information in the calibration. Since its barrier levels are inside the corridor (L,U), the change of dynamics in the spot FX rate at the first exit time from the corridor cannot affect its value. As we can see from table 1, the fit to the DNT options is excellent.

Table 1. USD/STG 31 May 2007. Double-no-touch (DNT) option prices. USD/STG 31/05/2007. Spot FX rate 1.97575.

Maturity	Model price	Mid-market	Bid	Offer	Using implied parameters from vanillas in second experiment
	Barrier levels 1.92/2.02				
1 m	0.7542	0.76	0.745	0.775	0.8309
6 w	0.6886	0.695	0.68	0.71	0.7787
3 m	0.3268	0.34	0.325	0.355	0.4429
6 m	0.0916	0.09	0.075	0.105	0.1660
9 m	0.0256	0.045	0.03	0.06	0.0621
	Barrier levels 1.95/2.00				
1 m	0.2466	0.245	0.23	0.26	0.3572

We used 20 vanilla options in the second stage of the calibration which consisted of options with five different strikes (in order of increasing strike, they corresponded, in line with the market convention (Carr and Wu 2007), to put options with deltas of −0.10 and −0.25, a call option at the strike corresponding to the delta-neutral straddle (which roughly equates to a delta of 0.5) and call options with deltas of 0.25 and 0.1) for options of four different maturities (six months, nine months, twelve months and two years). We could have used some shorter-dated options in the calibration but we surmised that the prices of very short-dated vanilla options would have only a relatively small sensitivity to the parameters of the process of the spot FX rate after the first exit time from the corridor.

The calibration to vanilla options for each specification is shown graphically in figures 1 to 4. Overall, the fits are qualitatively not quite as good as for the DNT options but are still very good. Note that the fits for each of the specifications (1), (2) and (3) are qualitatively quite similar—one has to view the figures quite closely to see the differences. The residual pricing errors (calculated as the sum of squares of proportional differences between model and market prices) were 0.0285, 0.0265 and 0.0220 for specifications (1), (2) and (3) respectively. Hence, the best overall fit is obtained with specification (3) (albeit it also has the most parameters). Carr *et al*. (2002, 2003) and Carr and Wu (2007) show how, in their ability to fit and explain implied volatility surfaces, finite activity Lévy processes are outperformed by infinite activity Lévy processes, which, in turn, are outperformed by time-changed Lévy processes. Our results are in broad agreement with this.

To illustrate the ability of our model to simultaneously fit the market prices of both DNT options and vanilla options, whilst also demonstrating how difficult this can be to do for other models, we performed the following two experiments.

- First experiment: In specification (1), the dynamics of the spot FX rate are that of a Kou (2002) DEJD model both before and after the first exit time from the corridor. Only the parameters of the process change. We took the parameters obtained from the first stage of the calibration and assumed that these parameters were unchanged after the first exit time from the corridor. We then re-priced the vanilla options. In other words, we re-price the vanilla options in the Kou (2002) DEJD model where the parameters were obtained by fitting the model to the market prices of DNT options. The residual pricing error was 0.717 (compared with 0.0285, 0.0265 and 0.0220 for specifications (1), (2) and (3) respectively). The fit to vanilla options is also displayed in figures 1 to 4 (labelled 'Using parameters implied from DNT options'). It is striking how poor the fit is in our experiment, in comparison with specifications (1), (2) and (3).
- Second experiment: We then performed almost the same experiment except in reverse. We calibrated a Kou (2002) DEJD model (without changing the dynamics at the first exit time from the corridor) to the market prices of vanilla options. Using the parameters obtained from this calibration, we then re-priced the six DNT options which we had used in the original calibration of our model. The DNT prices obtained are in table 1 (in the column labelled 'Using implied parameters from vanillas in second experiment'). The residual pricing error was more than 470 times greater than when we had actually calibrated to the market prices of DNT options. Again, we can see how poor the fit is in our experiment, compared to when we had actually calibrated to the DNT options.

5.2. *Calibration to market data as of 6 July 2007*

In this sub-section, we calibrate our modelling approach to the market prices of DNT options and vanilla options as of 6 July 2007. We used two different specifications of the dynamics of the spot FX rate before the first exit time from the corridor (labelled 2(i) and 2(ii)) and two different specifications of the dynamics of the spot FX rate after the first exit time from the corridor (labelled 2(1) and 2(2)).

Specifications 2(i) and 2(ii) (for the dynamics of the spot FX rate before the first exit time from the corridor) were as follows.

- 2(i): Our CEE2 process (equation (2.1)) with $M=6$ and the intensity rates and the diffusion volatility assumed to be constants (which is equivalent to the Markov chain only having one state). In order to make for a more parsimonious calibration, we assumed that $b_3=b_4$ and $b_1=b_6=3b_4$, $b_2=b_5=2b_4$ (there was no special reason for choosing these multiples but, taking our lead from Asmussen *et al*. (2007), they seemed pragmatic choices). Hence, there are eight parameters to calibrate, namely b_4, $a_1(1)$, $a_2(1)$, $a_3(1)$, $a_4(1)$, $a_5(1)$, $a_6(1)$, $\sigma(1)$.

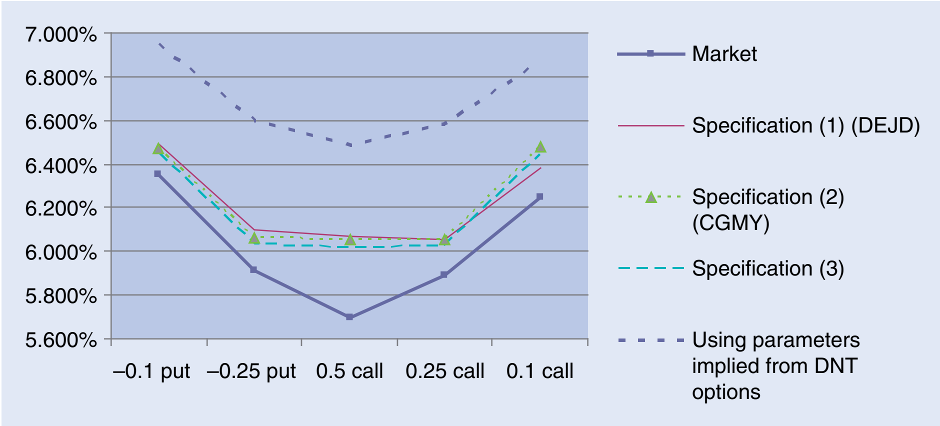

Figure 1. USD/STG 31 May 2007. Implied volatilities (in percent) of vanilla options with a maturity of six months.

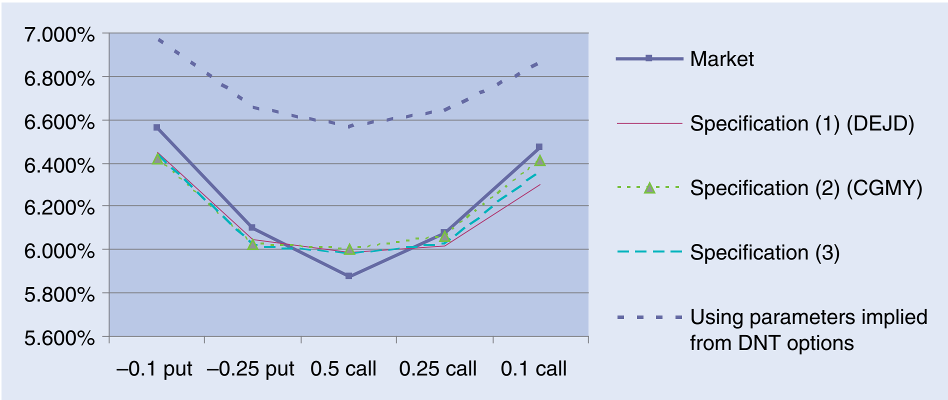

Figure 2. USD/STG 31 May 2007. Implied volatilities (in percent) of vanilla options with a maturity of nine months.

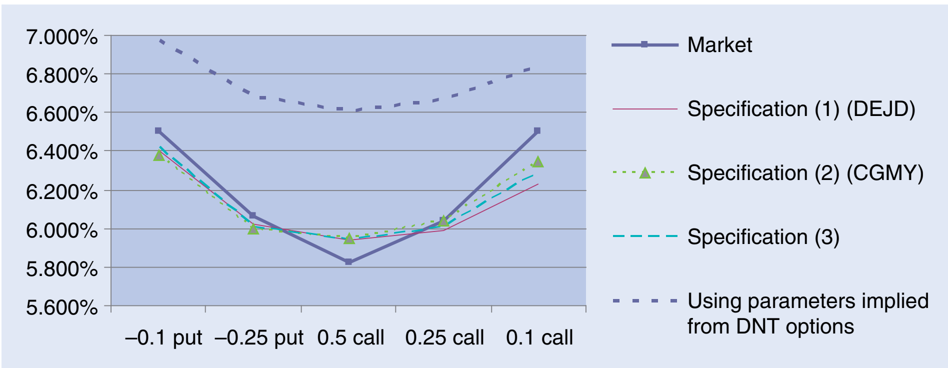

Figure 3. USD/STG 31 May 2007. Implied volatilities (in percent) of vanilla options with a maturity of 12 months.

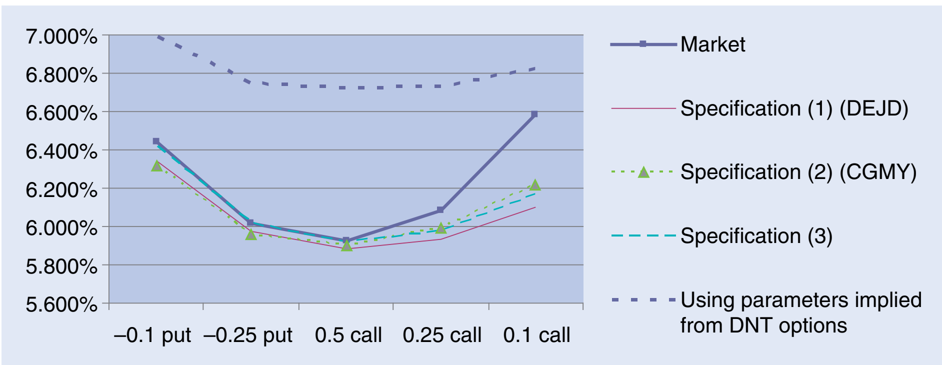

Figure 4. USD/STG 31 May 2007. Implied volatilities (in percent) of vanilla options with a maturity of two years.

