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ABSTRACT   *The fair price of a financial option on an asset that follows a Poisson jump diffusion process satisfies a partial integro-differential equation. When numerical methods are used to solve such equations the integrals are usually evaluated using either quadrature methods or fast Fourier methods. Quadrature methods are expensive since the integrals must be evaluated at every point of the mesh. Though less so, Fourier methods are also computationally intensive since in order to avoid wrap around effects they require enlargement of the computational domain. They are also slow to converge when the parameters of the jump process are not smooth, and for efficiency require uniform meshes. We present a different and more efficient class of methods which are based on the fact that the integrals often satisfy differential equations. Depending on the process the asset follows, the equations are either ordinary differential equations or parabolic partial differential equations. Both types of equations can be accurately solved very rapidly. We discuss the methods and present results of numerical experiments.*

KEY WORDS: Jump diffusion process, option pricing, differential equations

1.   Introduction

The fact that pure diffusion models are insufficient for representing the risk due to large market movements is well known. In addition to the introduction of other possible approaches, these difficulties have led to the development of option pricing models containing jumps. That is, a variety of models known as jump diffusion models have been proposed in which large asset returns are represented as price discontinuities. In the earliest of these, due to Merton (1976), the process the asset follows is assumed to be the sum of a Wiener process and a jump process where the jumps are lognormally distributed with constant parameters. Whenever the Black–Scholes price for a European option is known there is an analytic formula for pricing the option in this model as well. Furthermore, by choosing the parameters of the jump process appropriately different volatility smiles and skews can be generated. Models where the process is a pure jump process have also been developed. For example, Madan *et al.* (1998) more recently proposed an infinite activity Variance Gamma model. Other models which combine the deterministic volatility function approach originally developed by Dupire (1994), Derman and Kani (1994) and Rubinstein (1994) with jump processes have also been used for pricing options (Andersen and Andreasen,

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2000). Kou (2002) has suggested a model with double exponential distribution for the log jump size. Although analytic formulas exist for pricing certain options in the Kou model (continuously monitored barriers, perpetual Americans), numerical methods are required for pricing the most common ones. Similarly, computational methods are needed for pricing most options in other jump diffusion models, especially when the options are path dependent.

The majority of numerical evaluation methods rely on the fact that the option prices satisfy partial integro-differential equations (PIDEs). That is, in addition to (possibly degenerate) differential operators, the pricing equations contain a non-local integral term. Various methods have been used to solve these equations. In particular, in an early paper, Amim (1993) used an explicit multinomial tree based approach. D'Halluin *et al.* (2004, 2005b) implemented implicit methods for evaluating vanilla European options, barrier options, and American options. They also showed that when a log spaced grid is used with a Crank Nicolson discretization on a problem with constant parameters the resulting scheme is unconditionally strictly stable. In addition, they showed that the simple Picard iteration scheme (also suggested by Tavella and Randall (2000)) for solving the discretized equations is globally convergent. Specifically, they reported that when they priced options in the Merton model the error was reduced by two orders of magnitude at each iteration for typical values of the time step size and Poisson arrival intensity. More recently, d'Halluin *et al.* (2005a) presented a semi-Lagrangian approach for pricing American Asian options under jump diffusion processes. Andersen and Andreasen (2000) derived a forward equation describing the evolution of European call options as functions of strike and maturity, and discussed its application to the problem of fitting the stock process to option prices in the market. They also presented a second order accurate unconditionally stable operator splitting (ADI) method for pricing options which does not require iterative solution of an algebraic equation at each time step. (Unfortunately, it is not clear how to extend their method to the valuation of American options while retaining second order accuracy.) Cont and Voltchkova (2003) used a discretization that is implicit in the differential terms and implicit in the integral term, and showed that it converges to a viscosity solution. Their method extends to infinite activity models, and does not require the diffusion part of the equation to be non-degenerate. These partial integro-differential equations have also been solved by many others. See, for example, Zhang (1993) and Matache *et al.* (2002).

Although the pricing equations have often been solved numerically, because of the integrals in the equations the methods have proven relatively expensive. The obvious discretizations of the pricing equations combine standard discretization methods for the differential terms with quadrature methods such as Simpson's rule or Gaussian quadrature for evaluating the integral term. This approach is computationally expensive since the integral must be approximated at each point of the mesh used for discretizing the the differential terms. The difficulties are greater if an implicit discretization of both the integral and the differential terms is used. The expense of evaluating the integral at all points of the computational grid can, however, be reduced by making the same exponential change of variables often used when solving the Black–Scholes differential equation when there is no jump process. This converts the integral term into a correlation integral which can be evaluated at all the mesh points simultaneously using the Fast Fourier Transform. This approach has been suggested by many authors (Wilmott, 1998; Tavella and Randall, 2000; Andreasen and Anderson, 2000).

There are, however, several problems associated with Fourier methods. The main problem, of course, is that since the integrals are not periodic it is necessary to extend the computational region to avoid "wrap around" effects. Enlarging the grid is expensive, especially in higher dimensions. In some cases it is possible to reduce this cost somewhat. For example, when pricing options using the constant parameter Merton model, Anderson and Andreasen assume that the option price is

linear outside of a grid of size equal to a number of standard deviations of the underlying process. They solve the linear part in closed form, and solve the rest of the problem using an FFT-ADI method. Still, they have to extend the computational region.

We also note that Fourier methods may be slow to converge if the probability density is not smooth (the Gibbs phenomenon), as it is in the Kou double exponential model. This was noted by d'Halluin *et al.*

In this paper we develop more efficient methods for evaluating the correlation jump integrals. The basic idea is the following. In many cases the correlation integral is the solution of an ordinary or partial differential equation. These equations can often be solved more rapidly than the integrals can be accurately evaluated by quadrature, even using Fourier methods.

In particular, we note that in the Merton model the integral in the PIDE is at time $t = t_s$ equal to a translation of a solution of a heat equation with initial values equal to the solution of the PIDE at time t_s. The evaluation time τ of the solution of the heat equation is equal to half the variance of the Poisson process, and the translation amount is equal to the expected value of the Poisson process. (We can understand why this is so, since after an exponential change of variables the density in the Merton model is a Gaussian.) Therefore, the correlation integral can be approximated by solving the heat equation numerically. Since the variance of the Poisson process is normally a small number, not many time steps are needed to solve the heat equation very accurately. Fourier methods require $Cn \log n$ operations where typically C is 10. This fact, and the fact that the region must be extended when using Fourier methods mean that our methods are potentially much faster. In fact, numerical experiments indicate that they are sometimes nearly an order of magnitude faster. Our methods also extend to problems with similar density functions. For example, we can use differential methods to evaluate integrals when the density is the product of a polynomial with a Gaussian, or the product of a polynomial, a Gaussian and an exponential. The methods also extend to higher dimensions.

