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Bounded Brownian Motion

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Abstract: Diffusions are widely used in finance due to their tractability. Driftless diffusions are needed to describe ratios of asset prices under a martingale measure. We provide a simple example of a tractable driftless diffusion which also has a bounded state space.

Keywords: standard Brownian motion; Brownian martingale; diffusion coefficient

1. Introduction

Standard Brownian motion (SBM) is the most widely studied stochastic process because it serves as a highly tractable model of both a martingale and a Markov process. In finance, the martingale property describes asset prices relative to some numeraire under the assumption of no arbitrage. The Markov property can also describe some asset prices when markets for them are thought to be semi-strong form efficient. However, for limited liability assets, such as stocks, it is well known that SBM cannot describe their prices because prices are non-negative, while the state space of an SBM is the whole real line.

The observation lead Samuelson and Osborne to propose that arbitrage-free asset prices, relative to a numeraire, be modeled as a geometric Brownian martingale (GBM). As is well known, this Markovian martingale is obtained from standard Brownian motion by exponentiating. The convexity of the exponential introduces a positive drift, so one can restore martingality by introducing time decay. One can alternatively create a GBM by starting from a linear Brownian motion with constant drift and then evaluating the scale function of the drifting Brownian motion on the drifting Brownian motion. The resulting GBM has state space $(0, \infty)$, making it suitable to describe arbitrage-free prices of a limited liability asset relative to a numeraire.

While the use of a GBM addresses the lower bound constraint imposed by limited liability, it does not address the upper bound. SBM and GBM can both achieve arbitrarily high positive values. This renders them unsuitable to describe the prices of assets with a finite number of payouts, each of which is bounded above. Examples would include coupon bonds and derivative securities with a finite number of bounded payouts (e.g., a binary option). Exchange rates describe the price of one currency in terms of another and are often legally manipulated by governments to lie between two positive bounds. One may also wish to describe a financial concept other than a price with a stochastic process. For example, stochastic processes are used to describe interest rates, variance rates, and hazard rates. Historical data on these observables is typically confined to a band, which often leads to the imposition of mean reversion in the dynamics. However, it would not be unreasonable to use the historical band as a guide to setting a future band that these rates cannot escape.

Many concepts in probability have evolved into financial concepts. For example, a probability can be used to describe the price of a binary option Taleb (2017), while a correlation is used to describe the swap rate in a correlation swap Jacquier and Slaoui (2010). Since both of these probability concepts are bounded, it becomes natural to consider bounded processes to describe all of these concepts. In this paper, we construct a stochastic process S with the following properties:

1. The process S is obtained by evaluating the scale function of a mean-repelling Ornstein Uhlenbeck (OU) process. As a result, S is a time-homogeneous driftless diffusion. Furthermore, since the scale function can be evaluated in closed form, the transition probabilities of S are known in closed form.
2. The state space of the process S is $(0, H)$, for some $H > 0$. In words, the process S is bounded below and above and its natural boundaries are zero and a positive constant H. Since S has zero drift and can't explode, it is a martingale.
3. The diffusion coefficient of S is positive and bounded above by a positive constant h.
4. Aside from its starting value $S_0 \in (0, H)$, the process S has two free parameters H and h which respectively describe the maximum value and maximum (normal) volatility of S.

Analogous with the term geometric Brownian motion, we christen this process "bounded Brownian motion".

2. Applications

2.1. Managed Currency

Consider a forward exchange rate in a setting when a monetary authority is able to manage the money supply or interest rates such that a given exchange rate stays between two positive barriers over the life of the forward contract. By adding a positive constant to the bounded Brownian motion S, we synthesize the risk-neutral dynamics of the forward exchange rate process. Papers modeling the spot FX process between bands include Carr and Kakuschadze (2017), Hui et al. (2008), Ingersoll (1997), and Rady (1997).

2.2. Correlation Swap

In its simplest form, a correlation swap designates two underlying assets and a fixed maturity date. At maturity, the dollar payoff is affine in the realized correlation of returns between two underlying assets. The slope of this affine relation is the notional of the correlation swap which is determined at inception. The ratio of the intercept to this notional is the correlation swap rate for maturity T, which is also determined at inception. As we move through calendar time, the conditional expected value of the floating leg of a seasoned correlation swap is a martingale fluctuating in the interval $[-1, 1]$. By multiplying S by $\frac{2}{H}$ and subtracting one, we obtain such a process.