- 2(ii): Our CEE2 process (equation (2.1)) with $M = 4$. We assumed that the Markov chain was in state 1 at time t_0, i.e.$\Psi(t_0) = 1$. In order to mimic a stochastic time-change (see remark 2.3), we assumed that

$$\frac{\sigma^2(1)}{\sigma^2(2)} = \frac{a_1(1)}{a_1(2)} = \frac{a_2(1)}{a_2(2)} = \frac{a_3(1)}{a_3(2)} = \frac{a_4(1)}{a_4(2)}.$$

Furthermore, we assumed that $b_2 = b_3$ and $b_1 = b_4 = 2b_3$ and further that $\varepsilon_{12} = \varepsilon_{21}$. Hence, there are again eight parameters to estimate, namely $a_1(1)$, $a_2(1)$, $a_3(1)$, $a_4(1)$, $\varepsilon_{12} = \varepsilon_{21}$, b_3, $\sigma(1)$ and $\sigma^2(1)/\sigma^2(2)$.

We now describe the calibration to the market prices of DNT options for each of the specifications 2(i) and 2(ii). The spot FX rate (mid-market) was 2.0060. We calibrated to a total of 12 DNT options. The barrier levels of eight of the DNT options were 1.9500 and 2.0500. These correspond to the levels L and U at which the dynamics of the spot FX rate change. The maturities of these eight DNT options were one month (1 m), two months (2 m), three months (3 m), four months (4 m), five months (5 m), six months (6 m), nine months (9 m) and twelve months (12 m). In addition, we used two DNT options with maturities of one month and three months with barrier levels at 1.9700 and 2.0400 and two further DNT options with maturities of one week (1 w) and one month with barrier levels at 1.9800 and 2.0300, again, in order to provide some additional information in the calibration. As we can see from table 2(i) (for specification 2(i)) and table 2(ii) (for specification 2(ii)), the fit to the DNT options is excellent for both specifications. However, we can see that the fit with specification 2(ii) is much better than for specification 2(i) (the residual error for the latter is more than 3.3 times that for the former). Since both specifications have the same number of parameters, it suggests that allowing for a stochastic time-change (as in specification 2(ii)) as well as for jumps significantly improves the accuracy of our calibration. This is in broad agreement with calibrations to vanilla options by, for example, Carr *et al.* (2003).

Since specification 2(ii) performed much better than specification 2(i), we will focus on the former for the rest of this sub-section. We took the parameters from the calibration to DNT options for specification 2(ii) as given and then calibrated to the market prices of vanilla options for the two specifications, 2(1) and 2(2), of the dynamics of the spot FX rate after the first exit time from the corridor. Specifications 2(1) and 2(2) were constructed as follows.

- 2(1): Our CEE2 process with $M = 4$ and the same parameter restrictions as in specification 2(ii) that we used above (but with different parameters compared to the dynamics of the spot FX rate before the first exit time from the corridor). We assumed that the Markov chain started off in state 1 at the instant after the first exit time from the corridor, i.e. $\Psi(\tau) = 1$. This specification has eight parameters.

Table 2(i). USD/STG 6 July 2007, specification 2(i). Double-no-touch (DNT) option prices. USD/STG 06/07/2007. Spot FX rate 2.0060.

Maturity	Model price	Mid-market	Bid	Offer
	Barrier levels 1.95/2.05			
1 m	0.79926	0.765	0.75	0.78
2 m	0.52767	0.5	0.485	0.515
3 m	0.34054	0.325	0.31	0.34
4 m	0.22900	0.22	0.205	0.235
5 m	0.15190	0.15	0.135	0.165
6 m	0.09807	0.1	0.085	0.115
9 m	0.02825	0.05	0.035	0.065
12 m	0.00813	0.03	0.015	0.045
	Barrier levels 1.97/2.04			
1 m	0.55560	0.515	0.5	0.53
3 m	0.10412	0.115	0.1	0.13
	Barrier levels 1.98/2.03			
1 w	0.87202	0.85	0.825	0.875
1 m	0.26594	0.245	0.23	0.26

Table 2(ii). USD/STG 6 July 2007, specification 2(ii). Double-no-touch (DNT) option prices. USD/STG 06/07/2007. Spot FX rate 2.0060.

Maturity	Model price	Mid-market	Bid	Offer	Using implied parameters from vanillas in second experiment
	Barrier levels 1.95/2.05				
1 m	0.75156	0.765	0.75	0.78	0.87750
2 m	0.48389	0.5	0.485	0.515	0.68817
3 m	0.31510	0.325	0.31	0.34	0.51895
4 m	0.21993	0.22	0.205	0.235	0.40033
5 m	0.15559	0.15	0.135	0.165	0.30634
6 m	0.11013	0.1	0.085	0.115	0.23070
9 m	0.04496	0.05	0.035	0.065	0.10390
12 m	0.01954	0.03	0.015	0.045	0.04715
	Barrier levels 1.97/2.04				
1 m	0.51154	0.515	0.5	0.53	0.74133
3 m	0.10707	0.115	0.1	0.13	0.26720
	Barrier levels 1.98/2.03				
1 w	0.84506	0.85	0.825	0.875	0.94998
1 m	0.24514	0.245	0.23	0.26	0.50202

- 2(2): This specification is the same as that in specification (3) that we used in the previous sub-section. This specification also has eight parameters.

As in the previous sub-section, we used 20 vanilla options which consisted of options with five different strikes (as before) for options of four different maturities (as before, six months, nine months, twelve months and two years). The results of the calibration are shown in figures 5 to 8. Overall, the fits are very good. The residual pricing errors were 0.0222 and 0.0266 for specifications

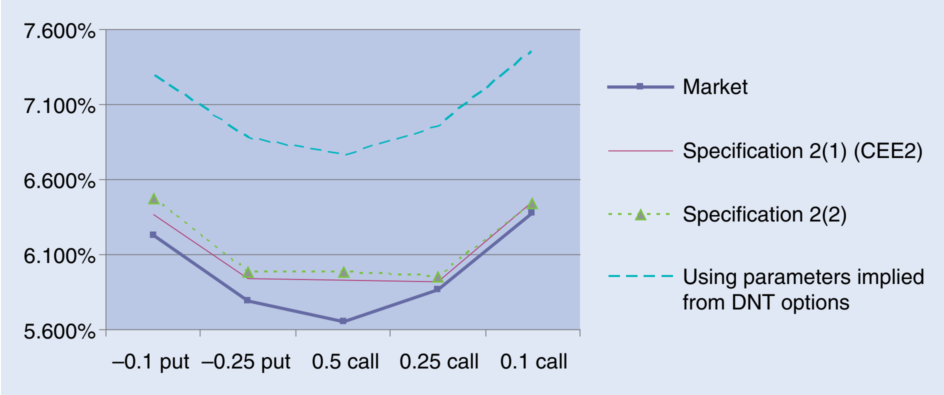

Figure 5. USD/STG 6 July 2007. Implied volatilities (in percent) of vanilla options with a maturity of six months.

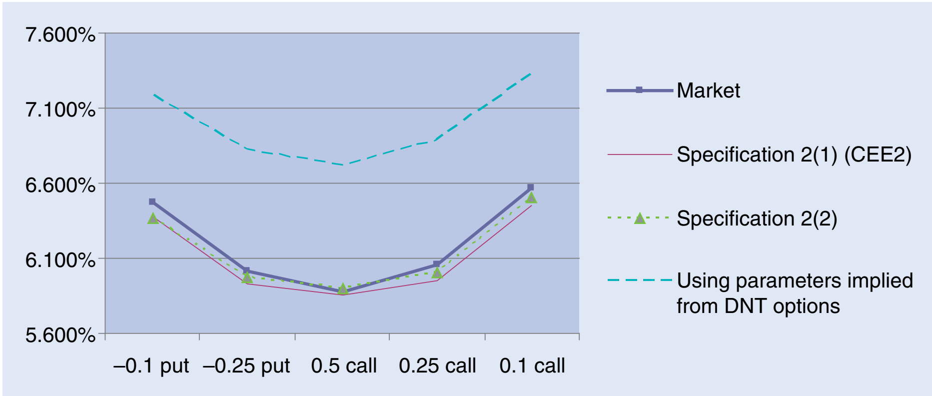

Figure 6. USD/STG 6 July 2007. Implied volatilities (in percent) of vanilla options with a maturity of nine months.

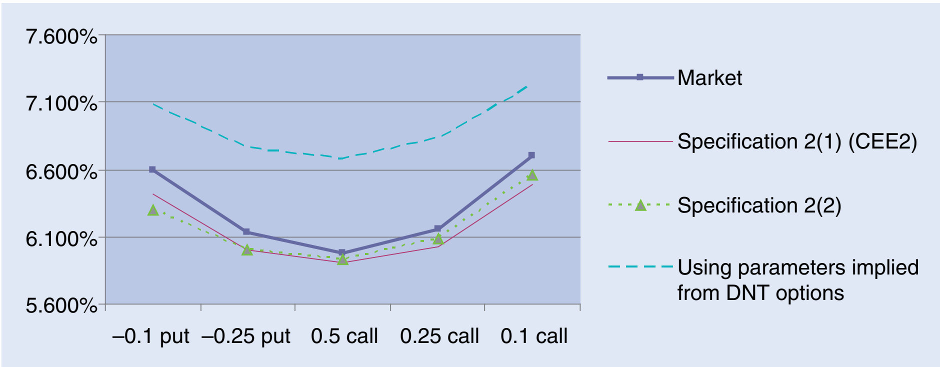

Figure 7. USD/STG 6 July 2007. Implied volatilities (in percent) of vanilla options with a maturity of 12 months.

2(1) and 2(2) respectively. Hence, the best overall fit is obtained by specification 2(1).

We also repeated the two experiments we did in the previous sub-section.

