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BESSEL PROCESSES, THE INTEGRAL OF GEOMETRIC BROWNIAN MOTION, AND ASIAN OPTIONS*

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Abstract. This paper is motivated by questions about averages of stochastic processes which originate in mathematical finance, originally in connection with valuing the so-called Asian options. Starting with [M. Yor, *Adv. Appl. Probab.*, 24 (1992), pp. 509–531], these questions about exponential functionals of Brownian motion have been studied in terms of Bessel processes using the Hartman–Watson theory of [M. Yor, *Z. Wahrsch. Verw. Gebiete*, 53 (1980), pp. 71–95]. Consequences of this approach for valuing Asian options proper have been spelled out in [H. Geman and M. Yor, *Math. Finance*, 3 (1993), pp. 349–375] whose Laplace transform results were in fact regarded as a significant advance. Unfortunately, a number of difficulties with the key results of this last paper have surfaced which are now addressed in this paper. One of them in particular is of a principal nature and originates with the Hartman–Watson approach itself: this approach is in general applicable without modifications only if it does not involve Bessel processes of negative indices. The main mathematical contribution of this paper is the development of three principal ways to overcome these restrictions, in particular by merging stochastics and complex analysis in what seems a novel way, and the discussion of their consequences for the valuation of Asian options proper.

Key words. Asian options, integral of geometric Brownian motion, Bessel processes, Laplace transform, complex analytic methods in stochastics

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1. Introduction.

This paper addresses questions about exponential functionals of Brownian motion and the integral of geometric Brownian motion in particular. These questions reduce to the study of the quadratic variation processes $A^{(\nu)}$ of geometric Brownian motion which for any real drift ν are explicitly given by the integrals over time

$$A_t^{(\nu)} = \int_0^t e^{2(\nu w + B_w)}\, dw, \qquad t \in [0, \infty),$$

with B a standard Brownian motion. These processes have both a surprisingly rich theory and manifold applications ranging from the physics of random media to mathematical finance and insurance. In fact, the insurance-motivated study of certain perpetuities in [8] seems to have initiated this line of research. Here, the above integrals over the whole time axis are considered and shown to be distributed as the reciprocals of certain gamma variables. Drawing on his probabilistic interpretation of the Hartman–Watson identities in [27], Yor was able to extend this work and to determine the law of the processes $A^{(\nu)}$ in [29]. This approach, using the Laplace transform and based on Bessel processes, has been found by Yor to open many surprising vistas on the processes $A^{(\nu)}$ and their applications; see in particular [32]. The interest in these processes $A^{(\nu)}$ is partially due to their importance in mathematical finance, in particular for understanding the so-called Asian options.

Asian options are widely used financial derivatives. As discussed in Part I of this paper, these options provide in general nonlinear payoffs on the arithmetic average

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of the price of an underlying asset. A common objective in their valuation is to derive an explicit expression for a certain functional of $A^{(\nu)}$. The pursuit of this objective has evolved over the last fifteen years as an interplay between theoretical and computational perspectives; see [20] or [9], for instance. Yor's work in [29] clarified the structure of the Black–Scholes prices of Asian options by expressing them as certain triple integrals. In contrast to this result, however, it was the Laplace transform approach of [12], also based on the Hartman–Watson theory, which had far-reaching consequences for the way Asian option valuation is seen today, and on which we focus here from Part II onwards. This is essentially because a very explicit expression in terms of Kummer's confluent hypergeometric function resulted from this approach for what has been regarded as the Laplace transform of the value of such options. From a numerical point of view, this expression has proved to be amenable to computation by numerical inversion; see [11] for a recent example. Also it seems fair to say that this has led financial mathematicians toward a new interest in developing and applying these techniques.

Unfortunately, some difficulties with this Laplace transform approach to valuing Asian options have emerged. First, it turned out that the Laplace transforms computed in [12] are not those of the Asian options' value. This appeared to decrease the relevance of this result to financial application. Luckily, it turned out that there is a reduction of the original problem of valuing Asian options to the one considered in [12]. All of this is discussed in Part II of this paper.

A difficulty of a more serious and more principal nature, however, originates with the Hartman–Watson approach on which the Laplace transform computations of [12] are based. Its idea is to analyze $A^{(\nu)}$ using Bessel processes of indices ν, and its applicability has limits if ν is negative because of the pathologies which Bessel processes of such negative indices develop. Together with background material about Bessel processes, we have thus given in Part III a new exposition of the analysis in [12], which explicitly takes care of the nonnegativity restriction on the index ν. Hereby, we have been encouraged by the kind support of Yor, and we have tried to incorporate his tutorials and suggestions. In terms of financial mathematics, our extension to negative ν in fact extends the analysis of Asian options from the zero dividend situation originally considered in [12] to one of general real dividend yields, and thus to one with general real risk-neutral drifts. In this setting, the condition that ν is not negative translates into the postulation that risk-neutral drift is not less than half the squared volatility. Unfortunately, this lower bound on the drift restricts the financial applicability of the results. In fact, the greater the volatility is, the greater the range of parameters is in which the nonnegativity condition on ν is violated and in which the approach does not give the Laplace transforms. Unfortunately, it is precisely due to high volatility of the underlying asset that Asian options are used in the first place.

Thus, the third contribution of our paper is to remove these restrictions by showing the existence of the desired Laplace transforms. In fact, we develop three ways for removing these restrictions. More precisely, we develop three principal ways of coping with the difficulties caused by Bessel processes of negative indices in the Hartman–Watson approach. This can be seen as the main mathematical contribution of the paper. The approaches of Parts IV and VI are based on an analysis of Bessel processes in the spirit of Yor. Our first approach extends the analysis of [12] to the missing cases using deeper properties of Bessel processes of negative indices. Our second approach gives a new uniform proof using zero index Bessel processes only. Our third approach, in Part V, extends that of [2] and merges stochastic and complex analytic techniques. This appears to be a rather novel and promising line of attack as we are able to largely dispense with Bessel processes and focus instead on Brownian motion.

Apart from all this, our extension of the Laplace transform approach of [12] has made possible advances in valuing Asian options, some of which are sketched in Part VII. Hence, it may be fair to say that the Laplace transform approach of [12] has proved to be a rich source for new results and insights in both finance and mathematics.

Part I. Prologue

2. Black–Scholes modeling. The results to be discussed originate from the so-called risk-neutral approach to the valuation of contingent claims. General equilibrium treatments for this and other notions developed for the analysis of financial markets and instruments can be found, for instance, in [6], [7], [13, Chaps. 1–4]. This analysis is based on models of security markets, and this section aims to recall the most fundamental of these, the Black–Scholes model of security markets.

In fact, we need only that particular case of the Black–Scholes model in which there are only two securities, and we understanding that these are traded on markets in which their prices are determined by equating demand and supply. First, there is a riskless security, a bond, whose price β grows at the continuously compounding positive interest rate r, i.e., for which we have $\beta_t = \exp(rt)$ at any time t in $[0,\infty)$. Then there is a risky security. The fundamental idea is that all uncertainties affecting its price S yield a certain probability space. In fact, consider for this a complete probability space equipped with the standard filtration of a standard Brownian motion on the time set $[0,\infty)$. Giving expression to the fact that S comes as an equilibrium price, we have the risk-neutral measure $\mathbf{Q}$ on this filtered space, a probability measure equivalent to the given one. Also, with B being any standard $\mathbf{Q}$-Brownian motion, the exact modeling then is that S is the strong solution of the following stochastic differential equation:

$$dS_t = \varpi S_t\,dt + \sigma S_t\,dB_t, \qquad t \in [0,\infty),$$

or equivalently, using Itô calculus,

$$S_t = S_0 e^{(\varpi-\sigma^2/2)\,t+\sigma B_t}, \qquad t \in [0,\infty).$$

The positive constant σ is the volatility of S. The specific form of the otherwise arbitrary constant ϖ depends on the nature of the security modeled, which could be a stock, a currency, a commodity, etc. For example, if S is the price of a stock paying a dividend continuously so as to have constant dividend yield δ, then we have $\varpi = r - \delta$.

3. Asian options and their equilibrium pricing. In the Black–Scholes framework of section 2, fix any time t_0 and consider the process J given for any time t by

$$J(t) = \int_{t_0}^{t} S_u\,du.$$

The *arithmetic-average Asian option* written at time t_0 with maturity T and strike price K is then the stochastic process on the closed time interval from t_0 to T paying $(J(T)/(T-t_0)-K)^+ := \max\{0, J(T)/(T-t_0)-K\}$ at time T and paying nothing at all other times. As such it is a contingent claim on the time interval from t_0 to T with payoff $(J(T)/(T-t_0)-K)^+$.

It is one of the fundamental insights that, in the equilibrium framework of the Black–Scholes model, any such contingent claim on a risky security has an equilibrium price which is equal to the discounted expectation of its payoff with respect to the risk-neutral measure conditional on today's information; see, for instance, [6, section 22K, (47)], [7, section 8A], or [16, Corollary 5.1.1]. Applying this *arbitrage pricing principle*, the price C_t of the Asian option at any time t between t_0 and T is given by the discounted $\mathbf{Q}$-expectation conditional on the information $\mathcal{F}_t$ available at time t:

$$C_t = e^{-r(T-t)}\mathbf{E}^{\mathbf{Q}}\left[\left(\frac{J(T)}{T-t_0}-K\right)^+ \Big| \mathcal{F}_t\right].$$

However, following [12, section 3.2], we do not focus on this price but instead normalize the valuation problem as follows. On factoring out the reciprocal of the length $T-t_0$ of the time period, we split the integral $J(T)$ into two integrals, one of which is deterministic by time t and the other of which is random. We then couple the deterministic integral with the new strike. For the random integral, we restart the Brownian motion driving the underlying asset at time t, and then, using the scaling property of Brownian motion, we change time so as to normalize its coefficient in the new time scale to two. The precise result is the factorization

$$C_t = \frac{e^{-r(T-t)}}{T-t_0}\,\frac{4S_t}{\sigma^2}\,C^{(\nu)}(h,q),$$

which reduces the general valuation problem to computing

$$C^{(\nu)}(h,q) = \mathbf{E}^{\mathbf{Q}}\left[(A_h^{(\nu)}-q)^+\right],$$

the *normalized time-t price* of the Asian option. Herein, $A^{(\nu)}$ is Yor's process

$$A_h^{(\nu)} = \int_0^h e^{2(B_w+\nu w)}\,dw,$$

and the normalized parameters are as follows:

$$\nu = \frac{2\varpi}{\sigma^2}-1, \quad h = \frac{\sigma^2}{4}(T-t), \quad q = kh+q^*,$$

where

$$k = \frac{K}{S_t}, \qquad q^* = q^*(t) = \frac{\sigma^2}{4S_t}\left(K\,(t-t_0) - \int_{t_0}^t S_u\,du\right).$$

To interpret these quantities, ν is the *normalized adjusted interest rate*, h is the *normalized time to maturity*, which is nonnegative, and q is the *normalized strike price*. On a conceptual level, notice that valuing any Asian option in this way is reduced to computing a single function $C^{(\nu)}$, and that a similar notion of normalized hedging of Asian options can be developed along these lines. On a structural level, moreover, notice how q becomes affine linear in the time variable h with coefficients k and q^* depending only on quantities known at time t.

