Optimal Positioning in Derivative Securities

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Abstract

We consider a simple single period economy in which agents invest so as to maximize expected utility of terminal wealth. We assume the existence of three asset classes, namely a riskless asset (the bond), a single risky asset (the stock), and European options of all strikes (derivatives). By restricting investor beliefs and preferences, we explicitly solve for the optimal position for each investor in the three asset classes. In contrast to previous literature, our analysis is conducted in a general equilibrium setting in which positions are determined simultaneously with asset prices. We find that heterogeneity in preferences or beliefs induces investors to hold derivatives individually, even though derivatives are not held in aggregate. Under heterogeneous lognormal beliefs, we find that the risk-neutral density is not lognormal. We also determine who buys and who sells options in general equilibrium and we derive some new separation results.

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1 Introduction

It is widely accepted that the positions taken in derivative securities reflect prices, preferences, and prior beliefs. Thus, it is commonly argued that investors who believe that future volatility will exceed current implied volatility should buy options, while those who believe otherwise should sell. Similarly, market wisdom has it that investors who tend to be more risk averse than average should buy options, while only the most aggressive investors should sell them naked.

It is less clear what position should be taken when beliefs and preferences conflict in their recommendation. For example, it is not clear whether a conservative investor who believes that volatility will be high should buy options or sell them. The resolution of this problem requires a model for determining optimal positions in derivative securities. Fortunately, some outstanding work has been done in this area, most notably the work of Brennan and Solanki[3] and of Leland[9]. This work shows for example that the conservative investor who believes volatility should be high should sell at-the-money options, but then protect against severe losses by buying out-of-the-money puts and calls.

While these insights resolve much of the confusion surrounding optimal positioning, some interesting issues issues remain. For example, since these models all assume that asset prices are given, it is not clear that an investor always stands ready to take the opposite side of the transaction contemplated by the decision maker. If all investors simultaneously conclude that the time is ripe for selling at-the-money options, then clearly supply will not equal demand. Clearly, a general equilibrium framework is needed which respects the zero net supply conditions of derivatives markets.

The purpose of this paper is to study the determinants of optimal positioning in derivative securities in a general equilibrium framework. For this purpose, we consider a simple single period economy in which agents invest so as to maximize expected utility of terminal wealth. We assume the existence of three asset classes, namely a riskless asset (the bond), a single risky asset (the stock), and European options of all strikes (derivatives). By restricting investor beliefs and preferences, we explicitly solve for the optimal position for each investor in the three asset classes. We find that heterogeneity in preferences or beliefs induces investors to hold derivatives individually, even though derivatives are not held in aggregate. Under heterogeneous lognormal beliefs, we find that the combination of universal risk aversion and the stock supply condition causes the risk-neutral density to display more negative skewness than the lognormal. Under certain assumptions, we also find that optimal positions decompose into a finite number of funds, most of which involve nonlinear payoffs. This observation has important implications for optimal security design and for optimal market structure.

The structure of this paper is as follows. The next section describes our model and then reviews certain well known results on spanning and on optimal positioning in a partial equilibrium setting. The third section derives general results on optimal positioning in a general equilibrium setting. In order to derive closed form results for prices and positions, the next two sections restrict preferences to generalized log utility and to negative exponential utility respectively. The final section summarizes the paper and suggests avenues for future research. Three appendices contain technical results whose inclusion in the body of the paper would distract the reader.

2 Assumptions and Literature Review

This section describes our market structure and then reviews certain well known results on spanning in complete markets and on optimal positioning in derivative securities in a partial equilibrium setting.

2.1 Market Structure

Consider a one period model in which investments are made at time 0 with all payoffs being received at time 1. There is a riskless asset costing B_0 initially and paying unity at time 1, which we call the bond. There is also a single risky asset, costing S_0 initially and paying the random amount S at time 1, which we call the stock. In addition, we also assume that markets exist for out-of-the-money European puts and calls of all strikes. While this assumption is not standard, it allows us to examine the question of optimal positioning in a complete market without requiring the heavy machinery of continuous time mathematics. We note that the assumption of a continuum of strikes is essentially the counterpart of the standard assumption of continuous trading. Just as the latter assumption is frequently made as a reasonable approximation to an environment where investors can trade frequently, we take our assumption as a reasonable approximation when there are a large but finite number of option strikes (eg. for the S&P500). In each case, the assumption adds analytic tractability without representing a large departure from reality.

2.2 Spanning

It is well known that our market structure implies the existence of a unique risk-neutral density that may be identified from option prices (see Breeden and Litzenberger[1]). It is also well known that investors can achieve any smooth function of the underlying stock price by taking a static position at time 0[1]. However, the previous literature does not explicitly identify the position that must be taken in order to achieve a given payoff. In contrast, Appendix 1 proves that any twice continuously differentiable function, $f(S)$, of the terminal stock price S, can be replicated by a unique initial position of $f(S_0) - f'(S_0)S_0$ unit discount bonds, $f'(S_0)$ shares, and $f''(K)dK$ out-of-the-money options of all strikes K:

$$\begin{aligned} f(S) &= [f(S_0) - f'(S_0)S_0] + f'(S_0)S \\ &\quad + \int_0^{S_0} f''(K)(K-S)^+ dK + \int_{S_0}^{\infty} f''(K)(S-K)^+ dK. \end{aligned} \quad (1)$$

The positions in the bond and the stock create a tangent to the payoff at the initial stock price. The positions in the out-of-the-money options are used to bend the tangent line so as to match the payoff at all price levels[2]. In the remainder of this paper, we derive optimal payoff functions for investors in a general equilibrium context. For brevity, we leave it to the reader to use (1) to determine the exact positions in the available assets.

Since the payoff $f(S)$ is linear in the payoffs from the available assets, the same linear relationship must prevail among the initial values. Specifically, letting $V_0[f]$ denote the initial value of the arbitrary[3] payoff $f(\cdot)$, and letting B_0, $P_0(K)$, and $C_0(K)$ denote the initial prices of the bond, put, and call respectively, then it follows from (1) and the no arbitrage condition that:

$$\begin{aligned} V_0[f] &= [f(S_0) - f'(S_0)S_0]B_0 + f'(S_0)S_0 \\ &\quad + \int_0^{S_0} f''(K)P_0(K)dK + \int_{S_0}^{\infty} f''(K)C_0(K)dK. \end{aligned} \quad (2)$$

[1]This observation was first noted in Breeden and Litzenberger[1] and established formally in Green and Jarrow[8] and Nachman[10].

[2]An important special case of (1) is put-call-parity, which arises when $f(S)$ is the payoff of an in-the-money call $(S-K_0)^+$ for $K_0 < S_0$. In this case $f'(S_0) = 1$, while $f''(K_0)$ is a delta function centered at K_0. See Richards and Youn[11] for an accessible introduction to generalized functions such as delta functions.

[3]We require that the payoff be twice differentiable and that the integrals in (2) not diverge.

Appendix 2 shows that (2) directly implies that the initial value of an arbitrary payoff $f(\cdot)$ can also be expressed as:

$$V_0[f] = B_0 \int_0^\infty f(K)q(K)dK, \tag{3}$$

where the state pricing density $B_0 q(K)$ may be recovered from option prices by the relation:

$$B_0 q(K) = \begin{cases} \frac{\partial^2 P_0(K)}{\partial K^2} & \text{for } K \leq S_0; \\ \frac{\partial^2 C_0(K)}{\partial K^2} & \text{for } K > S_0. \end{cases} \tag{4}$$

The result (4) is of course well known from Breeden and Litzenberger [1] and is here seen as a simple consequence of the replication strategy (1).

