Put-Call Symmetry: Extensions and Applications

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This version:‡ December 7, 2007

Abstract

Classic put-call symmetry relates the prices of puts and calls at strikes on opposite sides of the forward price. We extend put-call symmetry in several directions. Relaxing the assumptions, we generalize to unified local/stochastic volatility models and time-changed Lévy processes, under a symmetry condition. Further relaxing the assumptions, we generalize to various *asymmetric* dynamics. Extending the conclusions, we take an arbitrarily given payoff of European style or single/double/sequential-barrier style, and we construct a conjugate European-style claim of equal value, and thereby a semi-static hedge of the given payoff.

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‡We thank David Bates, Tomasz Bielecki, George Constantinides, Raphael Douady, Bruno Dupire, George Papanicolaou, Liuren Wu, and especially the referees for very helpful comments.

1 Introduction

Classic put-call symmetry (Bates [3] and Bowie-Carr [4]), relates the prices of puts and calls at strikes that are unequal but equidistant logarithmically to the forward price. For example, it implies that if a forward price M follows geometric Brownian motion under an appropriate pricing measure, and $M_0 = 100$, then a 200-strike call on M has time-0 price equal to 2 times the price of the 50-strike put at the same expiry.

We extend put-call symmetry in several directions. Relaxing the assumptions, we generalize to unified local/stochastic volatility models and time-changed Lévy processes, under a symmetry condition. Further relaxing the assumptions, we generalize to various *asymmetric* dynamics. Extending the conclusions, we take an arbitrarily given payoff of European style or single/double/sequential-barrier style, and we construct a conjugate European-style claim of equal value, and thereby a semi-static hedge of the given payoff.

Specifically, generalizing Bates [3] and Schroder [20], we prove, for any positive $\mathbb{P}$-martingale M, the equivalence of four conditions: first, the symmetry of the implied volatility skew generated by M; second, the equality of the distributions of M_T/M_0 under $\mathbb{P}$ and M_0/M_T under $\mathbb{M}$ where $d\mathbb{M}/d\mathbb{P} = M_T/M_0$; third, the equality $\mathbb{E}_0 G(M_T) = \mathbb{E}_0[(M_T/M_0)G(M_0^2/M_T)]$ for arbitrary payoff functions G; fourth, the symmetry of the $\mathbb{H}$-distribution of $\log(M_T/M_0)$, where $d\mathbb{H}/d\mathbb{P} = M_T^{1/2}/\mathbb{E}M_T^{1/2}$. If any of (hence all of) these conditions hold, we say that M satisfies PCS. Moreover, this equivalence generalizes to stopping times τ, in place of time 0.

Then we find conditions on the M dynamics sufficient for PCS to hold. Extending Carr-Ellis-Gupta [7], Schroder [20], and Fajardo-Mordecki [11], we treat cases which include local volatility diffusions, stochastic volatility diffusions, stochastically time-changed Lévy processes, and combinations thereof. Moreover, in the case of stochastic volatility, we find a *necessary* and sufficient condition, by proving a converse to the Renault-Touzi theorem [19] that independent stochastic volatility implies symmetry of the volatility skew.

Next we develop consequences and applications of PCS. Immediate corollaries include the classic put-call symmetry, obtained by taking G be a call payoff. Then, extending Bowie-Carr [4], Carr-Ellis-Gupta [7], and Carr-Chou [5, 6], we replicate – and thus price – single barrier, double barrier, and sequential barrier options having general payoffs. The replication strategies are semi-static (by trading only at first passage times) and model-independent (by assuming only PCS). We also explicitly extract first-passage-time densities given vanilla option prices.

We extend the scope of these applications to processes which do *not* satisfy PCS, by applying various transformations in single and multiple variables. Thus our pricing/hedging results extend to non-martingale and skewed-volatility dynamics, including asymmetric local volatility (with contributions from Forde [12]), asymmetric CGMY, and asymmetric jump diffusions, all with drift. This extension has practical significance in equity markets, which typically exhibit asymmetry.

We view these results as part of a broad program which aims to use European options – whose values are determined by *marginal* distributions – to extract information about *path-dependent* risks, and to hedge those risks robustly.

2 Put-Call Symmetry

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a complete probability space with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ contains all the null sets of $\mathbb{F}$. Let $\mathbb{E}_t$ denote $\mathcal{F}_t$-conditional $\mathbb{P}$-expectation. Let $T > 0$.

Definition 1. Define a *payoff* function to be a nonnegative Borel function on $\mathbb{R}$.

2.1 Arithmetic PCS

Let X be an adapted process. The following proposition needs no proof.

Proposition 2.1. *The time-*0 *distribution of* $X_T - X_0$ *is symmetric if and only if*

$$\mathbb{E}_0 G(X_T - X_0) = \mathbb{E}_0 G(X_0 - X_T) \tag{2.1}$$

for any payoff function G.

Note that both sides of (2.1) always exist, in the extended reals.

Definition 2. If the condition in Proposition 2.1 holds, then we say that ***arithmetic*** $\boldsymbol{PCS}_0$ holds for $(X, \mathcal{F}, \mathbb{P})$. Any of the 0, X, $\mathcal{F}$, $\mathbb{P}$ may be suppressed if they are clear from the context.

2.2 Geometric PCS

Of more importance to us than arithmetic PCS is *geometric* PCS, which can be defined using any of four equivalent conditions.

Throughout this paper, let M be a positive càdlàg martingale; this implies in particular that $\mathbb{E}_0 M_T = M_0 < \infty$. A standard example takes M to be a T-forward price, and $\mathbb{P}$ to be "domestic" T-forward risk-neutral measure.

Define the "foreign" risk-neutral measure $\mathbb{M}$ by the Radon-Nikodym derivative

$$d\mathbb{M}/d\mathbb{P} := M_T/M_0.$$

Define the "half" measure $\mathbb{H}$ by the Radon-Nikodym derivative

$$d\mathbb{H}/d\mathbb{P} := M_T^{1/2}/\mathbb{E}M_T^{1/2}.$$

Let $\mathbb{E} \equiv \mathbb{E}^{\mathbb{P}}$ and $\mathbb{E}^{\mathbb{M}}$ denote expectations under $\mathbb{P}$ and $\mathbb{M}$ respectively. Write $\mathbb{E}_t$ and $\mathbb{E}_t^{\mathbb{M}}$ for the $\mathcal{F}_t$-conditional expectations, and write $\mathbb{P}_t$ and $\mathbb{M}_t$ for the regular conditional probabilities given $\mathcal{F}_t$.

Definition 3. For $t < T$, the time-t expiry-T *implied volatility skew*, with respect to a measure $\mathbb{Q}$, of an integrable positive random variable U, is defined for each $x \in \mathbb{R}$ as the unique $I_t^{\mathbb{Q},U}(x)$ such that

$$\mathbb{E}_t^{\mathbb{Q}}(U - \kappa(x))^+ = (\mathbb{E}_t^{\mathbb{Q}} U) N\left(\frac{-x}{I_t^{\mathbb{Q},U}(x)\sqrt{T-t}} + \frac{I_t^{\mathbb{Q},U}(x)\sqrt{T-t}}{2}\right) - \kappa(x) N\left(\frac{-x}{I_t^{\mathbb{Q},U}(x)\sqrt{T-t}} - \frac{I_t^{\mathbb{Q},U}(x)\sqrt{T-t}}{2}\right)$$

where $\kappa(x) := e^x \mathbb{E}_t^{\mathbb{Q}} U$. For $t = T$, our convention is to define the time-T expiry-T implied volatility skew to be zero for all x.

So $I^{\mathbb{P}} := I^{\mathbb{P},M_T}$ is the Black implied volatility of options on M_T, as a function of log-moneyness.

Write $I^{\mathbb{M}} := I^{\mathbb{M},1/M_T}$. If one interprets M as the dollar-denominated price of a Euro, then $I^{\mathbb{M}}$ is the implied volatility of Euro-denominated options on the dollar.

The conditions in the following proposition define PCS.

Theorem 2.2. *The following conditions are equivalent; if one holds a.s. on an event, then all do.*

*(a) The time-*0 *implied volatility skew at expiry T is symmetric in log-moneyness (i.e. $I_0^{\mathbb{P}}$ is even).*

*(b) The time-*0 *distribution of M_T/M_0 under $\mathbb{P}$ is identical to the time-*0 *distribution of M_0/M_T under $\mathbb{M}$.*

(c) We have

$$\mathbb{E}_0 G(M_T) = \mathbb{E}_0\left[\frac{M_T}{M_0} G\left(\frac{M_0^2}{M_T}\right)\right] \tag{2.2}$$

for any payoff function G.

*(d) The time-*0 *distribution of $X_T := \log(M_T/M_0)$ under $\mathbb{H}$ is symmetric.*

Definition 4. If any of the time-0 conditions (a, b, c, d) hold, then we say that [geometric] $\boldsymbol{PCS}_0$ holds for $(M, \mathcal{F}, \mathbb{P})$. Any of the 0, M, $\mathcal{F}$, $\mathbb{P}$ may be suppressed if they are clear from the context. "PCS" means geometric PCS, unless "arithmetic" is explicitly specified.

Definition 5. We call the function $m \mapsto (m/M_0)G(M_0^2/m)$ the *conjugate* of G, with respect to M_0.

Proof. Condition (b) is clearly equivalent to the following condition, which we designate (b'): The time-0 distribution of M_T under $\mathbb{P}$ is identical to the time-0 distribution of M_0^2/M_T under $\mathbb{M}$. In turn $(b') \Leftrightarrow (c)$, because the right-hand side of (2.2) is just $\mathbb{E}_0^{\mathbb{M}} G(M_0^2/M_T)$. Hence $(b) \Leftrightarrow (c)$.

To establish $(a) \Leftrightarrow (b)$, observe that Lee [15] Thm 4.1 proves the following model-independent fact (which in particular does not assume any of the conditions $(a) - (c)$): For all $x \in \mathbb{R}$,

$$I_0^{\mathbb{P}}(-x) = I_0^{\mathbb{M}}(x). \tag{2.3}$$

Therefore

$$(a) \iff \text{for all } x,\ I_0^{\mathbb{P}}(x) = I_0^{\mathbb{P}}(-x) \iff \text{for all } x,\ I_0^{\mathbb{P}}(x) = I_0^{\mathbb{M}}(x) \iff (b)$$

where the middle step is from (2.3), and the last step is because the time-0 implied volatility skew $I_0^{\mathbb{P}}$ (resp. $I_0^{\mathbb{M}}$) determines the time-0 distribution of M_T/M_0 (resp. M_0/M_T) under $\mathbb{P}$ (resp. $\mathbb{M}$).

To establish $(d) \Leftrightarrow (c)$, first note that $d\mathbb{H}/d\mathbb{P} = e^{X_T/2}/\mathbb{E}e^{X_T/2}$, so

$$\mathbb{E}_0 G(M_T) = \mathbb{E}_0^{\mathbb{H}} \frac{\mathbb{E}_0 e^{X_T/2}}{e^{X_T/2}} G(M_0 e^{X_T}) \quad \text{and} \quad \mathbb{E}_0\left[\frac{M_T}{M_0} G\left(\frac{M_0^2}{M_T}\right)\right] = \mathbb{E}_0^{\mathbb{H}} \frac{\mathbb{E}_0 e^{X_T/2}}{e^{-X_T/2}} G(M_0 e^{-X_T}).$$

If (d) holds, then X_T and $-X_T$ have the same time-0 $\mathbb{H}$-distribution, which proves (c). If (c) holds, then for any real constant p, taking $G(y) = (\pm \mathrm{Re}[(y/M_0)^{ip+1/2}])^+$ and $G(y) = (\pm \mathrm{Im}[(y/M_0)^{ip+1/2}])^+$ shows that $\mathbb{E}_0(M_T/M_0)^{ip+1/2} = \mathbb{E}_0(M_T/M_0)^{-ip+1/2}$, hence that X_T and $-X_T$ have the same time-0 $\mathbb{H}$-characteristic function. ☐

Remark 2.3. If $(M, \mathcal{H})$ is a positive martingale where $\mathcal{H}_t \subseteq \mathcal{F}_t$, then PCS holding for $(M, \mathcal{F})$ implies that it holds for $(M, \mathcal{H})$, by iterated expectations.

Remark 2.4. Condition (d) relates [geometric] PCS to arithmetic PCS. Specifically, $(M, \mathbb{P})$ satisfies PCS if and only if $(X, \mathbb{H})$ satisfies arithmetic PCS, where $X_t := \log(M_t/M_0)$.

Each choice of G gives rise to a symmetry. If $\mathbb{P}$ is T-forward martingale measure, then these symmetries are indeed statements about European option prices, after multiplication on both sides by the price of a discount bond maturing at T.

Corollary 2.5 (Classic Put-Call Symmetry)**.** *If PCS holds, then for all $K > 0$,*

$$\mathbb{E}_0(M_T - K)^+ = \frac{K}{M_0}\mathbb{E}_0(M_0^2/K - M_T)^+.$$

Thus a call struck at K has price equal to K/M_0 puts struck at M_0^2/K.

Proof. By Theorem 2.2,

$$\mathbb{E}_0(M_T - K)^+ = \mathbb{E}_0 \left[\frac{M_T}{M_0}(M_0^2/M_T - K)^+\right] = \frac{K}{M_0}\mathbb{E}_0(M_0^2/K - M_T)^+.$$

This is put-call symmetry in the sense of Bates [3] and Bowie-Carr [4]. See Figure 1. □

Corollary 2.6 (Power Symmetry)**.** *If PCS holds, then for all $p \in \mathbb{R}$,*

$$\mathbb{E}_0 M_T^p = M_0^{2p-1}\mathbb{E}_0 M_T^{1-p}.$$

Rewritten more symmetrically,

$$\mathbb{E}_0(M_T/M_0)^p = \mathbb{E}_0(M_T/M_0)^{1-p}.$$

Thus, for all p, claims on the pth and $(1-p)$th powers of price relative have the same value.

Proof. Let $G(y) = y^p$ in Theorem 2.2. See Figure 2. □

Corollary 2.7 (Variance-Entropy Symmetry)**.** *If PCS holds, then*

$$-\mathbb{E}_0 \log\left(\frac{M_T}{M_0}\right) = \mathbb{E}_0\left[\frac{M_T}{M_0}\log\left(\frac{M_T}{M_0}\right)\right]$$

provided that either expectation exists.