4. A first reduction of the normalized valuation. There is now a dichotomy in computing the normalized time-t price

$$C^{(\nu)}(h,q) = \mathbf{E}^{\mathbf{Q}}\left[\left(A_h^{(\nu)} - q\right)^+\right]$$

of the Asian option according to whether or not the normalized strike price q is positive. Indeed, if q is nonpositive, Asian options lose their option feature. Computing the values $C^{(\nu)}(h,q)$ is straightforward.

LEMMA. *If $q \leqq 0$, we have $C^{(\nu)}(h,q) = \mathbf{E}^{\mathbf{Q}}[A_h^{(\nu)}] - q$ with $\mathbf{E}^{\mathbf{Q}}[A_h^{(\nu)}] = e^{2h(\nu+1)} - 1/(2(\nu+1))$ for any real ν, and it is thus sufficient to compute $C^{(\nu)}(h,q)$ if $q > 0$.*

This can be proved on applying Fubini's theorem, and the last expectation is seen to be analytic in ν with its value at $\nu = -1$ equal to h. It should be mentioned that formulas for all moments of $A^{(\nu)}$ have been derived at various instances over the last fifty years; see for example [32, section 2.4.1, (4.d′), p. 33 and Postscript 3b, p. 54].

5. Yor's integral representation for Asian option values. A closed form for the normalized time-t prices $C^{(\nu)}$ of Asian options can be obtained as a consequence of Yor's triple integral representation [29, (6.e), p. 528]. Recall the latter is based on Yor's characterization of the law of $A^{(\nu)}$ in [29, (6.c), p. 527] and so is eventually based on the Hartman–Watson theory of [27]. Furnishing a measure for the difficulty of computing normalized prices, the precise form of Yor's closed form is as follows.

THEOREM. *For any reals h, $q > 0$, and ν, we have*

$$C^{(\nu)}(h,q) = c_{\nu,h} \int_0^\infty x^\nu \int_0^\infty e^{-(1+x^2)\,y/2} \left(\frac{1}{y} - q\right)^+ \psi_{xy}(h)\,dy\,dx.$$

Herein the function ψ_a, for any positive real number a, is given by the following integral:

$$\psi_a(h) = \int_0^\infty e^{-w^2/(2h)} e^{-a\,\mathrm{ch}(w)}\,\mathrm{sh}(w)\,\sin\!\left(\frac{\pi}{h}\,w\right) dw\,,$$

for any $h > 0$, and we abbreviate

$$c_{\nu,h} = \frac{1}{\pi\sqrt{2\pi^3 h}}\, e^{\pi^2/(2h) - \nu^2 h/2}.$$

While Yor's formula seems to require ν to be bigger than at least minus one, it is valid for all real ν. This is proved in [24, section 6].

One purpose of closed form expressions is to provide means for actually computing option prices. While Yor's results above are the key to many insights into the mathematical structure of $A^{(\nu)}$, Yor's formula of the theorem has a number of structural difficulties in this regard. First, it involves three integrations with seemingly no further structure or further possibilities for simplification. Thus, methods for explicitly computing it will necessarily be rather complex. However, we have noticed an apparently even bigger obstacle to computability. For instance, taking $t_0 = 0$ and $\varpi = r$ equal to 5%, we compute the factors $c_{\nu,h}$ as follows:

TABLE 1. *$c_{\nu,h}$ as function of T and σ.*

$c_{\nu,h}$	$\sigma = 20\%$	$\sigma = 30\%$	$\sigma = 40\%$
$T = 1$ year	2.627×10^{213}	2.265×10^{94}	4.816×10^{52}
$T = 6$ months	7.686×10^{427}	5.717×10^{189}	2.583×10^{106}

As the examples of section 8 and section 24 will show, normalized prices of Asian options are not too big. Yor's formula thus expresses them as the product of a big number times a triple integral. The latter has to be very small and must be computed with very high accuracy to get reasonably accurate results. The Laplace transform approach developed in [12] explicitly for the purpose of valuing Asian options was seen to offer a way out of these difficulties.

Part II. Laplace Transform Results

6. The Laplace transform in option valuation. Working with continuous functions on the nonnegative real line of at most exponential growth, the *Laplace transform* $\mathcal{L}(f)$ of any such function f is defined by

$$\mathcal{L}(f)(z) = \int_0^\infty e^{-zx} f(x)\, dx$$

for any complex number z in a half-plane contained sufficiently deep within the complex right half-plane.

The connection of this notion with option valuation in general and Asian option valuation in particular is as follows. Fix any option type, such as section 3's European-style Asian call option, on a certain stock with price S. It will depend on a number of parameters, such as strike price and maturity date. At a fixed point in time t, consider the family which consists of all options on the market with all such parameters fixed except maturity dates M. In the Asian option example thus consider the Asian options of all maturities available at time t which have the same strike price. In this way regard maturity date M as a real variable ranging from t to infinity. The value of the option thus becomes a function of M.

However, it is normalized prices $C^{(\nu)}$ we have to consider for the Asian option. The normalizations of section 3 in fact turn $C^{(\nu)}$ into a function of, in particular, normalized time to maturity. As a function of maturity date M, normalized time to maturity is explicitly given by $h(M) = (\sigma^2/4)\,(M - t)$. With M from t to infinity, $h(M)$ thus ranges from 0 to infinity. The normalized price of the Asian option so becomes a function on the nonnegative real line. Call it f_{AO} for the sake of emphasis. Recalling from section 3 how the normalized strike price q depends in an affine linear way on normalized time to maturity, we then have more precisely

$$f_{AO}(x) = \mathbf{E}^{\mathbf{Q}}\big[\big(A_x^{(\nu)} - (kx + q^*)\big)^+\big]$$

for any nonnegative real x.

At this point we should signal a difficulty with [12] which we have noticed in the Fall of 1999. It is not the functions f_{AO} this last paper is working with, but the functions $f_{GY,a}$ given for any positive real a by

$$f_{GY,a}(x) = \mathbf{E}^{\mathbf{Q}}\big[\big(A_x^{(\nu)} - a\big)^+\big]$$

for any nonnegative real x. Notice that the function f_{AO} has a nonconstant strike price $x \mapsto kx + q^*$, whereas any function $f_{GY,a}$ has the constant strike price a. So the value function of the Asian option f_{AO} is different from any function $f_{GY,a}$, which thus is the value function of a *non-Asian option*. Because of the injectivity of the Laplace transform on continuous functions, the Laplace transform $\mathcal{L}(f_{AO})$ is different from any Laplace transform $\mathcal{L}(f_{GY,a})$ too. The problem at this point concerns the relevance of mathematics to financial application proper: Is there a way of relating the valuation of Asian options to the valuation of the non-Asian options?

7. Valuing Asian options using families of non-Asian options. The basic idea is as follows. Try to reconstruct the normalized time-t price

$$C^{(\nu)}(h,q) = \mathbf{E}^{\mathbf{Q}}\Big[\big(A_x^{(\nu)} - (kx+q^*)\big)^+\Big]\Big|_{x=h}$$

of the Asian option from a family of auxiliary functions whose single members are unrelated to the problem of valuing the Asian option but amenable to the analysis of [12]. Given that the time dependency of the normalized strike price poses the problems, simply force this strike price to be constant. Thus we arrive, for any positive real a, at section 6's functions $f_{GY,a}$ on the positive real line recalled as given by $f_{GY,a}(x) = \mathbf{E}^{\mathbf{Q}}[(A_x^{(\nu)} - a)^+]$ for any positive real x. These are the functions considered in [12]. As we remarked in section 6, they are the values of certain non-Asian options and, taken individually, they cannot be used to value the original Asian option. However, our finding is that as a whole they allow one to recover the normalized time-t price. With the concepts of section 3 this *key reduction* is made precise in the following lemma.

LEMMA. *If $q = kh + q^*$ is positive, computing the normalized time-t price $C^{(\nu)}(h,q)$ of the Asian option reduces to computing all $f_{GY,a}$ with $a > 0$. More precisely, $C^{(\nu)}(h,q)$ is obtained by choosing the function $f_{GY,kh+q^*}$ and evaluating it at h.*

A moment's reflection will convince the reader that this is true by construction, and meanwhile, the particular case of the lemma, which we explained to Yor in May 2000, can be found in [32, pp. 95–96]. Notice that our construction works in the more general situation, where functions of the form $h(y) = f(y, \varphi(y))$ with a known map φ have to be computed: compute the functions f and then intersect with the graph of φ to get h. Also, to stress the main result again, as one consequence of this technique we have at this point reduced valuing Asian options to valuing the family of all non-Asian options.

8. Laplace transforms of the non-Asian option values $f_{GY,a}$. The key reduction of the preceding section shows the way to apply the Laplace transform of [12] to value Asian options. Moreover, adopting the notation of section 3, consider the family of all functions $f_{GY,a}$ with $a > 0$ that send any $x > 0$ to

$$f_{GY,a}(x) = \mathbf{E}^{\mathbf{Q}}\big[(A_x^{(\nu)} - a)^+\big].$$

Recall from section 7 that its single members are unrelated to valuing the Asian option, but that as a whole, the family allows reconstruction of the normalized time-t price $C^{(\nu)}(h,q)$ of the Asian option. As a first step in actually computing the $f_{GY,a}$, try to compute their Laplace transforms $F_{GY,a}$ given by

$$F_{GY,a}(z) = \int_0^\infty e^{-zx} f_{GY,a}(x)\,dx = \mathcal{L}(f_{GY,a})(z).$$

Here the complex number z is to be taken in a half-plane sufficiently deep within the right complex half-plane such that the integrals are finite. The function so obtained is analytic. The precise conditions under which these integrals are finite is part of our description of these *generalized Geman–Yor Laplace transforms* in the following theorem.