2.3 Optimal Positioning in Partial Equilibrium

The optimal positioning problem has been addressed in the partial equilibrium context by Brennan and Solanki [3] and by Leland[9] among others. Following these authors, we suppose that there are n investors in the economy, indexed by $i = 1, \ldots, n$. Each investor is endowed with β_i shares, where $\sum_{i=1}^{n} \beta_i = 1$. For an initial stock price of S_0, the initial value of the endowment is the investor's initial wealth $W_0^i \equiv \beta_i S_0$. Each investor's preferences are characterized by an increasing concave utility function U_i defined over their random terminal wealth W_i. Each investor's beliefs are characterized by a probability density function $p_i(S)$, defined on the entire positive half line with $p_i(S) > 0$ for $S > 0$. Each investor is assumed to maximize expected utility $\int_0^\infty U(W_i)p_i(S)dS$. Since all terminal wealth is consumed, W_i can be replaced by a function $f(S)$ relating terminal wealth to the terminal stock price S. The completeness of the market allows investors to maximize expected utility by choice of any function $f(S)$ that they can afford:

$$\max_{f(\cdot)} \int_0^\infty U_i[f(S)]p_i(S)dS. \tag{5}$$

The affordability of the payoff $f(\cdot)$ is captured by requiring that the initial value, $V_0[f]$, of the payoff, $f(S)$, must be less than or equal to the investor's initial wealth, W_0^i. In our complete market, the initial value of the payoff can be calculated using the risk-neutral density, and thus the budget constraint is:

$$B_0 \int_0^\infty f(S)q(S)dS \leq W_0^i. \tag{6}$$

In a partial equilibrium, the prices of the bond, the stock, and the options are taken as given. Thus, the risk-neutral density used in (6) can be implied from option prices and is in effect specified directly. Following Brennan and Solanki [3], consider the Lagrangean for this constrained optimization problem (5) and (6), given by:

$$\mathcal{L} = \int_0^\infty U_i[f_i(S)]p_i(S)dS - \lambda_i \left[\int_0^\infty f_i(S)B_0q(S)dS - W_0^i \right]. \tag{7}$$

Differentiating with respect to the payoff function $f(\cdot)$ and setting the result to zero yields the first order condition determining the *optimal* payoff function $\phi_i(S)$:

$$\frac{p_i(S)}{B_0q(S)}U_i'[\phi_i(S)] = \lambda_i. \tag{8}$$

The optimal payoff can be determined explicitly by solving (8) for $\phi_i(S)$:

$$\phi_i(S) = (U_i')^{-1}\left(\lambda_i B_0 \frac{q(S)}{p_i(S)}\right). \tag{9}$$

This equation makes it clear that the optimal payoff depends on risk aversion and on the deviation of personal beliefs from the risk-neutral density. To quantify this further, let $T_i[\phi_i(S)] \equiv -\frac{U_i'[\phi_i(S)]}{U_i''[\phi_i(S)]}$ denote the investor's risk tolerance and let $D_i(S) \equiv \ln\left(\frac{p_i(S)}{q(S)}\right)$ measure the deviation of the investor's beliefs from the market. Taking the logarithmic derivative of both sides of the first order condition (8) with respect to the stock price yields the decomposition in Leland[9] of an investor's optimal *exposure* into the product of his preferences and beliefs:

$$\phi_i'(S) = T_i[\phi_i(S)]D_i'(S). \tag{10}$$

Thus, when one terminal price is compared to an adjacent one, the investor increases his payoff at the higher price if his personal density grows faster than the risk-neutral density, and decreases it otherwise. However, the lower is the risk tolerance, the smaller is the required response of payoffs to deviations in growth rates of personal probabilities over risk-neutral probabilities.

Equation (10) is an ordinary differential equation (o.d.e.) governing the optimal payoff $\phi_i(S)$. Under certain regularity conditions on the risk tolerance, this o.d.e. may be solved subject to the budget constraint:

$$\int_0^\infty \phi_i(S) B_0 q(S) dS = W_0^i. \tag{11}$$

The solution to this problem is given by (9), where the parameter λ_i is obtained by substituting the optimal payoff (9) in (11):

$$\int_0^\infty (U_i')^{-1} \left(\lambda_i B_0 \frac{q(S)}{p_i(S)} \right) B_0 q(S) dS = W_0^i. \tag{12}$$

Taking prices as given, one can develop fully explicit solutions for optimal payoffs for various preference and belief pairings. To illustrate, suppose for simplicity that the risk-neutral density is given by a lognormal density function:

$$q = \ell(r, \sigma) \equiv \frac{1}{\sqrt{2\pi}\sigma S} \exp\left\{ -\frac{1}{2} \left[\frac{\ln(S/S_0) - (r - \sigma^2/2)}{\sigma} \right]^2 \right\}, \tag{13}$$

where r is the continuously compounded riskfree rate and σ is the implied volatility. Further suppose that a given investor's beliefs are also characterized by a lognormal density function with a different mean μ_i[4]:

$$p_i = \ell(S; \mu_i, \sigma) \equiv \frac{1}{\sqrt{2\pi}\sigma S} \exp\left\{ -\frac{1}{2} \left[\frac{\ln(S/S_0) - (\mu_i - \sigma^2/2)}{\sigma} \right]^2 \right\}. \tag{14}$$

Finally suppose that this belief is paired with power utility $U(W_i) = \frac{\gamma_i}{\gamma_i - 1} W^{\frac{\gamma_i - 1}{\gamma_i}}$, so that marginal utility has the form:

$$U'(W_i) = W^{-\frac{1}{\gamma_i}}.$$

Substituting the inverse of this function in (12), solving for λ, and substituting the result, (13), and (14) in (9) implies that the optimal payoff has the form:

$$\phi_i(S) = \frac{W_0^i + B_0 \tau_i / \gamma_i}{V_0[S^{\gamma_i \mathcal{S}_i}]} S^{\gamma_i \mathcal{S}_i}, \tag{15}$$

[4]Note that μ_i is the log of the expected value of the periodically compounded gross return, $\mu_i = \ln E \frac{S}{S_0}$, while σ is the standard deviation of the continuously compounded return, $\sigma = \text{Std} \ln(S/S_0)$.

where $\mathcal{S}_i \equiv (\mu_i - r)/\sigma^2$ is the investor's Sharpe ratio and for the lognormal risk-neutral density, $V_0[S^{\gamma_i \mathcal{S}_i}] = S_0^{\gamma_i \mathcal{S}_i} B_0 e^{\gamma_i \mathcal{S}_i (r - \sigma^2/2) + \gamma_i^2 \mathcal{S}_i^2 \sigma^2/2}$. The optimality of this payoff in the continuous-time context is discussed in Cox and Huang[5]. Figure 1 shows the effect of varying the expected return on the optimal payoff. When $\mu_i > r + \gamma_i \sigma^2$ (eg. $\mu = 8.25\%$, the investor takes

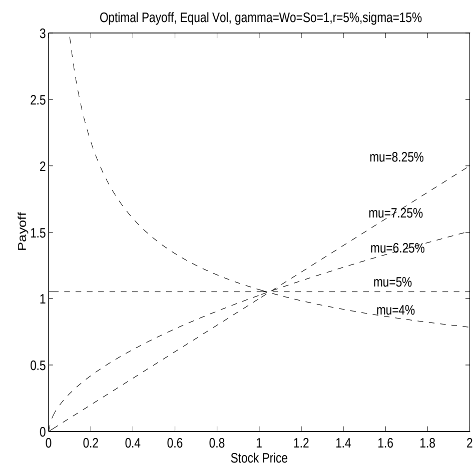

Figure 1: Optimal Payoff in Partial Equlibrium

a leveraged position in stock. He also buys puts to protect the downside and buys calls to convexify the upside. As expected return is lowered, the investor borrows less, buys less stock, and reduces his option purchases. When $\mu_i = r + \gamma_i \sigma^2$, the investor holds only stock. As expected return is lowered beyond this point, the investor starts to be long the riskless asset and continues to be long the stock. The investor now judiciously sells puts and calls. When $\mu_i = r$, the investor holds only the riskless asset. Finally, when $\mu_i < r$, the investor shorts the stock and buys puts to convexify the effect of stock price declines, while buying calls to protect against stock price rises.