Another numerical difficulty that arises when Fourier methods are used for evaluating the convolution integral in Merton-type models is related to the difficulty in computing solutions of the heat equation using quadrature methods. Such difficulties are well known (Strikwerda, 1989). Specifically, if the variance is large, then the kernel of the integral does not decay very quickly. Therefore, it is necessary to use a large computational region for approximating the convolution integral. In contrast, boundary conditions for evaluating the integral by solving the heat equation are easily determined. For example, for a call the values of the integral are close to zero when asset prices are sufficiently small, and they are linear in the asset price for prices that are sufficiently large. On the other hand, the smaller the variance of the process, the larger the (Gaussian) kernel of the integral is near the point at which the integral is being evaluated. When the kernel is large the error made in computing the integral is also large. That is, any errors in the function convolved with the kernel are magnified. This is independent of the method used to perform the summation when approximating the integral.

Our method for evaluating the convolution integral in the Kou double exponential model is simpler and faster. That is because after a linear change of variables the integrand becomes separable. Specifically, after the change of variables the value of the integral at a point x is the product of an exponential function of x and an integral where x only appears as the lower or upper limit of integration. Therefore, the integral can be evaluated at all n points of the grid using only Cn operations, where C is a small number, normally about 3. The evaluation of the integral can also be viewed as the solution of a first order ordinary differential equation. Again, our technique for evaluating the correlation integrals in the Kou model can be extended to evaluating the integrals in other similar models.

Another advantage of our method over Fourier methods is that we do not require a uniform mesh for evaluating the integral terms, nor do we obtain increased speed by requiring the number of mesh points to be a power of 2. In particular, we have used a very non-uniform grid when solving the pricing equation for digital options, concentrating the mesh points near the payoff discontinuity. This cannot be done as efficiently using Fourier methods.

We also note that our evaluation methods often extend to higher dimensions. For example, the correlation integrals in Merton-type models and exponential models can be viewed as solutions of two- and three-dimensional differential equations. Also, the density function in the Variance Gamma model is the fundamental solution of a Helmholtz equation in 3D. The integrals can therefore be evaluated by solving the Helmholtz equation in three dimensions.

We note that the Fast Gauss Transform (Greengard and Strain, 1991) has been suggested as a rapid method of evaluating correlation integrals in the Merton and Kou models. This is potentially an $O(n)$ algorithm for evaluating sums of Gaussians. However, the overhead for the method is large, and D'Halluin *et al.* (2005b) report that their preliminary tests pricing options in jump diffusion problems indicate the method does not perform as well as the Fast Fourier Transform for reasonable tolerance levels.

The rest of this paper is organized as follows. In Section 2 we present the equations used to price options in jump diffusion models, in Section 3 we describe the standard numerical methods for solving them, in Section 4 we give our method for evaluating the jump integrals in Merton-type models, in Section 5 we give our method for evaluating the jump integrals in exponential-type models, and in the last section we provide results of numerical tests.

2.   The Basic Equations

In this section we present the model of evolution of asset prices and the PIDEs which must be solved in order to value options on assets.

We assume that the movement of the asset price over time is given by a process of the form

$$\frac{\mathrm{d}S}{S} = \nu \mathrm{d}\tau + \sigma \mathrm{d}z + (\eta - 1)\mathrm{d}q,$$

where ν is the drift rate, σ is the volatility of the Brownian part of the process, and $\mathrm{d}q$ is a Poisson process. Here, $\mathrm{d}q = 0$ with probability $1 - \lambda \mathrm{d}t$, $\mathrm{d}q = 1$ with probability $\lambda \mathrm{d}t$, where λ is the Poisson arrival intensity (there is a probability $\lambda \mathrm{d}t$ of a jump in q in time step $\mathrm{d}t$), and $\eta - 1$ is an impulse function giving a jump from S to $S\eta$. The average relative jump size, $E(\eta - 1)$, is denoted by κ. We assume that the Poisson process $\mathrm{d}q$ is independent of the Wiener process $\mathrm{d}z$.

It is well known (Merton, 1976) that under the above assumptions the value of a contingent claim $V(S, \tau)$ depending on the asset price S and time τ satisfies the following PIDE:

$$V_t = \frac{\sigma^2 S^2}{2} V_{SS} + (r - \lambda\kappa)SV_S - (r + \lambda)V + \lambda \int_0^\infty V(S\eta, t)g(\eta)\,\mathrm{d}\eta, \qquad (2.1)$$

where $t = T - \tau$ is the time remaining until expiration at T, r is the risk-free interest rate, and $g(\eta)$ is the probability density function of the jump amplitude η.

Suppose $I(V(S, t))$ is the integral in the above equation:

$$I(V((S, t)) = \int_0^\infty V(S\eta, t)g(\eta)\mathrm{d}\eta.$$

By making the changes of variables

$$S = e^x, \quad \eta = e^y \text{ and } d\eta = e^y \, dy, \tag{2.2}$$

we see

$$I(V(S, t)) = I(V(x, t)) = \int_{-\infty}^{\infty} \tilde{V}(x + y, t) f(y) \, dy, \tag{2.3}$$

where $f(y) = g(e^y) e^y$ and $\tilde{V}(x + y, t) = V(e^{x+y}, t)$.

The function $f(y)$ is the probability density of a jump of size $y = \log(\eta)$.

The integral (2.3) is a correlation integral, and, as noted above, can therefore be evaluated using the Fast Fourier Transform. In the next sections we present alternate methods for evaluating the integral for several density functions f, and show why the methods can be more accurate and more efficient.

3.   Discretization and Iterative Solution of PIDE

In this section we describe the commonly used discretizations of the PIDE (2.1), and methods of solving the resulting systems of equations.

We first consider the differential terms. When solving (2.1) several authors (e.g., D'Halluin *et al.*) discretize the differential part directly.