2.3. Protection Leg of CDO

The protection leg of a CDO has a nonnegative value that fluctuates in the interval $[0, H]$. The value before the maturity date T is in $(0, H]$, while the value at T is in $[0, H]$. The dynamics presented here fluctuate in the interval $(0, H)$. However, we will show a way to change the domain to $(0, H]$.

3. Genesis

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. We will refer to $\mathbb{Q}$ as the risk-neutral measure. Let $X_0 \in \mathbb{R}$ and for positive constants H and h, consider the following[1] mean-repelling OU process:

$$dX_t = \frac{2\pi h^2}{H^2} X_t dt + h dW_t, \qquad t \geq 0, \tag{1}$$

where W is a $\mathbb{Q}$ standard Brownian motion.

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[1] The scaling factor 2π in the drift of X is entirely optional, but is inserted here so that the volatility of the bounded Brownian motion can later be expressed in terms of an un-normalized Gaussian function, rather than a normalized one.

Let $S(x) : \mathbb{R} \mapsto \mathbb{R}$ be a C^2 function which solves the linear second order ordinary differential equation (ODE):

$$\frac{h^2}{2}S''(x) + \frac{2\pi h^2}{H^2}xS'(x) = 0, \qquad x \in \mathbb{R}. \tag{2}$$

As this is a linear first order ODE in $S'(x)$, the general solution is given by:

$$S(x) = \int_a^x \exp\left[-\int_b^y \frac{2\pi}{H^2}zdz\right]dy, \tag{3}$$

where the lower integral limits a and b are arbitrary. Since S is increasing in x, it is generally referred to as a scale function of the diffusion process X.

Let $S_H(x)$ denote the particular increasing function obtained when $a = -\infty$ and $b = 0$:

$$S_H(x) \equiv \int_{-\infty}^x \exp\left[-\int_0^y \frac{2\pi}{H^2}zdz\right]dy. \tag{4}$$

Some easy calculus gives:

$$\begin{aligned} S_H(x) &= \int_{-\infty}^x \exp\left[-\frac{1}{2}\frac{2\pi y^2}{H^2}\right]dy \\ &= H\int_{-\infty}^{\frac{\sqrt{2\pi}x}{H}} \frac{\exp[-\frac{z^2}{2}]}{\sqrt{2\pi}}dz \\ &= HN\left(\frac{\sqrt{2\pi}x}{H}\right), \qquad t \geq 0, \end{aligned} \tag{5}$$

where $N(d) : \mathbb{R} \mapsto (0,1)$ is the standard normal cumulative distribution function (CDF). Since N maps $\mathbb{R}$ to $(0,1)$, (5) implies that S_H maps $\mathbb{R}$ to $(0, H)$.

Let $\{S_t; t \geq 0\}$ be the stochastic process obtained by evaluating the scale function S_H on the OU process X:

$$S_t \equiv S_H(X_t) = HN\left(\frac{\sqrt{2\pi}X_t}{H}\right), \tag{6}$$

from (5). It is well known that S is a time-homogeneous driftless diffusion. Since the range of the function $S_H(\cdot)$ is bounded, it is clear from (6) that the process S takes values on $(0, H)$. Since S is bounded, it is clearly a martingale and not just a local martingale.

4. Valuing Perpetual Claims Without Knowing Volatility

Consider perpetual claims written on the path of a single underlying asset. In this section, we show that we can value several such claims without knowing the volatility of the underlying asset. The price of the underlying asset will be a continuous martingale, but it need not be the particular continuous martingale S introduced in the last section. To distinguish general results applying to continuous martingales from particular results applying to the driftless bounded diffusion S defined by (6), we will denote the former process by M. We being with some observations about hitting probabilities of S and then generalize to M.