Proof. Apply Theorem 2.2 to the payoff functions $y \mapsto \log(y/M_0)^+$ and $y \mapsto [-\log(y/M_0)]^+$.

We call this "variance-entropy" symmetry because the RHS is the *relative entropy* of $\mathbb{M}$ with respect to $\mathbb{P}$; and if M is an Itô process, then Carr-Madan [9] shows the LHS is one-half of the *variance swap rate* (the expectation of *realized variance*, defined as the quadratic variation of $\log M_t$ on $[0, T]$). □

Figure 1: Classic put-call symmetry

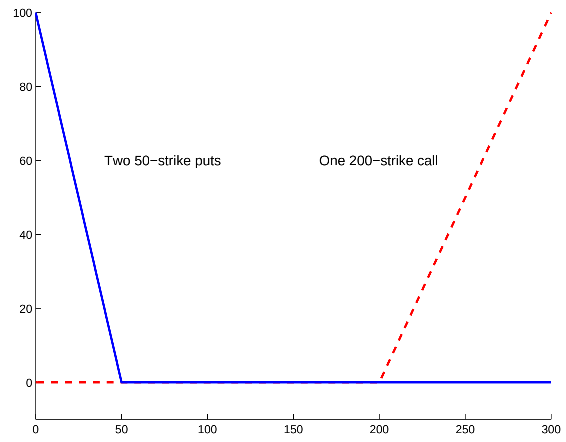


Figure 2: Power symmetry

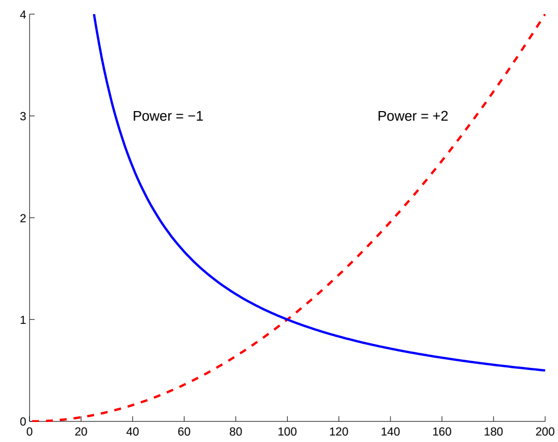
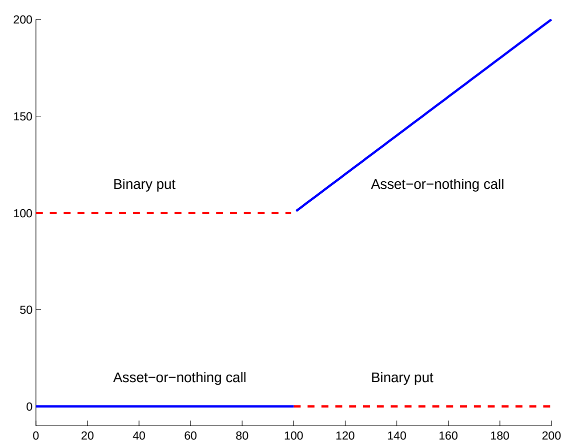


Figure 3: Asset-or-nothing/cash-or-nothing symmetry



Corollary 2.8 (Left/Right-truncation Symmetry)**.** *Assume PCS, and let $H > 0$. Then*

$$\mathbb{E}_0\big[G(M_T)\mathbb{I}(M_T < H)\big] = \mathbb{E}_0\left[\frac{M_T}{M_0}G\left(\frac{M_0^2}{M_T}\right)\mathbb{I}(M_T > M_0^2/H)\right].$$

for any payoff fucntion G.

Thus truncating the right-hand tail of a payoff at H and truncating the left-hand tail of the conjugate payoff at M_0^2/H have the same effect on price.

Proof. Apply Theorem 2.2 to $G(y)\mathbb{I}(y > H)$. □

Corollary 2.9 (Cash-or-nothing / Asset-or-nothing Symmetry)**.** *Assume PCS, and let $H > 0$. Then*

$$\mathbb{E}_0(M_0\mathbb{I}(M_T < H)) = \mathbb{E}_0(M_T\mathbb{I}(M_T > M_0^2/H)).$$

Thus M_0 cash-or-nothing puts struck at H have value equal to an asset-or-nothing call struck at M_0^2/H.

Proof. Let $G(y) = M_0$ in Corollary 2.8. This is binary symmetry in the sense of Carr-Ellis-Gupta [7]. See Figure 3. □

3 Sufficient Conditions for SDEs

We give symmetry conditions on stochastic differential equations sufficient for the PCS conditions (a, b, c, d) to hold. (If *asymmetry* conditions hold, Sections 6 and 7 show how to modify the PCS conclusions.)

3.1 Unified Local and Stochastic Volatility

Consider a "unified" volatility model, which combines local and stochastic volatility, by allowing volatility to depend jointly on spot and a second factor. Under symmetry conditions on the coefficients of the driving SDE, we prove that distributional symmetry (b) holds, and hence PCS holds. We thereby obtain, in a transparent and probabilistically intuitive way, generalizations of the symmetry results of Lee [14] and Renault-Touzi [19]. (Recent independent work by Poulsen [17] uses similar reasoning to show that (c) holds under Geometric Brownian motion – a special case of our local-stochastic volatility setting.)

Theorem 3.1 (Unified local and stochastic volatility)**.** *Suppose that the two-dimensional process $(\log(M_t/M_0), V_t)$ satisfies*

$$d\begin{pmatrix} X_t \\ V_t \end{pmatrix} = \begin{pmatrix} -f^2(X_t, V_t, t)/2 \\ \alpha(X_t, V_t, t) \end{pmatrix} dt + \begin{pmatrix} f(X_t, V_t, t) & 0 \\ 0 & \beta(X_t, V_t, t) \end{pmatrix} d\begin{pmatrix} W_{1t} \\ W_{2t} \end{pmatrix} \tag{3.1}$$

where (W_1, W_2) is a standard Brownian motion and the functions $\alpha(x, v, t)$, $\beta(x, v, t)$, and $f(x, v, t)$ are even in x and imply weak uniqueness for (3.1)*.*

Then PCS holds.

Proof. We have

$$\begin{aligned} d(-X_t) &= \frac{1}{2} f^2(X_t, V_t, t)dt - f(X_t, V_t, t)dW_{1t} \\ &= -\frac{1}{2} f^2(X_t, V_t, t)dt - f(X_t, V_t, t)dW_{1t}^{\mathbb{M}} \\ &= -\frac{1}{2} f^2(-X_t, V_t, t)dt + f(-X_t, V_t, t)d\tilde{W}_{1t}^{\mathbb{M}}, \end{aligned} \tag{3.2}$$

because Girsanov's Theorem implies that $(W_{1t}^{\mathbb{M}}, W_{2t})$ is a Brownian motion under $\mathbb{M}_0$, where

$$W_{1t}^{\mathbb{M}} := W_{1t} - \int_0^t f(X_s, V_s, s)ds,$$

and hence so is

$$(\tilde{W}_{1t}^{\mathbb{M}}, W_{2t}) := (-W_{1t}^{\mathbb{M}}, W_{2t}). \tag{3.3}$$

Moreover, by the evenness of α and β in x,

$$dV_t = \alpha(-X_t, V_t, t)dt + \beta(-X_t, V_t, t)dW_{2t}. \tag{3.4}$$

By (3.1) and (3.2) and (3.4), both $((X, V); (W_1, W_2); \mathbb{P}_0)$ and $((-X, V); (\tilde{W}_1^{\mathbb{M}}, W_2); \mathbb{M}_0)$ solve the SDE (3.1). By weak uniqueness, X_T under $\mathbb{P}_0$ has the same distribution as $-X_T$ under $\mathbb{M}_0$, which is the desired condition (b). □

For conditions on α, β, and f sufficient for weak uniqueness to hold, see sources such as [2], Theorem 2.1.20. Note that the PDE approach to symmetry, as in [7], implicitly requires similar assumptions to ensure the uniqueness of PDE solutions.

As a first corollary, we generalize Lee [14] who showed that symmetric local volatility implies the smile symmetry property (a).

Corollary 3.2 (Symmetric local volatility)**.** *Suppose that* $X_t := \log(M_t/M_0)$ *satisfies*

$$dX_t = -\frac{1}{2}\sigma^2(X_t, t)dt + \sigma(X_t, t)dW_t \tag{3.5}$$

where the function σ *is such that* $\sigma(x, t) = \sigma(-x, t)$ *and weak uniqueness holds for* (3.5).

Then PCS holds.

Proof. This follows immediately from Theorem 3.1. □

As a second corollary, we generalize Renault-Touzi [19], who showed that independent Markovian stochastic volatility implies the smile symmetry property (a).

Corollary 3.3 (Independent Markovian stochastic volatility)**.** *Suppose that the two-dimensional process* $(\log(M_t/M_0), V_t)$ *satisfies*

$$d \begin{pmatrix} X_t \\ V_t \end{pmatrix} = \begin{pmatrix} -V_t/2 \\ \alpha(V_t, t) \end{pmatrix} dt + \begin{pmatrix} \sqrt{V_t} & 0 \\ 0 & \beta(V_t, t) \end{pmatrix} d \begin{pmatrix} W_{1t} \\ W_{2t} \end{pmatrix} \tag{3.6}$$

where (W_1, W_2) *is a standard Brownian motion and the functions* α *and* β *are such that weak uniqueness holds for* (3.6).

Then PCS holds.

Proof. This follows immediately from Theorem 3.1. □

3.2 *Necessary* and Sufficient Conditions under Stochastic Volatility

We formulate and prove a converse to Corollary 3.3.

Renault and Touzi [19] showed that in standard stochastic volatility models, zero correlation between price and volatility shocks implies a symmetric smile in implied volatility. Here we show that symmetric smile implies zero correlation.

Theorem 3.4. *Suppose that the two-dimensional process* $(\log M_t, V_t)$ *satisfies*

$$d\begin{pmatrix} X_t \\ V_t \end{pmatrix} = \begin{pmatrix} -V_t/2 \\ \alpha(V_t,t) \end{pmatrix} dt + \begin{pmatrix} \sqrt{V_t} & 0 \\ \rho\beta(V_t,t) & \bar{\rho}\beta(V_t,t) \end{pmatrix} d\begin{pmatrix} W_{1t} \\ W_{2t} \end{pmatrix} \tag{3.7}$$

where $\bar{\rho} := \sqrt{1-\rho^2}$ *and* (W_1, W_2) *is a standard Brownian motion. Assume that* $\alpha(v,t)$ *and* $\beta(v,t)$ *satisfy global Lipschitz conditions in* v*, uniformly in* t*. Assume that* V_t *and* β *are positive.*

Then $\rho = 0$ if and only if PCS_0 *holds for all* T *in some nonempty interval* $(T_*, T^*]$.

Proof. The "only if" direction is by Corollary 3.3. For the "if," note that by the same manipulations as in (3.2),

$$d(-X_t) = -\frac{1}{2}V_t dt + \sqrt{V_t}d\tilde{W}^{\mathbb{M}}_{1t}, \tag{3.8}$$

but the V dynamics are

$$dV_t = [\alpha(V_t,t) + \rho\beta(V_t,t)\sqrt{V_t}]dt - \rho\beta(V_t,t)d\tilde{W}^{\mathbb{M}}_{1t} + \bar{\rho}\beta(V_t,t)dW_{2t}. \tag{3.9}$$

where $(\tilde{W}^{\mathbb{M}}_{1t}, W_{2t})$ is the standard $\mathbb{M}$-Brownian motion, defined as in (3.3) with $f(x,v,t) := \sqrt{v}$.

We are given that for all $T \in (T_*, T^*]$, the $\mathbb{P}$-distribution of $X_T - X_0$ is identical to the $\mathbb{M}$-distribution of $X_0 - X_T$. Therefore

$$\mathbb{E}_0 \int_0^T V_t dt = -2\mathbb{E}_0(X_T - X_0) = -2\mathbb{E}^{\mathbb{M}}_0(X_0 - X_T) = \mathbb{E}^{\mathbb{M}}_0 \int_0^T V_t dt$$

where the first step is by (3.7) and the last step is by (3.8). So, for all $T \in (T_*, T^*]$, we have

$$\mathbb{E}_0 \int_T^{T^*} V_t dt = \mathbb{E}^{\mathbb{M}}_0 \int_T^{T^*} V_t dt.$$

By continuity of the V paths, $\mathbb{E}_0 V_{T^*} = \mathbb{E}^{\mathbb{M}}_0 V_{T^*}$. However, by applying to (3.9) the SDE comparison Theorem V.54 in Protter [18], combined with a coupling argument, we see that if $\rho \gtrless 0$, then $\mathbb{E}^{\mathbb{M}}_0 V_{T^*} \gtrless \mathbb{E}_0 V_{T^*}$.

We conclude that $\rho = 0$. □

4 Sufficient Conditions for Time-Changed Lévy Processes

We begin with a known result about Lévy processes; then we introduce multiple stochastic time changes.

4.1 Exponential Lévy Processes

Theorem 4.1 (Fajardo-Mordecki [11])**.** *Suppose that* $X_t := \log(M_t/M_0)$ *is a Lévy process whose Lévy measure* ν *satisfies*

$$\nu(dy) = e^{-y}\nu(-dy), \tag{4.1}$$

meaning that for all intervals $Y \subseteq \mathbb{R}$ *we have* $\nu(-Y) = \int_Y e^y d\nu(y)$*. Then* (b) *holds.*

Proof. We briefly review the known proof. Let X have $\mathbb{P}$-characteristic function $\psi(z)$. Then $-X$ has $\mathbb{M}$-characteristic function $\psi(-z-i)$. Now substitute $-z-i$ into the Lévy-Khinchin representation of ψ, rearrange terms, and conclude that $-X$ has $\mathbb{M}$-Lévy measure $e^{-y}\nu(-dy)$. □

A corollary is that PCS holds, by Theorem 2.2.

4.2 Stochastically Time-Changed Products of Exponential Lévy Processes

Now we introduce multiple stochastic clocks.