3 Optimal Positioning in General Equilibrium

While the risk-neutral density can be implied from option prices, it is by no means clear that this density should be in the same parametric class as the investor's prior beliefs. It is also not clear whether in equilibrium, someone is always available to take the other side of the investor's optimal derivatives position. For example, we cannot have all investors protecting their portfolios with puts as puts are in zero net supply. To address this issue, we consider the solution of a general equilibrium model where the risk-neutral distribution and optimal positions are determined simultaneously.

Consider an economy in which n investors simultaneously optimize their holdings. We require that the equilibrium risk-neutral density appearing in (9) and (12) must re-price the bond, the stock, and all options. The bond repricing condition is:

$$B_0 \int_0^\infty 1 q(S) dS = B_0, \tag{16}$$

or equivalently, the risk-neutral density $q(\cdot)$ integrates to one. We assume that bonds and options are in zero net supply and thus in aggregate, only the stock is held:

$$\sum_{i=1}^{n} \phi_i(S) = S, \tag{17}$$

which implies that the sum of the exposures is unity:

$$\sum_{i=1}^{n} \phi_i'(S) = 1. \tag{18}$$

The above equations imply that the the risk-neutral expected return on the stock is the riskless rate. To see this, recall that each investor is endowed with β_i shares, where $\sum_{i=1}^{n} \beta_i = 1$. Since $W_0^i \equiv \beta_i S_0$, initial wealths sum to the initial stock price:

$$\sum_{i=1}^{n} W_0^i = S_0.$$

Substituting in the budget constraint (11) and interchanging summation and integration implies:

$$\int_0^\infty B_0 \sum_{i=1}^{n} \phi_i(S) q(S) dS = S_0.$$

Finally, substituting in (17) gives the desired result:

$$B_0 \int_0^\infty S q(S) dS = S_0. \tag{19}$$

3.1 The Risk-Neutral Density in Equilibrium

In any equilibrium model, only relative prices are determined. Thus, in this subsection, we will take S_0 as given, and solve for the risk-neutral density $q(S)$ and the bond price B_0 in terms of S_0. In the next subsection, we will determine optimal payoffs $\phi(S)$ in terms of S_0. To obtain an expression for the risk-neutral density in general equilibrium, recall the multiplicative decomposition (10) of exposures into beliefs and preferences:

$$\phi_i'(S) = T_i[\phi_i(S)]D'(S) = T_i[\phi_i(S)]\frac{d}{dS}\ln\left[p_i(S)/q(S)\right]. \tag{20}$$

Summing over i implies:

$$\sum_{i=1}^{n}\phi_i'(S) = \sum_{i=1}^{n}T_i[\phi_i(S)]\left[\frac{d}{dS}\ln p_i(S) - \frac{d}{dS}\ln q(S)\right] = 1, \tag{21}$$

from (18). Solving for $q(S)$ gives our first general equilibrium result

Theorem 1: *In a general equilibrium pure exchange economy, the risk-neutral density satisfies:*

$$q(S) = q(0)\exp\left\{-\int_0^S \frac{1}{T(Z)}dZ\right\}\exp\left\{\int_0^S \sum_{i=1}^{n}\frac{T_i[\phi_i(Z)]}{T(Z)}\frac{p_i'(Z)}{p_i(Z)}dZ\right\}, \tag{22}$$

where $T(S) \equiv \sum_{i=1}^{n} T_i[\phi_i(S)]$ *is the total risk tolerance in state* S*.*

Since the optimal payoff ϕ_i depends on q, (22) is not an explicit expression for the risk-neutral density. Nonetheless, (22) indicates that the equilibrium risk-neutral density is the product of a factor reflecting total risk tolerance i.e. $\exp\left\{-\int_0^S \frac{1}{T(Z)}dZ\right\}$ and a factor reflecting the personal beliefs, which we term the market view. The greater the risk tolerance of a given investor, the more his probability density gets reflected in the market view.

Corollary 1: *In a general equilibrium pure exchange economy with homogeneous beliefs, the risk-neutral density satisfies:*

$$q(S) = q(0)\exp\left\{-\int_0^S \frac{1}{T(Z)}dZ\right\}p(S). \tag{23}$$

Corollary 1 is obtained from (22) by setting $p_i(S) = p(S)\forall i$. The first factor in (23) is a positive declining function of S which changes the mean in the market view to the riskless rate, and may add negative skewness. For example, if $p(S)$ is a normal density and aggregate risk tolerance is constant, then

$q(S)$ is also normal but with a shifted mean. However, if $p(S)$ is lognormal and risk tolerance is constant, then $q(S)$ is not in the lognormal family, but is skewed to the left with the density having a fatter left tail. For linear aggregate risk tolerance, similar results hold. If aggregate risk tolerance is infinite or equivalently, if there exist individuals with zero risk aversion, then $q(S) = p(S)$. It follows that the disparity between the risk-neutral density and the density describing homogeneous beliefs is a consequence of universal risk aversion and the requirement that the risky stock be held in equilibrium.

To determine the equilibrium bond price, note that the constant q_0 appearing in Theorem 1 and Corollary 1 is determined by the requirement (16) that the density integrates to one. Multiplying (22) by S and integrating S from 0 to ∞ gives the forward price $\frac{S_0}{B_0}$. Solving this expression for B_0 gives the equilibrium bond price. As the result is complicated, we defer the statement of this result until we have specialized preferences.

3.2 The Optimal Payoffs in General Equilibrium

To obtain the optimal payoffs in our general equilibrium, we substitute (22) into (20) to express the optimal exposure in terms of preferences and beliefs:

Theorem 2: *In a general equilibrium pure exchange economy, the optimal exposure satisfies:*

$$\phi_i'(S) = \frac{T_i[\phi_i(S)]}{T(S)} + T_i[\phi_i(S)] \quad \frac{d \ln p_i(S)}{dS} - \sum_{i=1}^{n} \frac{T_i[\phi_i(S)]}{T(S)} \frac{d \ln p_i(S)}{dS} \quad . \tag{24}$$

The solution to this system of nonlinear ODE's gives the optimal payoff. While one cannot express the solution in general, this expression shows the determinants of the optimal exposure in general equilibrium. The first term in (24) reflects the investor's risk tolerance relative to the population total. The second term is a composite of the investor's risk tolerance and the extent to which the investor's beliefs differ from a risk tolerance weighted average of the beliefs of other investors in the economy.

Corollary 2: *In a general equilibrium pure exchange economy with homogeneous beliefs, the optimal exposure satisfies:*

$$\phi_i'(S) = \frac{T_i[\phi_i(S)]}{T(S)}. \tag{25}$$

Corollary 2 is obtained from (24) by setting $p_i(S) = p(S) \forall i$. Since the right side of (25) is positive, homogeneous beliefs imply that all investors must have an increasing payoff. The greater the investor's risk tolerance relative to the total, the greater the exposure of the investor's position.

The results of Cass and Stiglitz[4] imply that investors with homogeneous beliefs and linear risk tolerances with identical cautiousness (i.e. $T(W) = \tau_i + \gamma W$) will not hold derivatives. This raises the question of sufficient conditions under which investors will hold derivatives. It also raises the question of the shape of the optimal payoff when derivatives are held. The next two sections show that under generalized log utility or negative exponential utility, heterogeneity in beliefs induces investors to hold derivatives. We will derive the optimal payoff in each case. We now address whether heterogeneity in *preferences* can induce demand for derivatives and what the form of the optimal payoff looks like. It is difficult to solve for the optimal payoff in general. However, the following corollary shows that derivatives are held in an economy with two linear risk tolerance (LRT) investors with homogeneous beliefs, but *opposite* cautiousness.