- First experiment: We re-price the vanilla options with our CEE2 process (assuming absolutely no change in the stochastic process† at the first exit time from the corridor) where the parameters were obtained by fitting the model to the market prices of DNT options (with specification 2(ii)). The results (labelled 'Using parameters implied from DNT options') are displayed in figures 5 to 8, where, again, we see a very poor fit in our experiment.
- Second experiment: We calibrated our CEE2 process (assuming absolutely no change in the stochastic process at the first exit time from the corridor and using specification 2(ii) with the same parameter restrictions) to the

†This is a stronger statement than simply assuming that the parameters do not change. We are also saying that the Markov chain does not switch state purely as a result of the spot FX rate exiting from the corridor.

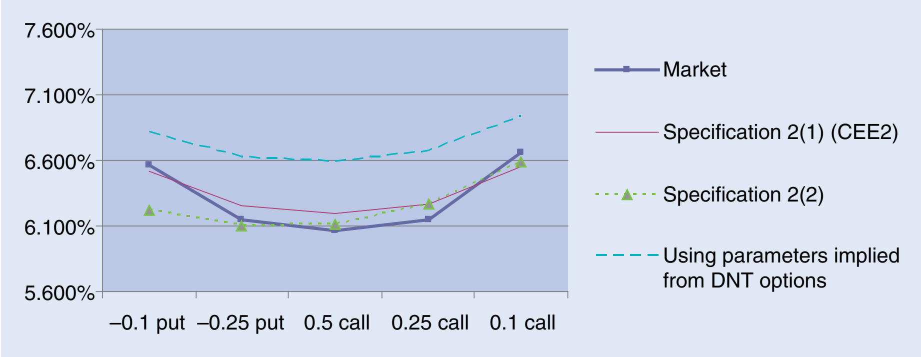

Figure 8. USD/STG 6 July 2007. Implied volatilities (in percent) of vanilla options with a maturity of two years.

market prices of vanilla options. Using the parameters obtained from this calibration, we then re-priced the 12 DNT options which we had used in the calibration above. The DNT prices obtained are in table 2(ii) (in the column labelled 'Using implied parameters from vanillas in second experiment') and we can see that, in all cases, they are much greater than the market prices and, in some cases, they are greater than the market prices by a factor of 2. The residual pricing error was more than 280 times greater in our experiment than when we had actually calibrated to the market prices of DNT options which, again, shows how poor the fit is in our experiment.

5.3. *Further discussion of our illustrative calibrations*

We have already remarked that our calibrations were designed to be illustrative rather than exhaustive. With only two days of data, it would be premature to draw definitive conclusions. What we can say is that the parameters obtained were, essentially, insensitive to the starting point of the calibration and that, inspecting the values of the parameters (see the on-line supplement (Ambrose *et al*. 2008)), we consider them to be economically plausible. Furthermore, when existing models in the literature are used, it is worthy of note that the poor fit in all our experiments suggests how difficult it is to calibrate effectively to the prices of both vanilla and DNT options. By contrast, our model, by design, fits well to both DNT and vanilla options.

We make one important comment on the second experiment which we performed for each set of market data. The price of a DNT option is essentially (modulo discounting) the risk-neutral probability of not hitting either of the barriers before maturity. In each of our second experiments (see the right-hand most columns (labelled 'Using implied parameters from vanillas in second experiment') of table 1 and table 2(ii)), the prices of DNT options are all much higher than the market prices. This suggests that traders price DNT options in the market as if the (risk-neutral) probability of hitting either of the barriers prior to maturity is, in fact, much higher than would be implied from the market prices of vanilla options. Intuitively, this suggests that traders believe that sudden moves in the spot FX rate are more likely to happen than is implied by the market prices of vanilla options, even in a Kou (2002) DEJD model or with our CEE2 process which already account for the possibility of such sudden moves by incorporating jumps. This may provide tentative evidence to suggest that traders price DNT options as if the risk-neutral dynamics of the spot FX rate are different within the corridor (L,U) or before the first exit time from the corridor compared to after the first exit time from the corridor. This, in turn, may provide tentative evidence to suggest that traders (perhaps, unknowingly, or perhaps, knowingly, based on heuristics and their own intuition) price FX options as if the risk-neutral dynamics do change at the first exit time from the corridor, as we assumed in section 2.

6. Conclusions

We have introduced a modelling framework which, by design, can be efficiently calibrated to the market prices of DNT options (or other types of barrier options) and vanilla options. The framework is very flexible in that it can allow for a reasonably wide-range of underlying stochastic processes. The most important assumption is that, at the instant that the spot FX rate first touches or breaches either a lower barrier level or an upper barrier level (i.e. at the first exit time from the corridor), the risk-neutral dynamics of the spot FX rate can change. Although we resist the temptation to draw definitive conclusions from a limited set of data, we have provided some evidence (at least working with our CEE2 process or the Kou (2002) double-exponential jump-diffusion (DEJD) model which is a special case of it) that traders (perhaps, unknowingly, or perhaps, knowingly, based on their own intuition) price FX options as if the risk-neutral dynamics do change.

Finally, we briefly mention two possible areas for future research. Firstly, we have assumed that the risk-neutral dynamics of the spot FX rate can change at the first exit time from the corridor (L,U). It would be possible to introduce multiple corridor levels

$$0 < L_N < \cdots < L_2 < L_1 < S(t_0) < U_1 < U_2 < \cdots < U_N < \infty,$$

and every time the spot FX rate first exits from one of the corridor levels, the dynamics can change. For each of the corridors $(L_1, U_1), (L_2, U_2), \ldots, (L_N, U_N)$, one could assume that the spot FX rate follows our CEE2 process but with different parameters. After the first exit time from the outer-most corridor levels, one could assume that the spot FX rate follows a stochastic process in accordance with assumption 2.5. We believe that our modelling framework would still retain a measure of tractability, although at possible risk of over-complication and over-parameterization.

Secondly, one could apply the ideas of our modelling framework to the simultaneous pricing of credit default swaps, equity default swaps (which are a type of single-touch barrier option) and other credit sensitive instruments. In the same vein, one could also apply them to the simultaneous pricing of sovereign credit default swaps and barrier and vanilla FX options on emerging market currencies. However, for the sake of brevity, we leave these ideas for future research.

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Appendix A

The appendix is divided into four sections. In appendices A1 and A2, we will evaluate the $2(M+2)$ Laplace transform terms on the RHS of equation (4.6). In appendix A3, we will evaluate the prices of barrier options when the dynamics of the spot FX rate follow our CEE2 process (equation (2.1)). In appendix A4, we will briefly consider how we can incorporate term structures of interest rates into our modelling framework and, also, how we can price other types of exotic options.

Appendix A1

In appendix A1, we focus on the $(M+2)$ 'option price like' terms on the RHS of each line of equation (4.6). Define

$$I_0 \equiv V(U, K, \tau),$$

and for each i, $1 \le i \le M/2$,

$$I_{i,1\le i\le M/2} \equiv \int_{0+}^{\infty} b_i \exp((1-\rho_i b_i)x) V(U, K\exp(-x), \tau)\,\mathrm{d}x,$$

and for each i, $1+M/2 \le i \le M$,

$$I_{i,1+M/2\le i\le M} \equiv \int_{-\infty}^{0-} b_i \exp((1-\rho_i b_i)x) V(L, K\exp(-x), \tau)\,\mathrm{d}x,$$

$$I_{M+1} \equiv V(L, K, \tau).$$

We will firstly consider how we can evaluate $I_{i,1\le i\le M/2}$, for each i, $1 \le i \le M/2$. Define the characteristic function $\Phi(z) \equiv E^{Q}_{t_0}[\exp(\bar{j}z\log(S_2(\tau)/S_2(t_0)))]$, where $\bar{j} = \sqrt{-1}$ (we have used the engineers' notation $\bar{j}$ for the imaginary unit to avoid potential confusion with the subscript index i). Note that $S_2(t)$ is as defined in assumption 2.5. In other words, $\Phi(z)$ is the characteristic function for the process after the first exit time from the corridor. Then using results of Lipton (2001), Lewis (2001) and Kangro *et al.* (2004), and recalling that $V(U, K\exp(-x), \tau)$ represents the *undiscounted* price, we can write

$$\begin{aligned} &I_{i,1\le i\le M/2} \\ &= \int_{0+}^{\infty} b_i \exp((1-\rho_i b_i)x) \\ &\times \left\{ \left(\frac{1+\varphi}{2}\right) U\exp(-(r_f - r_d)\tau) + \left(\frac{1-\varphi}{2}\right) K\exp(-x) \right\} \mathrm{d}x \\ &- \int_{0+}^{\infty} \frac{b_i}{\pi} \exp((1-\rho_i b_i)x) \\ &\times \mathrm{Re}\left(\int_0^{\bar{j}\mathrm{Im}(z)+\infty} \Phi(-z) S(t_0) \frac{(K\exp(-x)/S(t_0))^{\bar{j}z+1}}{z^2 - \bar{j}z}\,\mathrm{d}z \right) \mathrm{d}x \end{aligned},$$

where $\varphi = 1$ if the vanilla option is a call and $\varphi = -1$ if the vanilla option is a put, Re and Im denote real and imaginary parts respectively and where z is a complex number whose imaginary part satisfies $0 < \mathrm{Im}(z) < 1$. We can change the order of integration and evaluate the integral with respect to x analytically. Then, by changing the order of integration again, we can express the Laplace Transform of $I_{i,1\le i\le M/2}$ in the form

$$\begin{aligned} &L[I_{i,1\le i\le M/2}] \\ &= \left(\frac{1+\varphi}{2}\right) \frac{b_i U}{(\rho_i b_i - 1)} \left(\frac{1}{\alpha + r_f - r_d}\right) + \left(\frac{1-\varphi}{2}\right) \frac{K}{\rho_i} \left(\frac{1}{\alpha}\right) \\ &- \frac{S(t_0) b_i}{\pi} \mathrm{Re}\left(\int_0^{\bar{j}\mathrm{Im}(z)+\infty} \Omega(z) \frac{(K/S(t_0))^{\bar{j}z+1}}{(z^2 - \bar{j}z)(\rho_i b_i + \bar{j}z)}\,\mathrm{d}z \right), \end{aligned}$$

where $\Omega(z) \equiv \int_0^\infty \exp(-\alpha\tau)\Phi(-z)\,\mathrm{d}\tau$ is the Laplace transform of the characteristic function and is available in closed form for Lévy models and also for some time-changed Lévy models.