Theorem 4.2. *For* $j = 1, 2, \dots, J$*, let* $(H^j_s, \mathcal{H}^j_s)$ *be a positive martingale such that* $\log(H^j_s/H^j_0)$ *is a* $\mathcal{H}^j_s$*-Lévy process, with Lévy measure* ν^j *satisfying* (4.1)*.*

Let each θ^j *be a right-continuous increasing process, and let* $M_t := \prod_j H^j_{\theta^j_t}$*.*

Let $\Theta := \sigma(\{\theta^j_t : 1 \le j \le J,\ t \le T\})$ *and assume that* $\Theta, \mathcal{H}^1, \dots, \mathcal{H}^J$ *are independent.*

Then PCS holds for $(M_t, \mathcal{F}_t)$*, if* $\mathcal{F}_t = \bigvee_j \hat{\mathcal{H}}^j_{\theta^j_t}$*, where* $\hat{\mathcal{H}}^j_s := \mathcal{H}^j_s \vee \Theta$*.*

Proof. Let $\tau := 0$. It suffices to show that, conditional on $\mathcal{F}_\tau$, the $\mathbb{P}$-characteristic function of $X_T := \log(M_T/M_\tau)$ is identical to the $\mathbb{M}$-characteristic function of $-X_T$. This holds because, for all real z,

$$\mathbb{E}^{\mathbb{M}}[e^{-izX_T} | \mathcal{F}_\tau] = \mathbb{E}[e^{X_T}e^{-izX_T} | \mathcal{F}_\tau] = \mathbb{E}[e^{izX_T} | \mathcal{F}_\tau]$$

where the remainder of the proof will justify the last equality.

Let $\mathcal{H}^{(j)}_s := \mathcal{F}_{A^j_s}$, where $A^j_s := \inf\{t : \theta^j_t > s\}$.

For each j, we have that $\log(H^j_s/H^j_0)$ is a $\mathcal{H}^{(j)}_s$-Lévy process, and θ^j_τ is a $\mathcal{H}^{(j)}_s$-stopping time, hence each

$$Y^j_s := \log(H^j_{s+\theta^j_\tau}/H^j_{\theta^j_\tau}) \text{ is a } \mathcal{H}^{(j)}_{s+\theta^j_\tau}\text{-Lévy process.} \tag{4.2}$$

Applying Theorem 4.1, under $\mathcal{F}_\tau$-conditional probability measure, to the process $(Y^j_s, \mathcal{H}^{(j)}_{s+\theta^j_\tau})$, on the time interval $s \in [0,\ \theta^j_T - \theta^j_\tau]$, we have

$$\mathbb{E}\ e^{izX^j_T} | \mathcal{F}_\tau\ = \mathbb{E}\ e^{X^j_T}e^{-izX^j_T} | \mathcal{F}_\tau\ .$$

where $X^j_T := \log(H^j_{\theta_T}/H^j_0)$. Moreover, the X^j_T for $j = 1, \dots, J$ are independent given $\mathcal{F}_\tau$. Taking the product over $j = 1, \dots, J$ completes the proof. □

Remark 4.3. By Remark 2.3, PCS also holds for smaller filtrations, such as $\mathcal{F}_t = \bigvee_j \hat{\mathcal{G}}^j_{\theta^j_t}$ where $\hat{\mathcal{G}}^j_s := \mathcal{H}^j_s \vee \sigma(A^j_u : u \le s)$, or such as $\mathcal{F}_t = \sigma(M_u : u \le t)$.

Remark 4.4. We do not assume that the stochastic clocks $\theta^1, \ldots, \theta^J$ are independent of each other, only that they are independent of the Lévy processes. The clocks could be mutually independent, or correlated, or identical.

In the following corollary, the continuous part of a jump-diffusion runs on one clock, while the jumpy part runs on another clock.

Corollary 4.5 (Jump diffusion)**.** *Let σ_t and λ_t be nonnegative processes, predictable with respect to some filtration $\mathcal{R}_t \subseteq \mathcal{F}_t$, and such that $\int_0^t \sigma_u^2 du$ and $\int_0^t \lambda_u du$ are a.s. finite for all t. Let*

$$dM_t/M_{t-} = \sigma_t dW_t + dZ_t \tag{4.3}$$

where $(W, \mathcal{F})$ is Brownian motion and $(Z, \mathcal{F})$ is a compensated compound Cox process [equivalently, a compensated doubly-stochastic compound Poisson process] driven by $\mathcal{R}$, with intensity λ_t, such that at jump times, $\log(1+\Delta Z)$ has distribution ν satisfying

$$\nu(dy) = e^{-y}\nu(-dy).$$

Assume that W and $\mathcal{R}$ are independent, and W and Z are conditionally independent given $\mathcal{R}$.

Then PCS holds.

Proof. Let W' be a Brownian motion and Z' a unit-intensity compensated compound Poisson process with jump distribution ν, such that W' and Z' and $\mathcal{R}_\infty$ are independent. Let

$$\theta_t^1 := \int_0^t \sigma_s^2 ds \qquad \theta_t^2 := \int_0^t \lambda_s ds.$$

and let $\mathcal{G}_t := \mathcal{F}^{W'}_{\theta_t^1} \vee \mathcal{F}^{Z'}_{\theta_t^2} \vee \mathcal{R}_\infty$.

Conditional on $\mathcal{F}_0 \vee \mathcal{R}_\infty$, the process $Y_t := \int_0^t \sigma dW$ is Gaussian with instantaneous variance σ_t^2, independent of Z which is inhomogeneous compound Poisson with intensity λ_t. So the $(\mathcal{F}_0 \vee \mathcal{R}_\infty)$-conditional distribution of the process (Y_t, Z_t) matches the $\mathcal{G}_0$-conditional distribution of the process $(W'_{\theta_t^1}, Z'_{\theta_t^2})$.

Letting $\mathcal{E}$ denote stochastic exponential, then, the $(\mathcal{F}_0 \vee \mathcal{R}_\infty)$-conditional distribution of the process

$$M_t = M_0\ \mathcal{E}\big(Y+Z\big)_t = M_0\ \mathcal{E}\big(Y\big)_t \mathcal{E}\big(Z\big)_t$$

matches the $\mathcal{G}_0$-conditional distribution of the process

$$M'_t := M_0\ \mathcal{E}\big(W'\big)_{\theta_t^1} \mathcal{E}\big(Z'\big)_{\theta_t^2}.$$

By Theorem 4.2, PCS holds for $(M', \mathcal{G})$. So PCS holds for $(M, \mathcal{F} \vee \mathcal{R}_\infty)$, and hence for $(M, \mathcal{F})$. $\square$

Remark 4.6. Taking $Z = 0$ and σ^2 to be a one-dimensional diffusion recovers Corollary 3.3. Corollary 4.5 generalizes by allowing jumps in the volatility and price processes, and allowing multi-dimensional or non-Markovian volatility dynamics.

Remark 4.7. Special cases of Corollary 4.5 appear in Bates ([3], Proposition 2 and Corollaries 1,2), Schroder ([20] Example 8), and Andreasen [1]. The first two assumed that λ and σ solve SDEs driven by Brownian motion, and the last assumed no price jumps. Our approach extends into the general setting of time-changed Lévy processes (and combinations thereof), provides a unified transparent proof, and places the conclusions into the multifaceted PCS framework.

5 Applications to Barrier Options

We give applications to the pricing and semi-static hedging of barrier options. "Semi-static hedging" here means replication of the barrier contract by trading European-style claims at no more than two times after inception. Compared to dynamic hedges, semi-static hedges avoid the costs of frequent trading, and often avoid dependence on specific modeling assumptions – in our setting, this means assuming PCS conditions instead of specifying the exact form of the underlying dynamics. For hedging of barrier options, particular cases of these semi-static strategies have been shown to outperform dynamic delta-hedging strategies in Carr-Wu's [10] empirical tests on JPY-USD and GBP-USD data, and in Nalholm-Poulsen's [16] simulations of four skew-consistent models.

Although we will focus on barrier contracts, let us remark that the pricing and replication of barriers has fundamental relevance to other path-dependent contracts that decompose into barriers. For example, by Bowie-Carr [4], a fixed-strike lookback option is equivalent to a strip of one-touch barrier options, with barriers ranging across all price levels beyond the strike.

We will show that a barrier-contingent claim paying $\chi G(M_T)$, where χ is the indicator of some barrier event and G is some payoff function, is hedged initially by a European-style claim paying some function $\Gamma(M_T)$, where we will compute explicitly the function Γ in terms of G. We assume the availability of the claim on the Γ payoff.

In typical cases, continuous payoffs Γ defined on some interval I can, in turn, be synthesized if we have T-expiry puts and calls at arbitrary strikes in I, along with T-maturity bonds. More precisely, if $\Gamma : I \to \mathbb{R}$ is a difference of convex functions, then for any $K_0 \in I$, we have for all $x \in I$ the representation

$$\Gamma(x) = \Gamma(K_0) + \Gamma'(K_0)(x - K_0) + \int_{K\in I:\ K\geq K_0} \Gamma''(K)(x-K)^+ dK + \int_{K\in I:\ K<K_0} \Gamma''(K)(K-x)^+ dK \tag{5.1}$$

where Γ' is the left-derivative of Γ, and Γ'' is the second derivative, which exists as a generalized function; see Carr-Madan [9].

In typical cases, discontinuous payoffs Γ can likewise be approximated using T-expiry puts, calls, and bonds. More precisely, let $I = [0, \infty)$ or $I = (0, \infty)$ be partitioned into intervals by a finite or infinite sequence $\{\cdots < x_{-1} < x_0 < x_1 < \cdots\} \subset I$ that has no limit point inside I. Assume that the restrictions of $\Gamma : I \to \mathbb{R}$ to each of those intervals (excluding the points $\{x_j\}$) are differences of convex functions, but allow Γ to be discontinuous at $\{x_j\}$. Construct an approximation Γ_ε as follows. There exist $\{a_j\}$ such that the intervals $(x_j - a_j, x_j + a_j)$ are nonempty and disjoint. For each $\varepsilon < 1$, define $I_\epsilon := \bigcup_j (x_j - \varepsilon a_j, x_j + \varepsilon a_j)$. Define $\Gamma_\varepsilon : I \to \mathbb{R}$ to be the continuous function such that

$$\Gamma_\varepsilon(x) := \Gamma(x) \quad \text{if } x \notin I_\epsilon \text{ or } x \in \{x_j\}$$

$$\Gamma''_\varepsilon(x) = 0 \quad \text{if } x \in I_\epsilon \backslash \{x_j\}.$$

Then Γ_ε is a difference of convex functions on I, and therefore has put/call/bond representation (5.1); and it approximates Γ in the sense that $\Gamma_\varepsilon \to \Gamma$ pointwise on I, as $\varepsilon \to 0$.

In the remainder of this paper, we will want to interpret expectations as prices (with respect to T-maturity bond numeraire), so let $\mathbb{P}$ be T-forward measure.

5.1 PCS at Stopping Times

First we reformulate arithmetic PCS for stopping times $\tau \le T$. The following needs no proof.

Proposition 5.1. *For any stopping time $\tau \le T$, the following conditions are equivalent; if one holds a.s. on an event, then both do.*

$(i)_\tau$ *The time-τ distribution of $X_T - X_\tau$ is symmetric*

$(ii)_\tau$ *We have*

$$\mathbb{E}_\tau G(X_T - X_\tau) = \mathbb{E}_\tau G(X_\tau - X_T) \tag{5.2}$$

for any payoff function G.

Definition 6. If condition $(i)_\tau$ or $(ii)_\tau$ holds, then we say that ***arithmetic PCS***$_\tau$ holds for $(X, \mathcal{F}, \mathbb{P})$. Any of the X, $\mathcal{F}$, $\mathbb{P}$ may be suppressed if they are clear from the context.

Now we reformulate geometric PCS for stopping times $\tau \le T$.

Theorem 5.2. *For any stopping time $\tau \le T$, the following conditions are equivalent; if one holds a.s. on an event, then all do.*

$(a)_\tau$ *The time-τ implied volatility skew at expiry T is symmetric in log-moneyness.*

$(b)_\tau$ *The time-τ conditional distribution of M_T/M_τ under $\mathbb{P}$ is identical to the time-τ conditional distribution of M_τ/M_T under $\mathbb{M}$.*

$(c)_\tau$ *We have*

$$\mathbb{E}_\tau G(M_T) = \mathbb{E}_\tau \left(\frac{M_T}{M_\tau} G \left(\frac{M_\tau^2}{M_T} \right) \right) \tag{5.3}$$

for any payoff function G.

$(d)_\tau$ *The time-τ conditional distribution of $X_T := \log(M_T/M_\tau)$ under $\mathbb{H}$ is symmetric.*

Definition 7. If any of the conditions $(a, b, c, d)_\tau$ hold, then we say that [geometric] ***PCS***$_\tau$ holds for $(M, \mathcal{F}, \mathbb{P})$. Any of the M, $\mathcal{F}$, $\mathbb{P}$ may be suppressed if they are clear from the context.

Proof. Follow the proof of Theorem 2.2, with τ in place of each 0. ☐

Some sufficient conditions are as follows.

Theorem 5.3. *Let $\tau \le T$ be a stopping time. Assume the two-dimensional process* $(\log(M_t/M_\tau),\ V_t)$ *satisfies for $t \in [\tau, T]$ the SDE*

$$d \begin{pmatrix} X_t \\ V_t \end{pmatrix} = \begin{pmatrix} -f^2(X_t, V_t, t)/2 \\ \alpha(X_t, V_t, t) \end{pmatrix} dt + \begin{pmatrix} f(X_t, V_t, t) & 0 \\ 0 & \beta(X_t, V_t, t) \end{pmatrix} d \begin{pmatrix} W_{1t} \\ W_{2t} \end{pmatrix}$$

where (W_1, W_2) is $\mathbb{P}$-BM and the functions $\alpha(x, v, t)$, $\beta(x, v, t)$, and $f(x, v, t)$ are even in x and imply weak uniqueness for the SDE. Then PCS_τ holds.