Corollary 3: *In a general equilibrium pure exchange economy with homogeneous beliefs, suppose that risk tolerances are given by $T_1[W_1] = \tau_1 + \gamma W_1$ and $T_2[W_2] = \tau_2 - \gamma W_2$. Then, optimal payoffs are given explicitly by:*

$$\begin{aligned} \phi_1(S) &= \frac{S}{2} - \frac{\tau}{2\gamma} + \sqrt{\left(\frac{S}{2} + \frac{\tau_2 - \tau_1}{2\gamma}\right)^2 + k^2} \\ \phi_2(S) &= \frac{S}{2} + \frac{\tau}{2\gamma} - \sqrt{\left(\frac{S}{2} + \frac{\tau_2 - \tau_1}{2\gamma}\right)^2 + k^2}, \end{aligned} \tag{26}$$

where $\tau \equiv \tau_1 + \tau_2$ and k is an arbitrary constant.
This corollary is proved in Appendix 3. In this simple economy, a three fund separation occurs in which each investor holds equal positions in the stock and offsetting positions in the bond and the derivative[5]. The optimal derivative security is the square root of the sum of a positive constant and a squared linear position in the stock. Figure 2 graphs the optimal payoffs.

Although the results of Corollary 3 pertain to only two investors, it is worth quoting from Dumas[6][6]:

> The two-investor equilibrium is as basic to financial economics as is the two-body problem in mechanics.

In order to obtain explicit solutions for the optimal payoff in an n investor economy, we next restrict preferences. In particular, the next section as-

[5] Appendix 3 also shows that if $k = 0$, then the investors no longer hold derivatives.

[6] In a highly original paper, Dumas[6] numerically solves for an equilibrium without derivatives in an intertemporal setting with two investors with different utility functions.

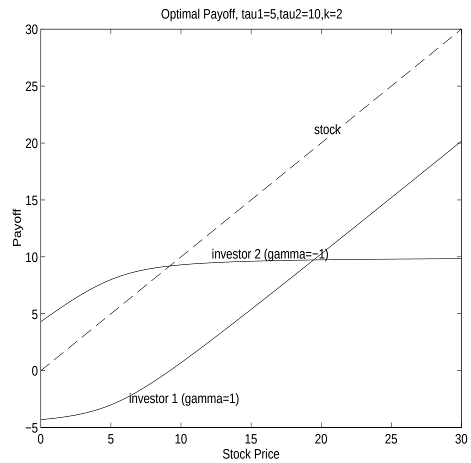

Figure 2: Optimal Payoff Under Homogeneous Beliefs and Opposite Cautiousness.

sumes generalized logarithmic utility[7], $U_i(W) = \ln(\tau_i + W)$, while the following section considers negative exponential utility $U_i(W) = -\tau_i \exp\left(-\frac{W}{\tau_i}\right)$. In both cases, we examine the cross-section of investor beliefs and preferences in order to explain the positions held.

4 Generalized Logarithmic Utility

If each investor's utility function has the form:

$$U_i(W) = \ln(\tau_i + W),$$

then marginal utility is:

$$U_i'(W) = \frac{1}{\tau_i + W}, \tag{27}$$

and risk tolerance is linear with unitary cautiousness:

$$T_i[W_i] \equiv -\frac{U_i'(W)}{U_i''(W)} = \tau_i + W_i. \tag{28}$$

[7] See Rubinstein[13] for further motivation for the use of this preference structure.

Substituting the inverse of the marginal utility function function (27) in (12), solving for λ, and substituting in (9) implies[8] that generalized log utility investors prefer a payoff of the form:

$$\phi_i(S) = -\tau_i + \frac{R_0^i}{B_0}\frac{p_i(S)}{q(S)}, \tag{29}$$

where:

$$R_0^i \equiv W_0^i + B_0\tau_i \geq 0 \tag{30}$$

is defined as the risk capital of investor i. Note that the greater is τ_i, the greater is the risk tolerance $T_i[W_i] = \tau_i + W_i$ and the greater is the risk capital R_0^i. Summing (29) over investors and invoking the market clearing condition (17) gives:

$$S = -\tau + \sum_{i=1}^{n} \frac{R_0^i}{B_0}\frac{p_i(S)}{q(S)}, \tag{31}$$

where $\tau \equiv \sum_{i=1}^{n} \tau_i$. Solving (31) for the risk-neutral density gives:

$$q(S) = \sum_{i=1}^{n} \frac{R_0^i}{B_0}\frac{p_i(S)}{S+\tau}. \tag{32}$$

In order that $q(S)$ be nonnegative for all nonnegative S, we require $\tau \geq 0$, i.e. that the aggregate floor $-\tau$ cannot be positive. Thus, positive floors on the part of some must be compensated for by negative floors on the part of others. The greater is the risk capital R_0^i of an investor, the more impact his beliefs have on the risk-neutral density.

To obtain equilibrium bond prices, integrate (32) over S from 0 to ∞, impose (16) and (30), and solve for B_0:

$$B_0 = \frac{\int_0^\infty \frac{S}{S+\tau}\sum_{i=1}^n W_0^i p_i(S)dS}{1 - \int_0^\infty \frac{S}{S+\tau}\sum_{i=1}^n \tau^i p_i(S)dS}.$$

We summarize our first results under generalized logarithmic utility in the following theorem:

Theorem 3: *In a general equilibrium pure exchange economy with generalized log utility, the risk-neutral density and bond price are given by:*

$$q(S) \quad = \quad \sum_{i=1}^{n} \frac{R_0^i}{B_0}\frac{p_i(S)}{S+\tau}, \quad \tau \geq 0$$

[8] The solutions can also be obtained by substituting (28) in the o.d.e. (10) and solving this *linear* o.d.e. subject to (11).

$$B_0 = \frac{\int_0^\infty \frac{S}{S+\tau} \sum_{i=1}^n W_0^i p_i(S) dS}{1 - \int_0^\infty \frac{S}{S+\tau} \sum_{i=1}^n \tau^i p_i(S) dS}.$$

Substituting (32) in (29) gives the optimal payoff:

Theorem 4: *In a general equilibrium pure exchange economy with generalized log utility, the optimal payoff is given by:*

$$\phi_i(S) = -\tau_i + \frac{R_0^i p_i(S)\tau}{\sum_{i=1}^n R_0^i p_i(S)} + \frac{R_0^i p_i(S) S}{\sum_{i=1}^n R_0^i p_i(S)}. \tag{33}$$

Thus, each investor first establishes a floor at $-\tau_i$ and then invests all remaining wealth in two customized derivative funds. The holdings in the riskless fund and the first customized derivative sum to zero across investors, while the holdings in the second customized derivative sum to the stock price. The higher is τ_i, the higher is the investor's risk tolerance and risk capital, and the larger is his position in each customized derivative. Note that each customized derivative is a limited liability claim. The first customized derivative is also bounded above by the absolute value of the aggregate floor τ, while the second customized derivative is bounded above by the stock price. If the two customized payoffs are made available to investors, then no one holds the stock directly, although holdings must sum to the stock.

4.1 Generalized Logarithmic Utility and Homogeneous Beliefs

Under generalized log utility, derivatives are not held if beliefs are homogeneous:

Corollary 4: *In a general equilibrium pure exchange economy with homogeneous beliefs and generalized log utility, the optimal payoff is given by:*

$$\phi_i(S) = -\tau_i + \frac{R_0^i \tau}{\sum_{i=1}^n R_0^i} + \frac{R_0^i S}{\sum_{i=1}^n R_0^i}. \tag{34}$$

To create this payoff, each investor holds the riskless asset and a long position in the stock. The higher is τ_i or W_0^i, the higher is the investor's risk tolerance and risk capital, and the larger is the position in stock. Thus, for generalized log utility investors with homogeneous beliefs, differences in risk tolerance do not induce demand for derivatives, but instead only affect the division between the riskless asset and the stock[9].