Others first make a change of variables $S = S(x)$ which reduces (2.1) to

$$\frac{\partial V}{\partial t} = \frac{\sigma^2 S^2(x)}{2J(x)} \frac{\partial}{\partial x} \frac{1}{J(x)} \frac{\partial V}{\partial x} + (r - \lambda\kappa) \frac{S(x)}{J(x)} \frac{\partial V}{\partial x} - rV + \lambda \int_{-\infty}^{\infty} f(y) V(x + y, t) \, dy, \tag{3.1}$$

where

$$J(x) = \frac{dS(x)}{dx}.$$

In particular, as noted in the previous section, it is common to make an exponential change of variables

$$S = e^x, \tag{3.2}$$

which reduces the equation to

$$\frac{\partial V}{\partial t} = \frac{\sigma^2}{2} V_{xx} + (r - \lambda\kappa) \frac{\partial V}{\partial x} - rV + \lambda \int_{-\infty}^{\infty} f(y) V(x + y, t) \, dy. \tag{3.3}$$

This equation was used by Andreasen and Andersen and by Cont and Voltchkova.

It is also often possible to increase accuracy by further concentrating grid points where the solution varies most rapidly. Specifically, suppose the option price is being approximated on the interval $S_m \leq S \leq S_M$, and B is the point where additional resolution is needed. The transformation

$$S(\tilde{x}) = B + \alpha(c_2 \sinh(c_2\tilde{x} + c_1(1 - \tilde{x}), \tag{3.4}$$

where

$$c_1 = \sinh^{-1}\left(\frac{S_m - B}{\alpha}\right),$$

and

$$c_2 = \sinh^{-1}\left(\frac{S_M - B}{\alpha}\right)$$

maps the interval $0 \leq \tilde{x} \leq 1$ to the interval $S_m \leq S \leq S_M$. If a uniform mesh in $\tilde{x}$ is used, and α is much smaller than $S_M - S_m$, then a very non-uniform grid in S concentrated near B results. See Tavella and Randall (2000).

In order to allow large time steps the Crank Nicolson method with uniform spacing in space and time is often used to discretize the differential part of the above PIDEs:

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} = L\left(\frac{V_i^{n+1} + V_i^n}{2}\right) + \lambda I(V(x,t)). \tag{3.5}$$

Here, $V^n = (V_0^n, V_1^n, \ldots, V_M^n)$ denotes the solution of the PIDE at the nth time step at the discretization points $\{x_i\}$, $L(V_i^n)$ is the discretization of the differential terms, $I(V(x,t))$ is the integral in (3.1), and Δt is the time step.

In order to solve the PIDEs the integrals must be evaluated at the differential discretization points. One can choose the quadrature points $\{y_j\}$ to be the same points or they can be different, in which case it is necessary to interpolate the values of the integral onto the differential points. This second approach was taken by d'Halluin *et al.* (2005b).

In any case, in order to evaluate the integral term the integration region is truncated to a finite interval, $y_{\min}$, $y_{\max}$, and the integral is approximated by a sum.

In particular, when the density can be integrated analytically the approximation

$$I(V^n(y_i, t)) = \sum_{j=0}^{M} V_{i+j}^n f_j, \tag{3.6}$$

where

$$f_j = \int_{y_{j-1/2}}^{y_{j+1/2}} f(x) \, dx,$$

and $y_{j+1/2}$ is the point midway between the quadrature points y_j and y_{j+1}, is often used.

The most obvious way of including the integral when solving (3.2) is by "lagging", or evaluating it at the previous time step:

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} = \frac{1}{2}(L(V_i^{n+1} + V_i^n) + \lambda I(V_i^n).$$

The advantage of this approach is that, at each time step, a tridiagonal linear system of equations results. The drawback, of course, is that the truncation error of the scheme is then $O(\Delta t)$ instead of $O(\Delta t^2)$.

In order to retain the formal second order accuracy in time of the Crank Nicolson method it is necessary to solve the equation

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} = \frac{1}{2}(L(V_i^{n+1} + V_i^n) + \lambda \frac{I(V_i^{n+1}) + I(V_i^n)}{2},$$

where the integral is evaluated at the current time step as well.

There are several ways to do this. The simplest is by using straightforward (Picard) iteration with respect to k:

$$\frac{V_i^{[k+1]} - V_i^n}{\Delta t} - \frac{1}{2}(L(V_i^{[k+1]} + V_i^n) = \lambda \frac{I(V_i^{[k]}) + I(V_i^n)}{2}. \tag{3.7}$$

At the start of the nth time step the iteration begins with $V^{[1]} = V^n$, and continues until the difference between two successive iterates $V^{[k+1]} - V^{[k]}$ satisfies some convergence criterion, for example

$$\frac{||V^{[k+1]} - V^{[k]}||}{||V^{[k+1]}||} \leq \epsilon, \tag{3.8}$$

where the norm is the maximum norm. D'Halluin *et al.* showed that this iteration is globally convergent when combined with a Crank Nicolson discretization of (2.1). More specifically, they showed that if r and λ are non-negative the following three conditions are sufficient to guarantee convergence of the iteration.

1. The off-diagonal terms in the matrix equation arising from the discretization of the differential terms in (2.1) are positive.
2.

$$\sum_{j=0}^{M} f_j \leq 1. \tag{3.9}$$

3.

$$f_j \geq 0, \tag{3.10}$$

where f_j are given in (3.6)

As noted by d'Halluin *et al.*, if one is pricing single factor options using typical grids and parameter values the first condition is usually satisfied when central differences are used to approximate the first derivatives in (2.1). Indeed, in all our calculations we found that this was the case. However, as d'Halluin *et al.* remarked, when this is not the case, one can enforce the condition by instead using one sided (upwind) differences to approximate the first derivatives. It was also noted in that paper that since upwind differences are normally only used at a few points the overall rate of convergence is not affected.

As for the other two conditions, if one approximates the integral (2.3) by a standard quadrature method, then conditions (3.9) and (3.10) are easily satisfied. It is not immediately apparent they are satisfied when differential methods are used. We discuss these conditions in Section 4.

In our numerical experiments we used this iterative procedure, but combined it with a Crank Nicolson discretization of (3.1) for vanilla call options, and a Crank Nicolson discretization of (3.2) after making the change of variable (3.4) for pricing digital options.

In contrast, Andersen and Andreasen used an ADI method to solve (3.1). Specifically, letting D be the term involving derivatives with respect to x,

$$D = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + (r - \lambda\kappa)\frac{\partial}{\partial x},$$

they wrote (3.1) as

$$\frac{\partial V}{\partial t} - DV = -rV + \lambda f * V,$$

where $*$ denotes convolution.