Proof. By the strong Markov property, (W_1, W_2) is still $\mathcal{F}_\tau$-conditionally a Brownian motion on $[\tau, T]$. So we may follow the proof of Theorem 3.1, with τ in place of each 0. □

Example (Independent stochastic volatility). In particular, taking f, α, β to depend on (v, t) only, Theorem 5.3 implies that the models specified in Corollary 3.3 satisfy PCS_τ for *all* $\tau \le T$.

Example (Local volatility symmetric about a barrier). Let $H > M_0$ be a constant; think of it as the particular barrier level in some knock-in contract. Let $\tau_H := \inf\{t : M_t \ge H\}$. If $dM_t = \sigma(|\log(M_t/H)|)M_t dW_{1t}$ where σ is any sufficiently regular function, then Theorem 5.3, with $f(x, v, t) = \sigma(|x|)$, implies that M satisfies $\text{PCS}_{\tau_H \wedge T}$. This will suffice to derive hedges for the barrier contract. We will not assume that PCS_τ holds for *all* $\tau \le T$.

Theorem 5.4. *For $j = 1, 2, \ldots, J$, let $(H^j_s, \mathcal{H}^j_s)$ be a positive martingale such that* $\log(H^j_s/H^j_0)$ *is a $\mathcal{H}^j_s$-Lévy process, with Lévy measure ν^j satisfying* (4.1).

Let each θ^j be a right-continuous increasing process, and let $M_t := \prod_j H^j_{\theta^j_t}$.

Let $\Theta := \sigma(\{\theta^j_t : 1 \le j \le J,\ t \le T\})$ and assume that $\Theta, \mathcal{H}^1, \ldots, \mathcal{H}^J$ are independent.

If $\mathcal{F}_t = \bigvee_j \hat{\mathcal{H}}^j_{\theta^j_t}$, where $\hat{\mathcal{H}}^j_s := \mathcal{H}^j_s \vee \Theta$, and $\tau \le T$ is any stopping time, then PCS_τ holds for $(M_t, \mathcal{F}_t)$.

Proof. Each step in the proof of Theorem 4.2 still holds for general $\tau \le T$, instead of $\tau = 0$.

In particular, (4.2) is by the strong Markov property. The $\mathcal{F}_\tau$-conditional independence of $X^1_T, \ldots, X^J_T$ is straightforward to verify for τ taking only countably many possible values; for general τ, construct a sequence of stopping times $\tau_n \downarrow \tau$, where each τ_n takes countably many values, and apply the backwards martingale convergence theorem. □

Each hedging application will depend only on PCS_τ holding for a *particular* stopping time τ, not *all* stopping times τ. Each theorem will specify the relevant τ to be the passage time to the particular barrier(s) in the contract.

5.2 Semi-Static Hedging of Single Barriers

In order to maintain our focus on [geometric] PCS, we will prove just the basic hedging result under arithmetic PCS, and then proceed with the full development for the geometric case.

Theorem 5.5. *Let X be a martingale. Let τ_H be the first passage time to the barrier $H \neq X_0$; thus $\tau_H := \inf\{t : \eta X_t \geq \eta H\}$, where $\eta := \operatorname{sgn}(H - X_0) \in \{-1, 1\}$. Let $\chi := \mathbb{I}(\tau_H \leq T)$.*

Assume that $X_{\tau_H} = H$ in the event that $\tau_H \leq T$; a sufficient condition is that X has continuous paths. If X satisfies arithmetic PCS$_{\tau_H \wedge T}$, then for any payoff function G with $\mathbb{E}G(X_T) < \infty$, the following semi-static strategy replicates a knock-in claim on $\chi G(X_T)$.

At time 0, hold a European-style claim on

$$\Gamma(X_T) = G(X_T)\mathbb{I}(\eta X_T \geq \eta H) + G(2H - X_T)\mathbb{I}(\eta X_T > \eta H). \tag{5.4}$$

If and when the barrier knocks in, exchange the $\Gamma(X_T)$ claim for a claim on $G(X_T)$, at zero cost.

Proof. On the event that $\tau_H > T$, the barrier never knocks in, and the claim on $\Gamma(X_T)$ expires worthless, as desired. On the event that $\tau_H \leq T$, we need to show the zero cost of the exchange from the $\Gamma(X_T)$ claim to the $G(X_T)$ claim:

$$\begin{aligned} \mathbb{E}_\tau G(X_T) &= \mathbb{E}_\tau G(X_T)\mathbb{I}(\eta X_T \geq \eta H) + \mathbb{E}_\tau G(X_T)\mathbb{I}(\eta X_T < \eta H) \\ &= \mathbb{E}_\tau G(X_T)\mathbb{I}(\eta X_T \geq \eta H) + \mathbb{E}_\tau G(2X_\tau - X_T)\mathbb{I}(\eta(2X_\tau - X_T) < \eta H) \\ &= \mathbb{E}_\tau[G(X_T)\mathbb{I}(\eta X_T \geq \eta H) + G(2H - X_T)\mathbb{I}(\eta X_T > \eta H)] \end{aligned} \tag{5.5}$$

as desired, where the middle step is by Proposition 5.1 and the last step is because $X_\tau = H$ on the event $\tau_H \leq T$. □

Henceforth we will devote our attention to the case of geometric PCS.

Let τ_H be the first-passage time of M to the barrier $H \neq M_0$, where M is a positive martingale. Thus $\tau_H := \inf\{t : \eta M_t \geq \eta H\}$ where $\eta := \operatorname{sgn}(H - M_0) \in \{-1, 1\}$.

Theorem 5.6 (Single barrier)**.** *Fixing any expiry T and barrier $H \neq M_0$, let $\chi := \mathbb{I}(\tau_H \leq T)$. Assume that $M_{\tau_H} = H$ in the event that $\tau_H \leq T$; a sufficient condition is that M has continuous paths. If M satisfies PCS$_{\tau_H \wedge T}$, then for any payoff function G with $\mathbb{E}G(M_T) < \infty$, the following semi-static strategy replicates a knock-in claim on $\chi G(M_T)$.*

At time 0, hold a European-style claim on

$$\Gamma_{\text{ki}}(M_T) = G(M_T)\mathbb{I}(\eta M_T \geq \eta H) + (M_T/H)G(H^2/M_T)\mathbb{I}(\eta M_T > \eta H). \tag{5.6}$$

If and when the barrier knocks in, exchange the $\Gamma_{\text{ki}}(M_T)$ claim for a claim on $G(M_T)$, at zero cost.

Proof. On the event that $\tau_H > T$, the barrier never knocks in, and the claim on $\Gamma(M_T)$ expires worthless, as desired. On the event that $\tau_H \leq T$, we need to show the zero cost of the exchange from the $\Gamma(M_T)$ claim to the $G(M_T)$ claim:

$$\begin{aligned} \mathbb{E}_\tau G(M_T) &= \mathbb{E}_\tau G(M_T)\mathbb{I}(\eta M_T \geq \eta H) + \mathbb{E}_\tau G(M_T)\mathbb{I}(\eta M_T < \eta H) \\ &= \mathbb{E}_\tau G(M_T)\mathbb{I}(\eta M_T \geq \eta H) + \mathbb{E}_\tau (M_T/M_\tau)G(M_\tau^2/M_T)\mathbb{I}(\eta(M_\tau^2/M_T) < \eta H) \\ &= \mathbb{E}_\tau[G(M_T)\mathbb{I}(\eta M_T \geq \eta H) + (M_T/H)G(H^2/M_T)\mathbb{I}(\eta M_T > \eta H)] \end{aligned} \tag{5.7}$$

as desired, where the middle step is by where the middle step is by Theorem 5.2 and the last step is because $M_\tau = H$ on the event $\tau_H \leq T$. □

Figure 4: We hedge this up-and-in payoff. Barrier is at 100, and $M_0 < 100$.

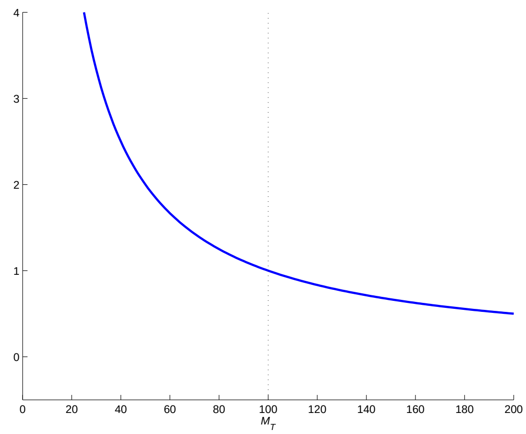


Figure 5: Decompose into sub-barrier and super-barrier payoffs, and conjugate the sub-barrier piece

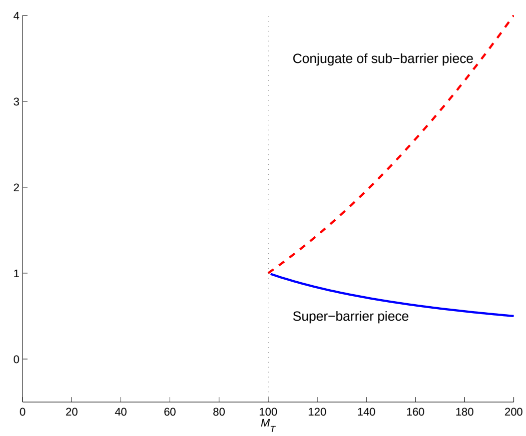
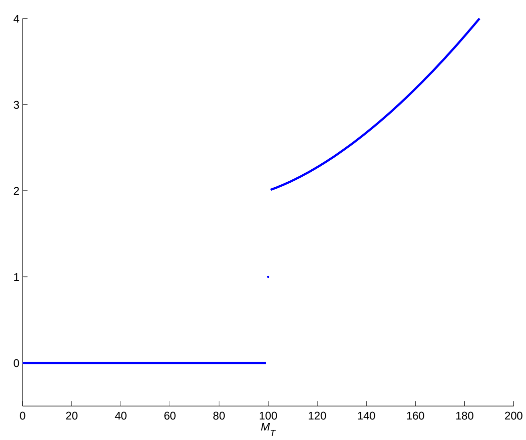


Figure 6: Sum the super-barrier and conjugated sub-barrier pieces. This is the semi-static hedge.



Figures 4 to 6 illustrate the construction of Γ_{ki}.

Remark 5.7. Theorems 5.6 and 5.14 extend the special cases treated (previously or concurrently) in Bowie-Carr [4], Carr-Ellis-Gupta [7], Carr-Chou [5, 6], Andreasen [1], Forde [12], and Poulsen [17]. Our approach provides a proof valid for all payoff functions, and for all PCS processes, including those with jumps, provided that the jumps cannot cross the barrier.

Remark 5.8. If either $\mathbb{P}(M_T = H) = 0$ or $G(H) = 0$, then the hedge payoff (5.6) can be rewritten

$$[G(M_T) + (M_T/H)G(H^2/M_T)]\mathbb{I}(\eta M_T > \eta H). \tag{5.8}$$

Absent those conditions, the simplification need not hold. Consider, for example, a one-touch whose barrier H is an absorbing boundary for the process M. Then a claim on (5.8) would have the wrong payoff (zero) in the event the barrier is hit, but our strategy (5.6) has the correct payoff.

According to Theorems 5.3 and 5.4, either of the following is sufficient for the PCS_τ assumption to hold: a local volatility function symmetric (logarithmically) about the barrier, or independent stochastic volatility. Under such conditions, Theorem 5.6 shows how to replicate barrier options using strategies which require trading only at the first passage time. Some examples follow.

Example 5.9 (Up-and-in put). Consider an up-and-in put struck at $K < H$. The hedge is a claim on

$$(K - M_T)^+\mathbb{I}(M_T \geq H) + (M_T/H)(K - H^2/M_T)^+\mathbb{I}(M_T > H) = (K/H)(M_T - H^2/K)^+.$$

So hold K/H calls struck at H^2/K. If M never hits H, the calls expire worthless, as desired. If and when M hits H, exchange the calls for a put struck at K. This example extends the validity of Carr-Ellis-Gupta's [7] hedging strategy to all continuous PCS processes.

Example 5.10 (One-touch with up-barrier). Consider a one-touch paying $\chi \times 1$ at expiry, where $H > M_0$. Then the hedge is a claim on

$$1 \times \mathbb{I}(M_T \geq H) + \frac{M_T}{H} \times 1 \times \mathbb{I}(M_T > H) = 2h_H(M_T) + \frac{1}{H}(M_T - H)^+,$$

where $h_H(y) := \frac{1}{2}\mathbb{I}(y = H) + \mathbb{I}(y > H)$. So the hedge is 2 "symmetric" binary calls plus $1/H$ vanilla calls, all struck at H, where we define a symmetric binary call to pay $1/2$ if it expires at-the-money.

If and when M hits H, exchange the position for a bond paying 1 at expiry.

Example 5.11 (Down-and-in power). For a claim paying χM_T^p, the hedge is a claim on

$$M_T^p\mathbb{I}(M_T \leq H) + H^{2p-1}M_T^{1-p}\mathbb{I}(M_T < H).$$

If and when M hits H, exchange the position for a claim on M_T^p.

Remark 5.12. A knock-*out* option paying $(1-\chi)G(M_T) = G(M_T) - \chi G(M_T)$ is identical to the difference between a European and a knock-in, so a semi-static hedge for the knock-out follows easily from the semi-static hedge for the knock-in.

Remark 5.13. If M has up (down) jumps, then the Theorem 5.6 hedging strategy for the up-and-in (down-and-in) option may no longer replicate, because the equality of $\mathbb{E}_{\tau_H}\Gamma(M_T)$ and $\mathbb{E}_{\tau_H}G(M_T)$ assumes that $M_{\tau_H} = H$. However, suppose that a.s.

$$\mathbb{E}_{\tau_H} \; \Gamma(M_T) - G(M_T) \; = f(M_{\tau_H}) \tag{5.9}$$

where f is some increasing (decreasing) function. Then the hedging portfolio *superreplicates* the knock-in, because $f(M_{\tau_H}) \geq f(H) = 0$.

A sufficient condition for (5.9) to hold in the case of the up (down) barrier is that $\Gamma - G$ is an increasing (decreasing) function and M is an exponential Lévy process.

5.3 Extracting First-Passage-Time Distributions

We extract from vanilla options prices the distribution of the first passage time τ_H where $H > M_0$.