Let ℓ and r be two real-valued constants satisfying:

$$-\infty < \ell < X_0 < r < \infty. \tag{7}$$

In words, ℓ and r are finite and bracket X_0. Let σ_ℓ and σ_r respectively denote the first passage times of X to ℓ and r. Let $S_0 \in (0, H)$ be the initial value of the S process:

$$S_0 \equiv S_H(X_0) = HN\left(\frac{\sqrt{2\pi}X_0}{H}\right), \tag{8}$$

from (6). Let D and U be defined by:

$$D \equiv S_H(\ell) \qquad U \equiv S_H(r). \tag{9}$$

Since $S_H(\cdot)$ is increasing, (7) implies that D and U satisfy:

$$0 < D < S_0 < U < H. \tag{10}$$

In words, D and U are both in $(0, H)$ and bracket S_0. Let τ_u and τ_d respectively denote the first passage times of S to U and D. Since S is a process in natural scale, the probability that S hits U before D starting from S_0 is given by:

$$\mathbb{Q}\{\tau_u < \tau_d | S_0 = S\} = \frac{S - D}{U - D}, \qquad S \in (D, U). \tag{11}$$

In fact, (11) is a standard result that holds for any continuous martingale with unbounded quadratic variation. We will soon discuss results for such processes, but, for now, we confine our explorations to the driftless bounded diffusion S defined by (6).

Consider a claim that pays one dollar if the process S hits U before D and zero otherwise. If interest rates are zero, then (11) gives the value of such a claim as a function of the starting point S of the underlying. Notice that given a direct observation of the starting value S, this value is invariant to the volatility of the process S. Since $\sigma_\ell = \tau_d$ and $\sigma_r = \tau_u$, we can make the corresponding statements for hitting probabilities and claim values when X is the underlying.

In fact, the affine form of the hitting probabilities of S generalize to any continuous martingale M, whose quadratic variation becomes infinite as the horizon become infinite:

$$\lim_{T \uparrow \infty} \langle M \rangle_T = \infty. \tag{12}$$

This condition is needed to rule out continuous martingales which absorb at spatial boundaries placed between the starting value and the barrier of interest. Let $(-\infty, \infty)$ be the state space of the continuous martingale. Suppose again that D and U are both in $(-\infty, \infty)$ and bracket M_0:

$$-\infty < D < M_0 < U < \infty. \tag{13}$$

Consider the random payoff from a "perpetual" claim on M that pays R_d at time τ_d if M hits D first and R_u at time τ_u if M hits U first. Assuming zero interest rates and no dividends from the underlying asset before the first exit, the initial value of this claim is:

$$V_0 = \frac{M_0 - D}{U - D} R_u + \frac{U - M_0}{U - D} R_d. \tag{14}$$

By setting $R_d = 0$ and $R_u = 1$, one obtains the following generalization of (11):

$$\mathbb{Q}\{\tau_u < \tau_d | \mathcal{F}_0\} = \frac{M_0 - D}{U - D}, \tag{15}$$

where we assume that the initial filtration $\mathcal{F}_0$ contains M_0. By setting $R_d = D$ and $R_u = U$, one can interpret R_d and R_u as liquidating dividends from the asset whose price is M. In this case, no assumption on interest rates is needed since the replicating strategy just holds one share.

For the above claims, monitoring of the barriers begins at time $t = 0$. Consider instead a more general claim where monitoring of the exit barriers starts from some fixed time $T \geq 0$. We claim that static positions in European options maturing at T can be used to span this payoff. In particular, consider a butterfly spread with strikes D, K, and U where $-\infty < D < K < U < \infty$. Suppose that the positions in European options are chosen so that the butterfly spread pays off one dollar if $S_T = K$ at T. Then we claim the value of this butterfly spread on M is the joint risk-neutral probability that M hits K after T before it hits D or U after T.

Butterfly spreads can be synthesized using puts or calls. We will use puts and hence let $P_0(K, T)$ denote the initial price of a European put of strike $K \in \mathbb{R}$ and maturity $T \geq 0$. Consider the following butterfly spread payoff:

$$BS_T = \frac{(U - M_T)^+ - (K - M_T)^+}{U - K} - \frac{(K - M_T)^+ - (D - M_T)^+}{K - D}. \tag{16}$$

The initial cost of forming this butterfly spread is:

$$BS_0 = \frac{P_0(U, T) - P_0(K, T)}{U - K} - \frac{P_0(K, T) - P_0(D, T)}{K - D}. \tag{17}$$

Since the payoff in (16) is bounded between 0 and 1, no arbitrage forces the value in (17) to also be bounded between 0 and 1.