[9] This is a special case of the Cass and Stiglitz[4] result that two fund monetary separation is a consequence of homogeneous beliefs and linear risk tolerance with identical cuatiousness.

4.2 Generalized Logarithmic Utility and Derivative Fund Theorems

Under certain conditions, the customized derivatives optimal for a given investor can be decomposed into a linear combination of payoffs of universal interest. Under heterogeneous beliefs, an m fund separation arises if each investor's density can be represented as:

$$p_i(S) = p(S) \sum_{k=1}^{m} c_{ik} f_k(S), \quad i = 1, \ldots, n, \tag{35}$$

where $p(S)$ is the unknown true density and $\{f_k(S), k = 1, \ldots, m\}$ is a collection of basis functions. In words, each investor's density differs from the true density by a multiplicative error, which can be represented by a finite number of basis functions. When (35) holds, we have:

$$\begin{aligned} R_0^i p_i(S) &= p(S) \sum_{k=1}^{m} R_0^i c_{ik} f_k(S), \text{ and} \\ \sum_{i=1}^{n} R_0^i p_i(S) &= p(S) \sum_{k=1}^{m} \theta_k f_k(S), \end{aligned}$$

where $\theta_k \equiv \sum_{i=1}^{n} R_0^i c_{ik}$. Substituting into (33) determines the optimal payoff:

Corollary 5: *In a general equilibrium pure exchange economy with generalized log utility and beliefs satisfying (35), the optimal payoff is given by:*

$$\phi_i(S) = -\tau_i + \sum_{k=1}^{m} R_0^i c_{ik} \frac{f_k(S)\tau}{\sum_{k=1}^{m} \theta_k f_k(S)} + \sum_{k=1}^{m} R_0^i c_{ik} \frac{f_k(S) S}{\sum_{k=1}^{m} \theta_k f_k(S)}. \tag{36}$$

Thus, each investor's holdings separate into $2m + 1$ funds. The first fund is the riskless fund, which is used to establish the floor of $-\tau_i$. Each investor holds $R_0^i c_{ik}$ units of each of the $2m$ derivative funds, where the funds have a payoff of $\left\{ \frac{(\tau)^{1-l} S^l f_k(S)}{\sum_{k=1}^{m} \theta_k f_k(S)}, k = 1, \ldots, m, l = 0, 1 \right\}$. No one holds the stock individually, although the collective holdings sum to the stock.

4.3 Lognormal Beliefs and Zero Aggregate Floor

Under further restrictions on preferences and beliefs, we can obtain explicit formulas for the risk-neutral density, for the bond price, and for the optimal payoffs in equilibrium. We assume that each generalized log utility investor has lognormal beliefs, $\ell(\mu_i, v_i)$, with heterogeneous means μ_i and volatilities v_i. Substituting (14) in (32) and setting the aggregate floor of $-\tau$ to zero gives a risk-neutral density of:

$$q(S) = \sum_{i=1}^{n} \frac{R_0^i}{B_0} \frac{\ell(S; \mu_i, v_i)}{S}. \tag{37}$$

Multiplying (37) by SB_0 and integrating over S implies that capital at risk aggregates to the initial stock price:

$$\sum_{i=1}^{n} R_0^i = S_0, \tag{38}$$

from[10] (19). Completing the square in the lognormal in (37) implies that the risk-neutral density can also be written as:

$$q(S) = \sum_{i=1}^{n} \frac{R_0^i}{B_0} \frac{e^{-\mu_i + v_i^2}}{S_0} \ell(S; \mu_i - v_i^2, v_i). \tag{39}$$

Integrating over S and invoking (16) gives the bond pricing equation:

$$B_0 = \sum_{i=1}^{n} \frac{R_0^i}{S_0} e^{-\mu_i + v_i^2}. \tag{40}$$

Thus, from (38), the bond price is a risk-capital weighted average of each investor's expectation of $\frac{S_0}{S}$. Recalling that the risk-capital $R_0^i \equiv W_0^i + B_0 \tau_i$ depends on the bond price, substitution gives an explicit expression:

$$B_0 = \frac{\sum_{i=1}^{n} \beta_i e^{-\mu_i + v_i^2}}{1 - \sum_{i=1}^{n} \frac{\tau_i}{S_0} e^{-\mu_i + v_i^2}}. \tag{41}$$

This expression simplifies if we further assume that $\tau_i = 0 \; \forall i$:

$$B_0 = \sum_{i=1}^{n} \beta_i e^{-\mu_i + v_i^2}. \tag{42}$$

Thus, each investor's expectation of $\frac{S_0}{S}$ is now weighted by their initial stock endowment. We summarize these results in the following theorem:

[10] From (32), (38) holds for any density when $\tau = 0$.

Theorem 5: *In a general equilibrium pure exchange economy with generalized log utility, lognormal beliefs, the risk-neutral density and bond price are explicitly given by:*

$$q(S) = \sum_{i=1}^{n} \frac{R_0^i}{B_0} \frac{e^{-\mu_i + v_i^2}}{S_0} \ell(S; \mu_i - v_i^2, v_i) \tag{43}$$

$$B_0 = \sum_{i=1}^{n} \frac{R_0^i}{S_0} e^{-\mu_i + v_i^2}. \tag{44}$$

Theorem 5 indicates that the risk-neutral density is not lognormal, even though each investor has lognormal beliefs. However, under homogeneous beliefs, the risk-neutral density is lognormal:

Corollary 6: *In a general equilibrium pure exchange economy with generalized log utility, aggregate floor of zero, and homogeneous lognormal beliefs, the risk-neutral density is:*

$$q(S) = \ell(S; \mu - v^2, v). \tag{45}$$

The requirement (19) that the stock's risk-neutral expected return be the riskfree rate implies that $\mu - v^2 = r$, so that the Black Scholes formula holds for options, as shown in Rubinstein[12]. However, no one holds any options in this economy, as shown in (34).

Returning to the case of heterogeneous lognormal beliefs, substituting (14) in (33) determines the optimal payoff in our present setting:

Theorem 6: *In a general equilibrium pure exchange economy with generalized log utility, aggregate floor of zero, and lognormal beliefs, the optimal payoff is explicitly given by:*

$$\phi_i(S) = -\tau_i + \frac{R_0^i \ell(S; \mu_i, v_i) S}{\sum_{i=1}^n R_0^i \ell(S; \mu_i, \nu_i)}, \tag{46}$$

Figure 3 shows the optimal payoffs for a two investor economy when the investors have the same initial wealths, the same floor of zero, and agree on the mean. Investor 1 believes volatility is 10%, while investor 2 thinks it is 20%. The optimal payoff for investor 1 resembles a bell shaped curve, consistent with his low volatility view and his floor of zero. The optimal payoff of investor 2 accomodates the payoff of investor 1 and the requirement that payoffs sum to the stock price.

Assuming that investors all agree on volatility, then the optimal payoff simplifies:

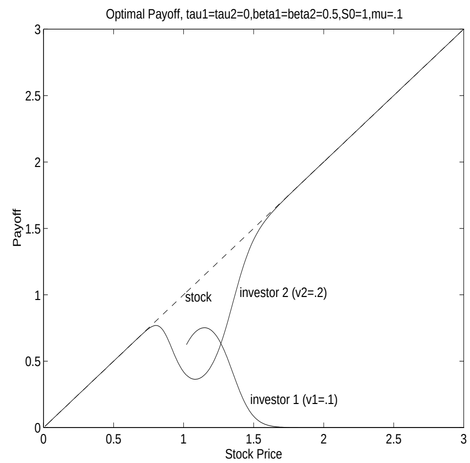

Figure 3: Optimal Payoffs When Means Agree.