Let $B_0(T)$, $C_0^b(K,T)$, $C_0(K,T)$ be the time-0 prices of respectively the bond, the symmetric binary call, and the vanilla call, with strike K and expiry T. In Example 5.10 we have shown

$$B_0(T)\mathbb{P}(\tau_H \leq T) = 2C_0^b(H,T) + \frac{1}{H}C_0(H,T),$$

so

$$\mathbb{P}(\tau_H \leq T) = \frac{2C_0^b(H,T)}{B_0(T)} + \frac{C_0(H,T)}{HB_0(T)}. \tag{5.10}$$

Note that $C_0^b(H,T)$ is implied by the prices of vanillas:

$$C_0^b(H,T) = \lim_{\varepsilon \to 0} \frac{C_0(H-\varepsilon,T) - C_0(H+\varepsilon,T)}{2\varepsilon}.$$

So, in principle, the first passage time distribution can be extracted from the prices of standard European calls.

If interest rates are deterministic, then the forward measures coincide for all T; if also (5.10) is differentiable in T, then the risk-neutral density $p_{\tau_H}(T)$ of the first passage time can be extracted from the forward prices of standard European calls struck around H and T:

$$p_{\tau_H}(T) = \frac{\partial}{\partial T} \;\; \frac{2C_0^b(H,T)}{B_0(T)} + \frac{C_0(H,T)}{HB_0(T)} \;\; . \tag{5.11}$$

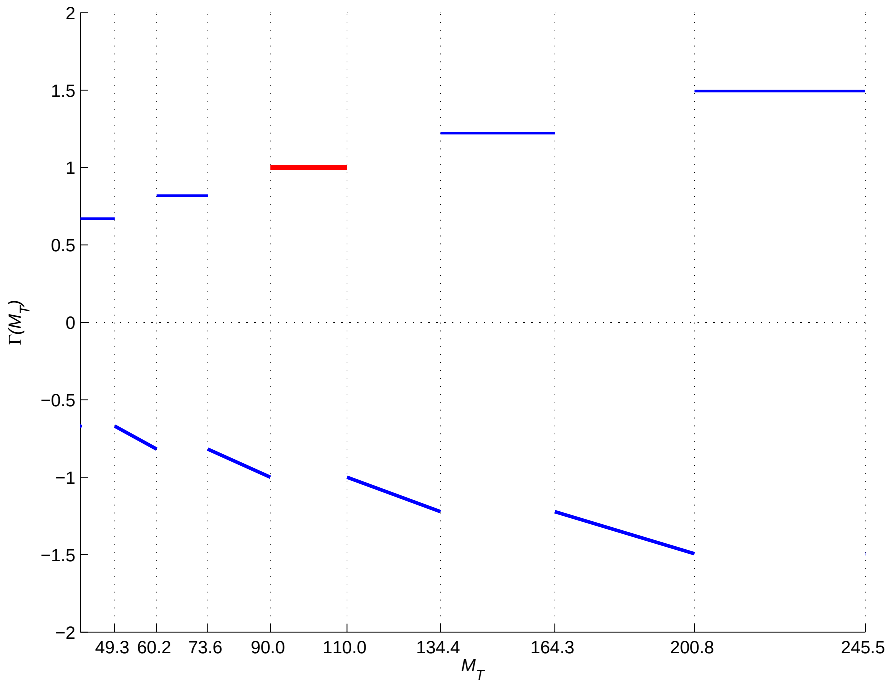
Thus the symmetry property enables inference of the distribution of first passage time (a path-dependent random variable) from European option prices (which are determined by marginal distributions).

5.4 Semi-Static Hedging of Double Barriers

Let $\tau_U := \inf\{t : M_t \geq U\}$ and $\tau_L := \inf\{t : M_t \geq L\}$ be the first passage times of M to, respectively, the upper barrier U and lower barrier L.

Special cases of the following result appear in some of the papers cited in Remark 5.7.

Figure 7: Payoff of the hedge of a double-no-touch with barriers at 90 and 110



Theorem 5.14 (Double barrier)**.** *Fixing any expiry T and barriers L, U with $0 < L < M_0 < U$, let $\chi := \mathbb{I}(\tau_L \wedge \tau_U \leq T)$. Assume that $M_{\tau_U} = U$ in the event $\tau_U \leq T$ and $M_{\tau_L} = L$ in the event $\tau_L \leq T$; a sufficient condition is that M has continuous paths. If $PCS_{\tau_L \wedge \tau_U \wedge T}$ holds, then for any payoff function G bounded on (L, U), the following semi-static strategy replicates a double-knock-out claim on $(1-\chi)G(M_T)$.*

At time 0, hold a European-style claim on

$$\Gamma_{\text{dko}}(M_T) := \sum_{n=-\infty}^{\infty} \frac{L^n}{U^n} G^*\Big(\frac{U^{2n} M_T}{L^{2n}}\Big) - \frac{L^{n-1} M_T}{U^n} G^*\Big(\frac{U^{2n}}{L^{2n-2} M_T}\Big) \;, \tag{5.12}$$

where $G^*(m) := G(m)\mathbb{I}(m \in (L, U))$. If and when the barrier knocks out, close the $\Gamma_{\text{dko}}(M_T)$ claim at zero cost.

Proof. For each M_T, at most one term in the infinite sum is nonzero, because G^* vanishes outside (L, U). Moreover, note that each term has absolute value bounded by a constant times $\sqrt{M_T}$:

If $L < \frac{U^{2n}}{L^{2n-2} M_T} < U$ then $\frac{U^{2n-1}}{L^{2n-2}} < M_T < \frac{U^{2n}}{L^{2n-1}}$, hence

$$\frac{L^{n-1} M_T}{U^n} \Big| G^*\Big(\frac{U^{2n}}{L^{2n-2} M_T}\Big)\Big| < C \frac{U^n}{L^n} < \frac{C\sqrt{U}}{L} \sqrt{M_T}.$$

If $L < \frac{U^{2n} M_T}{L^{2n}} < U$ then $\frac{L^n}{U^n} < \sqrt{M_T / L}$, hence

$$\frac{L^n}{U^n} \Big| G^*\Big(\frac{U^{2n} M_T}{L^{2n}}\Big)\Big| < \frac{C}{\sqrt{L}} \sqrt{M_T}.$$

It follows that

$$\sum_{n=1}^{\infty}\mathbb{E}_\tau \;\frac{L^n}{U^n}\, G^* \;\frac{U^{2n}M_T}{L^{2n}} \quad + \frac{L^{n-1}M_T}{U^n}\, G^* \;\frac{U^{2n}}{L^{2n-2}M_T} \qquad \le \frac{C\sqrt{U}}{L}\mathbb{E}_\tau \quad \overline{M_T} < \infty$$

because $\mathbb{E}_\tau M_T < \infty$ under our standing assumption that M is a positive martingale. Therefore, we may freely interchange expectation and summation, and telescope the sum if necessary.

If the barrier never knocks out, then the claim on $\Gamma_{\text{dko}}(M_T)$ expires worth $G^*(M_T) = G(M_T)$, as desired. So we need only establish the zero value, at the knock-out time, of the $\Gamma_{\text{dko}}(M_T)$ claim. On the event $\tau = \tau_U$,

$$\mathbb{E}_\tau\, \Gamma_{\text{dko}}(M_T) = \mathbb{E}_{\tau_U} \sum_{n=-\infty}^{\infty} \;\frac{L^n}{U^n} G^* \;\frac{U^{2n}M_T}{L^{2n}} \quad - \frac{L^{n-1}}{U^{n-1}} G^* \;\frac{U^{2n-2}M_T}{L^{2n-2}} \qquad = 0,$$

by Theorem 5.2. Likewise, on the event $\tau = \tau_L$,

$$\mathbb{E}_\tau\, \Gamma_{\text{dko}}(M_T) = \mathbb{E}_{\tau_L} \sum_{n=1}^{\infty} \;\frac{L^n}{U^n} G^* \;\frac{U^{2n}M_T}{L^{2n}} \quad - \frac{L^n}{U^n} G^* \;\frac{U^{2n}M_T}{L^{2n}} \qquad = 0,$$

by Theorem 5.2. □

Example 5.15. Figure 7 shows the semi-static hedge of a double-no-touch (a double-knock-out with $G = 1$). It also gives intuition for the construction (5.12). On $(L, U) = (90, 110)$, assign to Γ_{dko} the no-knockout value G. Moreover, to make $\mathbb{E}_{\tau_U} = \mathbb{E}_{\tau_L} = 0$, start by defining $\Gamma_{\text{dko}}|_{(73.6,90)}$ and $\Gamma_{\text{dko}}|_{(110,134.4)}$ to be the negatives of the conjugates of $\Gamma_{\text{dko}}\mathbb{I}_{(90,110)}$ with respect to 90 and 110 respectively. Then define $\Gamma_{\text{dko}}|_{(60.2,73.6)}$ and $\Gamma_{\text{dko}}|_{(134.4,164.3)}$ to be the negatives of the conjugates of $\Gamma_{\text{dko}}\mathbb{I}_{(110,134.4)}$ and $\Gamma_{\text{dko}}\mathbb{I}_{(73.6,90)}$ with respect to 90 and 110 respectively. Iterate this process.

Remark 5.16. A double-knock-*in* option paying $\chi G(M_T) = G(M_T) - (1-\chi)G(M_T)$ is identical to the difference between a European and a double-knock-out, so a semi-static hedge for the double-knock-in follows easily from the semi-static hedge for the double-knock-out.

Example 5.17. By taking G to be constant, we replicate a one-touch with payment at expiry. Moreover, under deterministic interest rates, a one-touch with payment at *hit* – such as the protection leg of an equity default swap – can be replicated by a continuum of our one-touches with payment-at-expiry.

5.5 Semi-Static Hedging of Sequential Barriers

Let $\tau_U := \inf\{t : M_t \ge U\}$ be the time of first-U-passage, and $\tau_{UL} := \inf\{t \ge \tau_U : M_t \le L\}$ be the time of first-L-passage-subsequent-to-τ_U, where $L, M_0 \in (0, U)$.

Let $\chi_{UL} := \mathbb{I}(\tau_{UL} \le T)$ and $\chi_U := \mathbb{I}(\tau_U \le T)$.

Consider the following example of a *sequential* barrier option: an up-and-in down-and-out claim which pays $\chi_U(1-\chi_{UL})G(M_T)$. Thus, the claim knocks in upon passage to U, becoming a down-and-out claim on $G(M_T)$ with knockout barrier at L, in effect from time τ_U to T.

Figure 8: We hedge this UI DO call, with up-barrier at 120, down-barrier at 90, and $K = 100$.

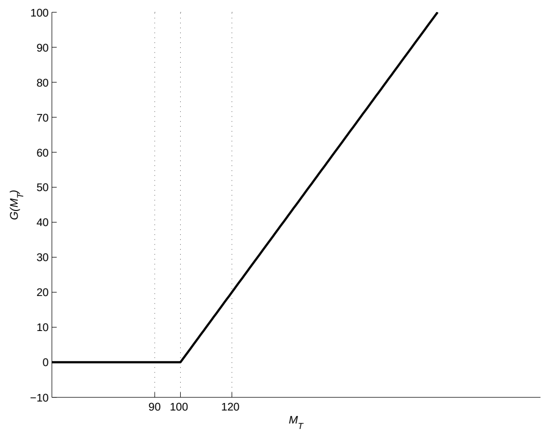


Figure 9: The semi-static hedge of a DO call with down-barrier 90 and $K = 100$.

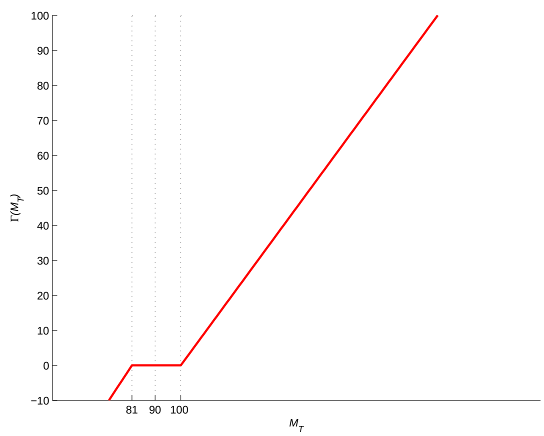
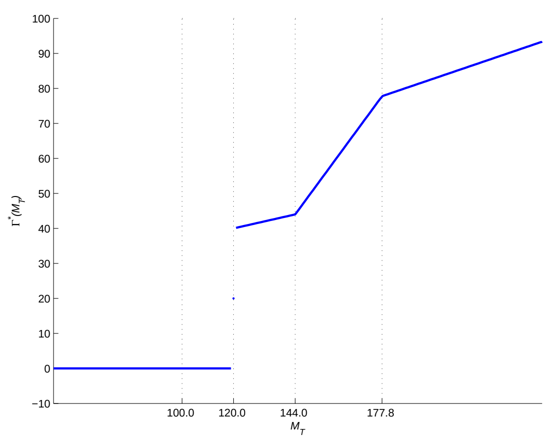


Figure 10: The semi-static hedge of the UI DO call.



Theorem 5.18 (Sequential barrier)**.** *Fixing any expiry T and barriers L, U with $L, M_0 \in (0, U)$, assume that $M_{\tau_U} = U$ in the event $\tau_U \leq T$ and that $M_{\tau_{UL}} = L$ in the event $\tau_{UL} \leq T$; a sufficient condition is that M has continuous paths. If PCS_{τ_U} and $PCS_{\tau_{UL}}$ hold, then for any payoff function G with $\mathbb{E}G(M_T) < \infty$, the following semi-static strategy replicates a claim on $\chi_U(1 - \chi_{UL})G(M_T)$. Let*

$$\Gamma(M_T) := G(M_T) - G(M_T)\mathbb{I}(M_T \leq L) - (M_T/L)G(L^2/M_T)\mathbb{I}(M_T < L). \tag{5.13}$$

At time 0, hold a European-style claim on

$$\Gamma^*(M_T) := \Gamma(M_T)\mathbb{I}(M_T \geq U) + (M_T/U)\Gamma(U^2/M_T)\mathbb{I}(M_T > U). \tag{5.14}$$

At time τ_U, convert this $\Gamma^*(M_T)$ claim to a claim on $\Gamma(M_T)$, at zero cost. Then, at time τ_{UL}, close out the $\Gamma(M_T)$ claim, at zero cost.