As in Lewis (2001) and Kangro *et al.* (2004), we can evaluate the integral with respect to z by integrating along the line $z = u + \bar{j}/2$, where u is real. This means that we can evaluate $L[I_{i,1\le i\le M/2}]$ with a single numerical integration.

We can derive expressions for $L[I_0]$, $L[I_{i,1+M/2\le i\le M}]$ and $L[I_{M+1}]$ in an almost identical fashion. We obtain

$$\begin{aligned} L[I_0] &= \left(\frac{1+\varphi}{2}\right)\left(\frac{U}{\alpha + r_f - r_d}\right) + \left(\frac{1-\varphi}{2}\right)\left(\frac{K}{\alpha}\right) \\ &- \frac{S(t_0)}{\pi} \mathrm{Re}\left(\int_0^{\bar{j}\,\mathrm{Im}(z)+\infty} \Omega(z) \frac{(K/S(t_0))^{\bar{j}z+1}}{(z^2 - \bar{j}z)}\,\mathrm{d}z \right), \end{aligned}$$

$$\begin{aligned} &L[I_{i,1+M/2\le i\le M}] \\ &= \left(\frac{1+\varphi}{2}\right) \frac{b_i L}{(1-\rho_i b_i)} \left(\frac{1}{\alpha + r_f - r_d}\right) + \left(\frac{1-\varphi}{2}\right) \frac{K}{(-\rho_i)} \left(\frac{1}{\alpha}\right) \\ &- \frac{S(t_0) b_i}{\pi} \mathrm{Re}\left(\int_0^{\bar{j}\mathrm{Im}(z)+\infty} \Omega(z) \frac{(K/S(t_0))^{\bar{j}z+1}}{(z^2 - \bar{j}z)(-\rho_i b_i - \bar{j}z)}\,\mathrm{d}z \right), \end{aligned}$$

$$\begin{aligned} L[I_{M+1}] &= \left(\frac{1+\varphi}{2}\right)\left(\frac{L}{\alpha + r_f - r_d}\right) + \left(\frac{1-\varphi}{2}\right)\left(\frac{K}{\alpha}\right) \\ &- \frac{S(t_0)}{\pi} \mathrm{Re}\left(\int_0^{\bar{j}\mathrm{Im}(z)+\infty} \Omega(z) \frac{(K/S(t_0))^{\bar{j}z+1}}{(z^2 - \bar{j}z)}\,\mathrm{d}z \right). \end{aligned}$$

Notice how all $(M+2)$ integrals with respect to z have a very similar form. We can exploit this fact to optimize our computer code. Note that, if we allow for the possibility that the dynamics change to a different process if the upper barrier U is exited first compared to if the lower barrier L is exited first (as in specifications (3) and 2(2) of section 5), we simply use the respective two different characteristic functions in the above equations.

The above results enable us to evaluate the $(M+2)$ 'option price like' terms on the RHS of each line of equation (4.6) whenever the dynamics of the spot FX rate after the first exit time from the corridor are such that we know the Laplace transform of the characteristic function. However, it will be necessary to perform the above integrals numerically, which experience leads us to believe is the most CPU-intensive part of the algorithm. A major simplification occurs when the dynamics of the spot FX rate after the first exit time from the corridor are those of our CEE2 process (with different parameters and, possibly, a different value of M compared to those before the first exit time from the corridor) or special cases of it such as the Kou (2002) DEJD model. In this case, we can compute the Laplace transforms of I_0, $I_{i,1\le i\le M/2}$, for each i, $1\le i\le M/2$, $I_{i,1+M/2\le i\le M}$, for each i, $1+M/2\le i\le M$, and I_{M+1} in closed form (see appendix A3 and especially remark A.7 for more details).

Appendix A2

In appendix A2, we focus on the terms on the LHS of each line of the RHS of equation (4.6), i.e. we focus on the $(M+2)$ 'probability like' terms. We will also prove lemma 4.1. Before we give the derivations, we need to define some additional notation and prove a preliminary lemma.

Firstly, for notational convenience, we define $N\equiv 2M+2$. In the following two lines, j indicates the state of the Markov chain:

$$\mu(j)\equiv r_d-r_f-\frac{1}{2}\sigma^2(j)-\sum_{i=1}^{M}\frac{a_i(j)}{(1-\rho_i/b_i)}+\sum_{i=1}^{M}a_i(j),\quad \text{for } j=1,2,$$

$$f(\psi,j)\equiv\frac{1}{2}\sigma^2(j)\psi^2+\mu(j)\psi-\alpha-\sum_{i=1}^{M}a_i(j)+\sum_{i=1}^{M}\frac{a_i(j)}{(1-\psi\rho_i/b_i)},\quad \text{for } j=1,2. \tag{A.1}$$

We will need the following preliminary lemma.

Lemma A.1: *Let ψ_i, $i=0,1,\ldots,N,N+1$, be the $(N+2)$ roots of the $(N+2)$th-order polynomial equation in ψ,*

$$(f(\psi,1)-\varepsilon_{12})(f(\psi,2)-\varepsilon_{21})=\varepsilon_{12}\varepsilon_{21}. \tag{A.2}$$

Suppose (solely for the purpose of proving this lemma—this is not necessary anywhere else in this paper) that we order (without loss of generality) the mean jump amplitudes of the Poisson processes $N_i(t)$, those producing up jumps $(1\le i\le M/2)$ and those producing down jumps $(1+M/2\le i\le M)$, in such a way that

$$-\infty<-b_M<\cdots<-b_{M/2+1}<0<b_{M/2}<\cdots<b_1<\infty.$$

Then the roots of the polynomial in equation (A.2) are all real and can be ordered as follows:

$$\begin{aligned}&-\infty<\psi_{N+1},\psi_N<-b_M<\psi_{N-1},\psi_{N-2}\\&<\cdots<\psi_{N/2+4},\psi_{N/2+3}<-b_{M/2+1}<\psi_{N/2+2},\psi_{N/2+1}\\&<0<\psi_{N/2},\psi_{N/2-1}<b_{M/2}<\psi_{N/2-2},\psi_{N/2-3}\cdots<\psi_3,\psi_2\\&<b_1<\psi_1,\psi_0<\infty.\end{aligned}$$

Proof: Firstly, we can show (by a modest extension of the arguments in lemma 2.1 of Kou and Wang (2003)) that the polynomials $f(\psi,1)-\varepsilon_{12}=0$ and $f(\psi,2)-\varepsilon_{21}=0$ each have $M+2$ real roots ordered as

$$\begin{aligned}&-\infty<\psi^1_{M+1}<-b_M<\psi^1_M<\cdots<\psi^1_{M/2+2}<-b_{M/2+1}\\&<\psi^1_{M/2+1}<0<\psi^1_{M/2}<b_{M/2}<\psi^1_{M/2-1}\cdots<\psi^1_1\\&<b_1<\psi^1_0<\infty,\end{aligned}$$

and as

$$\begin{aligned}&-\infty<\psi^2_{M+1}<-b_M<\psi^2_M<\cdots<\psi^2_{M/2+2}<-b_{M/2+1}\\&<\psi^2_{M/2+1}<0<\psi^2_{M/2}<b_{M/2}<\psi^2_{M/2-1}\cdots<\psi^2_1\\&<b_1<\psi^2_0<\infty,\end{aligned}$$

respectively. Hence we must be able to write equation (A.2) in the form

$$g(\psi)\equiv k\frac{\left\{\begin{array}{c}(\psi-\psi^1_0)(\psi-\psi^2_0)(\psi-\psi^1_1)(\psi-\psi^2_1)\ldots\\(\psi-\psi^1_{M+1})(\psi-\psi^2_{M+1})\end{array}\right\}}{\left[\prod_{i=1}^{M}(1-\psi\rho_i/b_i)\right]^2}=\varepsilon_{12}\varepsilon_{21},$$

for some positive constant k.

If we consider firstly the interval $[b_1,\infty)$, then it is clear that $g(\psi)\to\infty$ when $\psi\to\infty$ and that $g(\psi)\to\infty$ when $\psi\to b_1+$. Suppose that ψ^1_0 and ψ^2_0 are not equal, then $g(\psi)$ must be negative for $\min(\psi^1_0,\psi^2_0)<\psi<\max(\psi^1_0,\psi^2_0)$. Hence, since $\varepsilon_{12}\varepsilon_{21}$ is positive, the polynomial $g(\psi)=\varepsilon_{12}\varepsilon_{21}$ must have two real roots in the interval $[b_1,\infty)$. In the event that ψ^1_0 and ψ^2_0 are equal, then the polynomial $g(\psi)=0$ must have repeated roots, i.e. $g(\psi)$ touches zero at $\psi=\psi^1_0=\psi^2_0$. Hence, again, the polynomial $g(\psi)=\varepsilon_{12}\varepsilon_{21}$ must again have two real roots in the interval $[b_1,\infty)$.

We can apply a similar line of reasoning to each interval $[b_2,b_1],\ldots,[b_{M/2},b_{M/2-1}],[0,b_{M/2}],[-b_{M/2+1},0],\ldots,(-\infty,-b_M]$ in turn and conclude that in each of these $M+2$ intervals, the polynomial $g(\psi)=\varepsilon_{12}\varepsilon_{21}$ must have two real roots. But the polynomial $g(\psi)=\varepsilon_{12}\varepsilon_{21}$ is of order $N+2=2(M+2)$ and so it has exactly $2(M+2)$ roots. Hence, the lemma is proven. □

We now proceed to evaluate the $(M+2)$ 'probability like' terms on the LHS of each line of the RHS of equation (4.6). We calculate them by, firstly, computing,

$$L[\Pr(X(\tau)\ge u\,\&\,\tau\in ds|\Psi(t_0)=j_0)],$$

for each i, $1 \le i \le M/2$,

$$L[\Pr(\Delta N_i(\tau) = 1 \,\&\, X(\tau) = u + x \,\&\, \tau \in \mathrm{d}s | \Psi(t_0) = j_0)],$$

where $x > 0$, for each i, $1 + M/2 \le i \le M$,

$$L[\Pr(\Delta N_i(\tau) = 1 \,\&\, X(\tau) = l + x \,\&\, \tau \in \mathrm{d}s | \Psi(t_0) = j_0)],$$

where $x < 0$,

$$L[\Pr(X(\tau) \le l \,\&\, \tau \in \mathrm{d}s | \Psi(t_0) = j_0)]$$

in theorem A.2. Note that the first (respectively, the last) term is the Laplace transform of the first passage time density to the upper (respectively, lower) barrier regardless of whether the process $X(t)$ diffuses through or jumps through (i.e. overshoots) the upper (respectively, lower) barrier.