THEOREM. *If the normalized strike price q is positive, the integrals $F_{GY,a}$ are finite for any complex number z with $\mathrm{Re}(z) > \max\{0, 2(\nu+1)\}$, and we have*

$$F_{GY,a}(z) = \frac{D_\nu(a,z)}{z(z-2(\nu+1))},$$

where on choosing the principal branch of the logarithm

$$D_\nu(a,z) = \frac{e^{-1/(2a)}}{a} \int_0^\infty e^{-x^2/(2a)} x^{\nu+3} I_{\sqrt{2z+\nu^2}}\left(\frac{x}{a}\right) dx.$$

Herein I_μ is the modified Bessel function with complex order μ, as discussed in [15, Chap. 5]. Generalizing [12, (3.9), p. 363] the integral $D_\nu(a,z)$ can be expressed using the confluent hypergeometric function Φ discussed in [15, Chap. 9] as follows.

COROLLARY. *For any complex* z *with* $\mathrm{Re}(z) > \max\{0, 2(\nu+1)\}$, *we have*

$$\begin{aligned} D_\nu(a,z) = {} & \Gamma\left(\frac{\nu+4+\mu(z)}{2}\right) \frac{1}{\Gamma(\mu(z)+1)} \\ & \times \Phi\left(\frac{\nu+4+\mu(z)}{2},\ \mu(z)+1;\ \frac{1}{2a}\right) \frac{(2a)^{(\nu+2-\mu(z))/2}}{e^{1/(2a)}} \end{aligned}$$

on setting $\mu(z) = \sqrt{2z+\nu^2}$.

We again stress that these Laplace transforms are not those of the Asian option price, but rather the Laplace transforms of the prices of auxiliary options. To obtain Asian option prices, we have to invert these Laplace transforms and then proceed using section 7's lemma; formally speaking, $C^{(\nu)}(h,q) = \mathcal{L}^{-1}(F_{GY,q})(h)$. In full mathematical generality, analytic inversion has been achieved in [21], and we come back to this in section 23. Numerical inversions have also been accomplished. For example, Fu, Madan, and Wang [11] computed the following seven cases as reproduced in [9, Table 7.1]:

TABLE 2. *Prices* $2C^{(\nu)}$ *for* $K = 2.0$ *and* $t_0 = t = 0$ *using numerical Laplace inversion.*

Case	r	σ	T	S_0	ν	$2C^{(\nu)}$
1	2%	10%	1	2.0	3	0.056
2	18%	30%	1	2.0	3	0.219
3	1.25%	25%	2	2.0	-0.6	0.172
4	5%	50%	1	1.9	-0.6	0.194
5	5%	50%	1	2.0	-0.6	0.247
6	5%	50%	1	2.1	-0.6	0.307
7	5%	50%	2	2.0	-0.6	0.352

Obtaining our two results proceeds in two steps. In an initial probabilistic step, the arguments of [12] are adapted to compute the modified Geman–Yor transforms $F_{GY,a}$. We give in Part III a new exposition of the argument incorporating the tutorials and kind suggestions of Yor. The key idea is to factorize the geometric Brownian motion of the underlying asset over a Bessel process of index ν. Pertinent notions are discussed in section 9. This makes time stochastic in such a way that Yor's process $A^{(\nu)}$ now takes the double role of both a stochastic clock and a control variable for the Asian option. At first sight, this appears to complicate the original valuation problem. However, this double role of $A^{(\nu)}$ is especially suited to the Laplace transform. Indeed, in contrast to the situation for the Asian option, the strike price a of the non-Asian option with value function $f_{GY,a}$ is independent of time. This makes it possible to reduce the computation of the Laplace transform $F_{GY,a}$ to the following problem: obtain explicit expressions for the Bessel semigroup of index ν and for a certain conditional expectation involving first passage times of Yor's process $A^{(\nu)}$.

However, such results are available for both concepts only if the index ν is nonnegative, an assumption which is explicit in [12, section 2]. This nonnegativity condition, however, translates into the condition that the risk-neutral drift is not less than half the squared volatility. Unfortunately, this places restrictions on the financial applicability of the result. For example, if volatility is 30%, the arguments of [12, section 3] are not valid if the difference between the risk-free rate and the dividend yield is less than 4.5%, and then we do not have the Laplace transforms of the non-Asian options either. Worse yet, the greater the volatility is, the greater is this range of parameters in which we do not have the Laplace transforms. Unfortunately, it is precisely due to high volatility that Asian options are used in the first place.

We have been a bit disconcerted by these findings. Luckily, however, we found that in particular those results of Table 2, where ν is negative, were reproduced in [9] using an alternative approach. Also, corroborating Weil's [26, p. 457] dictum that "theorems are proved by those who believe in them", we are now able to discuss in what follows three different ways for establishing the theorem and its corollary for arbitrary real risk-neutral drifts ν.

The first of these, as discussed in Part IV, is inspired by [29], and we think Yor could have given this argument had he been aware of the financial motivation for letting the parameter ν be negative. In fact, the key idea is to try to bypass the difficulties of Bessel processes of negative index ν by Girsanov transforming to the simpler Bessel processes of index zero. This theme is developed also in the third approach discussed in Part VI. Here, the idea is not to Girsanov transform Bessel processes derived from the geometric Brownian motion driving the underlying asset. Instead, Girsanov transform this geometric Brownian motion itself — an idea we distilled from [27]. The effect of this is that zero drift Bessel processes enter right from the beginning, and the result is a uniform argument based on these most natural Bessel processes.

In comparison to these two approaches, our second approach discussed in Part V seems somewhat novel. The idea is to combine stochastic methods with complex analytic ones. The net effect here is that with an input of some standard result from the latter area, such as the identity theorem, it is not Bessel processes which now have to be dealt with but Brownian motion. For this we are, moreover, not required to work on the process level, as it is sufficient to work on the level of expectations. So this second approach seems to be an example of a rather promising methodology for solving problems, which is to systematically enhance stochastic techniques with complex analytic ones.

Part III. Laplace Transforms if $\nu \geqq 0$

9. Preliminaries on Bessel processes. As a preliminary to establishing the Laplace transforms of section 8, this section collects a number of pertinent facts from the theory of Bessel processes. This theory is patterned after the example of the Bessel processes of integer dimension $\delta \geqq 2$, which are defined by taking the Euclidean distance from the origin of a Brownian motion in dimension δ. Applying Itô's lemma, their infinitesimal generator $\mathcal{A}$ is seen to be given by

$$\mathcal{A}f(x) = \frac{1}{2}\, f''(x) + \frac{2\nu+1}{2x}\, f'(x)$$

for any function f in $C_b^2(\mathbf{R}_{>0})$. This notion makes sense for any real number δ, and the real-valued diffusion associated to $\mathcal{A}$ using the Volkonskii construction (see for instance [14, Theorem 4.3.3, p. 91]) is the *Bessel process* $R^{(\nu)}$ on $[0,\infty)$ with *index* $\nu = (\delta/2) - 1$. While Bessel processes of nonnegative indices ν remain positive if begun with a positive value at time zero, Bessel processes of negative indices ν develop some pathologies. As explained in [19, XI, section 1], in this case they hit zero. If $-1 < \nu < 0$, they are thereupon instantaneously reflected and never become negative. For $\nu = -1$ they continue at zero.

This matters for the second way of defining Bessel processes of arbitrary dimension δ. In fact, the focus here is on squares of Bessel processes. Applying Itô's lemma, they are the continuous strong solutions of the stochastic differential equation

$$d\rho_t = 2\delta\, dt + 2\sqrt{|\rho_t|}\, dB_t, \qquad \rho_0 = 1$$

[19, XI, section 1]. These stochastic differential equations make sense for any real number δ and have a unique continuous strong solution also if δ is smaller than two. The obtained processes are studied in [30, section 3]. For nonnegative indices ν, they coincide with the squares of the corresponding Bessel processes of index ν. They develop some pathologies for negative indices ν. In this case, they hit zero if begun with a positive value at time zero, and if $\nu < -1$, they even continue negative. Notice that in such situations their square roots are purely imaginary and so cannot coincide with any of the Bessel processes constructed above. However, these two ways of extending the notion of Bessel processes do coincide for processes begun at time zero at a positive value up to the first time zero is hit. This essentially is the reason behind the following *Lamperti identity*, which may nevertheless be surprising.

LEMMA. *For the index ν any real number, we have the factorization*

$$e^{B_t+\nu t} = R^{(\nu)}\big(A_t^{(\nu)}\big)$$

for any $t > 0$, where $A_t^{(\nu)} = \int_0^t e^{2(B_w+\nu w)}\, dw$ is Yor's process.

For $\nu \geqq 0$ a proof is given in [28, section 2] while the general case is now contained as exercise XI (1.28) on page 452 of the third edition of [19]. We are indebted to Yor for this and for kindly supplying us with the following argument.

The idea is to apply the Itô rule to the square Z_t of $Y_t = \exp(\nu t + B_t)$ to get

$$Z_t = 2(\nu+1)\int_0^t Z_w\, dw + 2\int_0^t Z_w\, dB_w.$$

Time change the process using the inverse function $\tau(t) = \inf\{u | \int_0^u Z_w\, dw > t\}$ to Yor's process $A^{(\nu)}$ to get

$$Z_{\tau(t)} = 2(\nu+1)\, t + 2\int_0^{\tau(t)} Z_w\, dB_w.$$

To interpret the stochastic integral in this sum, apply the basic time change formalism for stochastic processes as in [17, section 8.5] to obtain

$$\int_0^{\tau(t)} Z_w\, dB_w = \int_0^t Z_w \sqrt{\tau'(w)}\, dW_w,$$

where W_t is defined as the stochastic integral $W_t = \int_0^{\tau(t)} \sqrt{Z_w}\, dB_w$ and is a Brownian motion. Using the inverse function theorem of calculus, the derivative of τ is equal to the reciprocal of the derivative with respect to time of Yor's process $A^{(\nu)}$ at time w.