Proof. The terminal value of the semi-static strategy is $\chi_U(1 - \chi_{UL})G(M_T)$, so we need only check that each exchange occurs at zero cost. On the event $\tau_U \leq T$ we have

$$\mathbb{E}_{\tau_U}\Gamma^*(M_T) = \mathbb{E}_{\tau_U}\Gamma(M_T).$$

by PCS_{τ_U}. On the event $\tau_{UL} \leq T$ we have

$$\mathbb{E}_{\tau_{UL}}\Gamma(M_T) = 0$$

by $\text{PCS}_{\tau_{UL}}$ and Remark 5.12. □

Example 5.19. Figures 8 to 10 illustrate the strategy for an up-and-in down-and-out (UI DO) call, paying $\chi_U(1 - \chi_{UL})(M_T - 100)^+$ where $L = 90$ and $U = 120$ and $K = 100$. The initial hedge is a claim on the Figure 10 payoff. If M does not hit 120, the hedge expires worthless, as desired. Otherwise, when it hits 120, the UI DO call becomes a down-and-out (DO) call, so we should exchange the hedge costlessly for a claim on the Figure 9 payoff. If subsequently M does not hit 90, then the Figure 9 payoff coincides with the Figure 8 payoff of the $K = 100$ call, as desired; otherwise, when it hits 90, the DO call knocks out, so we should close the hedge at zero cost.

Remark 5.20. As motivation for the introduction of sequential barrier options, consider an investor who plans to engage in a dynamic option trading strategy. Sequential barrier options in theory allow the investor to lock in at time 0 the cost of those trades. For example, suppose the strategy is to go long a call if S rallies to U but close the position if S subsequently declines to $U - a$; the (possibly positive or negative) trading costs of this strategy have a market value which can be locked in by purchasing at time 0 a combination long a UI call with barrier U and short a sequential UI DO call.

6 Techniques for *Asymmetric* Dynamics

In this section we exhibit techniques for deriving symmetries and hedges, if the underlying does *not* satisfy PCS. These results have practical significance because, empirically, the implied volatility symmetry condition typically does not hold in equity markets.

Throughout this section, let S be an adapted process. The general strategy is to recognize that even if this underlying process S does not satisfy PCS, we may introduce a related *auxiliary process* M which does satisfy PCS. Then (c) holds for M, and converting back to the original variable S produces an identity for the original underlying.

In Theorems 6.1 to 6.3, let $S = \phi(M)$ where $\phi : (0, \infty) \to \mathbb{R}$ is continuous and strictly monotonic. Hence ϕ has an inverse ψ defined on an interval containing all possible values of S.

Theorem 6.1 (Transformed PCS)**.** *If $M = \psi(S)$ satisfies PCS then*

$$\mathbb{E}_0 G(S_T) = \mathbb{E}_0 \quad \frac{M_T}{M_0} G \circ \phi(M_0^2/M_T) \quad = \mathbb{E}_0 \quad \frac{\psi(S_T)}{\psi(S_0)} G \circ \phi \quad \frac{\psi(S_0)^2}{\psi(S_T)} \tag{6.1}$$

for any payoff function G.

Proof. Apply the PCS relationship (c) to the function $G \circ \phi$ and the martingale M. □

Theorem 6.2 (Single barrier)**.** *Fixing any expiry T and barrier $H = S_0$, let $\tau_H := \inf\{t : \eta S_t \geq \eta H\}$ where $\eta := \operatorname{sgn}(H - S_0)$. Let $\chi := \mathbb{I}(\tau_H \leq T)$. Assume that $S_{\tau_H} = H$ in the event $\tau_H \leq T$; a sufficient condition is that S has continuous paths. If $M = \psi(S)$ satisfies PCS$_{\tau_H \wedge T}$, then for any payoff function G with $\mathbb{E}G(S_T) < \infty$, the following semi-static strategy replicates a knock-in claim on $\chi G(S_T)$. At time 0, hold a European-style claim on*

$$\Gamma_{\text{ki}}(S_T) := G(S_T)\mathbb{I}(\eta S_T \geq \eta H) + \frac{\psi(S_T)}{\psi(H)} G \circ \phi \quad \frac{\psi(H)^2}{\psi(S_T)} \quad \mathbb{I}(\eta S_T > \eta H). \tag{6.2}$$

If and when the barrier knocks in, exchange the $\Gamma_{\text{ki}}(S_T)$ claim for a claim on $G(S_T)$, at zero cost.

Proof. The events $\eta S_t \geq \eta H$ and $\eta^* M_t \geq \eta^* \psi(H)$ are equivalent, where $\eta^* := \operatorname{sgn}(\psi(H) - M_0)$.

So apply Theorem 5.6 to the function $G \circ \phi$, the martingale M, and the barrier $\psi(H)$. □

Theorem 6.3 (Double barrier)**.** *Fixing any expiry T and barriers L, U with $c < L < S_0 < U$, let $\tau_U := \inf\{t : S_t \geq U\}$ and $\tau_L := \inf\{t : S_t \leq L\}$. Let $\chi := \mathbb{I}(\tau_L \wedge \tau_U \leq T)$.*

Assume that $S_{\tau_U} = U$ in the event $\tau_U \leq T$, and that $S_{\tau_L} = L$ in the event $\tau_L \leq T$; a sufficient condition is that S has continuous paths. If $M = \psi(S)$ satisfies PCS$_{\tau_L \wedge \tau_U \wedge T}$, then for any payoff function G bounded on (L, U), the following semi-static strategy replicates a double-knock-out claim paying $(1 - \chi)G(S_T)$. At time 0, hold a European-style claim on

$$\Gamma_{\text{dko}}(S_T) := \sum_{n=-\infty}^{\infty} \quad \frac{\psi(L)^n}{\psi(U)^n} G^* \circ \phi \quad \frac{\psi(U)^{2n}\psi(S_T)}{\psi(L)^{2n}} \quad - \frac{\psi(L)^{n-1}\psi(S_T)}{\psi(U)^n} G^* \circ \phi \quad \frac{\psi(U)^{2n}}{\psi(L)^{2n-2}\psi(S_T)} \quad , \tag{6.3}$$

where $G^*(s) := \mathbb{I}(s \in (L, U))G(s)$. If and when a barrier knocks out, close the $\Gamma_{\text{dko}}(S_T)$ claim at zero cost.

Proof. If ϕ is increasing, then the events $S_t \geq U$ and $M_t \geq \psi(U)$ are equivalent, as are the events $S_t \leq L$ and $M_t \leq \psi(L)$. So apply Theorem 5.14 to the payoff function $G \circ \phi$, the martingale M, the lower barrier $\psi(L)$, and upper barrier $\psi(U)$.

If ϕ is decreasing, do the same, but for lower barrier $\psi(U)$ and upper barrier $\psi(L)$. Reindexing the resulting sum yields (6.3). □

In the following sections, we consider two families of choices for the ϕ function: displacements and power transformations. Then we extend to multivariate functions $\phi(M, Z)$.

6.1 Displacement

Let $S = M + c$.

Corollary 6.4 (Displacement)**.** *Under the assumptions of Theorems 6.1 to 6.3 respectively, let $S_t = \phi(M_t) := M_t + c$. Then the conclusions hold with*

$$\mathbb{E}_0 G(S_T) = \mathbb{E}_0 \left(\frac{S_T - c}{S_0 - c} G\left(c + \frac{(S_0 - c)^2}{S_T - c} \right) \right) \tag{6.4}$$

$$\Gamma_{\text{ki}}(S_T) = G(S_T)\mathbb{I}(\eta S_T \geq \eta H) + \left(\frac{S_T - c}{H - c} G\left(c + \frac{(H - c)^2}{S_T - c} \right) \right) \mathbb{I}(\eta S_T > \eta H) \tag{6.5}$$

$$\Gamma_{\text{dko}}(S_T) = \sum_{n=-\infty}^{\infty} \left(\frac{(L - c)^n}{(U - c)^n} G\left(c + \frac{(U - c)^{2n}(S_T - c)}{(L - c)^{2n}} \right) - \frac{(L - c)^{n-1}(S_T - c)}{(U - c)^n} G\left(c + \frac{(U - c)^{2n}}{(L - c)^{2n-2}(S_T - c)} \right) \right) \tag{6.6}$$

respectively.

Proof. Apply Theorems 6.1/6.2/6.3 with $\psi(s) = s - c$. □

Example 6.5. Let $K > c$.

Under the hypotheses of Theorem 6.1, a call (put) on $S = M + c$ struck at K has the same value as $(K - c)/(S_0 - c)$ puts (calls) on S struck at $(S_0 - c)^2/(K - c) + c$.

Under the hypotheses of Theorem 6.2, an up-and-in put on $S = M + c$ struck at $K < H$ can be replicated by $(K - c)/(H - c)$ calls on S struck at $(H - c)^2/(K - c) + c$, to be exchanged for the put at time τ_H.

Corollaries 6.6 and 6.8 give examples of S dynamics representable as displaced PCS processes.

Corollary 6.6 (Affine diffusion coefficient; Carr-Lee and independently Forde [12])**.** *Suppose that S is a martingale satisfying*

$$dS_t = (aS_t + b)\sqrt{V_t}\,dW_t, \tag{6.7}$$

where a, b are constants, and V, W are independent, where V is adapted and a.s. time-integrable. Fix any expiry T and barrier $H = S_0$. Let $\tau_H := \inf\{t : \eta S_t \geq \eta H\}$ where $\eta := \operatorname{sgn}(H - S_0)$. Let G be any payoff function with $\mathbb{E}G(S_T) < \infty$.

If $a = 0$ then with $c := -b/a$, the displaced symmetry (6.4) *holds; moreover, the semi-static strategy* (6.2)*, in the specific form* (6.5)*, replicates a knock-in paying $\mathbb{I}(\tau_H \leq T)G(S_T)$.*

If $a = 0$ then with $X := S$, the arithmetic symmetry (2.1) *holds; moreover, the semi-static strategy* (5.4) *replicates a knock-in paying $\mathbb{I}(\tau_H \leq T)G(S_T)$.*

Figure 11: Displacing a PCS GBM generates an implied volatility skew.

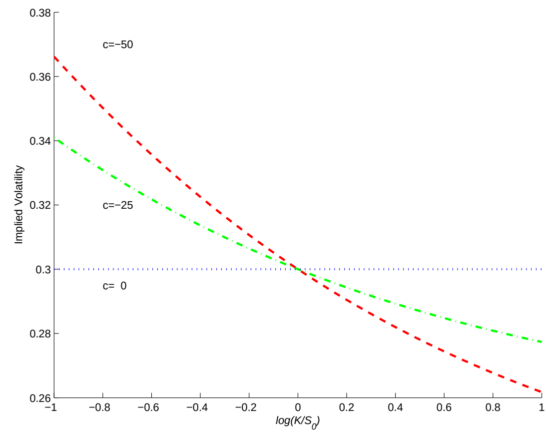


Figure 12: Displacing a PCS stochastic volatility diffusion generates a smiling skew.

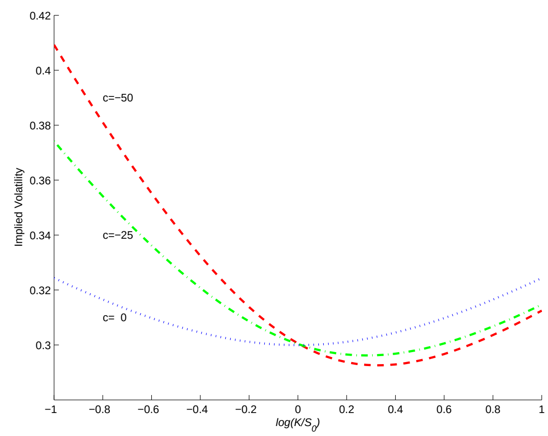
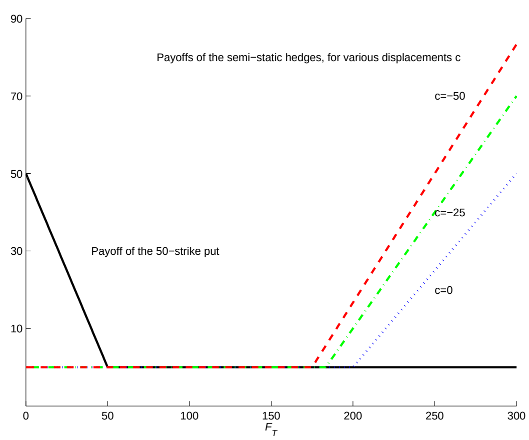


Figure 13: Hedges for *arbitrary* PCS dynamics displaced by 0, -25, -50, where $S_0 = 100$



Proof. If $a = 0$, then $M := S - c$ has dynamics

$$dM_t = dS_t = (aS_t + b)\sqrt{V_t}dW_t = (a(M_t + c) + b)\sqrt{V_t}dW_t = aM_t\sqrt{V_t}dW_t$$

which satisfy PCS and $\text{PCS}_{\tau_H \wedge T}$. If $a = 0$, then S satisfies arithmetic PCS and arithmetic $\text{PCS}_{\tau_H \wedge T}$. The conclusions follow. ☐

Implementing the hedge does not require knowing the dynamics of V nor the individual values of a and b; knowing the ratio b/a suffices.

Example 6.7. Figures 11 and 12 show implied volatility skews for $T = 1$, as a function of log-moneyness relative to $S_0 = 100$, where S arises from displacing, by 0 or -25 or -50, a PCS process M. In Figure 11, M is geometric Brownian motion; we take $V = 1$ in (6.7). In Figure 12, M is an independent stochastic volatility process; we take a and V in (6.7) such that realized volatility $(\int_0^T V_t dt)^{1/2}$ is lognormal with standard deviation 0.04. Both figures take a such that at-the-money implied volatility is 0.3. Thus, a given displacement can generate volatility skews having various convexities; but each hedge in Figure 13 is valid for *any* of the volatility skews associated with that displacement, regardless of convexity.

The following allows diffusion coefficients more general than affine.