Corollary 7: *In a general equilibrium pure exchange economy with generalized log utility, aggregate floor of zero, and lognormal beliefs with equal volatility v, the optimal payoff is explicitly given by:*

$$\phi_i(S) = -\tau_i + \frac{\hat{R}_0^i (S/S_0)^{p_i} S}{\sum_{i=1}^n \hat{R}_0^i (S/S_0)^{p_i}}, \tag{47}$$

where $\hat{R}_0^i \equiv R_0^i e^{p_i(\mu_i - v)/2}$ and $p_i \equiv \frac{\mu_i}{v^2}$.

Thus, the optimal customized payoff when means differ is a power of the stock price divided by a sum of powers. Figure 4 shows the optimal payoffs for a two investor economy when the investors have the same initial wealths, the same floor of zero, and agree on the volatility. Investor 1 believes the expected return is 10%, while investor 2 thinks it is 0%. Both payoffs would be synthesized using long positions in the stock. However, the more bullish investor 1 borrows at the riskfree rate and buys options while the less bullish investor 2 lends at the riskfree rate and sells options. Thus, in contrast to the case with homogeneous beliefs, options are used in the optimal portfolio, even if investors agree on volatility.

To obtain a separation result for the n investors case with equal volatility, let us further suppose that the n investors select their means from among $m < n$ possible values $\mu_1, \cdots \mu_m$. In particular, if investor i believes that the mean is μ_j, where μ_j is one of the m possible values, then his optimal payoff

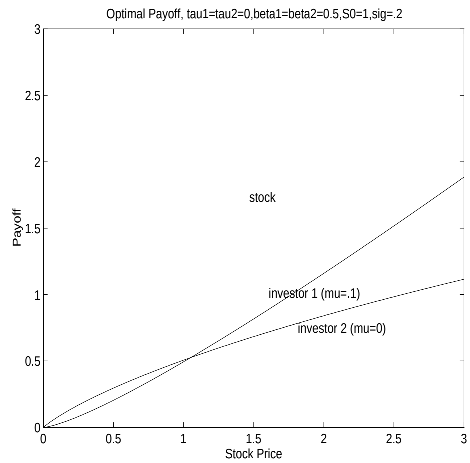

Figure 4: Optimal Payoffs When Volatilities Agree.

is:

$$\phi_i(S) = -\tau_i + \frac{\widehat{R}_0^j (S/S_0)^{p_j} S}{\sum_{k=1}^{m} \widehat{R}_0^k (S/S_0)^{p_k}}.$$

In aggregate, the n investors hold m risky funds, although each investor only has non-zero holdings in one risky fund.

To summarize the results of this section, investors with generalized log utility do not hold derivatives if beliefs are homogeneous, while derivatives are held when beliefs differ. When investors have lognormal beliefs, the risk-neutral distribution is not lognormal. The conclusion that derivatives serve a useful economic role under heterogeneous beliefs holds even when investors agree on the volatility. The next section shows that these conclusions also hold for negative exponential utility.

5 Negative Exponential Utility

Consider an n person economy in which all investors have negative exponential utility:

$$U_i(W) = -\tau_i \exp\left(-\frac{W}{\tau_i}\right), \forall i.$$

It follows that all investors have constant risk tolerances:

$$T_i[\phi(S)] \equiv -\frac{U_i'[\phi_i(S)]}{U_i''[\phi_i(S)]} = \tau_i \ \forall i.$$

Thus, from (24), each investor has an optimal exposure of the form:

$$\phi_i'(S) = \frac{\tau_i}{\tau} + \tau_i \quad \frac{d \ln p_i(S)}{dS} - \sum_{i=1}^{n} \frac{\tau_i}{\tau} \frac{d \ln p_i(S)}{dS} \quad , \tag{48}$$

where now $\tau \equiv \sum\limits_{i=1}^{n} \tau_i$ is the total risk tolerance. Integration gives the optimal payoff in terms of bonds, stocks, and derivatives:

Theorem 7: *In a general equilibrium pure exchange economy with negative exponential utility, the optimal payoff has the form:*

$$\phi_i(S) = \kappa_i + \frac{\tau_i}{\tau} S + \tau_i d_i(S), \tag{49}$$

where $d_i(S) \equiv \ln p_i(S) - \sum\limits_{i=1}^{n} \frac{\tau_i}{\tau} \ln p_i(S)$.

The constant of integration κ_i in (49) is determined by substituting (49) in the budget constraint (11).

$$\kappa_i = \frac{W_0^i - \frac{\tau_i}{\tau} S_0 - \tau_i V_0[d_i]}{B_0}.$$

In this economy, each investor's stock and derivatives position does not depend on his initial wealth. Thus, the bond position is used to finance the positions in stocks and derivatives. The magnitude of this position in stock and derivatives depends on their risk tolerance. The greater the risk tolerance, the greater the exposure to stocks and derivatives. Each investor's stock position does not depend on his beliefs. In contrast, each investor's derivatives position depends mainly on the extent to which his beliefs differ from those in the market. Thus, the open interest in derivatives markets is primarily a reflection of the heterogeneity of beliefs. If investors have homogeneous beliefs but differing risk aversion, then they do not hold derivatives[11]. Differences in risk aversion under homogeneous beliefs affect only the division between the riskless asset and the stock.

[11] This is again a consequence of the Cass and Stiglitz[4] result that two fund monetary separation holds when investors have homogeneous beliefs and linear risk tolerance with identical cautiousness.

To obtain separation results under constant risk tolerance and heterogeneous beliefs, note from (14) that the log of the lognormal density is a linear combination of the log price relative and its square. Suppose more generally that the log of each personal density can be written as a linear combination of basis functions:

$$\ln p_i(S) = \sum_{k=1}^{m} c_{ik} f_k(S). \tag{50}$$

Then (49) implies that the optimal payoff separates into $m + 2$ funds:

Corollary 8: *In a general equilibrium pure exchange economy with negative exponential utility and beliefs satisfying (50), the optimal payoff is given by:*

$$\phi_i(S) = \kappa_i + \frac{\tau_i}{\tau} S + \tau_i \sum_{k=1}^{m} \left(c_{ik} - \sum_{i=1}^{n} \frac{\tau_i}{\tau} c_{ik} \right) f_k(S) \quad . \tag{51}$$

In this m fund separation, the m derivative funds are the m basis functions which make up the log of the density. The optimal holding in the $k-th$ fund is $\tau_i \; c_{ik} - \sum_{i=1}^{n} \frac{\tau_i}{\tau} c_{ik}$. Thus, if investors agree on the coefficient of $\ln p$ on the $j-th$ basis function i.e. $c_{ij} = c_j$, then that fund is not held by anyone.

Under constant risk tolerance, the risk-neutral density given in (22) simplifies:

Theorem 8: *In a general equilibrium pure exchange economy with negative exponential utility, the risk neutral density is given by:*

$$q(S) = \kappa \exp(-S/\tau) \prod_{i=1}^{n} [p_i(S)]^{\frac{\tau_i}{\tau}}, \tag{52}$$

The constant κ in Theorem 8 is a normalizing constant given by the requirement that q integrates to 1. Thus, the market view is a risk-tolerance weighted geometric average of the individual densities. Given a specification of probability beliefs and an array of risk tolerances, it is straightforward to use (52) to value an option or any other derivative.

Note that under homogeneous beliefs, (52) simplifies:

Corollary 9: *In a general equilibrium pure exchange economy with negative exponential utility and homogeneous beliefs, the risk-neutral density is given by:*

$$q(S) = \kappa \exp(-S/\tau) p(S). \tag{53}$$

Thus, if $p(S)$ is normal, then $q(S)$ is also normal with the same variance and with mean equal to the forward price as shown in Brennan[2]. The next subsection assumes that p is lognormal, and shows that q ends up in a different class than p. We also allow for heterogeneity in means and volatilities.