Hence it is equal to the reciprocal of Z_w. On substitution we so identify the time-changed process Z as a continuous solution to the stochastic differential equation for the square of the Bessel process of index ν:

$$Z_{\tau(t)} = 2(\nu+1)\,t + 2\int_0^t \sqrt{Z_w}\,dW_w.$$

Using the uniqueness of the solution of these stochastic differential equations, the time-changed process Z is the square of the Bessel process of index ν. Reversing the time change, this translates into

$$Y_t^2 = (R_t^{(\nu)})^2\,(A_t^{(\nu)}).$$

To establish the identity of the lemma, we have to take square roots. This is not a problem if ν is nonnegative since then the Bessel process takes nonnegative values only. It does pose a problem if ν is negative. In this case, however, recall that the Bessel process starts at time zero with a positive value. Since it is continuous by hypothesis, it will remain positive until it first hits zero at time $t^* > 0$. Since the process $A^{(\nu)}$ begins at zero at time zero, there is a latest point in time t^{**}, infinity admitted, such that $A^{(\nu)}$ is smaller than t^* at all points in time t smaller than t^{**}. Thus we have the required identity at least for all points in time t smaller than t^{**}. Now Y_t is never zero. Since the processes on both sides of the identity are continuous in time, t^{**} must be infinity, and the proof is complete.

10. Computing Laplace transforms if $\nu \geqq 0$.

This section is the first step in the proof of the integral representation of section 8's theorem for the Laplace transform

$$F_{GY,a}(z) = \int_0^\infty e^{-zx} f_{GY,a}(x)\,dx = \mathcal{L}(f_{GY,a})(z),$$

where, with the concepts of sections 3 and 6, we have $f_{GY,a}(x) = \mathbf{E}^{\mathbf{Q}}[(A_x^{(\nu)} - a)^+]$ for any positive real numbers a and x. We now explain why one needs to restrict the probabilistic arguments of [12] and apply them mutatis mutandis in order to arrive at the following lemma.

LEMMA. *The assertions of section* 8*'s theorem are valid if* $\nu = 2\sigma^{-2}\varpi - 1 \geqq 0$.

We are very indebted to Yor for correspondence and discussions about this result, and are very grateful for his kind support. In what follows, we want to indicate the key steps of the proof following [12], while trying to incorporate his suggestions. We hope that no pitfalls have remained undetected.

The basic idea is to make time stochastic using the Lamperti identity

$$e^{\nu w + B_w} = R^{(\nu)}(A_w^{(\nu)})$$

for all positive real numbers w, as has been discussed in the preceding section. Here, $R^{(\nu)}$ is the Bessel process of index ν with $R^{(\nu)}(0) = 1$.

On applying this Lamperti identity, $A^{(\nu)}$ has the double role of both control variable and stochastic clock. That the "strike price" a is independent of time now becomes essential. It makes possible the transcription of the condition on the control variable so that it is bigger than a as the inverse time change

$$\tau_{\nu,a} = \inf\{u \mid A_u^{(\nu)} > a\}$$

for the stochastic clock. This is the key idea for obtaining the representation

$$f_{GY,a}(w) = \mathbf{E}^{\mathbf{Q}}\left[\frac{e^{2(\nu+1)[w-\tau_{\nu,a}]^+}-1}{2(\nu+1)}\,(R_a^{(\nu)})^2\right]$$

for all $w > 0$. Indeed, fix any positive real number x, and consider the process $A^{(\nu)}$ at x on the set of all events, where $\tau_{\nu,a}$ takes values less than or equal to x. Break the integral defining $A^{(\nu)}(x)$ at $\tau_{\nu,a}$. The first summand then is $A^{(\nu)}$ at time $\tau_{\nu,a}$ and so is equal to a. In the second summand, restart the Brownian motion in the exponent of the integrand at $\tau_{\nu,a}$, shifting the variable of integration accordingly. The second integral then is the product of $\exp(2\,(B(\tau_{\nu,a})+\nu\tau_{\nu,a}))$ times $A^{(\nu)}$ at $x-\tau_{\nu,a}$, by abuse of language after having applied the strong Markov property. This last process is such that it is independent of the information at time $\tau_{\nu,a}$. Unraveling the definition of $\tau_{\nu,a}$, the first above factor is thus the square of the Bessel process $R^{(\nu)}$ at time a. Now taking the expectation conditional on the information at $\tau_{\nu,a}$, we thus get

$$\mathbf{E}^{\mathbf{Q}}\left[\left(A_x^{(\nu)}-a\right)^+\,\middle|\,\mathcal{F}_{\tau_{\nu,a}}\right] = \left(R_a^{(\nu)}\right)^2\mathbf{E}^{\mathbf{Q}}\left[A^{(\nu)}_{[x-\tau_{\nu,a}]^+}\right].$$

On substitution for the expectation of $A^{(\nu)}(w)$ from section 4's lemma or using [29, section 4], the required expression for $f_{GY,a}$ follows.

At first sight this appears to complicate the problem. However, it is just what is especially suited to the Laplace transform $F_{GY,a}$ of $f_{GY,a}$, now given by

$$F_{GY,a}(z) = \int_0^\infty e^{-zw}\mathbf{E}^{\mathbf{Q}}\left[\frac{e^{2(\nu+1)[w-\tau_{\nu,a}]^+}-1}{2(\nu+1)}\,(R_a^{(\nu)})^2\right]dw.$$

Still, for computing this integral one wants to interchange the Laplace integral with the expectation $\mathbf{E}^{\mathbf{Q}}$. If z is real, it seems best to follow Yor's proposal for justifying this. Indeed, with the integrand of the double integral in question positive and measurable, apply Tonelli's theorem now but justify only in a later step that any of the resulting integrals are finite. The case of a general argument z is reduced to this case by considering the absolute value of the integrand, and the result is the identity

$$F_{GY,a}(z) = \frac{1}{z(z-2(\nu+1))}\mathbf{E}^{\mathbf{Q}}\left[e^{-z\tau_{\nu,a}}(R^{(\nu)})^2\right]$$

of measurable functions for any complex number z with $\mathrm{Re}(z) > 2(\nu+1)$. The idea for identifying the expectation in the numerator as $D_\nu(a,z)$ then is to condition on the Bessel process to obtain

$$D_\nu(a,z) = \int_0^\infty x^2\,\mathbf{E}^{\mathbf{Q}}\left[e^{-z\tau_{\nu,a}}\,\middle|\,R_a^{(\nu)}=x\right]p_{\nu,a}(1,x)\,dx,$$

where $p_{\nu,a}$ is the time-a semigroup density of the Bessel process of index ν starting at 1 at time zero. Following [12, p. 362] we make this integral explicit by making the single factors of its integrand explicit. For this, work with the results recalled in [12, section 2]. With respect to the conditional expectation factor, under the hypothesis $\nu \geqq 0$, Yor has computed it at positive real arguments z in [27, Théorème 4.7, p. 80] (see also [12, Lemma 2.1 and Proposition 2.6]). Using analytic continuation, the validity of his result can be seen to extend to the arguments z in the right half-plane required in the present situation. This then gives for the conditional expectation factor in $D_\nu(a,z)$ the following expression as a quotient of I-Bessel functions:

$$\mathbf{E}^{\mathbf{Q}}\big[e^{-z\,\tau_{\nu,a}}\,\big|\,R^{(\nu)}(a)=w\big]=\frac{I_{\sqrt{2z+\nu^2}}}{I_\nu}\left(\frac{w}{a}\right).$$

Explicit expressions for the Bessel semigroups $p_{\nu,a}$ have been known for $\nu>-1$ for some time; see [27, (4.3), p. 78] or [12, Proposition 2.2]. The density $p_{\nu,a}(1,w)$ of the time-a Bessel semigroup with index ν and starting point 1 is

$$p_{\nu,a}(1,w)=\frac{w^{\nu+1}}{a}\,e^{-(1+w^2)/(2a)}\,I_\nu\left(\frac{w}{a}\right).$$

Nothing seemed to have been known about such densities if $\nu\leqq-1$ before [30]. However, the results proved there for $\nu<-1$ still need to be handled with care, as will be explained in section 14. The upshot is that, in accordance with [12, section 2], the above decomposition of $D_\nu(a,z)$ seems to give explicit results without further qualifications only if $\nu\geqq 0$. Then, however, we have the required result

$$D_\nu(a,z)=\frac{e^{-1/(2a)}}{a}\int_0^\infty e^{-x^2/(2a)}\,x^{\nu+3}\,I_{\sqrt{2z+\nu^2}}\left(\frac{x}{a}\right)dx.$$

There is a further technical point to be taken care of herein: choose the principal branch of the logarithm to define the square root on the complex plane with the nonpositive real line deleted.

To complete the Tonelli argument proposed to us by Yor and to complete the proof, we have to establish the finiteness of this integral for any fixed complex number z with $\mathrm{Re}(z)>2(\nu+1)$. This is a consequence of the convergence analysis of these integrals by the proposition in section 11, and granting this result, the proof of the lemma is complete.

11. Integrability analysis.

In terms of the concepts of section 3, this section studies finiteness of the integrals

$$D_\nu(a,z)=\frac{e^{-1/(2a)}}{a}\int_0^\infty e^{-x^2/(2a)}x^{\nu+3}\,I_{\sqrt{2z+\nu^2}}\left(\frac{x}{a}\right)dx$$

for any real $a>0$ in terms of their complex parameters ν and z. The precise result is the following proposition.

Proposition. *Let $\varepsilon\geqq 0$ be any real number. If $|\,\mathrm{Im}(\nu)|\leqq\varepsilon$, the integrals $D_\nu(a,z)$ are finite for any complex z with real part $\mathrm{Re}(z)>2\varepsilon^2$.*

The proposition depends on the following result about the complex square root associated to the principal branch of the complex logarithm.