They then solved this equation using the operator splitting method:

$$\left(\frac{2}{\Delta t} - D_h\right)V^{n+(1/2)} = \left(\frac{2}{\Delta t} - r + \lambda f*\right)V^n,$$

$$\left(\frac{2}{\Delta t} + r - \lambda f*\right)V^{n+1} = \left(\frac{2}{\Delta t} + D_h\right)V^{n+(1/2)},$$

where D_h is the usual centered difference approximation to D.

This method is second order accurate in space and time, and should be significantly faster for pricing European options. However, it is not clear how to modify it for pricing American options while retaining second order accurate convergence in time.

In any case, once the discretization of the equation and iterative strategy have been chosen, the question of how to evaluate the integral term remains. Since it is a correlation integral, one way to perform the computation is by using the Fast Fourier Transform. That is, the integral can be approximated by

1. computing the Fourier transform $\hat{V}$ of V_i^n,
2. multiplying $\hat{V}$ by the Fourier transform of f, i.e. form $R = \hat{V}\hat{f}$,
3. computing the inverse Fourier transform of R.

Thus, the cost of this method of calculation is essentially the cost of two FFTs. However, since the integral is not periodic, in order to avoid wrap around problems the computational region and function V must be extended beyond the region where an accurate approximation is needed. For example, when pricing a call, one can assume that the option value is linear in S for S sufficiently large, and close to 0 for S sufficiently small.

In the next two sections we present alternate, more rapid ways of evaluating the integrals for Merton-type and exponential-type densities.

4.   Merton Model and Extensions

In the model originally presented by Merton (1976) the probability density function of the jumps is

$$g(\eta) = \frac{e^{(-[\log(\eta) - \mu]^2/2\gamma^2)}}{\sqrt{2\pi}\,\gamma\,\eta}.$$

The expected relative change in the stock price is $\kappa = E[\eta - 1] = e^{\mu + \gamma^2/2} - 1$.

When μ, σ and γ are constant the price of a European non-path-dependent option can be expressed as an infinite sum:

$$\sum_{n=0}^{\infty} \frac{1}{n!} e^{\lambda'(T-t)} (\lambda'(T-t))^n V_{\text{BS}}(S, t, \sigma_n, r_n),$$

where

$$\lambda' = \lambda(1+\kappa), \quad \sigma_n^2 = \sigma^2 + \frac{n\gamma^2}{T-t}, \quad r_n = r - \lambda k + \frac{n \log(1+\kappa)}{T-t},$$

and V_{BS} is the Black–Scholes formula for the option value when there are no jumps. This formula is easily implemented, and normally only the first five or six terms in the sum are needed to obtain six digits of accuracy in the option price.

However, when the Black–Scholes price is not known, numerical methods are generally needed to price the option.

We can, of course, use the methods described in the previous section. However, the cost of computing the correlation integral by Fourier methods at each time step is significant. We now show how to decrease the cost of computing the integrals and avoid other numerical difficulties.

We first note that after making the exponential change of variables $S = e^x$ in the density function g, the integral in the pricing equation can be written

$$I(V(x, t)) = \int_{-\infty}^{\infty} V(x+y, t) \frac{e^{-(y-\mu)^2/2\gamma^2}}{\sqrt{2\pi}\gamma} \, dy.$$

By further letting $s = x + y$ we obtain

$$I(V(x, t)) = \frac{1}{\sqrt{2\pi}\gamma} \int_{-\infty}^{\infty} V(s, t) e^{-(s-x-\mu)^2/2\gamma^2} \, ds. \tag{4.1}$$

The kernel in (4.1) is a translation of the fundamental solution of the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{4.2}$$

on the line $-\infty \le x \le \infty$, where τ is an artificial variable corresponding to half of the variance of the jump process, $\gamma^2/2$. (That is, we view τ as a continuous variable, $0 \le \tau \le \gamma^2/2$.) It follows that, at any fixed time t, the integral (4.1) is a solution of the heat equation (4.2) with initial data $V(x, t)$. This solution is evaluated at $\tau = \gamma^2/2$, and then translated by the amount μ.

When the option price does not have steep gradients (e.g., vanilla calls) we evaluate the integral by solving the heat equation using a discretization that is fourth order accurate in space and second order accurate in τ.

The discretization combines the usual centered difference in time and a weighted average of the centered differences in space at the current and the previous step in τ:

$$\frac{u_i^{n+1} - u_i^n}{\Delta \tau} = \theta \delta_x^2 u_i^{n+1} + (1-\theta) \delta_x^2 u_i^n, \tag{4.3}$$

where

$$\delta_x^2 u_i^n = \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2},$$

and the weight $\theta = (1/2) - [(\Delta x)^2/12\Delta\tau]$ depends on the relative mesh spacing. We note that the scheme is stable, i.e. all the eigenvalues of the iteration matrix are non-negative (Richtmeyer and Morton, 1967).

We also note that if we use this differential method of approximating $I(V_i)$, conditions (3.9) and (3.10), which are sufficient for the convergence of the Picard iteration scheme (3.7), can be satisfied. Condition (3.9) follows if the scheme used to solve the heat is monotone, i.e. $u_i^{n+1} \geq 0$ for all i if $u_i^n \geq 0$ for all i. If one is solving an initial value problem for the heat equation with uniform mesh it is known (Dautray, 1990) that the scheme (4.3) is of positive type if

$$4\frac{\Delta\tau}{(\Delta x)^2} \leq \frac{2 - \theta}{(1 - \theta)^2}.$$

This equation is satisfied if $18\alpha^2 - 21\alpha \leq 1$ where $\alpha = \Delta\tau/(\Delta x)^2$. Thus, for a given mesh width in x, if the steps in τ are sufficiently small the scheme will be monotone. Since we normally used large mesh widths in x, in the majority of our numerical experiments, this was the case. However, even when the condition was not satisfied, the Picard iteration (3.7) was still rapidly convergent.

In order to show (3.10) is satisfied by our method, we let $E_{i,j}^{n_\tau}$ be the solution at the ith mesh point and the n_τth time step of the discrete heat equation (2.5) with initial data $u_j^0 = 1$, $u_k^0 = 0$ for $k \neq j$. Here, n_τ is the total number of steps in τ used to solve the heat equation.

We note that $\sum_{j=1}^{M} E_{i,j}^{n_\tau}$ is the solution at the ith mesh point of the discrete heat equation with initial values 1 at all mesh points, and is therefore less than or equal to 1. (This follows by direct calculation or the discrete maximum principle.) Since, by linearity, our approximation to $I(V_i)$ is equal to $\sum_{j=1}^{M} V_j E_{i,j}^{n_\tau}$, we see that condition (3.10) is satisfied.