Let τ_B^T be the first time after T that the continuous martingale M touches a barrier B. If the martingale never touches B after T, then $\tau_B^T = \infty$. Let $\tau_{DU}^T \equiv \tau_D^T \wedge \tau_U^T$ be the first time after T that the martingale touches either D or U. If the martingale never touches D or U after T, then $\tau_{DU}^T = \infty$.

Theorem 1. No arbitrage and zero interest rates implies:

$$BS_0 = \mathbb{Q}\{\tau_K^T < \tau_{DU}^T\}. \tag{18}$$

Proof. We need to show that BS_0 is the initial cost of a strategy that pays one dollar if $\tau_K^T < \tau_{DU}^T$ and which pays zero otherwise. Consider the following trading strategy. At time 0, the investor buys the butterfly spread by:

1. buying $\frac{1}{U-K}$ puts struck at U
2. selling $\frac{1}{U-K} + \frac{1}{K-D}$ puts struck at K
3. buying $\frac{1}{K-D}$ puts struck at L.

□

The net cost is given in (17). The put portfolio is held static to T. If $S_T < D$ or $S_T > U$, then the portfolio expires worthless. This matches the payoff of the desired claim since if $S_T < D$, then we must have $\tau_K^T \geq \tau_D^T = \tau_{DU}^T$ and similarly, if $S_T > U$, then we must have $\tau_K^T \geq \tau_U^T = \tau_{DU}^T$.

If $S_T \in (D, U)$, then use the payoff from the portfolio to finance the following positions:

1. if $S_T \in (D, K)$ buy $\frac{1}{K-D}$ shares and borrow $\frac{D}{K-D}$ dollars.
2. if $S_T \in (K, U)$ short $\frac{1}{U-K}$ shares and lend $\frac{U}{U-K}$ dollars.

If $\tau_K^T > \tau_{DU}^T$, then at the first exit time τ_{DU}^T of the corridor (D, U), liquidate the stock bond portfolio for zero. Otherwise, if $\tau_K^T < \tau_{DU}^T$, then at the hitting time τ_K^T, liquidate the stock bond portfolio for one dollar. Since the quadratic variation of M grows without bound, the risk-neutral probability that τ_K^T and τ_{DU}^T are both infinite is zero. This concludes the proof. Q.E.D.

There is a second kind of butterfly spread with a probabilistic interpretation. We first consider the simpler spot starting case. Consider a perpetual claim written on the continuous martingale M satisfying both (12) and (13). Suppose again that the claim pays off at τ_{DU}. For $K \in (D, U)$, suppose that the payoff at time τ_{DU} is the *Local Time* of M at time τ_{DU}. Loosely speaking, the payoff at τ_{DU} accumulates over time twice the instantaneous variance experienced by the process M at K until τ_{DU}:

$$L^M_{\tau_{DU}}(K) \equiv 2\int_0^{\tau_{DU}} \delta(M_t - K)d\langle M\rangle_t. \tag{19}$$

From the Tanaka Meyer formula, the cost of creating this payoff is:

$$\hat{G}(M,K) = \begin{cases} 2\frac{(M-D)(U-K)}{U-D}, & -\infty < D < M < K < U < \infty \\ 2\frac{(K-D)(U-M)}{U-D}, & -\infty < D < K < M < U < \infty. \end{cases} \tag{20}$$

This can be written much more succinctly as:

$$\hat{G}(M,K) = 2\frac{[(M \wedge K) - D]^+[U - (K \vee M)]^+}{U-D}. \tag{21}$$

when graphed against either M or K, the function $\hat{G}$ is flat at zero outside (D, U) and triangular in between. When graphed against M, the kinks are at D,K, and U and the change in slope of $\hat{G}$ at K is two. Again, it is remarkable that the claim can be valued without knowledge of the volatility of M.

To value the forward-start version of the above claim, i.e., the claim paying:

$$2\int_T^{\tau^T_{DU}} \delta(M_t - K)d\langle M\rangle_t \tag{22}$$

at τ^T_{DU}, consider the following butterfly spread payoff:

$$BS_T = 2\frac{[(M_T \wedge K) - D]^+[U - (K \vee M_T)]^+}{U-D}. \tag{23}$$

This payoff can be synthesized by:

1. buying $2\frac{U-K}{U-D}$ puts struck at D
2. selling two puts struck at K
3. buying $2\frac{K-D}{U-D}$ puts struck at U.