Corollary 6.8 (A three-parameter family of diffusion coefficients; Forde [12])**.** *Let $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and c be constants. Fix any expiry T and constant $H > c$. Let $\tau_H := \inf\{t \geq 0 : \eta S_t \geq \eta H\}$ where $\eta := \text{sgn}(H - S_0)$. Let G be any payoff function with $\mathbb{E}G(S_T) < \infty$. Assume S satisfies*

$$dS_t = (S_t - c)\min\left(\gamma,\ \alpha + \beta\left(\log\frac{S_t - c}{H - c}\right)^2\right) dW_t, \qquad S_0 > c. \tag{6.8}$$

If $H = S_0$, then the displaced symmetry (6.4) *holds. If $H = S_0$ is some barrier, then the semi-static strategy* (6.2)*, in the specific form* (6.5)*, replicates a knock-in with payoff $\mathbb{I}(\tau_H \leq T)G(S_T)$.*

Proof. Note that $M := S - c$ satisfies $M_0 > 0$ and

$$dM_t = M_t \min\left(\gamma,\ \alpha + \beta\left(\log\frac{M_t}{M_{\tau_H \wedge T}}\right)^2\right) dW_t, \qquad t \in [\tau_H \wedge T, T],$$

So M is a positive martingale and $\text{PCS}_{\tau_H \wedge T}$ holds by Theorem 5.3. The conclusions now follow from Corollary 6.4. ☐

Remark 6.9 (General diffusion coefficients: an approximation)*.* Suppose that

$$dS_t = \sigma(S_t)dW_t, \qquad t \geq \tau_H$$

where the function σ is analytic in a neighborhood of the barrier H, and $\sigma'(H) = 0$.

Then Forde [12] observes that there exist α, β, and c, such that at $S = H$, the integrand of (6.8) agrees with σ, in the leading three terms of their Taylor expansions. Specifically, let us expand the integrand of (6.8) in powers of $(S - H)$, to obtain

$$\alpha(H - c) + \alpha(S - H) + \frac{\beta}{H - c}(S - H)^2 + O(S - H)^3,$$

which matches $\sigma(H) + \sigma'(H)(S-H) + \frac{1}{2}\sigma''(H)(S-H)^2 + O(S-H)^3$, if we choose

$$\begin{aligned} \alpha &:= \sigma'(H) \\ \beta &:= \sigma(H)\sigma''(H)/(2\sigma'(H)) \\ c &:= H - \sigma(H)/\sigma'(H). \end{aligned} \tag{6.9}$$

Hence the family of "exactly hedgeable" diffusions in (6.8) is rich enough to approximate, with error $O(S-H)^3$, a *general* local volatility function $\sigma(S)$, provided σ is well-behaved at H.

To obtain approximate semi-static hedges, use (6.5) and (6.6) with $c := H - \sigma(H)/\sigma'(H)$, from (6.9). Note that α and β do not affect the hedge construction, hence σ'' does not affect the hedge construction. So knowing *only* the level and slope of σ at H suffices to determine a hedge which works perfectly for a class of local volatility functions, including one that agrees with σ in level, slope, *and convexity* at H.

As a consistency check, take the affine case $\sigma(S) := aS + b$. We have from (6.9) the calibrated displacement $c = H - (aH + b)/a = -b/a$, in agreement with Corollary 6.6.

6.2 Power Transformation

Now let $M = S^p$.

Corollary 6.10 (Power transformation)**.** *Under the assumptions of Theorems 6.1 to 6.3 respectively, let $S_t = \phi(M_t) := M_t^{1/p}$, where $p = 0$. Then the conclusions hold with*

$$\mathbb{E}_0 G(S_T) = \mathbb{E}_0 \quad \frac{S_T}{S_0}^{\,p} G \quad \frac{S_0^2}{S_T} \tag{6.10}$$

$$\Gamma_{\text{ki}}(S_T) = G(S_T)\mathbb{I}(\eta S_T \geq \eta H) + \quad \frac{S_T}{H}^{\,p} G \quad \frac{H^2}{S_T} \quad \mathbb{I}(\eta S_T > \eta H) \tag{6.11}$$

$$\Gamma_{\text{dko}}(S_T) = \sum_{n=-\infty}^{\infty} \quad \frac{L^n}{U^n}^{\,p} G \quad \frac{U^{2n} S_T}{L^{2n}} \quad - \quad \frac{L^{n-1} S_T}{U^n}^{\,p} G \quad \frac{U^{2n}}{L^{2n-2} S_T} \tag{6.12}$$

respectively.

Proof. Apply Theorems 6.1/6.2/6.3 with $\psi(s) = s^p$. □

Example 6.11 (Geometric Brownian motion; Carr-Chou [5]). Given

$$dS_t = rS_t dt + \beta S_t dW_t$$

let

$$p := 1 - 2r/\beta^2.$$

Then $\psi(S_t) = S_t^p$ is driftless GBM which satisfies PCS. If $p = 0$ then (6.10) implies the call and put symmetries

$$\begin{aligned} \mathbb{E}_0(S_T - K)^+ &= \mathbb{E}_0 \quad \frac{S_T^p}{S_0^p} \quad \frac{S_0^2}{S_T} - K \quad ^+ \\ \mathbb{E}_0(K - S_T)^+ &= \mathbb{E}_0 \quad \frac{S_T^p}{S_0^p} \quad K - \frac{S_0^2}{S_T} \quad ^+ \end{aligned}$$

and (6.11) implies that an up-and-in put struck at $K < H$ can be replicated by a claim on

$$\frac{S_T^p}{H^p}\left(K - \frac{H^2}{S_T}\right)^+$$

to be exchanged for the put at time τ_H. Moreover, these conclusions still hold if $p = 0$, because in that case, $\log S$ satisfies arithmetic PCS and arithmetic $\text{PCS}_{\tau_H \wedge T}$.

For generalizations of these dynamics, see sections 7.1 and 7.2.

6.3 Multivariate Transformations

Now let $S_t = \phi(M_t, t)$ or more generally $S_t = \phi(M_t, Z_t)$ where Z is an adapted process independent of M, and M satisfies PCS.

We will derive pricing symmetries for S from the symmetries for M. In contrast to the symmetries of sections 6.1 and 6.2, these multivariate pricing symmetries typically do not lead to static hedges of barrier options, because the specification of the conjugate payoff will depend on value of the second factor t (or more generally Z_t), and the value of the second factor at the first passage time τ_H is typically unknown at time 0.

Theorem 6.12. *Let $\tau \leq T$ be a stopping time. Assume that $S_t = \phi(M_t, Z_t)$ where Z is an adapted process independent of M, and M satisfies PCS_τ. Then*

$$\mathbb{E}_\tau G(S_T) = \mathbb{E}_\tau \left(\frac{M_T}{M_\tau} G \circ \phi\left(\frac{M_\tau^2}{M_T}, Z_T\right)\right) \tag{6.13}$$

for all payoff functions G. In particular, if $S_t = M_t Z_t$, then

$$\mathbb{E}_\tau G(S_T) = \mathbb{E}_\tau \left(\frac{M_T}{M_\tau} G\left(\frac{M_\tau^2}{M_T} Z_T\right)\right) \tag{6.14}$$

and if moreover $Z_T = 1$ then

$$\mathbb{E}_\tau G(S_T) = \mathbb{E}_\tau \left(\frac{S_T}{S_\tau / Z_\tau} G\left(\frac{S_\tau^2 / Z_\tau^2}{S_T}\right)\right) \tag{6.15}$$

for all payoff functions G.

Proof. By independence, if we condition on Z_T, the M is still a martingale that satisfies condition $(b)_\tau$, hence $(c)_\tau$:

$$\mathbb{E}_\tau \left(G \circ \phi(M_T, Z_T) \mid Z_T \right) = \mathbb{E}_\tau \left(\frac{M_T}{M_\tau} G \circ \phi\left(\frac{M_\tau^2}{M_T}, Z_T\right) \middle| Z_T \right)$$

So (6.13) follows from iterated expectations. To obtain (6.14), take $\phi(m, z) := mz$. ☐

Example 6.13 (Spot prices). Spot prices are not generally martingales, but symmetries for spot prices follow from taking as the auxiliary process the forward price. Specifically, if $S = MZ$ is a spot price, and the forward price M satisfies PCS, and the discount factor is $Z_t = e^{-r(T-t)}$, then by (6.15),

$$\mathbb{E}_0 G(S_T) = \mathbb{E}_0 \; \frac{S_T}{S_0 e^{rT}} G \; \frac{S_0^2 e^{2rT}}{S_T} \quad . \tag{6.16}$$

In particular, taking G to be a call payoff gives

$$\mathbb{E}_0 (S_T - K)^+ = \frac{K}{S_0 e^{rT}} \mathbb{E}_0 \; \frac{S_0^2 e^{2rT}}{K} - S_T \; ^+$$

so a call on S struck at K has the same time-0 value as $K/(S_0 e^{rT})$ puts struck at $e^{2rT} S_0^2 / K$.

Example 6.14 (Spot prices, with jump to zero). Let $S = MZ$ where M satisfies PCS, and let $Z_t := \mathbb{I}(t < \tau_0) e^{-(r+\lambda)(T-t)}$ where τ_0 has exponential distribution with parameter λ, independent of M. Assume also that $G(0) = 0$. Then by (6.14),

$$\mathbb{E}_0 G(S_T) = \mathbb{E}_0 \; \frac{M_T}{M_0} G \; \frac{M_0^2}{M_T} \; \mathbb{I}(Z_T > 0) \; = e^{-\lambda T} \mathbb{E}_0 \; \frac{S_T}{S_0 e^{(r+\lambda)T}} G \; \frac{S_0^2 e^{2(r+\lambda)T}}{S_T} \tag{6.17}$$

In particular, taking G to be a call payoff gives

$$\mathbb{E}_0 (S_T - K)^+ = \frac{K e^{-\lambda T}}{S_0 e^{(r+\lambda)T}} \mathbb{E}_0 \; \frac{S_0^2 e^{(2r+2\lambda)T}}{K} - S_T \; ^+ \tag{6.18}$$

so a call on S struck at K has the same time-0 value as $K e^{-(2\lambda+r)T}/S_0$ puts struck at $e^{2(r+\lambda)T} S_0^2 / K$.

Corollary 6.15. *Let*

$$dS_t = \sigma_t S_{t-} dW_t + S_{t-}(dY_t - \Lambda \mu dt), \qquad S_0 > 0$$

where Y_t is a compound Poisson process with arrival rate Λ and jump sizes in $(-1, \infty)$, with mean μ. Assume that σ, W, and Y are independent. Then

$$\mathbb{E}_0 S_T^p = \frac{\mathbb{E}_0 Z_T^p}{\mathbb{E}_0 Z_T^{1-p}} \mathbb{E}_0 S_T^{1-p}.$$

where

$$Z_t := S_0 e^{-\Lambda \mu t} \prod_{u \in (0,t]} \frac{S_u}{S_{u-}}.$$

Proof. Note that if the jump distribution is asymmetric, then S does not satisfy PCS. Nonetheless, the auxiliary process

$$M_u := \exp \; \int_0^u \sigma_t dW_t - \frac{1}{2} \int_0^u \sigma_t^2 dt$$

does satisfy PCS, by Corollary 4.5. Itô's rule shows that this process is related to S by $S = MZ$; in other words, factoring out the compensated jumps of S leaves an auxiliary process M which satisfies PCS.

By (6.14) and $M_0 = 1$ and independence,

$$\mathbb{E}_0 S_T^p = \mathbb{E}_0 M_T^{1-p} Z_T^p = \mathbb{E}_0 M_T^{1-p} \times \mathbb{E}_0 Z_T^p \times \frac{\mathbb{E}_0 Z_T^{1-p}}{\mathbb{E}_0 Z_T^{1-p}} = \frac{\mathbb{E}_0 Z_T^p}{\mathbb{E}_0 Z_T^{1-p}} \mathbb{E}_0 S_T^{1-p},$$

as desired. $\square$

Therefore, under these jump dynamics, one claim on the pth power has value equal to the weight $\mathbb{E}_0 Z_T^p / \mathbb{E}_0 Z_T^{1-p}$ times the value of the $(1-p)$th power claim. If the jump distribution is known, then the weight can be calculated (and in the special case of no jumps, the weight equals S_0^{2p-1}, in agreement with Corollary 2.6).

7 Further Examples of Asymmetric Dynamics

In this section we give three examples of asymmetric processes which relate to PCS processes via (combinations of) the transformation techniques of section 6. The examples are a general one-dimensional diffusion, geometric Brownian motion with jump to zero, and the CGMY model, all with drift.

7.1 One-dimensional Diffusion

This example combines the general transformation technique of Theorem 6.1 and the displacement method. Suppose that

$$dS_t = \alpha(S_t)dt + \beta(S_t)dW_t \tag{7.1}$$

where $\beta > 0$ and $|\alpha|/\beta^2$ is locally integrable. Let ζ be a *scale function*

$$\zeta(s) := \int^s \exp\left(-2\int^v \frac{\alpha(u)}{\beta^2(u)}du\right) dv. \tag{7.2}$$

With the freedom to choose two constants of integration, the scale function is determined only up to strictly increasing affine transformations.

Applying the scale function to the diffusion via

$$\tilde{S}_t := \zeta(S_t)$$

removes the drift; as detailed in sources such as Karatzas-Shreve [13], Itô's rule shows that

$$d\tilde{S}_t = \sigma(\tilde{S}_t)dW_t \tag{7.3}$$

where

$$\sigma(x) = \zeta'(\zeta^{-1}(x))\beta(\zeta^{-1}(x)). \tag{7.4}$$

Forde [12] finds that the displacement technique described in Remark 6.9 gives approximate semi-static hedges (which are exact in particular cases, for example if σ is such that (7.3) takes the form (6.8)). Indeed, given a barrier $H = S_0$, let $\tilde{H} := \zeta(H)$. Applying to (7.3)-(7.4) the displacement calibration formula (6.9), we have

$$c = \tilde{H} - \frac{\sigma(\tilde{H})}{\sigma'(\tilde{H})} = \zeta(H) - \frac{\zeta'(H)}{\frac{\beta'(H)}{\beta(H)} - \frac{2\alpha(H)}{\beta^2(H)}}. \tag{7.5}$$

The last step holds because differentiating (7.4) yields the relation

$$\frac{\sigma'(\tilde{H})}{\sigma(\tilde{H})} = \frac{1}{\zeta'}\left(\frac{\zeta''}{\zeta'} + \frac{\beta'}{\beta}\right) = \frac{1}{\zeta'}\left(\frac{\beta'}{\beta} - \frac{2\alpha}{\beta^2}\right),$$

where the middle and right-hand expressions are evaluated at $\zeta^{-1}(\tilde{H}) = H$. As a consistency check, in the case $\alpha = 0$, we may choose ζ to be the identity, and (7.5) reduces to $H - \beta(H)/\beta'(H)$, in agreement with (6.9).