When the option prices have large gradients we use the same mesh as we do for discretizing the differential terms. Specifically, we make a change of variables of the form (3.4), so that

$$\mathrm{e}^x = B + \alpha(c_2 \sinh(c_2\tilde{x} + c_1(1 - \tilde{x}))),$$

and use a uniform mesh in $\tilde{x}$. This corresponds to a non-uniform mesh in x. We thus solve the heat equation $u_\tau = u_{xx}$, but use a method that is second order accurate with respect to the mesh width in $\tilde{x}$. We note that since the integral $I(V)$ is not a convolution integral with respect to the variable $\tilde{x}$, Fourier methods cannot be used to compute its values using a uniform mesh in $\tilde{x}$. Of course, the parabolic equation that $I(V)$ satisfies with respect to $\tilde{x}$ does not have constant coefficients. It can, however, be solved to second order accuracy in $\tilde{x}$ by solving a tridiagonal linear system of equations at each step in τ.

In order to determine u we need to specify its boundary conditions at the edge of the computational region. In order to price call options we used homogeneous Dirichlet boundary conditions at the left end point, and required the solution to have zero curvature with respect to S at the right end point, that is we set $\partial^2 u/\partial S^2 = 0$ (d'Halluin *et al.*, 2005b).

It is important that these boundary conditions are accurate for S relatively small. In Figure 1 we have plotted $\partial I(S, t)/\partial S$ for the option with $\gamma = 0.35$, $\sigma = 0.15$, $r = 0.05$, $E = 100$, $\mu = -0.90$, $t = 0.25$ and $\lambda = 0.1$. Thus we see that $\partial^2 I/\partial S^2$ is close to constant for S relatively small.

Since at each step in τ the discretizations lead to tridiagonal linear systems of equations whose solution only requires three operations per point, the total cost of solving equation (4.2) is $3Nm$ where m is the number of mesh points in the x direction. This is in contrast to two Fourier transforms, i.e. a cost of approximately $10m \log m$ for computing the integral using Fourier methods. However, it is important to note that when we use differential methods we need not enlarge the
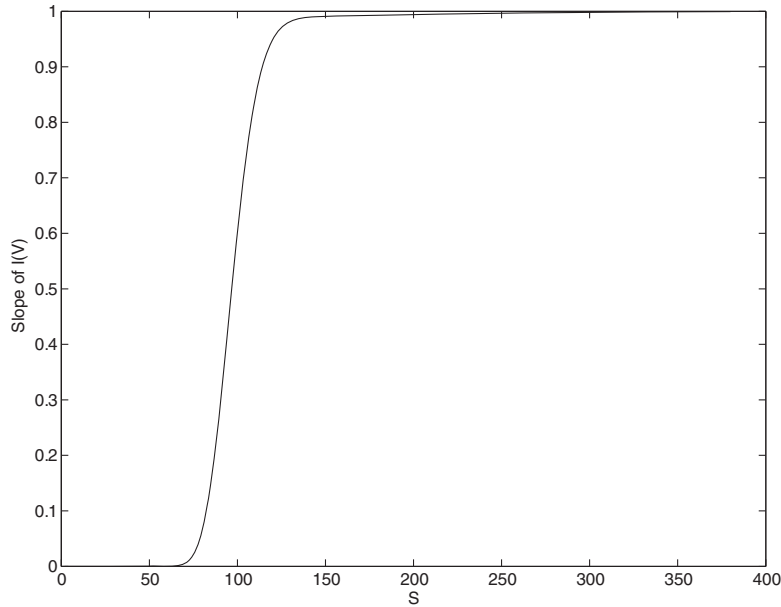
Figure 1. Slope of convolution integral for $\mu = -0.90$, $\gamma = 0.35$.

computational region to avoid problems with wrap around. That is, the number of mesh points in x is usually much smaller in our method of calculation. For details, see Section 6.

In addition to the expense of Fast Fourier Transform methods we encounter another problem when approximating the integral (4.1) by evaluating a sum

$$I(V_i^n) = \sum_{j=-\infty}^{\infty} V_j^n f_{i-j}, \qquad (4.4)$$

where

$$f_{i-j} = \frac{1}{\gamma\sqrt{2u\pi}} \int_{y_{j-1/2}}^{y_{j+1/2}} e^{-(y-y_i-\mu)^2/2\gamma^2} \, dy.$$

The problem is that the sum must be truncated. That is, we can only evaluate

$$\sum_{j=0}^{M} V_j^m f_{i-j},$$

for some M. However, if γ is large, the neglected terms involving f_{i-j} are not small unless $y_i - y_j$ is very large. This means that in order to obtain high accuracy, even near the center of the computational region, the integration region must be very large. On the other hand, if γ is small, f_{i-j} is large for i close to j, so any errors made in evaluating V_i will be magnified. In Figure 2 we have plotted $y = (1/\gamma\sqrt{2\pi})\,e^{-(x-x_0)^2/2\gamma^2}$ in a region around $x_0 = \log 100$ for $\gamma = 0.1$ (dotted line) and $\gamma = 0.45$.
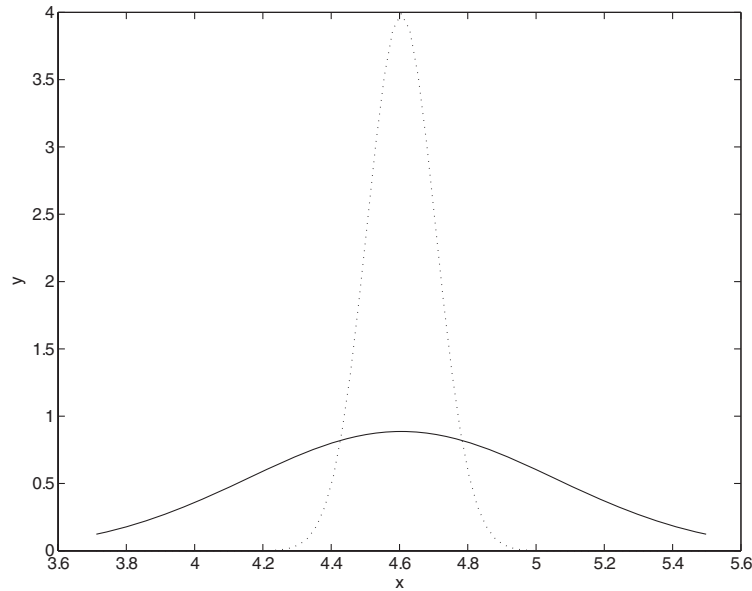
Figure 2. Gaussian density for $\gamma = 0.1$ and $\gamma = 0.45$.