The initial cost of forming this butterfly spread is:

$$BS_0 = 2\frac{U-K}{K-D}P_0(D,T) - 2P_0(K,T) + 2\frac{K-D}{U-K}P_0(U,T). \tag{24}$$

Suppose that this butterfly spread is held static to maturity. If $M_T < D$ or $M_T > U$, then the butterfly spread expires worthless. This matches the payoff since we already know at T that local time at K cannot increase from zero without M first hitting D or U.

If $M_T \in (D, U)$, then the payoff at T finances the initial position in the following trading strategy in stocks and bonds conducted over the period (T, τ^T_{DU}).

1. if $M_T \in (D, K]$, be long $2\frac{U-K}{U-D}$ shares and borrow $2\frac{U-K}{U-D}D$ dollars
2. if $M_T \in (K, U)$, be short $2\frac{K-D}{U-D}$ shares and lend $2\frac{K-D}{U-D}U$ dollars.

This strategy is self-financing except when the stock price is near the intermediate strike K. Using the Tanaka Meyer formula, one can show that this strategy generates the increase in the local time from T to τ^T_{DU}.

5. Diffusion Coefficient

So far we have been able to value various perpetual claims relative to either the price of the underlying asset or relative to European options written on that asset. For these claims, we have not needed to know the diffusion coefficient of the underlying asset. To value other kinds of claims (e.g., finite lived ones), it will be useful to examine the diffusion coefficient of S as we will show that it appears in the Jacobian when we develop the transition PDF of S.

To obtain the diffusion coefficient of S, we use Itô's formula on (6):

$$dS_t = \sqrt{2\pi} N'\left(\frac{\sqrt{2\pi} X_t}{H}\right) h dW_t, \qquad t \geq 0, \tag{25}$$

as we already know that S is driftless. Since $N'(\cdot)$ and h are positive, so is the diffusion coefficient of S. Evaluating (5) at X_t and inverting implies:

$$\frac{\sqrt{2\pi} X_t}{H} = N^{-1}\left(\frac{S_t}{H}\right), \tag{26}$$

where $N^{-1}(p) : (0,1) \mapsto \mathbb{R}$ is the inverse of the standard normal CDF. Substituting (26) in (25) gives the SDE followed by S:

$$dS_t = a(S_t) dW_t, \qquad t \geq 0, \tag{27}$$

where the diffusion coefficient (normal volatility) of S is given by:

$$a(S) \equiv \sqrt{2\pi} N'\left(N^{-1}\left(\frac{S}{H}\right)\right) h = e^{-\frac{\left[N^{-1}\left(\frac{S}{H}\right)\right]^2}{2}} h. \tag{28}$$

We note from (27) that S is indeed a time-homogeneous driftless diffusion. The diffusion coefficient given in (28) is proportional to a standard Gaussian function of $N^{-1}\left(\frac{S}{H}\right)$. As a consequence, movements in S become more certain when S is near 0 or H:

$$\lim_{S \downarrow 0} a(S) = 0 \qquad \lim_{S \uparrow H} a(S) = 0. \tag{29}$$

Besides its starting value $S_0 \in (0, H)$, the process S has two free parameters H and h. We have already seen that H defines the right end point of the domain of S. As S fluctuates through $(0, H)$, its volatility $\sqrt{2\pi} N'\left(N^{-1}\left(\frac{S_t}{H}\right)\right) h$ evolves as a positive stochastic process. We now show that h defines the right end point of the latter process' domain.