Lemma. *Let $\varepsilon\geqq 0$ be any real number. For any complex ν with $|\,\mathrm{Im}(\nu)|\leqq\varepsilon$ we have $\mathrm{Re}(\sqrt{2z+\nu^2})>|\,\mathrm{Re}(\nu)|$ for any complex z with $\mathrm{Re}(z)>2\varepsilon^2$.*

A proof of the lemma based on a close analysis of the square root can be found in [2, section 10]. To prove the proposition, finiteness of $D_\nu(a,z)$ under the conditions of the proposition follows by combining the above square root lemma with the asymptotic behavior of the Bessel function factor of its integrand near the origin and towards infinity. Indeed, setting

$$\mu=\sqrt{2z+\nu^2},$$

from [15, section 5.11] recall that I_μ is a continuous function on the positive real line, in particular whose asymptotic behavior for large real arguments is

$$I_\mu(\xi) \approx \frac{e^\xi}{\sqrt{2\pi\xi}} \qquad \text{as} \quad \xi \to \infty.$$

Hence the factor $\exp(-x^2/(2a))$ dominates the asymptotic behavior of the integrand of $D_\nu(a,z)$ with x to infinity, whence we obtain its integrability away from the origin. On the other hand, from [15, section 5.7] we have for real arguments near zero

$$I_\mu(\xi) \approx \frac{\xi^\mu}{2^\mu \Gamma(1+\mu)} \qquad \text{as} \quad \xi \downarrow 0.$$

Thus, if the real part of $\mu + \nu + 4$ is positive, or equivalently, if we have

$$\operatorname{Re}(\mu) > -(\operatorname{Re}(\nu) + 4),$$

no integrability problems arise for x near the origin. Under the conditions of the proposition, on the other hand, the above lemma gives $\operatorname{Re}(\mu) > |\operatorname{Re}(\nu)|$. Since this last inequality implies the former, the proof of the proposition is complete.

Part IV. Laplace Transforms in the General Case: Using Girsanov Transforms of Bessel Processes

12. Further preliminaries on Bessel processes. Our first way of extending the results of section 10 to negative indices ν requires further preliminaries on Bessel processes from [29, section 2]. Recall that Bessel processes, where the index ν is any real number, are the real-valued diffusions whose infinitesimal generators $\mathcal{A}$ are given by

$$\mathcal{A}f(x) = \frac{1}{2} f''(x) + \frac{2\nu+1}{2x} f'(x)$$

for any function f in $C_b^2(\mathbf{R}_{>0})$. To describe the law $P_{\mu,u}$ on $C(\mathbf{R}_{\geqq 0}, \mathbf{R}_{\geqq 0})$ of $R^{(\nu)}$ if this process begins at the nonnegative real u, let ρ be the canonical process on the space $C(\mathbf{R}_{\geqq 0}, \mathbf{R}_{\geqq 0})$; recall it operates as an evaluation map: $\rho_a(f) = f(a)$. If $\mathcal{R}$ is the canonical filtration with $\mathcal{R}_a$ equal to the sigma algebra generated by the ρ_s with $s \leqq a$, we then have the mutual absolute continuity relation.

Lemma. *If the Bessel process of any real index ν is begun at any nonnegative real u, its law is related to that of the zero index Bessel process started at u as follows*:

$$P_{\mu,u|\,\mathcal{R}_a \cap \{a<T_0\}} = \left(\frac{\rho_a}{u}\right)^\mu \exp\left(-\frac{\mu^2}{2}\int_0^a \frac{ds}{\rho_s^2}\right) P_{0,u\,|\mathcal{R}_a},$$

where T_0 is the first passage time of ρ to zero.

This is proved as an application of Girsanov's theorem by exchanging drifts in the stochastic differential equation of section 9, and it has the following corollary.

Corollary. *For any complex z with positive real part and any nonnegative real r,*

$$\mathbf{E}_u^0\left[\exp\left(-z\int_0^a \frac{ds}{\rho_s^2}\right)\Big|\,\rho_a = r\right] = \frac{I_{\sqrt{2z}}}{I_0}\left(\frac{ur}{a}\right),$$

where the expectation is taken with respect to the law $P_{0,u}$.

The corollary is proved in two steps. If z is any nonnegative real, $T_0 = \infty$, and we have the explicit expressions for the densities $p_{\mu,a}(u,r)$ of the Bessel semigroups already encountered in section 10 and related to the law via $P_{\mu,u}(a,dr) = p_{\mu,a}(u,r)\,dr$, i.e.,

$$p_{\mu,a}(u,r) = \left(\frac{r}{u}\right)^{\mu} \frac{r}{a} \exp\left(-\frac{1}{2a}\,(u^2+r^2)\right) I_\mu\left(\frac{ur}{a}\right),$$

then for any nonnegative reals u, $a > 0$ and r. Thus the corollary follows on taking expectations in the absolute continuity relation of the lemma. Observing that both sides of the identity to be proved are analytic functions in z on the right half-plane, the general case then follows by analytic continuation as a second step.

13. First proof of the Laplace transform using Bessel processes. Resuming the discussion of section 10, we are now able to complete the proof of section 8's theorem in the way it might have been envisaged by Yor: based on a careful analysis of Bessel processes. Recall that we still need to explicitly compute, for negative normalized risk-neutral drifts ν, the risk-neutral expectations

$$\mathbf{E}^{\mathbf{Q}}\left[e^{-z\tau_{\nu,a}}\left(R_a^{(\nu)}\right)^2\right],$$

where $a > 0$ and $\mathrm{Re}(z)$ is positive and sufficiently large in particular. From the time change part of the argument in section 9, recall that $\tau_{\nu,a}$ as the inverse time change of the process $A^{(\nu)}$ at time a is given by

$$\tau_{\nu,a} = \int_0^a \frac{ds}{(R_s^{(\nu)})^2}.$$

Again using section 9's Lamperti relation

$$e^{B_w+\nu w} = R^{(\nu)}\left(A_w^{(\nu)}\right),$$

which is valid for $w \geqq 0$, the point now is that a is smaller than the first passage time to zero T_0 of the Bessel process $R^{(\nu)}$. Put differently, this Bessel process, which starts in 1 at time zero, still is positive at time a. Using the absolute continuity relation of section 12's lemma, we have

$$\mathbf{E}^{\mathbf{Q}}\left[\left(R_a^{(\nu)}\right)^2 e^{-z\tau_{\nu,a}}\right] = \mathbf{E}_1^0\left[\rho_a^2 \exp\left\{-z\int_0^a \frac{ds}{\rho_s^2}\right\}\rho_a^\nu \exp\left\{-\frac{\nu^2}{2}\int_0^a \frac{ds}{\rho_s^2}\right\}\right].$$

Conditioning on the index-0 Bessel process thus gives

$$\begin{aligned}\mathbf{E}^{\mathbf{Q}}&\left[e^{-z\tau_{\nu,a}}\left(R_a^{(\nu)}\right)^2\right]\\ &= \int_0^\infty \mathbf{E}_1^0\left[\exp\left\{-\left(z+\frac{\nu^2}{2}\right)\int_0^a \frac{ds}{\rho_s^2}\right\}\,\middle|\,\rho_a=\rho\right]\rho^{\nu+2}\,p_{0,a}(1,\rho)\,d\rho.\end{aligned}$$

With the explicit form of the semigroup recalled in section 12, applying section 12's corollary gives

$$\mathbf{E}^{\mathbf{Q}}\left[e^{-z\tau_{\nu,a}}\left(R_a^{(\nu)}\right)^2\right] = \frac{1}{a}\,e^{-1/(2a)}\int_0^\infty I_{\sqrt{2z+\nu^2}}\left(\frac{\rho}{a}\right)\rho^{\nu+3}\,e^{-\rho^2/(2a)}\,d\rho$$

as desired. An application of section 11's proposition shows finiteness of this integral if $z_0 = \mathrm{Re}(z)$ is positive and larger than $2(\nu+1)$, and the first proof of section 8's theorem is complete. Note that meanwhile the argument of [32, pp. 97–99] seems to corroborate our statement that this proof is very much in the spirit of Yor.

14. Vista on the use of Bessel processes. The strategy of the preceding argument is to bypass the difficulties brought about by Bessel processes with negative indices by Girsanov transforming to Bessel processes of index zero. One could ask about a direct attack in the spirit of section 10. This would require possession of explicit expressions for both the densities of the respective Bessel semigroups and the pertinent conditional expectations of $\exp(-z\tau_{\nu,a})$. The recent work of Göing-Jaeschke and Yor in [30] in particular now provides certain analytic expressions for the densities. However, it is not Bessel processes, as considered in section 12, which are studied there, but rather processes obtained as strong solutions to the stochastic differential equations of squared Bessel processes, as mentioned in section 9. Recall that for negative indices, the latter processes become negative and their square roots, which should give the Bessel processes, are then purely imaginary. Still, the two notions of Bessel processes thus obtained coincide on their respective positive ranges. There, we have Bessel semigroup densities based on those derived in [30, section 3, Proposition 2, p. 21]. For indices $\nu < -1$ and time-0 starting values $y > 0$, they are given by

$$p_{\nu,t}(x,y) = h(x,y,\delta,t)\, \exp\left(\frac{y-x}{2t}\right) \int_0^1 \frac{(1-w)^{2(\mu-1)}}{w^{\mu}}\, \exp\left(\frac{1}{2t}\left(xw - \frac{y}{w}\right)\right) dw,$$

defining

$$h(x,y,\delta,t) = \frac{1}{\Gamma^2(\mu-1)}\, \frac{(xy)^{(\mu-1)}}{2^{2-\delta}(2-\delta)}\, t^{\delta-3},$$

and with $\delta = 2(1+\nu) < 0$ and $\mu = 1-\nu$. However, these densities are in terms of new classes of special functions. The clarification of their relations to those for Bessel processes of nonnegative indices is but one of the problems that require further study.

Part V. Laplace Transforms in the General Case: Combining Stochastics and Complex Analysis

15. Remarks on general philosophy. The discussion up to now has focused in particular on extension at the process level. The actual valuation problem, however, is not at the process level but at the expectation level. From this point of view, section 10 has identified two functions, f and g, in the variable ν and has proved them to be equal for ν nonnegative. One would like to have this equality also for negative ν, and thus extend the validity of the identity $f = g$ from the nonnegative real line to the whole real line. Such situations quite commonly appear in problems in analysis and are addressed there using analytic continuation. However, functions become amenable to complex analytic methods only on open subsets of the complex plane. Thus the *identity theorem* of complex analysis asserts that two functions on a connected open subset of the complex plane are equal if they are analytic and agree on a convergent sequence there only. So it is in fact no longer possible to stick to real numbers only. As a subset of the complex plane they are closed with an empty interior. At this stage, however, nothing is known about the functions of section 10 if ν is outside the nonnegative real line. In particular, it is not known if they exist at all, and this needs to be established together with their analyticity properties. Of these two functions, $D_\nu(a,z)$ is already given as an explicit analytic expression, while the other function is not. In fact, it is not explicit at all, as it is defined as the Laplace transform of a certain expectation. The question to be tackled here is then how to get explicit analyticity properties from such nonexplicit stochastic concepts. In the present situation, we attack this in two stages. First, establish analyticity properties of the expectation. Then, in a second step, study how these are preserved on taking Laplace transforms. As it turns out, using standard results from complex analysis, we find that it is not Bessel processes which enter into the analysis but simply Brownian motion. All of this may be regarded as an instance of why enhancing stochastics by complex analytic methods seems quite an interesting and promising line of thought.