In contrast, the values of the integral $I(V)$ are smooth and easily computed by differential methods. See Figure 3 for the graphs of $I(V)$ for $\gamma = 0.45$ (dotted line) and $\gamma = 0.1$, and option values time to expiration $T = 6$ months, volatility $\sigma = 0.15$, interest rate $r = 0.05$, strike price $K = 100$, $\mu = 0$, and $\lambda = 0.1$.

The same method of evaluating correlation integrals (4.1) can be used if the mean μ and variance γ of the jump process are time dependent since, at each time step, the parameter t is constant in the integral.

It is also possible to evaluate the jump integrals when the density function of the process has the form $f(y) = b(1 - cy)\,e^{-ay^2}$. In addition to integrals of the form (4.1) the correlation integrals then contain an integral of the form

$$I_1(y) = \int_{-\infty}^{\infty} V(s, t)c(s - y)\,e^{-a(s-y)^2}\,\mathrm{d}s.$$

To approximate such an integral we note that

$$I_1(y) = \frac{c}{2a}\frac{\partial I}{\partial y},$$

where

$$I(y) = \int_{-\infty}^{\infty} V(s, t)\,e^{-a(s-y)^2}\,\mathrm{d}s.$$

Therefore, the integral $I_1(y)$ is the derivative of a solution of a heat equation, and can thus be approximated by numerically solving the heat equation and differencing the results.
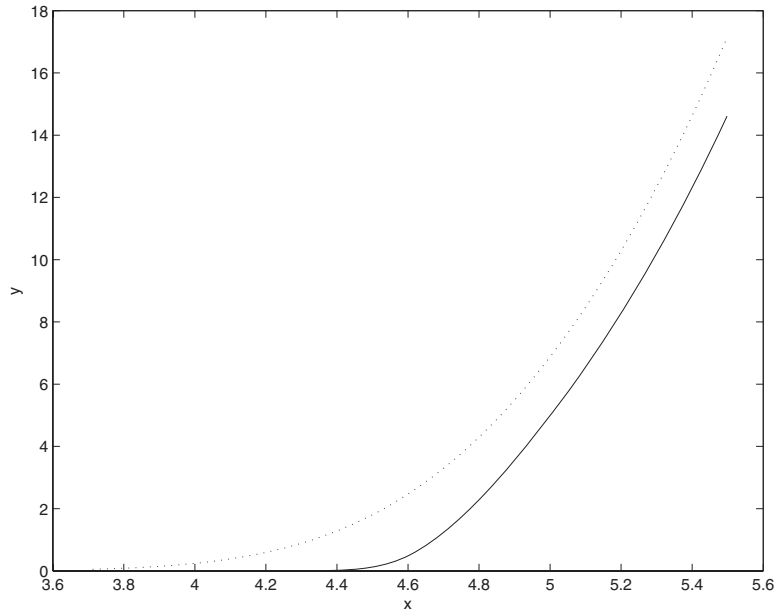
Figure 3. Correlation integral for $\gamma = 0.1$ and $\gamma = 0.45$.

Similarly, we can evaluate integrals with density functions of the form $f(y) = b(1 - cy^2)\,e^{-ay^2}$. In this case, in addition to integrals of the form (4.1) we need to compute integrals of the form

$$I_2(y) = \int_{-\infty}^{\infty} V(s, t)(s - y)^2\,e^{-a(s-y)^2}\,ds.$$

Now we note that

$$\frac{\partial I}{\partial y} = 2a \int_{-\infty}^{\infty} V(s, t)(s - y)\,e^{-a(s-y)^2}\,ds,$$

and

$$\frac{\partial^2 I}{\partial y^2} = 4a^2 \int_{-\infty}^{\infty} V(s, t)(s - y)^2\,e^{-a(s-y)^2}\,ds - 2a \int_{-\infty}^{\infty} V(s, t)\,e^{-a(s-y)^2}\,ds,$$

so

$$I_2(y) = \frac{1}{4a^2}\left(\frac{\partial^2 I(y)}{\partial y^2} + 2a I(y)\right).$$

Since we can compute approximations to the integrals $I(y)$ and their derivatives, we can approximate integrals of the form of $I_2(y)$, again essentially by solving the heat equation.

Similar calculations show that it is possible to evaluate integrals

$$\int_{-\infty}^{\infty} V(s, t)p(s - y)\,e^{-(s-y)^2}\,ds,$$

where $p(s - y)$ is any polynomial.

Finally, we consider density functions of the form $g(y) = c\,e^{-(ay^2+by+\eta)}$. To compute an integral

$$\int_{-\infty}^{\infty} V(s+y,t)\,e^{-(ay^2+by+\eta)}\,dy, \tag{4.5}$$

we note that

$$ay^2 + by = a\left(y + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2,$$

and so

$$\int_{-\infty}^{\infty} V(s+y,t)\,e^{-(ay^2+by+\eta)}\,dy = e^{(b/2a)^2-\eta} \int_{-\infty}^{\infty} V(s+y,t)\,e^{-a(y+(b/2a))^2}.$$

Thus, the integral (4.5) is also a translation of a solution of a heat equation which can be computed by the methods described in the beginning of this section.

5. Kou Model and Extensions

In Kou's (2002) model the density is a double exponential:

$$\bar{f} = p\eta_1\,e^{-\eta_1 x}\,H(x) + q\eta_2\,e^{\eta_2 x}\,H(-x),$$

where $\eta_1 > 1, \eta_2 > 0, p > 0, q = p - 1$, and H is the Heaviside function: $H(x) = -x$ for $x < 0$, $H(x) = x$ for $x \geq 0$.

As Kou notes, in order that the stock prices have finite expectation it is necessary that $\eta_1 > 1$. In this model

$$\kappa = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1.$$

By (3.1) it follows that the pricing PIDE contains integrals of the form

$$\int_{0}^{\infty} V(x+y,t)\,e^{-\eta_1 y}\,dx, \tag{5.1}$$

and

$$\int_{-\infty}^{0} V(x+y,t)\,e^{\eta_2 x}\,dx. \tag{5.2}$$

As noted, since these integrals are correlation integrals they are most commonly computed using the Fast Fourier Transform. As in the case of the Merton model, Fourier Transform methods require enlargement of the computational domain, and for efficiency normally require the number of mesh points to be a power of 2. Difficulties also arise because of the discontinuity of the density (the Gibb's phenomenon).