The standard normal density function $N'(\cdot)$ achieves its maximum value when its argument vanishes. The inverse normal CDF $N^{-1}(p)$ vanishes when its argument is $\frac{1}{2}$. Hence, we conclude that volatility is maximized when $S_t = \frac{H}{2}$, i.e., at the midpoint of its domain $(0, H)$. The maximum value of volatility achieved is:

$$\sqrt{2\pi} N'\left(N^{-1}\left(\frac{1}{2}\right)\right) h = h. \tag{30}$$

Now the standard normal probability density function (PDF) is even about zero which implies that its integral, $N(\cdot)$ is the sum of $1/2$ and a function which is odd about zero. It follows that $N^{-1}(\cdot)$ is odd about $1/2$. Since diffusion coefficient of S is proportional to the composition of the even function $N'(\cdot)$ with the function $N^{-1}(\frac{M}{H})$ of S which is odd about $\frac{H}{2}$, the diffusion coefficient of S is symmetric about $\frac{H}{2}$.

The speed density of a diffusion with scale density $s(x)$ and variance rate $\sigma^2(x)$ is $\frac{1}{\sigma^2(x)s(x)}$. Since S is a process in natural scale, $s(x) = 1$ and its speed density is simply the reciprocal of its variance rate. Hence, (27) implies that the speed density of S is given by:

$$m(S) \equiv \frac{1}{2\pi \left[N'\left(N^{-1}\left(\frac{S}{H}\right)\right)\right]^2 h^2} = \frac{e^{\left[N^{-1}\left(\frac{S}{H}\right)\right]^2}}{h^2}, \qquad S \in (0, H). \tag{31}$$

Let τ_{DU} be the first time that S exits the interval $[D, U]$. Suppose we consider how the mean of the random variable τ_{DU} behaves in the limit as we set $D = S - \epsilon$, $U = S + \epsilon$ and let $\epsilon > 0$ shrink down to zero. Clearly, the mean exit time approaches zero, but the question is at what speed. For a process in natural scale such as S, Karlin and Taylor (1981) show on page 197 that the mean exit time $E[\tau_{(S-\epsilon,S+\epsilon)}|S_0 = S]$ approaches zero like $O(\epsilon^2)$, where the coefficient is given by the speed function $m(S)$, i.e.,

$$m(S) = \lim_{\epsilon \downarrow 0} \frac{E[\tau_{(S-\epsilon,S+\epsilon)}|S_0 = S]}{\epsilon^2}. \tag{32}$$

This is the likely origin of the term "speed density". We note that the higher the diffusion coefficient of a process in natural scale, the lower its speed density. This observation prompted Rogers and Williams (1994) to jestingly suggest that $m(\cdot)$ alternatively be called a "sloth density". As indicated in (31), the speed density of the S process is inversely proportional to a Gaussian function of $N^{-1}\left(\frac{S}{H}\right)$. As a consequence, S exits intervals much more slowly on average when it is near zero or H than when it is near $H/2$.

Recall the rough interpretation of the local time of a continuous martingale as the amount of quadratic variation occurring at a point. More precisely, local time captures the stochastic rate at which the quadratic variation experienced below some point in space increases as we increase the point. Loosely speaking, the speed density of a time homogeneous one dimensional diffusion can be interpreted as the expected *calendar time* spent at a point, until the first time that the diffusion exits an interval. This rough interpretation is meant to be contrasted with the expected *quadratic variation* spent at a point until the first exit. Thus the speed density is used to convert the expected local time of a stochastic process into twice the spatial density of the occupation time.

To illustrate these points, let M now denote a time homogeneous one dimensional diffusion martingale:

$$dM_t = a(M_t)dW_t, \qquad t \geq 0. \tag{33}$$

From (19), the local time at K evaluated at the random time τ_{DU} is:

$$L^M_{\tau_{DU}}(K) \equiv 2\int_0^{\tau_{DU}} \delta(M_t - K)a^2(K)dt. \tag{34}$$

In contrast, twice the density of the occupation time at K until τ_{DU} is:

$$2\int_0^{\tau_{DU}} \delta(M_t - K)dt. \tag{35}$$

From the Tanaka Meyer formula, the cost of creating the payoff in (35) is:

$$G(M,K) = \begin{cases} 2\frac{(M-D)(U-K)}{U-D}m(K), & -\infty < D < M < K < U < \infty \\ 2\frac{(K-D)(U-M)}{U-D}m(K), & -\infty < D < K < M < U < \infty. \end{cases} \tag{36}$$