Some related results, in the case of a single upper barrier level and a Kou (2002) DEJD process, can be found in Kou and Wang (2003). We extend these results to the case when there is both a lower barrier level and an upper barrier level and the dynamics follow our CEE2 process in theorem A.2.

Firstly, we define a set of parameters $\bar{\varpi} = \{\varpi_0, \varpi_1, \ldots, \varpi_M, \varpi_{M+1}\}$, whose values will shortly be specified. The reader is warned that we are overloading our notation, so that the precise values of $\bar{\varpi} = \{\varpi_0, \varpi_1, \ldots, \varpi_M, \varpi_{M+1}\}$ will depend upon the relevant probability whose Laplace transform we are trying to calculate.

Then define the parameters $C_k(\bar{\varpi})$, $k = 0, 1, \ldots, 2N+2, 2N+3$, to be the solutions of the following $2N+4$ by $2N+4$ system of linear simultaneous equations:

$$\sum_{i=0}^{N+1} C_i(\bar{\varpi}) \exp(\psi_i u) = \varpi_0, \quad \sum_{i=0}^{N+1} C_{i+N+2}(\bar{\varpi}) \exp(\psi_i u) = \varpi_0, \tag{A.3}$$

$$\sum_{i=0}^{N+1} \frac{C_i(\bar{\varpi}) \exp(\psi_i u)}{((\psi_i/b_m) - \rho_m)} = \varpi_m,$$
$$\sum_{i=0}^{N+1} \frac{C_{i+N+2}(\bar{\varpi}) \exp(\psi_i u)}{((\psi_i/b_m) - \rho_m)} = \varpi_m, \quad \text{for each } m = 1, \ldots, M/2, \tag{A.4}$$

$$\sum_{i=0}^{N+1} \frac{C_i(\bar{\varpi}) \exp(\psi_i l)}{((\psi_i/b_m) - \rho_m)} = \varpi_m,$$
$$\sum_{i=0}^{N+1} \frac{C_{i+N+2}(\bar{\varpi}) \exp(\psi_i l)}{((\psi_i/b_m) - \rho_m)} = \varpi_m, \text{ for each } m = 1 + M/2, \ldots, M, \tag{A.5}$$

$$\sum_{i=0}^{N+1} C_i(\bar{\varpi}) \exp(\psi_i l) = \varpi_{M+1},$$
$$\sum_{i=0}^{N+1} C_{i+N+2}(\bar{\varpi}) \exp(\psi_i l) = \varpi_{M+1}, \tag{A.6}$$

$$C_k(\bar{\varpi})(f(\psi_k, 1) - \varepsilon_{12}) + \varepsilon_{12} C_{k+N+2}(\bar{\varpi}) = 0, \quad \text{for each } k, \ k = 0, 1, \ldots, N, N+1. \tag{A.7}$$

Theorem A.2: *For the case,* $L[\Pr(X(\tau) \ge u \,\&\, \tau \in \mathrm{d}s | \Psi(t_0) = j_0)]$, *set* $\varpi_0 = 1$,

$$\textit{and } \varpi_m = -1/\rho_m, \textit{ for } m = 1, \ldots, M/2 \textit{ and}$$
$$\varpi_m = 0, \textit{ for } m = 1 + M/2, \ldots, M \textit{ and } \varpi_{M+1} = 0. \tag{A.8}$$

For the case $L[\Pr(\Delta N_i(\tau) = 1 \,\&\, X(\tau) = u + x \,\&\, \tau \in \mathrm{d}s | \Psi(t_0) = j_0)]$, *for* $x > 0$, *for each* i, $1 \le i \le M/2$, *set* $\varpi_i = -b_i \exp(-\rho_i b_i x)$ *and*

$$\textit{for } m \ne i, \ \varpi_m = 0, \ m = 0, 1, \ldots, M, M+1. \tag{A.9}$$

For the case $L[\Pr(\Delta N_i(\tau) = 1 \,\&\, X(\tau) = l + x \,\&\, \tau \in \mathrm{d}s | \Psi(t_0) = j_0)]$, *for* $x < 0$, *for each* i, $1 + M/2 \le i \le M$, *set* $\varpi_i = b_i \exp(-\rho_i b_i x)$ *and*

$$\textit{for } m \ne i, \ \varpi_m = 0, \ m = 0, 1, \ldots, M, M+1. \tag{A.10}$$

For the case $L[\Pr(X(\tau) \le l \,\&\, \tau \in \mathrm{d}s | \Psi(t_0) = j_0)]$, *set* $\varpi_0 = 0$,

$$\textit{and } \varpi_m = 0, \textit{ for } m = 1, \ldots, M/2 \textit{ and } \varpi_m = -1/\rho_m,$$
$$\textit{for } m = 1 + M/2, \ldots, M \textit{ and } \varpi_{M+1} = 1. \tag{A.11}$$

Then, we have that

$$L[\Pr(X(\tau) \ge u \,\&\, \tau \in \mathrm{d}s | \Psi(t_0) = j_0)],$$

$$L[\Pr(\Delta N_i(\tau) = 1 \,\&\, X(\tau) = u + x \,\&\, \tau \in \mathrm{d}s | \Psi(t_0) = j_0)],$$

$$L[\Pr(\Delta N_i(\tau) = 1 \,\&\, X(\tau) = l + x \,\&\, \tau \in \mathrm{d}s | \Psi(t_0) = j_0)],$$

$$L[\Pr(X(\tau) \le l \,\&\, \tau \in \mathrm{d}s | \Psi(t_0) = j_0)],$$

are, in each case, equal to

$$\sum_{k=0}^{N+1} C_k(\bar{\varpi}) \quad \textit{if } j_0 = 1 \quad \textit{or} \quad \sum_{k=0}^{N+1} C_{k+N+2}(\bar{\varpi}) \quad \textit{if } j_0 = 2,$$

with the appropriate choices of $\bar{\varpi} = \{\varpi_0, \varpi_1, \ldots, \varpi_M, \varpi_{M+1}\}$ *as indicated above.*

Proof: It is conceptually somewhat similar to the case, in Kou and Wang (2003), of a single upper barrier when the dynamics follow a Kou (2002) DEJD process and so we simply outline the proof.

We consider, firstly, the case $L[\Pr(X(\tau) \ge u \,\&\, \tau \in \mathrm{d}s | \Psi(t_0) = j_0)]$. We write

$$L[\Pr(X(\tau) \ge u \,\&\, \tau \in \mathrm{d}s | \Psi(t_0) = j_0)] \equiv P^{j_0} \equiv P^{j_0}(X),$$

where j_0 is equal to 1 or 2.

We know (following Kou and Wang (2003) and Di Graziano and Rogers (2006)) that P^1 and P^2 must satisfy the following coupled system of OIDEs:

$$\frac{1}{2}\sigma^2(1)\mathrm{P}^1_{XX}+\mu(1)P^1_X-\left(\alpha+\sum_{i=1}^{M}a_i(1)\right)P^1+\varepsilon_{12}(P^2-P^1)$$
$$+\int_0^{\infty}\sum_{i=1}^{M/2}\mathrm{P}^1(X+J)a_i(1)b_i\exp(-\rho_i b_i J)\,\mathrm{d}J$$
$$+\int_{-\infty}^{0}\sum_{i=1+M/2}^{M}\mathrm{P}^1(X+J)a_i(1)b_i\exp(-\rho_i b_i J)\,\mathrm{d}J=0,$$
$$\frac{1}{2}\sigma^2(2)\mathrm{P}^2_{XX}+\mu(2)P^2_X-\left(\alpha+\sum_{i=1}^{M}a_i(2)\right)P^2+\varepsilon_{21}(P^1-P^2)$$
$$+\int_0^{\infty}\sum_{i=1}^{M/2}\mathrm{P}^2(X+J)a_i(2)b_i\exp(-\rho_i b_i J)\,\mathrm{d}J$$
$$+\int_{-\infty}^{0}\sum_{i=1+M/2}^{M}\mathrm{P}^2(X+J)a_i(2)b_i\exp(-\rho_i b_i J)\,\mathrm{d}J=0. \quad (A.12)$$

We postulate solutions of the form

$$P^1(X)=\sum_{k=0}^{N+1}C_k\exp(\psi_k X),\quad P^2(X)=\sum_{k=0}^{N+1}C_{k+N+2}\exp(\psi_k X), \quad (A.13)$$

in the region $l<X<u$, where the C_k terms are independent of X and are parameters to be determined. We impose the boundary conditions that when $X\geq u$, $P^1=1$, $P^2=1$ and that when $X\leq l$, $P^1=0$, $P^2=0$. These imply equations (A.3) and (A.6). Substituting our candidate solution into equation (A.12) and, computing the integral terms analytically, implies that

$$\sum_{k=0}^{N+1}\exp(\psi_k X)\left(C_k\left[\frac{1}{2}\sigma^2(1)\psi_k^2+\mu(1)\psi_k-\alpha-\sum_{i=1}^{M}a_i(1)-\varepsilon_{12}\right]\right.$$
$$\left.+\varepsilon_{12}C_{k+N+2}\right)+\sum_{k=0}^{N+1}\sum_{m=1+M/2}^{M}C_k a_m(1)\{\exp(\psi_k X)$$
$$-\exp(\psi_k l)\exp(-\rho_m b_m(l-X))\}/((\psi_k/b_m)-\rho_m)$$
$$+\sum_{k=0}^{N+1}\sum_{m=1}^{M/2}C_k a_m(1)\{\exp(\psi_k u)\exp(-\rho_m b_m(u-X))$$
$$-\exp(\psi_k X)\}/((\psi_k/b_m)-\rho_m)$$
$$+\sum_{m=1}^{M/2}a_m(1)\exp(-\rho_m b_m(u-X))/\rho_m=0$$

and

$$\sum_{k=0}^{N+1}\exp(\psi_k X)\left(C_{k+N+2}\left[\frac{1}{2}\sigma^2(2)\psi_k^2+\mu(2)\psi_k-\alpha\right.\right.$$
$$\left.\left.-\sum_{i=1}^{M}a_i(2)-\varepsilon_{21}\right]+\varepsilon_{21}C_k\right)$$
$$+\sum_{k=0}^{N+1}\sum_{m=1+M/2}^{M}C_{k+N+2}a_m(2)\left\{\exp(\psi_k X)-\exp(\psi_k l)\right.$$
$$\left.\times\exp(-\rho_m b_m(l-X))\right\}\Big/((\psi_k/b_m)-\rho_m)$$
$$+\sum_{k=0}^{N+1}\sum_{m=1}^{M/2}C_{k+N+2}a_m(2)\{\exp(\psi_k u)\exp(-\rho_m b_m(u-X))$$
$$-\exp(\psi_k X)\}/((\psi_k/b_m)-\rho_m)$$
$$+\sum_{m=1}^{M/2}a_m(2)\exp(-\rho_m b_m(u-X))/\rho_m=0.$$