16. First step: Analyticity of the function $D_\nu(a,z)$. As an initial step in the analytic continuation argument, this section studies the analytic properties of the generalized first Weber integral $D_\nu(a,z)$ of section 8's theorem. On choosing the square root associated to the principal branch of the logarithm, recall that

$$D_\nu(a,z) = \frac{e^{-1/(2a)}}{a} \int_0^\infty e^{-x^2/(2a)}\, x^{\nu+3}\, I_{\sqrt{2z+\nu^2}}\left(\frac{x}{a}\right) dx$$

is finite for any positive real number a and for any complex numbers z and ν such that the real part of $\nu + 4 + (2z+\nu^2)^{1/2}$ is positive as a consequence of the integrability analysis of section 11's proposition. Using the confluent hypergeometric function Φ discussed in [15, section 9.9], the precise analyticity result to be proved is the following.

PROPOSITION. *Let a be any positive real and ε any nonnegative real. For any complex number ν with $|\operatorname{Im}(\nu)| \leqq \varepsilon$, we have*

$$D_\nu(a,z) = \Gamma\left(\frac{\nu+4+\mu}{2}\right) \frac{1}{\Gamma(\mu+1)} \times \Phi\left(\frac{\nu+4+\mu}{2},\ \mu+1;\ \frac{1}{2a}\right) e^{-1/(2a)}\, (2a)^{(\nu+2-\mu)/2}$$

if z is any complex with $\operatorname{Re}(z) > 2\varepsilon^2$ setting $\mu = \sqrt{2z+\nu^2}$.

COROLLARY. *Let a be any positive real and let ε be any nonnegative real. For any complex number z with $\operatorname{Re}(z) > 2\varepsilon^2$, sending ν to $D_\nu(a,z)$ defines an analytic map on the set of all complex numbers ν with $|\operatorname{Im}(\nu)| \leqq \varepsilon$.*

Both results are based on the proposition of section 11, which gives finiteness of $D_\nu(a,z)$ under their conditions on ν, a, and z. Combining this with the analyticity properties of Φ discussed in [15, section 9.9] and those of the Gamma function, the corollary follows from the proposition. The proof of the proposition then reduces to explicitly computing $D_\nu(a,z)$. For this we modify the quite typical discussion in [25, section 13.3, p. 393f] of Hankel's generalization of Weber's first integral. The idea is to expand the modified Bessel function in the integrand of

$$I = \int_0^\infty e^{-x^2/(2a)}\, x^{\nu+3}\, I_\mu\left(\frac{x}{a}\right) dx$$

into its series of [15, section 5.7] and integrate term by term. Using [15, section 9.9] this is justified by the absolute convergence of the series for the resulting confluent hypergeometric series, and we get

$$I = \frac{1}{(2a)^\mu} \sum_{n=0}^\infty \frac{1}{\Gamma(\mu+1+n)} \frac{(2a)^{-2n}}{n!} \int_0^\infty e^{-x^2/(2a)} x^{\nu+3+\mu+2n}\, dx.$$

Changing variables $y = (2a)^{-1}x^2$, compute any nth integral as

$$\int_0^\infty e^{-x^2/(2a)} x^{\nu+3+\mu+2n}\, dx = \frac{1}{2}\,\Gamma\left(\frac{\nu+\mu+4}{2}+n\right) (2a)^{(\nu+\mu+4)/2+n}.$$

Extracting the series of the pertinent confluent hypergeometric function, we thus get

$$I = \frac{1}{2}\,\frac{\Gamma((\nu+4+\mu)/2)}{\Gamma(\mu+1)}\,\Phi\left(\frac{\nu+4+\mu}{2},\ \mu+1;\ \frac{1}{2a}\right)(2a)^{(\nu-\mu+4)/2}.$$

Multiplying this expression by $\exp(-(2a)^{-1})/a$, the proposition follows.

17. Second step: Analyticity of the functions $f_{GY,a}$. Establishing analyticity results in ν about the Laplace transform of the expectation defining non-Asian option prices combines insights from stochastics with insights of an analytic nature. We establish this result in two steps. First, in this section we consider any of the auxiliary functions $f_{GY,a}$ of section 7 as a function in the variable ν,

$$L(x,\nu) = \mathbf{E}^{\mathbf{Q}}\big[\big(A_x^{(\nu)} - a\big)^+\big],$$

for any fixed positive real numbers a and x. Using section 10's lemma, we know it is defined for nonnegative real numbers ν. However, this has been achieved in a very indirect way only: for these values of ν the Laplace transforms of the corresponding functions $f_{GY,a}$ have been shown to be finite. Now more is true indeed and we have the following statement.

LEMMA. *For any $x > 0$, the function $\nu \mapsto L(x,\nu)$ extends to a function on the complex plane, which is analytic at each point, and for which we have the majorization*

$$|L(x,\nu)| \leqq e^{x\ \mathrm{Im}^2(\nu)/2}\,\mathbf{E}^{\mathbf{Q}}[A_x^{(\mathrm{Re}(\nu))}].$$

For the proof of the lemma we now set $f(w) = (w-a)^+$. Applying Girsanov's theorem such that $W_x = \nu x + B_x$ becomes a standard Brownian motion, and dropping the reference to this new measure, we get

$$L(x,\nu) = \mathbf{E}\left[f\big(A_x^{(0)}\big)\,e^{\nu W_x}\right]\,e^{-x\nu^2/2}.$$

For establishing the analyticity statement of the lemma, it is thus sufficient to show that the expectation factor is analytic in any complex number ν. This is true by definition if we have the convergent series

$$\mathbf{E}\Big[f\big(A_x^{(0)}\big)\,e^{\nu W_x}\Big] = \sum_{m=0}^{\infty}\frac{\nu^m}{m!}\,\mathbf{E}\Big[f\big(A_x^{(0)}\big)\,W_x^m\Big]$$

for all ν. For this it is sufficient to show that the series is absolutely convergent for all ν. Using the Cauchy–Schwarz inequality, this is implied by the convergence of

$$\sum_{m=0}^{\infty}\frac{|\nu|^m}{m!}\,\sqrt{\mathbf{E}[f^2(A_x^{(0)})]}\sqrt{\mathbf{E}\big[W_x^{2m}\big]}$$

for all ν. Herein $\mathbf{E}[f^2(A_x^{(0)})]$ is majorized by the second moment of Yor's zero drift process $A^{(0)}$ at x and so is finite from [29, section 4]. Since the factors $\mathbf{E}[W_x^{2m}]$ are majorized by $\pi^{-1/2}\,(2x)^m\,m!$ for all $m \geqq 0$, convergence follows using the ratio test. Actually, we have established yet another upper bound to the price of the Asian option.

To establish the majorization of the lemma, taking absolute values inside the expectation in the above Girsanov representation of $L(x,\nu)$ gives

$$|L(x,\nu)| \leqq \mathbf{E}\Big[f\big(A_x^{(0)}\big)\,\big|e^{\nu W_x}\big|\Big]\,\big|e^{-x\nu^2/2}\big|.$$

The absolute value of the exponential factors are the exponentials of the real parts of the respective arguments. Majorizing the function in Yor's zero drift process by this process itself, we get

$$|L(x,\nu)| \leqq e^{x\,\mathrm{Im}^2(\nu)/2}\,\mathbf{E}\Big[A_x^{(0)} e^{\mathrm{Re}(\nu)W_x}\Big]e^{-x\,\mathrm{Re}^2(\nu)/2}.$$

Reversing the Girsanov transformation completes the proof of the lemma.

18. Third step: Analyticity of the transforms $F_{GY,a}$. While in the preceding section we studied the expectations

$$L(x,\nu) = \mathbf{E}^{\mathbf{Q}}\big[\big(A_x^{(\nu)} - a\big)^+\big]$$

for any fixed positive real numbers a and x as functions in the complex variable ν only, in this section we treat x as a variable. If ν is any nonnegative real number, we have from section 10's lemma that the integrals of the Laplace transforms

$$F(\nu)(z) = \int_0^\infty e^{-zx} L(x,\nu)\,dx$$

are finite if $\mathrm{Re}(z) > 2(\nu+1)$. In this section we give an independent proof of the following more general statement.

PROPOSITION. *For any complex number z with a positive real part, the map sending ν to $F(\nu)(z)$ is analytic in all complex numbers ν with real parts* $\mathrm{Re}(z) > \frac{1}{2}\,\mathrm{Im}^2(\nu) + 2(\mathrm{Re}(\nu)+1)$.

The proof of the proposition is based on the following lemma.

LEMMA. *For any complex ν, the Laplace transform $F(\nu)(z)$ is finite for any complex z with* $\mathrm{Re}(z) > \max\{0, \frac{1}{2}\,\mathrm{Im}^2(\nu) + 2(\mathrm{Re}(\nu)+1)\}$.

Proof of the lemma. As the first step in proving the lemma, we establish for any complex number ν the majorization

$$|F(\nu)(z)| \leqq \int_0^\infty \exp\left\{-\left(\mathrm{Re}(z) - \frac{1}{2}\,\mathrm{Im}^2(\nu)\right)x\right\}\mathbf{E}^{\mathbf{Q}}\left[A_x^{(\mathrm{Re}(\nu))}\right]dx$$

for any complex number z in the sense of measurable functions. Indeed, majorize the absolute value of $F(\nu)(z)$ by taking the absolute value inside the defining integral. The absolute value of the exponential factor then is equal to $\exp(-\mathrm{Re}(z)x)$. Majorizing the absolute value of $L(x,\nu)$ using section 17's lemma, the estimate follows.

Setting $\nu_0 = \mathrm{Re}(\nu)$, now let $\mathrm{Re}(z)$ be positive and such that $\xi_0 = \mathrm{Re}(z) - \mathrm{Im}^2(\nu)/2$ is larger than $2(\nu_0+1)$. We compute the Laplace transform of the right-hand side of the above inequality using section 4's lemma. If ν_0 is different from minus one, then

$$\int_0^\infty e^{-\xi_0 x}\mathbf{E}^{\mathbf{Q}}[A_x^{(\nu_0)}]\,dx = \frac{1}{\xi_0(\xi_0 - 2(\nu_0+1))},$$

using that ξ_0 is larger than $2(\nu_0+1)$ to compute the improper integrals. It converges to ξ_0^{-2} with ν_0 going to minus one. Thus it is seen to coincide with the Laplace transform for the case $\nu_0 = -1$.