5.1 Zero Cautiousness and Lognormal Beliefs

When all investors have lognormal beliefs, (14) implies that the log of each density is quadratic in $x \equiv \ln(S/S_0)$:

$$\ln \ell(S; \mu_i, v_i) = -\ln(\sqrt{2\pi} v_i S_0) - x - \frac{1}{2}\left(\frac{x - (\mu_i - v_i^2/2)}{v_i}\right)^2. \tag{54}$$

To obtain the optimal payoff under lognormal beliefs and constant risk tolerance of τ_i, substitute (54) in (49):

Theorem 9: *In a general equilibrium pure exchange economy with negative exponential utility and lognormal beliefs, the optimal payoff is given by:*

$$\phi_i(S) = \frac{W_0^i - m_i V_0[x] + p_i V_0[x^2/2] - S_0 \tau_i/\tau}{B_0} + \frac{\tau_i}{\tau} S + m_i x - p_i \frac{x^2}{2},$$

where $m_i \equiv \tau_i \left(\frac{\mu_i}{v_i^2} - \sum_{i=1}^n \frac{\tau_i}{\tau}\frac{\mu_i}{v_i^2}\right)$ and $p_i \equiv \tau_i \left(\frac{1}{v_i^2} - \sum_{i=1}^n \frac{\tau_i}{\tau}\frac{1}{v_i^2}\right)$.

Under constant risk tolerance and lognormal beliefs, the optimal payoff for each investor involves just two derivatives, one paying the log of the stock price and the other paying its square. The log contract is used to speculate on the ratio of mean to variance, while the squared log contract is used to speculate on variance. If investors agree on the ratio of the mean to the variance, then from the definition of m_i, they do not hold the log contract. Similarly, if investors agree on volatility (i.e. $v_i = v\ \forall i$), then they do not hold the log squared contract. If in addition, investors agree on the mean, (i.e. $\mu_i = \mu\ \forall i$), then no derivatives are held, consistent with (25).

In order to determine each investor's position in the riskless fund, the two derivative funds must be priced. Substituting (54) in (52) determines the equilibrium risk-neutral density:

$$q(S) = \kappa' \exp(-S/\tau) \ell(S; \hat{\mu}, \hat{v}), \tag{55}$$

where the aggregate precision $\frac{1}{\hat{v}^2} = \sum_{i=1}^n \frac{\tau_i}{\tau}\frac{1}{v_i^2}$ is a risk tolerance weighted average of the individual precisions, and $\hat{\mu} = \frac{\sum_{i=1}^n (\tau_i/v_i^2)\mu_i}{\sum_{i=1}^n (\tau_i/v_i^2)}$ is a weighted average of the individual means, where the weights are given by the ratio of the risk tolerance to the risk. The constant κ' is determined by requiring that q integrate to 1:

$$\kappa' = 1/ \int_0^\infty \exp(-S/\tau)\ell(S;\hat{\mu},\hat{v})dS \quad . \tag{56}$$

Unfortunately, the denominator is the Laplace transform of a lognormal density, which must be determined numerically. Once κ' is known, the values of the two derivatives funds are also obtained by quadrature:

$$V_0[x^j] = B_0 \int_0^\infty [\ln(S/S_0)]^j q(S)dS, j = 1, 2.$$

To obtain the bond price, multiply (55) by $B_0 S$ and integrate over S:

$$B_0 \int_0^\infty S\kappa' \exp(-S/\tau)\ell(S;\hat{\mu},\hat{v})dS = S_0,$$

from (19). Substituting $S\ell(S;\hat{\mu},\hat{v}) = S_0 e^{\hat{\mu}}\ell(\hat{\mu}+\hat{v}^2,\hat{v})$ gives the bond pricing equation:

$$B_0 = 1/ \int_0^\infty \kappa' \exp(\hat{\mu} - S/\tau)\ell(S;\hat{\mu}+\hat{v}^2,\hat{v})dS \quad .$$

We summarize these results with the following theorem:

Theorem 10: *In a general equilibrium pure exchange economy with negative exponential utility and lognormal beliefs, the equilibrium risk-neutral density and bond price are given by:*

$$q(S) = \kappa' \exp(-S/\tau)\ell(S;\hat{\mu},\hat{v}) \tag{57}$$

$$B_0 = 1/ \int_0^\infty \kappa' \exp(\hat{\mu} - S/\tau)\ell(S;\hat{\mu}+\hat{v}^2,\hat{v})dS \quad , \tag{58}$$

where κ' is given in (56).

In Theorem 10, we note that $q(S)$ is not a lognormal density even though each investor believes that the stock price is lognormally distributed. However, the market view is lognormal since it is a geometric average of the lognormal individual views. The negative exponential adds negative skewness to this lognormal density[12]. As a result, a graph of Black Scholes implied volatilities against strike prices will slope down, as is observed in equity index option markets.

[12] However, if one investor is risk-neutral, say the n-th investor, then aggregate risk tolerance is infinite, and $q(S) = \ell(S;\mu_n, v_n)$.

6 Summary and Future Research

Our primary contribution is the explicit delineation of prices and positions for investors in a general equilibrium context. Under linear risk tolerance with identical cautiousness, the results of Cass and Stiglitz[4] imply that homogeneous beliefs induce investors to shun derivatives, even though they differ in risk aversion. However, under heterogeneous beliefs or other preference specifications, investors optimally hold derivatives individually, even though they are not held in aggregate. In fact, under generalized logarithmic utility and lognormal beliefs, options are used in the optimal portfolio, even if investors agree on volatility. In this case, an m fund separation result holds if the n investors select their means from among $m < n$ possible values. This separation result hold even though investors differ in risk aversion and thus has important implications for optimal security design.

Similarly, under negative exponential utility and lognormal beliefs, a four fund separation result occurs. In addition to the bond and the stock, investors take positions in two other derivatives: one which pays the log of the price and the other which pays the square of the log. The log contract is primarily used to express views on the mean, whereas the squared log contract is a vehicle for trading volatility. If investors use options to create the squared log contract, then the discontinuity in slope at the current stock price induces relatively large positions in at-the-money options. In a multi-period setup, movement of the stock price would induce large trading volume in such options, a phenomena which is universally observed in listed options markets.

Future research should investigate more fully the relationship between the implications of heterogeneous beliefs and the consequences of background risk as studied in Franke, Stapleton, and Subrahmanyam [7]. For example, it would be interesting to investigate whether higher background risk corresponds to a larger effective volatility view held by an investor engaged in a buy and hold strategy. Other interesting directions for future research would be to extend these results to a multi-period or intertemporal setting. In a continuous time economy with continuous trading opportunities, jumps of random size would induce the demand for options demonstrated here. In the interests of brevity, an investigation of the properties of such an equilibrium is best left for future research.