Many terms cancel in view of equations (A.4) and (A.5) (with the relevant values of ϖ_j). If we equate coefficients of $\exp(\psi_k X)$, then, with some minor algebraic rearrangement, we have that, for each k, $k=0,1,\ldots,N,N+1$,

$$\left[\frac{1}{2}\sigma^2(1)\psi_k^2+\mu(1)\psi_k-\alpha-\sum_{i=1}^{M}a_i(1)+\sum_{i=1}^{M}\frac{a_i(1)}{(1-\psi_k\rho_i/b_i)}-\varepsilon_{12}\right]$$
$$\times C_k+\varepsilon_{21}C_{k+N+2}=0,$$
$$\varepsilon_{12}C_k+\left[\frac{1}{2}\sigma^2(2)\psi_k^2+\mu(2)\psi_k-\alpha-\sum_{i=1}^{M}a_i(2)\right.$$
$$\left.+\sum_{i=1}^{M}\frac{a_i(2)}{(1-\psi_k\rho_i/b_i)}-\varepsilon_{21}\right]C_{k+N+2}=0. \quad (A.14)$$

If we view equation (A.14) as a 2 by 2 system of linear simultaneous equations in C_k and C_{k+N+2}, then, for a solution to exist, its determinant must vanish which immediately yields that the ψ_k terms are indeed the roots of the characteristic polynomial in equation (A.2). Furthermore, we see that the C_k and C_{k+N+2} terms, for each k, $k=0,1,\ldots,N,N+1$, must satisfy equation (A.7). Hence, we have verified the validity of our candidate solution.

Since we normalized so that $X(t_0)\equiv 0$, we evaluate P^1 and P^2 at $X=0$. This yields the stated results for the case $L[\Pr(X(\tau)\geq u\,\&\,\tau\in \mathrm{d}s|\Psi(t_0)=j_0)]$.

The only significant difference to the above in computing

$$L[\Pr(\Delta N_i(\tau)=1\,\&\,X(\tau)=u+x\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)]$$
$$\equiv P^{j_0}\equiv P^{j_0}(X),$$

where j_0 is equal to 1 or 2, is to note that P^1 and P^2 must vanish at $X=u$ and for $X\leq l$. Furthermore, for $X>u$, P^1 and P^2 must equal $\delta(u+x-X)\delta(\Delta N_i(\tau)-1)$, where δ denotes the Dirac delta function. The remaining two cases can be handled analogously. □

We now state proposition A.3. This proposition will enable us to compute M of the 'probability like' terms in equation (4.6).

Proposition A.3: *For the case* $L[\Pr(\Delta N_i(\tau)=1\,\&\,X(\tau)>u\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)]$, *for each* i, $1\leq i\leq M/2$, *set* $\varpi_i=-1/\rho_i$ *and*

$$\textit{for all } m\neq i,\ \varpi_m=0,\ m=0,1,\ldots,M,M+1. \quad (A.15)$$

For the case $L[\Pr(\Delta N_i(\tau)=1\,\&\,X(\tau)<l\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)]$, *for each* i, $1+M/2\leq i\leq M$, *set* $\varpi_i=-1/\rho_i$ *and*

$$\textit{for all } m\neq i,\ \varpi_m=0,\ m=0,1,\ldots,M,M+1. \quad (A.16)$$

Then, for each i, $1\leq i\leq M/2$,

$$L[\Pr(\Delta N_i(\tau)=1\,\&\,X(\tau)>u\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)],$$

and for each i, $1+M/2\leq i\leq M$,

$$L[\Pr(\Delta N_i(\tau)=1\,\&\,X(\tau)<l\,\&\,\tau\in\mathrm{d}s|\Psi(t_0)=j_0)],$$

are, in each case, equal to

$$\sum_{k=0}^{N+1}C_k(\bar{\varpi})\quad \textit{if } j_0=1\quad \textit{or}\sum_{k=0}^{N+1}C_{k+N+2}(\bar{\varpi})\quad \textit{if } j_0=2.$$

Furthermore, for each i, $1 \le i \le M/2$,

$$L[\Pr(\Delta N_i(\tau) = 1\,\&\, X(\tau) = u + x\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0)] = b_i \exp(-\rho_i b_i x) L[\Pr(\Delta N_i(\tau) = 1\,\&\, X(\tau) > u\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0)]. \quad (A.17)$$

For each i, $1 + M/2 \le i \le M$,

$$L[\Pr(\Delta N_i(\tau) = 1\,\&\, X(\tau) = l + x\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0)] = b_i \exp(-\rho_i b_i x) L[\Pr(\Delta N_i(\tau) = 1\,\&\, X(\tau) < l\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0)]. \quad (A.18)$$

Proof: Because of the linearity of the Laplace transform operator,

$$L[\Pr(\Delta N_i(\tau) = 1\,\&\, X(\tau) > u\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0)] = \int_{0+}^{\infty} L[\Pr(\Delta N_i(\tau) = 1\,\&\, X(\tau) = u + x\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0)]\,\mathrm{d}x$$

and

$$L[\Pr(\Delta N_i(\tau) = 1\,\&\, X(\tau) < l\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0)] = \int_{-\infty}^{0-} L[\Pr(\Delta N_i(\tau) = 1\,\&\, X(\tau) = l + x\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0)]\,\mathrm{d}x.$$

Notice that, in equations (A.9) and (A.10), all the ϖ_m terms are equal to zero except for when $m = i$, in which case $\varpi_i = -b_i \exp(-\rho_i b_i x)$ (equation (A.9)) or $\varpi_i = b_i \exp(-\rho_i b_i x)$ (equation (A.10)). Then, since $\int_{0+}^{\infty} -b_i \exp(-\rho_i b_i x)\,\mathrm{d}x = -1/\rho_i$ and $\int_{-\infty}^{0-} b_i \exp(-\rho_i b_i x)\,\mathrm{d}x = -1/\rho_i$, equations (A.15) and (A.16) must follow from linearity.

By a similar line of reasoning,

$$\frac{L[\Pr(\Delta N_i(\tau) = 1\,\&\, X(\tau) = u + x\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0)]}{L[\Pr(\Delta N_i(\tau) = 1\,\&\, X(\tau) > u\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0)]} = \frac{-b_i \exp(-\rho_i b_i x)}{-1/\rho_i}$$

and

$$\frac{L[\Pr(\Delta N_i(\tau) = 1\,\&\, X(\tau) = l + x\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0)]}{L[\Pr(\Delta N_i(\tau) = 1\,\&\, X(\tau) < l\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0)]} = \frac{b_i \exp(-\rho_i b_i x)}{-1/\rho_i}.$$

Since, for each i, $1 \le i \le M/2$, $\rho_i = 1$ and for each i, $1 + M/2 \le i \le M$, $\rho_i = -1$, the last two equations are equivalent to equations (A.17) and (A.18) and hence the proposition is proven. □

We now state corollary A.4 which will enable us to compute the two remaining 'probability like' terms in equation (4.6).

Corollary A.4: *For the case* $L[\Pr(X(\tau) = u\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0)]$, set $\varpi_0 = 1$, *and*

$$\varpi_m = 0, \quad \textit{for all } m = 1, \ldots, M + 1. \quad (A.19)$$

For the case $L[\Pr(X(\tau) = l\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0)]$, set

$$\varpi_m = 0, \quad \textit{for all } m = 0, \ldots, M \textit{ and } \varpi_{M+1} = 1. \quad (A.20)$$

Then,

$$L[\Pr(X(\tau) = u\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0)]$$

and

$$L[\Pr(X(\tau) = l\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0)]$$

are, in each case, equal to

$$\sum_{k=0}^{N+1} C_k(\bar{\varpi}), \textit{ if } j_0 = 1 \quad \textit{or} \quad \sum_{k=0}^{N+1} C_{k+N+2}(\bar{\varpi}), \textit{ if } j_0 = 2.$$

Proof: Since

$$\Pr(X(\tau) = u\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0) = \Pr(X(\tau) \ge u\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0) - \sum_{i=1}^{M/2} (\Pr(\Delta N_i(\tau) = 1\,\&\, X(\tau) > u\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0))$$

and

$$\Pr(X(\tau) = l\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0) = \Pr(X(\tau) \le l\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0) - \sum_{i=1+M/2}^{M} (\Pr(\Delta N_i(\tau) = 1\,\&\, X(\tau) < l\,\&\,\tau \in \mathrm{d}s|\Psi(t_0) = j_0)),$$

we can exploit linearity and, straightforwardly, obtain the form of the parameters $\bar{\varpi} = \{\varpi_0, \varpi_1, \ldots, \varpi_M, \varpi_{M+1}\}$ from the form of those already calculated. We obtain equations (A.19) and (A.20).

Proof of lemma 4.1: It follows immediately from taking the inverse Laplace transform (which is unique for continuous functions) of both sides of equations (A.17) and (A.18). We remark that Kou and Wang (2003) prove the same result in the special case of a Kou (2002) DEJD process using a very intuitive argument based on the memory-less property of exponentially distributed random variables. Our extension to our CEE2 process has an equally intuitive probabilistic interpretation. Indeed, our lemma can be proven directly, without resorting to Laplace transform methods.

Appendix A3

In appendix A3, we will give the prices of barrier options when the dynamics of the spot FX rate follow our CEE2 process. In order to shorten our exposition, we will give the prices of several types of option simultaneously in proposition A.5.