The Laplace transforms are finite if ξ_0 is positive and larger than $2(\nu_0+1)$, and then $F(\nu)(z)$ is finite a fortiori. The proof of the lemma is complete.

Proof of the proposition. To prove the proposition, fix any complex z with a positive real part and then choose any complex ν_0 satisfying the resulting inequality of the proposition. In fact, the validity of this inequality then extends to all ν in a compact neighborhood V of ν_0. As a consequence of section 17's lemma, $f_z(x,\nu) = \exp(-zx)L(x,\nu)$ is, for any $x>0$, analytic in ν on V in particular. Since V is compact, the argument of the lemma, moreover, shows that its absolute value is majorized by an integrable function g on the real line. The function $\nu \mapsto F(\nu)(z)$ is thus continuous on V being obtained by integration of f_z over the positive real line. To show that it is analytic on the interior of V, we want to apply Morera's theorem [18, 10.17, p. 208] and to show that

$$\int_{\partial\Delta} F(\nu)(z)\,d\nu = 0$$

for any triangle Δ in the interior of V. Indeed, since we have shown f_z to be integrable, applying Fubini's theorem gives

$$\int_{\partial\Delta} F(\nu)(z)\,d\nu = \int_0^\infty \int_{\partial\Delta} f_z(x,\nu)\,d\nu\,dx.$$

Recalling that $\nu \mapsto f_z(x,\nu)$ is analytic from section 17's lemma, the inner integral herein is zero by *Cauchy's theorem* for Δ, see [18, 10.13, p. 205]. Thus the whole double integral is zero as shown. This completes the proof of the proposition.

19. Final step: Second proof of the Laplace transform using analytic extension. It remains to pull things together and show how the results of Part V give a second proof of the two results of section 8 by using analytic extension.

First notice that section 8's corollary is implied by section 8's theorem using the computation of $D_\nu(a,z)$ in section 16's proposition. Thus we are reduced to proving section 8's theorem.

The proof of section 8's theorem is by analytic continuation using the identity theorem, see [18, Corollary to 10.18, p. 209]. As a consequence of section 10's lemma it remains to establish the identity in section 8's theorem for negative indices ν only. It will turn out that the existence of the Laplace transform on all complex numbers z with positive real part required in the theorem then has essentially been proved in section 18's proposition.

For establishing the crucial Laplace transform identity of section 8's theorem, first let z be any complex number with real part $\mathrm{Re}(z)>4$ and choose $0<\varepsilon<1$. Using section 16's corollary, the generalized first Weber integral

$$D_\nu(a,z) = \frac{e^{-1/(2a)}}{a}\int_0^\infty e^{-x^2/(2a)}x^{\nu+3}\,I_{\sqrt{2z+\nu^2}}\left(\frac{x}{a}\right)dx$$

is analytic in ν on the ε-thickened real line A_ε, which consists of all complex numbers ν with $|\,\mathrm{Im}(\nu)|<\varepsilon$. Picture A_ε as the band of height 2ε symmetric with respect to the real axis. If we choose ε so small that $\mathrm{Re}(z)>2(2+2\varepsilon)$, we claim to have analyticity of the Laplace transform

$$F(\nu)(z) = \int_0^\infty e^{-zx}L(x,\nu)\,dx$$

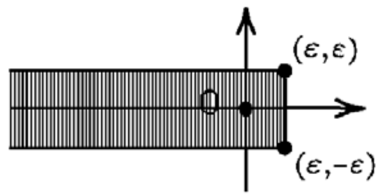

FIG. 1. *The ε-thickened half-line B_ε.*

as a function in ν on the ε-thickened half-line B_ε, which is the subset of A_ε consisting of all complex numbers ν with $|\operatorname{Im}(\nu)| < \varepsilon$ and $\operatorname{Re}(\nu) < \varepsilon$. Indeed, if $\operatorname{Re}(\nu) < \varepsilon$, we have

$$2(2 + 2\varepsilon) > 2\varepsilon + 2(\operatorname{Re}(\nu) + 2).$$

If $|\operatorname{Im}(\nu)| < \varepsilon$, we have $2\varepsilon > \operatorname{Im}^2(\nu)/2$ since $\varepsilon < 1$. Any ν in B_ε satisfies the inequality $\operatorname{Re}(z) > \operatorname{Im}^2(\nu)/2 + 2(\operatorname{Re}(z) + 2)$ of section 18's proposition, and the claim follows.

The functions $F(\nu)(z)$ and $D^*_\nu(a, z) = D_\nu(a, z)(z(z - 2(\nu + 1)))^{-1}$ are analytic as functions in ν on the ε-thickened half-line B_ε. It is now a consequence of section 10's lemma that we have

$$F(\nu)(z) = D^*_\nu(a, z) \qquad \text{for all} \quad \nu \geqq 0$$

such that $2(\nu + 1) < \operatorname{Re}(z)$. With $\operatorname{Re}(z) > 4$ this holds a fortiori for all $\nu \geqq 0$ such that $2(\nu + 1) < 4$, i.e., for all nonnegative real numbers ν smaller than 1. With ε smaller than 1 we have $F(\nu)(z) = D^*_\nu(a, z)$ for any ν in the subinterval $(0, \varepsilon)$ of B_ε. With B_ε open and connected, the identity theorem literally applies to give that $F(\nu)(z) = D^*_\nu(a, z)$ for all ν in B_ε. This identity then holds a fortiori for all real numbers ν in B_ε, i.e., for all $\nu < \varepsilon$. This gives section 8's theorem for any z with $\operatorname{Re}(z) > 4$.

To lift this last restriction on $\operatorname{Re}(z)$, notice that section 18's lemma implies $F(\nu)$ for any fixed real ν to be analytic on the half-plane $\{\operatorname{Re}(z) > 2(\nu + 1)\}$; this is seen by using a Morera-type argument, as used for proving section 18's proposition. On the other hand, applying section 16's corollary, $D^*_\nu(a, z)$ as a function of z is analytic on the intersection of this last half-plane with the right half-plane. The validity of the identity $F(\nu)(z) = D^*_\nu(a, z)$ thus can be analytically continued from complex numbers z with $\operatorname{Re}(z) > 4$ to complex numbers z with $\operatorname{Re}(z)$ positive and larger than $2(\nu + 1)$, and the second proof of section 8's theorem is complete.

Part VI. Laplace Transforms in the General Case: A Uniform Proof

20. Remarks on general philosophy. The question may arise of whether or not there is a uniform way to compute the Laplace transforms in section 8's theorem without requiring a two-step procedure. The Lamperti factorization

$$e^{B_w + \nu w} = R^{(\nu)}\big(A^{(\nu)}(w)\big)$$

of section 9's lemma may in fact offer a clue. The idea of the two arguments discussed up to now is to focus on the Bessel process side of this identity. It is the problems with Bessel processes that transcribe into their two-step approach. A possible remedy would be to bring in Bessel processes not at the earliest stage but as late as possible. The Lamperti factorization suggests focusing on the geometric Brownian motion of its left-hand side. While Yor has already made extensive use of this device, we explain in what follows what seems to be a most natural adaptation of the Girsanov technique to the Lamperti factorization approach for explicitly computing the Laplace transforms $F_{GY,a}$ with $a > 0$.

21. Third proof of the theorem: A uniform argument. This section sketches a uniform way to compute the Laplace transforms $F_{GY,a}$, with a any positive real, and thus provides a third proof of section 8's theorem. Recall that these transforms are defined by

$$F_{GY,a}(z) = \int_0^\infty e^{-zx} f_{GY,a}(x)\, dx$$

for any complex z, with $\mathrm{Re}(z)$ sufficiently large, and where $f_{GY,a}(x) = \mathbf{E}^{\mathbf{Q}}[f(A^{(\nu)}(x))]$ setting $f(x) = (x-a)^+$. The key idea is to apply Girsanov's theorem such that $W_x = \nu x + B_x$ becomes a Brownian motion and, by suppressing reference to this new measure, to transcribe $f_{GY,a}$ in terms of Yor's zero drift process $A^{(0)}$ as follows:

$$f_{GY,a}(x) = \mathbf{E}\left[f\big(A^{(0)}_x\big)\, e^{-\nu^2 x/2 + \nu W_x} \right].$$

With $\tau_{0,a} = \inf\{u \,|\, A^{(0)}(u) > 0\}$ being the inverse time change to $A^{(0)}$ at time zero, the computations of section 10 thus prove $f_{GY,a}(x) = 0$ on the set of all events where $x \leqq \tau_{0,a}$. Also on the set of all events, where $x \geqq \tau_{0,a}$ we now have the representation

$$f(A^{(0)}) = \big(R^{(0)}_a\big)^2 A^{(0)}_{x-\tau_{0,a}},$$

which applies to express $f_{GY,a}(x)$ as the following iterated expectation:

$$f_{GY,a}(x) = \mathbf{E}\left[\big(R^{(0)}_a\big)^2\, \mathbf{E}\left[A^{(0)}_{x-\tau_{0,a}} e^{-\nu^2 x/2 + \nu W_x} \,\Big|\, \mathcal{F}_{\tau_{0,a}} \right]\right].$$

Applying Laplace transforms to both sides of this identity then gives

$$F_{GY,a}(z) = \mathbf{E}\bigg[\big(R^{(0)}_a\big)^2 e^{-(z+\nu^2/2)\,\tau_{0,a}} \mathbf{E}\bigg[\int_0^\infty e^{-zx} A^{(0)}_x\, e^{-\nu^2 x/2 + \nu W_{x+\tau_{0,a}}}\, dx \,\Big|\, \mathcal{F}_{\tau_{0,a}} \bigg]\bigg].$$