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Appendix 1: Proof of Equation 1

The fundamental theorem of calculus implies that for any fixed F:

$$\begin{aligned} f(S) &= f(F) + 1_{S>F}\int_F^S f'(u)du - 1_{S<F}\int_S^F f'(u)du \\ &= f(F) + 1_{S>F}\int_F^S \left[f'(F) + \int_F^u f''(v)dv \right] du \\ & \quad -1_{S<F}\int_S^F \left[f'(F) - \int_u^F f''(v)dv \right] du. \end{aligned}$$

Noting that $f'(F)$ does not depend on u and applying Fubini's theorem:

$$f(S) = f(F) + f'(F)(S-F) + 1_{S>F}\int_F^S \int_v^S f''(v)dudv + 1_{S<F}\int_S^F \int_S^v f''(v)dudv.$$

Performing the integral over u yields:

$$\begin{aligned} f(S) &= f(F) + f'(F)(S-F) + 1_{S>F}\int_F^S f''(v)(S-v)dv + 1_{S<F}\int_S^F f''(v)(v-S)dv \\ &= f(F) + f'(F)(S-F) + \int_F^\infty f''(v)(S-v)^+dv + \int_0^F f''(v)(v-S)^+dv. \qquad (59) \end{aligned}$$

Setting $F = S_0$, the initial stock price, gives Theorem 1. Note that if $F = 0$, the replication involves only bonds, stocks, and calls:

$$f(S) = f(0) + f'(0)S + \int_0^\infty f''(v)(S-v)^+dv,$$

provided the terms on the right hand side are all finite. Similarly, for claims with $\lim_{F\uparrow\infty} f(F)$ and $\lim_{F\uparrow\infty} f'(F)F$ both finite, we may also replicate using only bonds, stocks, and puts:

$$f(S) = \lim_{F\uparrow\infty} f(F) + \lim_{F\uparrow\infty} f'(F)(S-F) + \int_0^\infty f''(v)(v-S)^+dv.$$

Appendix 2: Proof of Equations 3 and 4

Given the existence of options of all strikes, absence of arbitrage and (59) imply:

$$V_0[f] = [f(F) - f'(F)]B_0 + f'(F)S_0 + \int_F^\infty f''(v)(S-v)^+ dv + \int_0^F f''(v)(v-S)^+ dv. \tag{60}$$

Integrating (2) by parts gives:

$$\begin{aligned} V_0[f] &= [f(F) - f'(F)]B_0 + f'(F)S_0 f'(K)P(K) \mid_0^F \\ &\quad - \int_0^F f'(K)P'(K)dK + f'(K)C(K) \mid_F^\infty - \int_F^\infty f'(K)C'(K)dK. \end{aligned}$$

Since $P(0) = 0$ and $C(\infty) = 0$ and $C(F) - P(F) = S_0 - FB_0$, the second, third, and fifth terms cancel. Integrating by parts once again yields:

$$\begin{aligned} V_0[f] &= B_0 f(F) - f(K)P'(K) \mid_0^F + \int_0^F f(K)P''(K)dK \\ &\quad - f(K)C'(K) \mid_F^\infty + \int_F^\infty f(K)C''(K)dK. \end{aligned}$$

Noting that $C'(\infty) = P'(0) = 0$ and $P'(F) - C'(F) = B_0$ by differentiating put call parity, we observe that:

$$V_0[f] = B_0 \int_0^\infty f(K)q(K)dK,$$

where $q(K)$ is proportional to the second derivative with respect to strike of the option pricing function:

$$q(K) \equiv \begin{cases} \frac{1}{B_0}\frac{\partial^2 P(K)}{\partial K^2} & \text{for } K \leq F; \\ \frac{1}{B_0}\frac{\partial^2 C(K)}{\partial K^2} & \text{for } K > F. \end{cases}$$

Setting $F = S_0$ gives the desired result.

Appendix 3: Proof of Theorem 26

Recall that under homogeneous beliefs, the optimal exposure simplifies to:

$$\phi_i'(S) = \frac{T_i[\phi_i(S)]}{T(S)},$$

where $T(S) \equiv \sum_{i=1}^{n} T_i[\phi_i(S)]$. Suppose we have $n = 2$ investors with linear risk tolerance:

$$\begin{aligned} \phi_1'(S) &= \frac{\tau_1 + \gamma_1\phi_1(S)}{\tau + \gamma_1\phi_1(S) + \gamma_2\phi_2(S)} && (61) \\ \phi_2'(S) &= \frac{\tau_2 + \gamma_2\phi_2(S)}{\tau + \gamma_1\phi_1(S) + \gamma_2\phi_2(S)}, && (62) \end{aligned}$$

where $\tau \equiv \tau_1 + \tau_2$. This is a coupled system of nonlinear o.d.e.'s. Fortunately, it can be solved if we assume opposite cautiousness i.e. $\gamma_1 = -\gamma_2$. Without loss of generality, let $\gamma_1 = -\gamma_2 = \gamma \geq 0$. Dividing (61) by (62) implies:

$$\frac{\phi_1'(S)}{\phi_2'(S)} = \frac{\tau_1 + \gamma\phi_1(S)}{\tau_2 - \gamma\phi_2(S)}.$$

Re-arranging gives $\gamma[\phi_1(S)\phi_2'(S) + \phi_2(S)\phi_1'(S)] - \tau_2\phi_1'(S) + \tau_1\phi_2'(S) = 0$. Integrating both sides gives $\gamma\phi_1(S)\phi_2(S) - \tau_2\phi_1(S) + \tau_1\phi_2(S) = c$, where c is the constant of integration. Substituting $\phi_1(S) = S - \phi_2(S)$ gives a quadratic in ϕ_2:

$$\gamma[S - \phi_2(S)]\phi_2(S) - \tau_2[S - \phi_2(S)] + \tau_1\phi_2(S) = c,$$

Dividing by 2γ and re-arranging gives:

$$\frac{1}{2}\phi_2^2(S) - \left(\frac{S}{2} + \frac{\tau}{2\gamma}\right)\phi_2(S) + \frac{\tau_2 S + c}{2\gamma} = 0,$$

with solution:

$$\phi_2(S) = \frac{S}{2} + \frac{\tau}{2\gamma} - \sqrt{\left(\frac{S}{2} + \frac{\tau}{2\gamma}\right)^2 - \frac{\tau_2 S + c}{\gamma}}. \qquad (63)$$

Since $\phi_1(S) = S - \phi_2(S)$, we have:

$$\phi_1(S) = \frac{S}{2} - \frac{\tau}{2\gamma} + \sqrt{\left(\frac{S}{2} + \frac{\tau}{2\gamma}\right)^2 - \frac{\tau_2 S + c}{\gamma}}.$$

In order that both payoffs be real, we require:

$$\left(\frac{S}{2} + \frac{\tau}{2\gamma}\right)^2 - \frac{\tau_2 S + c}{\gamma} \geq 0.$$

Completing the square gives $\left(\frac{S}{2} - \frac{\tau_2 - \tau_1}{2\gamma}\right)^2 + \frac{\tau_1 \tau_2}{\gamma^2} - \frac{c}{\gamma} \geq 0$.
Thus, a necessary condition for real payoffs is that $c \leq \frac{\tau_1 \tau_2}{\gamma}$.
Choosing c so that this condition holds, define k^2 by:

$$c = \frac{\tau_1 \tau_2}{\gamma} - k^2 \gamma.$$

Then the optimal payoffs can be written as:

$$\begin{aligned}
\phi_1(S) &= \frac{S}{2} - \frac{\tau}{2\gamma} + \sqrt{\left(\frac{S}{2} - \frac{\tau_2 - \tau_1}{2\gamma}\right)^2 + k^2} \\
\phi_2(S) &= \frac{S}{2} + \frac{\tau}{2\gamma} - \sqrt{\left(\frac{S}{2} - \frac{\tau_2 - \tau_1}{2\gamma}\right)^2 + k^2}.
\end{aligned}$$

Note that if we set $k = 0$, then the payoffs are linear:

$$\phi_1(S) = S - \frac{\tau_2}{\gamma} \qquad \phi_2(S) = \frac{\tau_2}{\gamma}.$$

In any case, the positions sum to the stock as required. Furthermore, since $T_1[\phi_1(S)] = \tau_1 + \gamma \phi_1(S)$, we have:

$$T_1[\phi_1(S)] = \frac{\gamma S - (\tau_2 - \tau_1)}{2} + \sqrt{\left(\frac{\gamma S - (\tau_2 - \tau_1)}{2}\right)^2 + \gamma^2 k^2},$$

which is nonnegative. Similarly, since $T_2[\phi_2(S)] = \tau_2 - \gamma \phi_2(S)$, we have:

$$T_2[\phi_2(S)] = -\frac{\gamma S - (\tau_2 - \tau_1)}{2} + \sqrt{\left(\frac{\gamma S - (\tau_2 - \tau_1)}{2}\right)^2 + \gamma^2 k^2},$$

which is also nonnegative.