We wish to evaluate the price, at time t, of a double barrier knockout option whose payoff at time T, if the spot FX rate never exits the corridor (L,U), is $\max(\varphi(S(T) - K), 0)$ if the option has a vanilla-style payoff, or $I(\varphi S(T) \ge \varphi K)$ if the option has a binary cash-or-nothing-style payoff.

If the spot FX rate touches or breaches the lower barrier level L (respectively, upper barrier level U) first prior to maturity T, the option pays a rebate R_L (respectively, R_U) at time T. In the above, $I(\bullet)$ denotes the indicator function, $\varphi = 1$ if the option is a call and $\varphi = -1$ if the option is a put, K is the strike, $L \le K \le U$, and $L < S(t) < U$.

We set $s = T - t$, $y \equiv \log(S(t)/K)$, $y_l \equiv \log(L/K)$, $y_u \equiv \log(U/K)$, $j \equiv \Psi(t)$, where j is equal to 1 or 2. We write the price of the option, at time t, in the form $KP_d(t,T)H^j(y,s)$. In other words, $H^j(y,s)$ is the *undiscounted* price, normalized by the strike, if the Markov chain is in state j, at time t. We give the price of the option in proposition A.5.

Proposition A.5: *The price of the option, at time t, is* $KP_d(t,T)H^j(y,s)$, *where: in the case that* $j=1$: *If* $y<0$,

$$L[H^1(y,s)] = \sum_{i=0}^{N/2} D_i \exp(\psi_i y) + \sum_{i=0}^{N+1} E_i \exp(\psi_i y) + A_{y<0}\exp(y) + B_{y<0}, \tag{A.21}$$

else if $y \ge 0$,

$$L[H^1(y,s)] = \sum_{i=1+N/2}^{N+1} D_i \exp(\psi_i y) + \sum_{i=0}^{N+1} E_i \exp(\psi_i y) + A_{y\ge 0}\exp(y) + B_{y\ge 0}; \tag{A.22}$$

in the case that $j=2$: *If* $y<0$,

$$L[H^2(y,s)] = \sum_{i=0}^{N/2} D_{i+N+2} \exp(\psi_i y) + \sum_{i=0}^{N+1} E_{i+N+2} \exp(\psi_i y) + A_{y<0}\exp(y) + B_{y<0}, \tag{A.23}$$

else if $y \ge 0$,

$$L[H^2(y,s)] = \sum_{i=1+N/2}^{N+1} D_{i+N+2} \exp(\psi_i y) + \sum_{i=0}^{N+1} E_{i+N+2} \exp(\psi_i y) + A_{y\ge 0}\exp(y) + B_{y\ge 0}, \tag{A.24}$$

where D_k, $k = 0, 1, \ldots, 2N+2, 2N+3$, *solve the* $2N+4$ *by* $2N+4$ *system of linear simultaneous equations*

$$\sum_{i=0}^{N/2} D_i - \sum_{i=1+N/2}^{N+1} D_i = (A_{y\ge 0} - A_{y<0}) + (B_{y\ge 0} - B_{y<0}), \tag{A.25}$$

for each m, $m = 1, \ldots, M$,

$$\sum_{i=0}^{N/2} \frac{D_i}{((\psi_i/b_m) - \rho_m)} - \sum_{i=1+N/2}^{N+1} \frac{D_i}{((\psi_i/b_m) - \rho_m)} = \frac{(A_{y\ge 0} - A_{y<0})}{((1/b_m) - \rho_m)} - \frac{(B_{y\ge 0} - B_{y<0})}{\rho_m}, \tag{A.26}$$

$$\sum_{i=0}^{N/2} D_i \psi_i - \sum_{i=1+N/2}^{N+1} D_i \psi_i = (A_{y\ge 0} - A_{y<0}), \tag{A.27}$$

$$\sum_{i=0}^{N/2} D_{i+N+2} - \sum_{i=1+N/2}^{N+1} D_{i+N+2} = (A_{y\ge 0} - A_{y<0}) + (B_{y\ge 0} - B_{y<0}), \tag{A.28}$$

for each m, $m = 1, \ldots, M$,

$$\sum_{i=0}^{N/2} \frac{D_{i+N+2}}{((\psi_i/b_m) - \rho_m)} - \sum_{i=1+N/2}^{N+1} \frac{D_{i+N+2}}{((\psi_i/b_m) - \rho_m)} = \frac{(A_{y\ge 0} - A_{y<0})}{((1/b_m) - \rho_m)} - \frac{(B_{y\ge 0} - B_{y<0})}{\rho_m}, \tag{A.29}$$

$$\sum_{i=0}^{N/2} D_{i+N+2}\psi_i - \sum_{i=1+N/2}^{N+1} D_{i+N+2}\psi_i = (A_{y\ge 0} - A_{y<0}) \tag{A.30}$$

and

$$D_k(f(\psi_k, 1) - \varepsilon_{12}) + \varepsilon_{12} D_{k+N+2} = 0, \quad \text{for each } k, \; k = 0, 1, \ldots, N, N+1, \tag{A.31}$$

and where E_k, $k = 0, 1, \ldots, 2N+2, 2N+3$, *solve the* $2N+4$ *by* $2N+4$ *system of linear simultaneous equations*

$$\sum_{i=0}^{N+1} E_i \exp(\psi_i y_u) = \frac{R_u}{\alpha} - \sum_{i=1+N/2}^{N+1} D_i \exp(\psi_i y_u) - B_{y\ge 0} - A_{y\ge 0}\exp(y_u),$$

for each m, $m = 1, \ldots, M/2$,

$$\sum_{i=0}^{N+1} \frac{E_i \exp(\psi_i y_u)}{((\psi_i/b_m) - \rho_m)} = -\frac{R_u}{\alpha} - \sum_{i=1+N/2}^{N+1} \frac{D_i \exp(\psi_i y_u)}{((\psi_i/b_m) - \rho_m)} + B_{y\ge 0} - \frac{A_{y\ge 0}\exp(y_u)}{((1/b_m) - \rho_m)},$$

for each m, $m = M/2+1, M$,

$$\sum_{i=0}^{N+1} \frac{E_i \exp(\psi_i y_l)}{((\psi_i/b_m) - \rho_m)} = \frac{R_l}{\alpha} - \sum_{i=0}^{N/2} \frac{D_i \exp(\psi_i y_l)}{((\psi_i/b_m) - \rho_m)} - B_{y<0} - \frac{A_{y<0}\exp(y_l)}{((1/b_m) - \rho_m)},$$

$$\sum_{i=0}^{N+1} E_i \exp(\psi_i y_l) = \frac{R_l}{\alpha} - \sum_{i=0}^{N/2} D_i \exp(\psi_i y_l) - B_{y<0} - A_{y<0}\exp(y_l)$$

and

$$\sum_{i=0}^{N+1} E_{i+N+2} \exp(\psi_i y_u) = \frac{R_u}{\alpha} - \sum_{i=1+N/2}^{N+1} D_{i+N+2} \exp(\psi_i y_u) - B_{y\ge 0} - A_{y\ge 0}\exp(y_u),$$

for each m, $m = 1, \ldots, M/2$,

$$\sum_{i=0}^{N+1} \frac{E_{i+N+2} \exp(\psi_i y_u)}{((\psi_i/b_m) - \rho_m)} = -\frac{R_u}{\alpha} - \sum_{i=1+N/2}^{N+1} \frac{D_{i+N+2} \exp(\psi_i y_u)}{((\psi_i/b_m) - \rho_m)} + B_{y\ge 0} - \frac{A_{y\ge 0}\exp(y_u)}{((1/b_m) - \rho_m)},$$

for each m, $m = M/2+1, M$,

$$\sum_{i=0}^{N+1} \frac{E_{i+N+2}\exp(\psi_i y_l)}{((\psi_i/b_m)-\rho_m)} = \frac{R_l}{\alpha} - \sum_{i=0}^{N/2} \frac{D_{i+N+2}\exp(\psi_i y_l)}{((\psi_i/b_m)-\rho_m)} - B_{y<0} - \frac{A_{y<0}\exp(y_l)}{((1/b_m)-\rho_m)}$$

$$\sum_{i=0}^{N+1} E_{i+N+2}\exp(\psi_i y_l) = \frac{R_l}{\alpha} - \sum_{i=0}^{N/2} D_{i+N+2}\exp(\psi_i y_l) - B_{y<0} - A_{y<0}\exp(y_l)$$

and $E_k(f(\psi_k,1)-\varepsilon_{12})+\varepsilon_{12}E_{k+N+2}=0$, *for each* $k=0,1,\ldots,N,N+1$. *And where*:

If the option has a vanilla-style payoff $\max(\varphi(S(T)-K),0)$,

$$A_{y\geq 0} = \frac{(1+\varphi)}{2(\alpha+r_f-r_d)}, \quad A_{y<0} = \frac{(\varphi-1)}{2(\alpha+r_f-r_d)},$$

$$B_{y\geq 0} = -\frac{(1+\varphi)}{2\alpha}, \quad B_{y<0} = \frac{(1-\varphi)}{2\alpha},$$

and if the option has a binary cash-or-nothing-style payoff $I(\varphi S(T)\geq \varphi K)$,

$$A_{y\geq 0}=0, \quad A_{y<0}=0, \quad B_{y\geq 0}=\frac{(1+\varphi)}{2\alpha K}, \quad B_{y<0}=\frac{(1-\varphi)}{2\alpha K}.$$

In the special case that we are pricing an option where there are, in fact, no barriers present (i.e. the limit as $L\to 0$ *and* $U\to\infty$*), then the price of the option, at time t, is* $KP_d(t,T)H^j(y,s)$, *where* $L[H^j(y,s)]$ *is again as in equations* (A.21) *to* (A.24), *the* D_k *terms,* $k=0,1,\ldots,2N+2,2N+3$, *again solve equations* (A.25) *to* (A.31) *but now the* E_k *terms,* $k=0,1,\ldots,2N+2,2N+3$, *are all identically equal to zero.*

Proof: In the absence of arbitrage and following Sepp (2004) and Di Graziano and Rogers (2006), we know that $H^j \equiv H^j(y,s)$ satisfies the following coupled system of PIDEs:

$$\frac{1}{2}\sigma^2(1)H^1_{yy}+\mu(1)H^1_y-\left(\sum_{i=1}^{M}a_i(1)\right)H^1+\varepsilon_{12}(H^2-H^1)-H^1_s + \int_0^\infty \sum_{i=1}^{M/2} H^1(y+J)a_i(1)b_i\exp(-\rho_i b_i J)\,\mathrm{d}J + \int_{-\infty}^0 \sum_{i=1+M/2}^{M} H^1(y+J)a_i(1)b_i\exp(-\rho_i b_i J)\,\mathrm{d}J = 0,$$

$$\frac{1}{2}\sigma^2(2)H^2_{yy}+\mu(2)H^2_y-\left(\sum_{i=1}^{M}a_i(2)\right)H^2+\varepsilon_{21}(H^1-H^2)-H^2_s + \int_0^\infty \sum_{i=1}^{M/2} H^2(y+J)a_i(2)b_i\exp(-\rho_i b_i J)\,\mathrm{d}J + \int_{-\infty}^0 \sum_{i=1+M/2}^{M} H^2(y+J)a_i(2)b_i\exp(-\rho_i b_i J)\,\mathrm{d}J = 0. \quad \text{(A.32)}$$

We take the Laplace transform of equation (A.32) and substitute in our candidate solution (equations (A.21) to (A.24)). The rest of the proof closely follows theorem A.2 and so we omit the details, save to mention that equations (A.25), (A.27), (A.28) and (A.30) follow from the requirement that the solution be continuous and have a continuous first derivative at $y=0$. □

Remark A.6: Note that, in terms of our general pricing formula, we can price DNT options by treating them as having a binary cash-or-nothing-style payoff and setting $K=L$, $\varphi=1$ and $R_L=R_U=0$.