In the conditional expectation, restart the Brownian motion at time $\tau_{0,a}$. To the resulting additional νth power of $\exp(W_{\tau_{0,a}})$, apply the corresponding Lamperti factorization and interpret it as being equal to the νth power of the zero index Bessel process at time a. Collecting powers of this last process, we have

$$F_{GY,a}(z) = \mathbf{E}\bigg[\big(R^{(0)}_a\big)^{\nu+2} e^{-(z+\nu^2/2)\,\tau_{0,a}} \mathbf{E}\bigg[\int_0^\infty e^{-zx} A^{(0)}_x\, e^{-\nu^2 x/2 + \nu W_x}\, dx \,\Big|\, \mathcal{F}_{\tau_{0,a}} \bigg]\bigg].$$

The strong Markov condition used here also makes the whole integrand in the conditional expectation independent of time-$\tau_{0,a}$ information. Reversing the Girsanov transformation in this inner expectation, we have

$$F_{GY,a}(z) = \mathbf{E}\bigg[\big(R^{(0)}_a\big)^{\nu+2} e^{-(z+\nu^2/2)\,\tau_{0,a}} \mathbf{E}\bigg[\int_0^\infty e^{-(z+\nu^2/2)\,x} A^{(\nu)}_x\, dx \bigg]\bigg].$$

This puts us into the situation of section 10. Partially reversing the Tonelli argument, the inner expectation is equal to the first moment of the drift-ν process $A^{(\nu)}$ at time x recalled in section 4's lemma. Computing the Laplace transform, we thus arrive at

$$F_{GY,a}(z) = \frac{1}{2(\nu+1)} \left(\frac{1}{1-2(\nu+1)} - \frac{1}{z} \right) \mathbf{E}\left[\big(R^{(0)}_a\big)^{\nu+2} e^{-(z+\nu^2/2)\,\tau_{0,a}} \right]$$

if $\mathrm{Re}(z)$ is larger than at least $\max\{0, 2(\nu+1)\}$. In this way we are reduced to computing the expectation factor. This, however, proceeds as in section 10 by conditioning on the corresponding Bessel process. The main point is that this time conditioning is not on a Bessel process of an arbitrary index but on a Bessel process of index zero. Observing how ν enters now via the exponent of the index zero Bessel process and via a shifting factor for the time change $\tau_{0,a}$, we arrive at

$$\mathbf{E}\left[\left(R_a^{(0)}\right)^{\nu+2} e^{-(z+\nu^2/2)\,\tau_{0,a}}\right] = \frac{e^{-1/(2a)}}{a}\int_0^\infty e^{-x^2/(2a)}\,x^{\nu+3}\,I_{\sqrt{2z+\nu^2}}\left(\frac{x}{a}\right)dx$$

for any z with sufficiently large positive real part. Using section 11's proposition, this integral is finite if $\mathrm{Re}(z)$ is positive and larger than $2(\nu+1)$. So this third proof of section 8's theorem is complete.

Part VII. Epilogue

22. Consequences for Asian options: Hermite functions. Having persevered to this point, the reader may wonder about the nature and the quality of the implications of the mathematics developed up to now. Indeed, going back to the starting point of the journey, it seems that the Laplace transform approach makes possible significant improvements in understanding section 3's normalized prices $C^{(\nu)}(h,q)$ of Asian options themselves. These improvements are on a structural level and as such make possible advances on computing $C^{(\nu)}$ as a consequence. This is essentially by being able to establish new links of Asian option valuation with a well-studied class of special functions, i.e., the Hermite functions to be reviewed in this section.

Following [15, section 10.2ff], to which we refer for details, *Hermite functions* H_μ are analytic on the complex plane as functions of both their variable z and their degree μ. If the real part $\mathrm{Re}(\mu)$ of μ is larger than -1, they have the integral representation

$$H_\mu(z) = \frac{2^{\mu+1}}{\sqrt{\pi}}\,e^{z^2}\int_0^\infty e^{-x^2}x^\mu\,\cos\left(2zx - \frac{1}{2}\,\mu\pi\right)dx.$$

Thus they specialize to the μth Hermite polynomials if μ is any nonnegative integer, whence $H_0 = 1$, $H_1(z) = 2z$, $H_2(z) = 4z^2 - 2$, $H_3(z) = 8z^3 - 12z$, for example. If the real part of μ is negative, however, Hermite functions change their character. Then they have the integral representation

$$H_\mu(z) = \frac{1}{\Gamma(-\mu)}\int_0^\infty e^{-u^2-2zu}u^{-(\mu+1)}\,du$$

and specialize via $(2/\sqrt{\pi})H_{-1}(z) = \exp(z^2)\,\mathrm{Erfc}(z)$ to the complementary error function Erfc recalled as given by

$$\mathrm{Erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-\xi^2}\,d\xi.$$

For any complex μ, Hermite functions can be expressed in terms of the Kummer confluent hypergeometric function Φ by

$$H_\mu(z) = \frac{2^\mu\,\Gamma(1/2)}{\Gamma((1-\mu)/2)}\,\Phi\left(-\frac{\mu}{2},\ \frac{1}{2};\ z^2\right) + z\,\frac{2^\mu\,\Gamma(-1/2)}{\Gamma(-\mu/2)}\,\Phi\left(\frac{1-\mu}{2},\ \frac{3}{2};\ z^2\right)$$

for any complex z. From this representation one can see how Hermite functions are connected to the parabolic cylinder functions D_μ and to the Kummer confluent hypergeometric function of the second kind Ψ.

23. Consequences for Asian options: New integral representations. Hermite functions as recalled in the previous section appear naturally in the closed form solution for section 3's normalized price $C^{(\nu)}(h,q)$ of the Asian option we have developed in [21]. If the normalized strike price q is positive, it expresses this value as the sum of integral representations which have a product structure. They are obtained by integrating products of Hermite functions H_μ with weighted error functions as follows.

THEOREM. *If q is positive, the normalized price $C^{(\nu)}(h,q)$ of the Asian option is given by the following difference*:

$$C^{(\nu)}(h,q) = ce^{2h(\nu+1)}\, S_{\nu+2} - c\, S_\nu,$$

where the S_ξ are three-term sums

$$S_\xi = C_{\mathrm{trig},\xi}(\rho_\xi) + C_{\mathrm{hyp},\,\xi}(\rho_\xi) + C_{\mathrm{hyp},\,-\xi}\,(\rho_\xi)$$

whose single summands are integrals that depend on parameters $\rho_\xi \geqq 0$, but which as a whole are independent of these.

In terms of section 3's concepts, c is given by

$$c = c(\nu,q) = \frac{\Gamma(\nu+4)\,(2q)^{(\nu+2)/2}}{2\pi\,(\nu+1)\,e^{1/(2q)}},$$

recalling $\nu = 2\varpi/\sigma^2 - 1$. With ρ any nonnegative real, the *trigonometric terms* $C_{\mathrm{trig},\xi}(\rho)$ with ξ equal to ν or $\nu+2$ are the integrals

$$C_{\mathrm{trig},\xi}(\rho) = \int_0^{\pi/2} \mathrm{Re}\left(H_{-(\nu+4)}\left(-\frac{\mathrm{ch}(\rho+i\phi)}{\sqrt{2q}}\right) E_\xi(h)(\rho+i\phi)\right) d\phi$$

over the real parts of products of Hermite functions times certain functions $E_b(h)$, and the *hyperbolic terms* $C_{\mathrm{hyp},\,\xi}(\rho)$ with ξ equal to $\pm\nu$ or $\pm(\nu+2)$ are the integrals

$$C_{\mathrm{hyp},\,\xi}(\rho) = \int_\rho^{\infty} \mathrm{Im}\left(H_{-(\nu+4)}\left(-\frac{\mathrm{sh}(y)}{\sqrt{2q}}\,i\right) E_\xi(h)\left(y+i\frac{\pi}{2}\right)\right) dy$$

over the imaginary parts of such products. Herein $E_\xi(h)$ are the weighted complementary error functions for any complex w given by

$$E_\xi(h)(w) = e^{w\xi}\,\mathrm{Erfc}\left(\frac{w}{\sqrt{2h}} + \frac{\xi}{2}\sqrt{2h}\right).$$

Remark. If ρ equals zero, the trigonometric terms specialize to

$$C_{\mathrm{trig},\xi}(0) = 2\int_0^{\pi/2} H_{-(\nu+4)}\left(-\frac{\cos(\phi)}{\sqrt{2q}}\right)\cos(\xi\phi)\,d\phi.$$

Compared to the formula of section 5, the above formula is given as a sum of single integrals whose integrands have a structural interpretation as products of two functions. It identifies the higher transcendental functions occurring as factors in these products and shows how they are given by or built up from Hermite functions.

24. Epilogue. On a technical level, the differences just noted between section 5's and section 23's formulas can be regarded as consequences of the different mathematical approaches for proving the valuation formula. In fact, section 5's formula originates from Yor's direct attack on the law of the integral of geometric Brownian motion. In contrast, section 23's is eventually based on the indirect enveloping construction of section 7. In fact, to obtain Asian option prices, first analytically invert section 8 theorem's Laplace transforms $F_{GY,a}$, then proceed using the key reduction of section 7's lemma; formally speaking, $C^{(\nu)}(h,q) = \mathcal{L}^{-1}(F_{GY,q})(h)$.

However, there are not only structural differences between section 5's and section 23's formulas. Finally, deriving benchmarks for the normalized prices $C^{(\nu)}(h,q)$ of Asian options seems to be one of the main practical application of such formulas. As we have already mentioned, Yor's formula seems impracticable for this purpose in particular because of the gigantic size of the numbers it involves. For instance, consider valuing Asian options with annual interest rates r equal to 9%, with maturities of one year, with $K = S_0$, with $t_0 = 0$, and with a volatility of $\sigma = 30\%$. These values require coping with numbers of order 10^{100}. Our formula improves on this aspect as well. In the above situation, for instance, the four hyperbolic terms have orders 10^7 and the two trigonometric terms have orders 10^{-2}. Sharpening results of [20], it thus became possible to derive in [23, Chap. 5] the following first time benchmark values for normalized Asian option prices:

TABLE 3. *Normalized prices $C^{(\nu)}(h,q)$ of the Asian option for $T = 1$.*

σ	Maximal error	$C^{(\nu)}(h,q)$
20%	4.9727×10^{-16}	0.00074155998788343
30%	4.9687×10^{-16}	0.00217354504625037
40%	4.9157×10^{-16}	0.00478100328341654
50%	4.9461×10^{-16}	0.00890942045213227

All of this may be seen as a consequence of the Laplace transform approach to valuing non-Asian options initiated in [12]. In retrospect, [12] appears to be a rich source for new results in both mathematics and finance.

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