However, under a linear change of variables these and similar integrals can be expressed as products of exponentials and integrals of only one variable.

Specifically, under the change of variables $s = x + y$ the first integral becomes

$$e^{\eta_1 y} \int_y^\infty V(s, t) \, e^{-\eta_1 s} \, ds = e^{\eta_1 y} \, I_1(y), \tag{5.3}$$

and the second becomes

$$e^{-\eta_2 y} \int_{-\infty}^y V(s, t) \, e^{\eta_2 s} \, ds = e^{-\eta_2 y} \, I_2(y). \tag{5.4}$$

In order to compute the integral (5.3) we first approximate

$$I_0 = \int_0^\infty V(s, t) \, e^{-\eta_1 s} \, ds.$$

To do this we truncate the integration region to $(0, y_{max})$, where y_{max} is chosen so $|V(y_{max}, t) \, e^{-\eta_1 y_{max}}|$ is less than some tolerance ϵ, and use a uniformly spaced grid $(y_0, y_1, \ldots, y_M = y_{max})$:

$$I_0 \cong \sum_0^M w_i V(y_i, t) \, e^{-\eta_1 y_i},$$

where $\{w_i\}$ are the weights.

Next, to approximate $I_1(y)$ we use the fact that

$$I_1(y) = I_0 - \int_0^y V(s, t) \, e^{-\eta_1 s} \, ds. \tag{5.5}$$

The computational problem is thus reduced to the problem of computing the last integral. We do this inductively:

$$I_1(y_i) \cong I_1(y_{i-1}) + w_i V(y_i, t) \, e^{-\eta_1 y_i}.$$

The integral $I_2(y)$ is computed in the same manner. Thus, the entire calculation is linear in the number of evaluation points.

We can also view the integrals $I_1(y)$ and $I_2(y)$ as solutions of ordinary differential equations. For example, $I_1(y)$ satisfies the equation

$$\frac{dI}{dy} = e^{-\eta_1 y} \, V(y, t),$$

and $I_2(y)$ satisfies the equation

$$\frac{dI}{dy} = e^{\eta_2 y} \, V(y, t).$$

We can also compute the jump integrals when the density has components of the form $(1 - ax) \, e^{-\eta x}$ or $(1 - ax) \, e^{\eta x}$. In that case, in addition to integrals of the forms (5.1) and (5.2) we need to compute

integrals of the form

$$I(y) = \int_0^\infty V(x + y, t) a x \, e^{-\eta x} \, dx.$$

By the same change of variables $s = x + y$ such an integral can be expressed as

$$I(y) = a \int_y^\infty V(s, t)(s - y, t) \, e^{-\eta(s-y)} \, ds$$

$$= a \, e^y \left(\int_y^\infty V(s, t) s \, e^{-\eta s} \, ds - y \int_y^\infty V(s, t) \, e^{-\eta s} \, ds \right).$$

Since y only appears in the above integrals as the lower limit of integration the integrals can also be evaluated in $O(n)$ operations.

Similarly, we can evaluate integrals with density functions of the form $p(x) \, e^{ax}$ where p is any polynomial.

6. Numerical Results

In this section we report on results of numerical tests we performed solving the pricing PIDEs in the Merton and Kou jump diffusion models.

In our first experiments we solved the pricing equation in the Merton model for a vanilla call. In order to evaluate the correlation integrals we used both our differential method as well as the FFT method after extending the computational region. In both cases we made the exponential change of variables and solved the pricing PIDE using a Crank Nicolson discretization with a uniform mesh in x and time to expiration t. We used the same mesh points for evaluating the integral that we used for discretizing the differential part of the equation, and set the tolerance for the Picard iteration (3.8) to 10^{-7}. When we used the differential method we used 10 uniform steps in τ to to solve the heat equation.

Since the analytic solution of the pricing equation for the Merton model is known when the parameters are constant, we could determine the errors precisely.

The numbers in Tables 1 and 2 are results of our calculations pricing the European call option with time to expiration $T = 3$ months, volatility $\sigma = 0.25$, interest rate $r = 0.05$, strike price $K = 100$, $\mu = -0.90$, and $\lambda = 0.1$. Those in Table 1 were obtained when we used the differential method, and those in Table 2 were obtained when we used the Fourier method.

The numbers m in column 1 are the number of discretization points in x, the numbers nt in the second column are the number of time steps, the numbers in the third column are the asset prices, the numbers in the fourth column are the computed option values, the numbers in the fifth column are the errors, and the numbers in the last column are convergence ratio R, that is, the ratio of the error obtained using the given mesh to the error obtained using the mesh with half as many points in each direction.

Both methods appear to be almost second order accurate. The numbers also show that we could achieve the same level of accuracy using only half as many mesh points in x when we used the differential method as when we used the FFT based method. For example, we could use a mesh with 32 points in x in the differential method and obtain as accurate a solution as when we used a mesh with 64 points in x and the FFT method. The number of operations per time step per

Table 1. Values of a call option in the Merton model using the differential method

m	nt	S	Computed values	Error	R
64	30	90	1.862678	0.002424	
128	60	90	1.860879	0.000624	3.88
256	120	90	1.860419	0.000165	3.79
512	240	90	1.860297	0.000043	3.86
64	30	100	6.288013	0.006745	
128	60	100	6.282979	0.001711	3.94
256	120	100	6.281705	0.000437	3.92
512	240	100	6.281382	0.000114	3.85
64	30	110	13.620631	0.001634	
128	60	110	13.619413	0.000416	3.93
256	120	110	13.619107	0.000110	3.79
512	240	110	13.619025	0.000028	3.95

Table 2. Values of a call option in the Merton model using FFT

m	nt	S	Computed values	Error	R
128	60	90	1.863431	0.003176	
256	120	90	1.861055	0.000800	3.97
512	240	90	1.860454	0.000199	4.01
1024	480	90	1.860306	0.000051	3.92
128	60	100	6.288991	0.007723	
256	120	100	6.283223	0.001955	3.95
512	240	100	6.281770	0.000501	3.90
1024	480	100	6.281393	0.000125	4.01
128	60	110	13.621358	0.002361	
256	120	110	13.619609	0.000612	3.86
512	240	110	13.619155	0.000158	3.87
1024	480	110	13.619037	0.000040	3.97

iteration for the differential method is

$$3m + 3n_\tau m,$$

where n_τ is the number of time steps used to solve the heat equation. For the Fourier Transform method it is

$$3m + 10m \log m.$$

Since we used approximately the same number of iterations (3.7) per time step, the speedups, that is the ratios of the operation count using the Fourier method with $2m$ points in x to the operation count for the differential method with m points in x and $n_\tau = 10$, were 3.82 for $m = 32$, 4.42 for $m = 64$, 5.03 for $m = 128$, and 5.64 for $m = 256$.