Itô and McKean (1974) refer to this function as the Green's function. One can generalize by starting with a drifting process, but we do not explore that here. For the bounded Brownian Motion S, substituting (31) in (36) implies:

$$G^S(S,K) = \begin{cases} 2\frac{(S-D)(U-K)}{U-D}\frac{\exp\left\{\left[N^{-1}\left(\frac{S}{H}\right)\right]^2\right\}}{h^2}, & -\infty < D < S < K < U < \infty \\ 2\frac{(K-D)(U-S)}{U-D}\frac{\exp\left\{\left[N^{-1}\left(\frac{S}{H}\right)\right]^2\right\}}{h^2}, & -\infty < D < K < S < U < \infty. \end{cases} \tag{37}$$

6. Transition Density

To obtain the transition PDF of the time-homogeneous Markov process S, we first obtain the transition PDF of the process X. Suppose that we write the stochastic differential equation (SDE) (1) as:

$$dX_t = gX_t dt + hdW_t, \qquad t \geq 0, \tag{38}$$

where:

$$g \equiv \frac{2\pi h^2}{H^2} \geq 0 \tag{39}$$

is the expected relative growth rate in X. By standard calculations, one can show that X_T is Gaussian with mean:

$$E_0^{\mathbb{Q}} X_T = X_0 e^{gT}, \tag{40}$$

and variance:

$$V_x = h^2 \frac{e^{2gT}-1}{2g}. \tag{41}$$

We note that both the mean and the variance of X explode as $T \uparrow \infty$. Both results are a consequence of the fact that the process is mean-repelling, i.e., $g > 0$.

To obtain the PDF of S_T, first recall that the scale function $S_H(x)$ is defined in (5) as an increasing map from $\mathbb{R}$ to $(0, H)$. Let $x(S)$ be the inverse map from $(0, H)$ to $\mathbb{R}$:

$$x(S) \equiv \frac{H}{\sqrt{2\pi}} N^{-1}\left(\frac{S}{H}\right), \qquad S \in (0, H). \tag{42}$$

Since $S_H(\cdot)$ is increasing, so is $x(\cdot)$. In fact, from the inverse function theorem:

$$x'(S) = \frac{1}{S'_H(x)} = \frac{1}{\sqrt{2\pi}N'\left(\frac{\sqrt{2\pi}x}{H}\right)} = \frac{1}{\sqrt{2\pi}N'\left(N^{-1}\left(\frac{S}{H}\right)\right)}, \tag{43}$$

from (5). Using the definition of the standard normal density function $N'(\cdot)$, (43) simplifies to:

$$x'(S) = \exp\left\{\frac{1}{2}\left[N^{-1}\left(\frac{S}{H}\right)\right]^2\right\}, \tag{44}$$

for $S \in (0, H)$.

Let:

$$q(S_0, S; T) \equiv \frac{\mathbb{Q}\{S_T \in dS | S_0\}}{dS}, \qquad S \in (0, H), \tag{45}$$

be the transition PDF of S_T. Also let:

$$g(X_0, x; T) \equiv \frac{\mathbb{Q}\{X_T \in dx\}}{dx} = \frac{e^{-\frac{[x - X_0 e^{gT}]^2}{2V_x}}}{\sqrt{2\pi V_x}}, \qquad x \in \mathbb{R}, \tag{46}$$

be the transition PDF of X_T. By the change of variables theorem for densities:

$$q(S_0, S; T) = g(X_0, x(S); T)|x'(S)|, \qquad S \in (0, H). \tag{47}$$

Substituting (46) and (44) in (47) implies that the PDF of S_T is known in closed form:

$$\begin{aligned} q(S_0, S; T) &= \frac{e^{-\frac{[x(m)-X_0 e^{gT}]^2}{2V_x}}}{\sqrt{2\pi V_x}} e^{\frac{1}{2}\left[N^{-1}\left(\frac{S}{H}\right)\right]^2} \\ &= \frac{e^{-\frac{H^2}{4\pi V_x}\left[N^{-1}\left(\frac{m}{H}\right)-N^{-1}\left(\frac{S_0}{H}\right)e^{gT}\right]^2+\frac{1}{2}\left[N^{-1}\left(\frac{S}{H}\right)\right]^2}}{\sqrt{2\pi V_x}}, \qquad S \in (0, H), \end{aligned} \tag{48}$$

from (42), where g and V_x are given in (39) and (41) respectively. Not surprisingly, the horizon length T enters the PDF of S_T only through the mean and variance of X_T.