Remark A.7: Since (as we note at the end of the statement of proposition A.5) we can compute the Laplace transform of vanilla option prices, we can immediately compute $L[I_0]$ and $L[I_{M+1}]$ (see appendix A1), without the necessity of performing a Fourier inversion, when the dynamics of the spot FX rate after the first exit time from the corridor are those of our CEE2 process. In fact, in this special case, we can also analytically compute $L[I_{i,1\leq i\leq M/2}]$ and $L[I_{i,1+M/2\leq i\leq M}]$ (see appendix A1) by substituting the appropriate solution (equations (A.21) to (A.24)) and performing the relevant integrations analytically. We omit the full details since it only requires high-school calculus, save to mention that when $U<K$, for the case of computing $L[I_{i,1\leq i\leq M/2}]$, and when $L>K$, for the case of computing $L[I_{i,1+M/2\leq i\leq M}]$, we need to split the relevant integrals into two parts according to whether (in the notation of proposition A.5) $y<0$ or $y\geq 0$.

Appendix A4

As in the Black and Scholes (1973) world, introducing term structures of interest rates (and/or volatilities) into our model significantly complicates finding analytical results for the pricing of barrier options and for the distribution of first passage times. In appendix A4, we will briefly indicate how one might proceed to introduce term structures of interest rates as well as, possibly, time-dependent volatilities, Poisson jump intensity rates and Markov chain transition rates. In addition, we briefly examine how we can price other types of exotic options.

We consider the pricing of options with maturity T and fix a sequence of times $t_0\equiv T_0<T_1<T_2<\cdots<T_N\equiv T$. We assume that over each interval $[T_{n-1},T_n)$ for each $n=1,2,\ldots,N$, interest rates (consistent with the common market practice of using log-linear interpolation of discount factors), volatilities, Poisson jump intensity rates and Markov chain transition rates are all piecewise constant.

We can simulate the transition times of the Markov chain and then the jump times of the Poisson processes. We then form the superset of these simulated times and of $\{T_0,T_1,T_2,\ldots,T_N\}$. In between these times, the log of the spot FX rate evolves as Brownian motion (essentially the same idea can be found in Glasserman 2004). If we wish to price, for example, a DNT option with barrier levels at L and U (with continuously monitored barriers), we can use results in Potzelberger and Wang (2001) concerning the probability of a Brownian bridge process hitting either of the barrier levels. For pricing other types of option, we can use similar results to simulate whether the spot FX rate has exited the corridor (L,U) and, if so, simulate when it exited (see, for example, Buchmann 2004). If it has, then we can, starting from the simulated first exit time, simulate the process which we assume the FX rate follows after the first exit time from the corridor. Using these ideas, we can price a wide range of exotic options, even in the presence of time-dependent parameters. On the other hand, it would mean that we are completely reliant on Monte Carlo simulation for pricing. This in turn might make calibration rather slow. There is one special case in which we could have term structures of interest-rates and still retain analytic expressions (up to Laplace transform inversion) in our modelling framework.

This special case is as follows. We denote time-dependent parameters by using a superscript t. We are given term structures of interest rates where we assume that r_d^t and r_f^t are piecewise constants. Firstly, we consider the case where $t_0 \leq t < \tau$, i.e. when the spot FX rate has not exited from the corridor (L,U). Suppose that we assume that the diffusion volatility, the Poisson jump intensity rates and the Markov chain transition rates are such that there exist constants $\bar{r}_d$, $\bar{r}_f$, $\bar{\sigma}(1)$, $\bar{\sigma}(2)$, $\bar{\varepsilon}_{12}$, $\bar{\varepsilon}_{21}$ and $\bar{a}_i(1)$, $\bar{a}_i(2)$, for each $i=1,\ldots,M$, satisfying

$$\frac{r_d^t - r_f^t}{\sigma^{t2}(1)} = \frac{r_d^t - r_f^t}{\sigma^{t2}(2)} = \frac{\bar{r}_d - \bar{r}_f}{\bar{\sigma}^2(1)} = \frac{\bar{r}_d - \bar{r}_f}{\bar{\sigma}^2(2)},$$

$$\frac{r_d^t - r_f^t}{a_i^t(1)} = \frac{r_d^t - r_f^t}{a_i^t(2)} = \frac{\bar{r}_d - \bar{r}_f}{\bar{a}_i(1)} = \frac{\bar{r}_d - \bar{r}_f}{\bar{a}_i(2)}, \quad \text{for each } i=1,\ldots,M,$$

and

$$\frac{r_d^t - r_f^t}{\varepsilon_{12}^t} = \frac{r_d^t - r_f^t}{\varepsilon_{21}^t} = \frac{\bar{r}_d - \bar{r}_f}{\bar{\varepsilon}_{12}} = \frac{\bar{r}_d - \bar{r}_f}{\bar{\varepsilon}_{21}}. \tag{A.33}$$

In other words, all the parameters $\sigma^{t2}(1)$, $\sigma^{t2}(2)$, $a_i^t(1)$, $a_i^t(2)$, ε_{12}^t, ε_{21}^t have to be time-dependent in such a way that they always have a constant scaling to $r_d^t - r_f^t$. Then, we define a new time variable $\bar{s}$, via $\bar{\sigma}^2(1)\bar{s} \equiv \int_{t_0}^{s} \sigma^{u2}(1)\,\mathrm{d}u$, and define

$$\bar{\mu}(j) \equiv \bar{r}_d - \bar{r}_f - \frac{1}{2}\bar{\sigma}^2(j) - \sum_{i=1}^{M} \frac{\bar{a}_i(j)}{(1-\rho_i/b_i)} + \sum_{i=1}^{M} \bar{a}_i(j), \quad \text{for } j=1,2.$$

It is then easy to see that, for example, the coupled system of PIDEs in equation (A.32) with time-dependent parameters can be expressed in the form

$$\frac{1}{2}\bar{\sigma}^2(1)H_{yy}^1 + \bar{\mu}(1)H_y^1 - \left(\sum_{i=1}^{M}\bar{a}_i(1)\right)H^1 + \bar{\varepsilon}_{12}(H^2 - H^1) - H_{\bar{s}}^1 + \int_0^{\infty}\sum_{i=1}^{M/2} H^1(y+J)\bar{a}_i(1)b_i\exp(-\rho_i b_i J)\,\mathrm{d}J + \int_{-\infty}^{0}\sum_{i=1+M/2}^{M} H^1(y+J)\bar{a}_i(1)b_i\exp(-\rho_i b_i J)\,\mathrm{d}J = 0,$$

$$\frac{1}{2}\bar{\sigma}^2(2)H_{yy}^2 + \bar{\mu}(2)H_y^2 - \left(\sum_{i=1}^{M}\bar{a}_i(2)\right)H^2 + \bar{\varepsilon}_{21}(H^1 - H^2) - H_{\bar{s}}^2 + \int_0^{\infty}\sum_{i=1}^{M/2} H^2(y+J)\bar{a}_i(2)b_i\exp(-\rho_i b_i J)\,\mathrm{d}J + \int_{-\infty}^{0}\sum_{i=1+M/2}^{M} H^2(y+J)\bar{a}_i(2)b_i\exp(-\rho_i b_i J)\,\mathrm{d}J = 0.$$

We see that, with time-dependent parameters of the special form of equation (A.33), we have reduced the problem to one with constant parameters again. Hence, all our previous results, *mutatis mutandis*, are valid. The same conclusion holds for all our results in appendix A2.

Secondly, we consider the case where $\tau \leq t \leq T$, i.e. after the first exit time from the corridor (L,U). With the assumption of time-dependent, but piecewise constant, parameters, the results of appendix A1 are essentially still valid, albeit that the Laplace transform of the characteristic function will now have a rather more complicated form.

A final comment is in order. Rapisardi (2005) shows how, in the Black and Scholes (1973) world when valuing a barrier option with term structures of interest rates, one can, as a first-order approximation, use the spot interest rates to the maturity of the option. This is the approach we used in our illustrative calibrations in section 5. Whilst in practice, this should, intuitively, give a very good approximation to the price of a barrier option, it also introduces an internal inconsistency when pricing barrier options of different maturities. In our modelling framework, if we make all the relevant parameters time-dependent in such a way that they satisfy equation (A.33), this internal inconsistency is removed and we can, without approximation, account for term structures of interest rates in both the domestic and the foreign currency. However, the *quid pro quo* is that one has to artificially introduce time-dependent volatilities, Poisson jump intensity rates and Markov chain transition rates in order to retain analytical tractability. To the extent that using the spot interest rates to the maturity of the option is a good first-order approximation when valuing barrier options, one would expect that artificially introducing time-dependent volatilities, Poisson jump intensity rates and Markov chain transition rates would have little or no effect (either positive or negative) on the quality of the calibration to DNT options whilst exactly accounting for term structures of interest rates.