In Tables 3 and 4 we give results of pricing the digital call option (payoff $= 1$ if $S > K$) with $\sigma = 0.25$, $\gamma = 0.35$, strike $K = 100$, $r = 0.05$, $T = 0.25$, $\lambda = 0.1$, and $\mu = -0.90$.

When we used the differential method of evaluating the correlation integrals we made the change of variables (3.4) with $\alpha = 0.28$. We used this change of variables for evaluating both the integral and differential terms. This change of variable results in a mesh that is approximately

Table 3. Values of a digital option in the Merton model using the differential method

m	nt	S	Computed values	Error	R
32	20	90	0.2802168	0.0022490	
64	40	90	0.2819675	0.0004983	4.51
128	80	90	0.2823425	0.0001233	4.04
256	160	90	0.2824329	0.0000329	3.75
512	320	90	0.2824567	0.0000091	3.63
1024	640	90	0.2824658	0.0000000	3.76
32	20	100	0.6038454	0.0013994	
64	40	100	0.6028239	0.0003779	3.70
128	80	100	0.6025412	0.0000952	3.97
256	160	100	0.6024699	0.0000239	3.98
512	320	100	0.6024520	0.0000060	4.00
1024	640	100	0.6024475	0.0000015	4.05
32	20	110	0.8425873	0.0008047	
64	40	110	0.8419116	0.0001290	6.24
128	80	110	0.8418131	0.0000305	4.23
256	160	110	0.8417923	0.0000097	3.15
512	320	110	0.8417853	0.0000027	3.64
1024	640	110	0.8417833	0.0000007	3.80

Table 4. Values of a digital option in the Merton model using FFT

m	nt	S	Computed values	Error	R
32	20	90	0.2479166	0.0345492	
64	40	90	0.2724903	0.0099755	3.46
128	80	90	0.2787034	0.0037624	2.65
256	160	90	0.2810174	0.0014484	2.60
512	320	90	0.2816075	0.0008583	1.69
1024	640	90	0.2824658	0.0000000	4.64
32	20	100	0.6237558	0.0213098	
64	40	100	0.6079905	0.0055445	3.84
128	80	100	0.6036734	0.0012274	4.52
256	160	100	0.6027466	0.0003006	4.08
512	320	100	0.6025193	0.0000733	4.10
1024	640	100	0.6024646	0.0000186	3.94
32	20	110	0.9283726	0.0865900	
64	40	110	0.8626090	0.0208264	4.16
128	80	110	0.8516805	0.0098979	2.10
256	160	110	0.8473227	0.0055401	1.79
512	320	110	0.8432473	0.0014647	3.78
1024	640	110	0.8422568	0.0004742	3.09

five times as fine near the strike as it is at the edge of the computational region for the finest mesh (1024 points). We also placed the strike midway between mesh points, and set $n_\tau = 15$. We used the same change of variables and mesh location for discretizing the differential part of the PIDE when we used the FFT method for evaluating the integral term. That is, we used a non-uniform grid for discretizing the differential term, and a uniform grid for discretizing the integral term.

Table 5. Values of a call option in the Kou model

m	nt	S	Computed values	Error	R
100	30	90.00	0.674241	0.001564	
200	60	90.00	0.673120	0.000443	3.532
400	120	90.00	0.672795	0.000118	3.738
800	240	90.00	0.672710	0.000033	3.605
100	30	100.00	3.981782	0.008303	
200	60	100.00	3.975728	0.002249	3.692
400	120	100.00	3.974079	0.000600	3.749
800	240	100.00	3.973640	0.000161	3.715
1600	480	100.00	3.973527	0.000048	3.356
100	30	110.00	11.795925	0.001342	
200	60	110.00	11.794952	0.000369	3.633
400	120	110.00	11.794686	0.000103	3.595
800	240	110.00	11.794610	0.000027	3.780

We used linear interpolation to go from one set of mesh points to the other, and used the same number of grid points in both meshes.

In this case the increase in accuracy we obtained by using our differential method of evaluating the jump integrals compared with the Fourier method was greater than when we priced vanilla call options. Specifically, we obtained essentially the same level of accuracy at the points tested near the strike as we did using the FFT method with a mesh four times as fine. In particular, when we used 128 mesh points and the differential method for evaluating the correlation integral we obtained as accurate a solution as when we used 512 points with the Fourier method, so the speedup was 7.75. Similarly, when we used 256 points we could achieve the same level of accuracy as when we used 1024 points and the Fourier method. The ratio of operation counts was therefore 8.58.

In Table 5 we present results of pricing a European vanilla call option with double exponential probability density function using Simpson's rule as the quadrature formula. The input parameters we used are $\sigma = 0.15$, $r = 0.05$, $\lambda = 0.1$, $T = 0.25$, $K = 100$, $\eta_1 = 3.0465$, $\eta_2 = 3.0465$, and $p = 0.3445$. The exact value of the option at $S = 90$ is 0.672677, at $S = 100$ it is 3.973479, and at $S = 110$ it is 11.794583.

We note that d'Halluin *et al.*(2005b) also priced the same option. If we compare their results with ours we see that in order to obtain the same level of accuracy they used essentially the same number of mesh points as we did to discretize the differential part of the PIDE. However, for evaluating the integral they used a mesh with eight times the number of mesh points. Thus, in this case the differential method was significantly faster.

Finally, we note that because the Kou density does not decay as rapidly as the Gaussian density we needed to use more mesh points in x when pricing options in this model to achieve the same level of accuracy as when we priced options in the Merton model. However, in these calculations the number of operations per point was much smaller, so the entire calculation was still much less expensive.

7.   Conclusions

When the underlying asset follows a jump diffusion process the price of an option on that asset satisfies a partial integral differential equation. The primary difficulty encountered when trying to

solve such equations numerically is the presence of the non-local, integral terms. Most methods, including Fast Fourier Transform methods, that have been used to compute the integral term are relatively expensive. In this paper we have shown how to reduce the cost of evaluating such integrals when the density functions of the jump processes are products of polynomials, exponentials and Gaussians. In particular, we have used our methods to price options in the Merton model and in Kou's double exponential model and have found our differential methods to be up to almost an order of magnitude faster. In future work we plan to extend these methods to pricing options whose values depend on several assets.

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