The future level S enters the PDF of S_T only through the variable $N^{-1}\left(\frac{S}{H}\right)$. When the PDF of S_T is considered as a function of this latter variable, (48) indicates that it is proportional to the ratio of two Gaussian densities. The mean and standard deviation of the numerator Gaussian both increase with T, while the denominator Gaussian is a standard normal PDF. At short maturities, the graph of q against $N^{-1}\left(\frac{S}{H}\right)$ is dominated by the numerator Gaussian. Hence, the graph of q against S is an upside down U. As T increases, the numerator Gaussian tends to a uniform density. Hence, as T increases, the graph of q against $N^{-1}\left(\frac{S}{H}\right)$ tends to the reciprocal of a standard normal PDF.

An abrupt change in the shape of the PDF occurs at the maturity for which:

$$\frac{H^2}{4\pi V_x} = \frac{1}{2}. \tag{49}$$

Using (41), this critical level of T is easily found to be:

$$T^* = \frac{1}{2g} \ln\left(1 + \frac{gH^2}{\pi h^2}\right). \tag{50}$$

For $T < T^*$, the net coefficient of the quadratic in the argument of the exponential is negative. As a result, the graph of q against $N^{-1}\left(\frac{S}{H}\right)$ is Gaussian while the graph of q against S is an upside down U. When $T = T^*$, the coefficient on the quadratic vanishes and the PDF becomes exponential in the variable $N^{-1}\left(\frac{S}{H}\right)$ rather than Gaussian. When $T > T^*$, the coefficient on the quadratic becomes positive. As a result, the graph of q against $N^{-1}\left(\frac{S}{H}\right)$ is the reciprocal of a Gaussian while the graph of q against S is U shaped.

Consider a movie of the graph of the PDF q against the forward spatial variable S. As T increases, all of the probability mass moves out from around S_0 to around zero and around H. Intuitively, since S is a martingale, the mean of S_T must remain constant at S_0 while the variance of S_T must increase with the horizon length T. Since S is bounded, the only way its PDF can accommodate this behavior is to become U shaped in the interior.

Suppose one starts the martingale away from $H/2$ and stops it when it first hits $H/2$. When $S_0 = H/2$, the PDF of S is even in S about $H/2$. As a result, one knows the PDF and absorption probability of the stopped martingale. Hence if one wants a tractable bounded process with a lower natural barrier at zero and an absorbing upper barrier, simply start S in $(0, H/2)$ and stop it when it first hits $H/2$. This would describe the value of the protection leg of a CDO. Conversely, if one wants a tractable bounded process with an upper natural barrier and an absorbing lower barrier, simply start S in $(H/2, H)$ and stop it when it first hits $H/2$. This could describe a variation on the Black and Cox (1976) model for describing the dynamics of the value of a firm's assets when default is possible. Let S be the firm's asset value and suppose that H is a natural upper boundary. As in Black and Cox, we also suppose that default and liquidation occur at the first time that S hits $H/2$. It would be straightforward to value contingent claims written on the stopped process.

It is interesting to compare q with the lognormal PDF. If S is lognormally distributed, then in the lognormal PDF, $\ln(S/1)$ plays the same role as $N^{-1}\left(\frac{S}{H}\right)$ in q. The log function maps $\mathbb{R}^+$ to $\mathbb{R}$, while $N^{-1}(\cdot)$ maps $(0,1)$ to $\mathbb{R}$. Both functions asymptote to $\pm\infty$ at the endpoints of their domain. The standard parametrization of the PDF of geometric Brownian motion expresses the lognormal PDF in terms of the mean and variance of the corresponding drifting Brownian motion. Analogously, the above parametrization of the PDF of bounded Brownian motion expresses the PDF in terms of the mean and variance of the corresponding OU process.

A geometric Brownian martingale has a stability property. Specifically, the probability that the terminal level of the process is below any given positive level approaches one as the time horizon becomes arbitrarily large. We speculate that for bounded Brownian motion, the almost sure convergence is to the union of the two sets $(0,\epsilon)$ and $(H-\epsilon)$ for any $\epsilon > 0$.

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