Consider a knock-in option on S with payoff $\chi G(S_T)$ where

$$\chi := \mathbb{I}(\max_{t\in[0,T]} \eta S_t \geq \eta H) = \mathbb{I}(\max_{t\in[0,T]} \eta \tilde{S}_t \geq \eta \tilde{H}).$$

and $\eta := \operatorname{sgn}(H - S_0)$. Rewriting the payoff as $\chi\tilde{G}(\tilde{S}_T)$ where $\tilde{G} := G \circ \zeta^{-1}$, we apply (6.5) to obtain a semi-static hedge. At time 0, hold a European claim on

$$\Gamma_{\text{ki}} = G(S_T)\mathbb{I}(\eta S_T \geq \eta H) + \frac{\zeta(S_T) - c}{\zeta(H) - c} \, G \circ \zeta^{-1} \quad c + \frac{(\zeta(H) - c)^2}{\zeta(S_T) - c} \quad \mathbb{I}(\eta S_T > \eta H), \tag{7.6}$$

to be exchanged for a claim on $G(S_T)$ if and when the barrier knocks in. As a consistency check, note that affine transformations of ζ leave the solution (7.6) unchanged.

For knock-*out* options, Remark 5.12 applies.

7.2 Geometric Brownian Motion, with Jump to Zero

In this example we combine the power transformation and multivariate techniques, using a jump to zero as the second variable, in order to derive symmetries for Geometric Brownian motion with drift and with jump to zero. Let

$$\begin{aligned}
S_t &:= \tilde{S}_t Z_t \\
Z_t &:= \mathbb{I}(t < \tau_0) \\
d\tilde{S}_t &= (r - q + \lambda)\tilde{S}_t dt + \beta \tilde{S}_t dW_t, \qquad \tilde{S}_0 > 0,
\end{aligned}$$

where τ_0 is independent of W and is exponentially distributed with parameter $\lambda \geq 0$. Let

$$p := 1 - 2(r - q + \lambda)/\beta^2.$$

Then S satisfies the following symmetry.

Remark 7.1. Notation of the form $Y\mathbb{I}(X)$ or $\mathbb{I}(X)Y$, by definition, represents the expression Y if condition X holds, and 0 if condition X does not hold. So, for example, the expression $(1/x)\mathbb{I}(x > 0)$ is well-defined for all real x including 0.

Theorem 7.2. *Let G be a payoff function, let $\tau \leq T$ be a stopping time, and let*

$$\Gamma_\tau(S_T) := \begin{cases} G(0) & \text{if } S_T = 0 \\ (S_T^p/S_\tau^p)G(S_\tau^2/S_T) & \text{if } S_T > 0. \end{cases} \tag{7.7}$$

Then $\mathbb{E}_\tau G(S_T) = \mathbb{E}_\tau \Gamma_\tau(S_T)$.

Proof. If $p = 0$, then $\tilde{S}$ has scale function $s \mapsto s^p$. The auxiliary process $M_t := \psi(\tilde{S}_t) = \tilde{S}_t^p$ is a GBM hence satisfies PCS. By Theorem 6.12 with $\varphi(m, z) := G(m^{1/p} z)$, we have

$$\mathbb{E}_\tau G(S_T) = \mathbb{E}_\tau \; \frac{\tilde{S}_T^p}{\tilde{S}_\tau^p} G \; \frac{\tilde{S}_\tau^2}{\tilde{S}_T} Z_T \quad = \mathbb{E}_\tau \; G(0)\mathbb{I}(Z_T = 0) \; + \mathbb{E}_\tau \; \frac{\tilde{S}_T^p}{\tilde{S}_\tau^p} G \; \frac{\tilde{S}_\tau^2}{\tilde{S}_T} \; \mathbb{I}(Z_T = 1) \; = \mathbb{E}_\tau \Gamma_\tau(S_T)$$

because on the event $Z_T = 1$ we have $\tilde{S}_T = S_T$ and $\tilde{S}_\tau = S_\tau$.

If $p = 0$ then $\tilde{S}$ has scale function $s \mapsto \log s$. Now observe that $\log \tilde{S}$ satisfies arithmetic PCS_τ, and proceed similarly to the $p = 0$ case. ☐

As an application of this symmetry, we replicate a down-and-in claim.

Theorem 7.3 (Down barrier)**.** *Let $0 < H < S_0$ and $\tau_H := \inf\{t : S_t \leq H\}$. Let G be a payoff function with $\mathbb{E}G(S_T) < \infty$. Then the following semi-static strategy replicates a down-and-in claim on $\mathbb{I}(\tau_H \leq T)G(S_T)$. At time 0, hold a European-style claim on*

$$\Gamma_{\text{di}}(S_T) := \begin{cases} G(0) & \text{if } S_T = 0 \\ G(S_T) + \frac{S_T^p}{H^p} G(\frac{H^2}{S_T}) & \text{if } 0 < S_T < H \\ 0 & \text{if } S_T \geq H \end{cases}$$

If and when the barrier knocks in, exchange the Γ_{di} claim for a claim on $G(S_T)$, at zero cost.

Proof. On the event $\tau_H > T$, the option does not knock in, and the $\Gamma_{\text{di}}(S_T)$ claim expires worthless, as desired. On the event $\tau_H \leq T$, the $\Gamma_{\text{di}}(S_T)$ claim can be exchanged at zero cost to a claim on $G(S_T)$, because

$$\begin{aligned} \mathbb{E}_{\tau_H} G(S_T) &= \mathbb{E}_{\tau_H} G(S_T)\mathbb{I}(S_T \leq H) + \mathbb{E}_{\tau_H} G(S_T)\mathbb{I}(S_T > H) \\ &= \mathbb{E}_{\tau_H} G(S_T)\mathbb{I}(S_T \leq H) + \mathbb{E}_{\tau_H} (S_T^p / S_{\tau_H}^p) G(S_{\tau_H}^2 / S_T)\mathbb{I}(0 < S_T < H) \\ &= \mathbb{E}_{\tau_H} \Gamma_{\text{di}}(S_T) \end{aligned}$$

where the second step applies Theorem 7.2 to the function $G(s)\mathbb{I}(s > H)$ and stopping time $\tau_H \wedge T$, and the third step uses the fact that $S_{\tau_H} = H$ in the case $S_T > 0$.

We opted not to define $\Gamma_{\text{di}}(H) := G(H)$, because $S_T = H$ with zero probability. ☐

Likewise, an *up*-and-in claim can be semi-statically replicated. We impose the additional assumption that $G(0) = 0$, because S can hit zero without passing the up-barrier (unlike a down-barrier); in order for the hedge to handle this event correctly, we need $\Gamma(0) = 0$; but we already require $\Gamma(0) = G(0)$ because G and its conjugate in (7.7) agree at 0.

Theorem 7.4 (Up barrier)**.** *Let $H > S_0$ and $\tau_H := \inf\{t : S_t \geq H\}$. Let G be a payoff function with $G(0) = 0$ and $\mathbb{E}G(S_T) < \infty$. Then the following semi-static strategy replicates an up-and-in claim on $\mathbb{I}(\tau_H \leq T)G(S_T)$. At time 0, hold a European-style claim on*

$$\Gamma_{\text{ui}}(S_T) := \quad G(S_T) + \frac{S_T^p}{H^p} G \; \frac{H^2}{S_T} \quad \mathbb{I}(S_T \geq H).$$

If and when the barrier knocks in, exchange the Γ_{ui} claim for a claim on $G(S_T)$, at zero cost.

Proof. On the event $\tau_H > T$, the option does not knock in, and the European-style claim on Γ_{ui} expires worthless, as desired. On the event $\tau_H \leq T$, the Γ_{ui} claim can be converted at zero cost to a claim on $G(S_T)$, because

$$\begin{aligned}
\mathbb{E}_{\tau_H} G(S_T) &= \mathbb{E}_{\tau_H} G(S_T)\mathbb{I}(S_T \geq H) + \mathbb{E}_{\tau_H} G(S_T)\mathbb{I}(S_T < H) \\
&= \mathbb{E}_{\tau_H} G(S_T)\mathbb{I}(S_T \geq H) + \mathbb{E}_{\tau_H} \begin{cases} G(0) & \text{if } S_T = 0 \\ (S_T^p/S_{\tau_H}^p)G(S_{\tau_H}^2/S_T)\mathbb{I}(S_T > H) & \text{if } S_T > 0 \end{cases} \\
&= \mathbb{E}_{\tau_H} \Gamma_{\text{ui}}(S_T).
\end{aligned}$$

where the second step applies Theorem 7.2 to the function $G(s)\mathbb{I}(s < H)$ and stopping time $\tau_H \wedge T$, and the third step uses the $G(0) = 0$ assumption and the fact that $S_{\tau_H} = H$ in the case $S_T > 0$.

We opted not to define $\Gamma_{\text{ui}}(H) := G(H)$, because $S_T = H$ with zero probability. □

Likewise, a double-barrier option can be semi-statically replicated (without assuming $G(0) = 0$).

Theorem 7.5 (Double barrier)**.** *Let $\tau_U := \inf\{t : S_t \geq U\}$ and $\tau_L := \inf\{t : S_t \leq L\}$, where $0 < L < S_0 < U$. Let $\chi := \mathbb{I}(\tau_L \wedge \tau_U \leq T)$. Then the following semi-static strategy replicates a double-knock-out claim on $(1 - \chi)G(S_T)$, where G is a payoff function bounded on (L, U).*

At time 0, hold a European-style claim on

$$\Gamma_{\text{dko}}(S_T) := \mathbb{I}(S_T > 0) \sum_{n=-\infty}^{\infty} \quad \frac{L^n}{U^n}{}^{p} G^* \frac{U^{2n}S_T}{L^{2n}} - \frac{L^{n-1}S_T}{U^n}{}^{p} G^* \frac{U^{2n}}{L^{2n-2}S_T}$$

where $G^*(s) := G(s)\mathbb{I}(s \in (L, U))$. If and when a barrier knocks out, close the $\Gamma_{\text{dko}}(S_T)$ claim at zero cost.

Proof. If the barriers never knock out, then the claim on $\Gamma_{\text{dko}}(S_T)$ expires worth $G(S_T)$, as desired. So we need only establish the zero value, at the knock-out time, of a claim on $\Gamma_{\text{dko}}(S_T)$. Write $\tau := \tau_L \wedge \tau_U \wedge T$. On the event $\tau = \tau_U$,

$$\mathbb{E}_\tau \Gamma_{\text{dko}}(S_T) = \mathbb{E}_{\tau_U}\mathbb{I}(S_T > 0) \sum_{n=-\infty}^{\infty} \quad \frac{L^n}{U^n}{}^{p} G^* \frac{U^{2n}S_T}{L^{2n}} - \frac{L^{n-1}}{U^{n-1}}{}^{p} G^* \frac{U^{2n-2}S_T}{L^{2n-2}} \quad = 0$$

by Theorem 7.2. Likewise, on the event $\tau = \tau_L$,

$$\mathbb{E}_\tau \Gamma_{\text{dko}}(S_T) = \mathbb{E}_{\tau_L}\mathbb{I}(S_T > 0) \sum_{n=1}^{\infty} \quad \frac{L^n}{U^n}{}^{p} G^* \frac{U^{2n}S_T}{L^{2n}} - \frac{L^n}{U^n}{}^{p} G^* \frac{U^{2n}S_T}{L^{2n}} \quad = 0$$

by Theorem 7.2. □

Single knock-out and double knock-in options therefore also admit semi-static replication, by Remarks 5.12 and 5.16 respectively.

7.3 CGMY Model

In this example we combine the power transformation and multivariate techniques, here using time as the second variable.

Let $X_t = \log(S_t/S_0)$ be an Lévy process with $\mathbb{P}$-Lévy characteristics (μ, σ^2, ν), with respect to the constant "truncation" function 1, which is adequate because the tails of ν will decay exponentially. Let ν have density function

$$\nu(x) = \frac{C}{|x|^{1+Y}} e^{-\lambda_-|x|}\mathbb{I}(x < 0) + \frac{C}{|x|^{1+Y}} e^{-\lambda_+|x|}\mathbb{I}(x > 0)$$

where $C > 0$, $\lambda_- > 0$, $\lambda_+ > 0$, and $Y < 2$.

This is the *extended CGMY* process, where (borrowing terminology from [8]) "extended" refers to the lack of restrictions on μ and σ^2, so X may include a Brownian term with drift (but requiring that S discounted be a martingale would place a restriction on that drift). Let Ψ be the characteristic exponent of X; thus, for $Y = 0, 1$,

$$\Psi(u) = iu\mu - \frac{1}{2}u^2\sigma^2 + C\Gamma(-Y)[(\lambda_+ - iu)^Y - \lambda_+^Y + (\lambda_- + iu)^Y - \lambda_-^Y].$$

The extended CGMY process satisfies the following symmetry.

Theorem 7.6. *Let $p := \lambda_+ - \lambda_-$ and*

$$\alpha := \begin{cases} \Psi(-ip)/p & \text{if } p = 0 \\ \mu & \text{if } p = 0 \end{cases}$$

Then for any payoff function G,

$$\mathbb{E}G(S_T) = \mathbb{E} \quad \frac{S_T}{S_0 e^{\alpha T}} \quad ^p G \quad \frac{S_0^2 e^{2\alpha T}}{S_T} \quad .$$

Proof. If $p = 0$, then $X_t + \mu(T - t)$ satisfies arithmetic PCS. So, as claimed,

$$\mathbb{E}G(S_T) = \mathbb{E}G(S_0 e^{X_T}) = \mathbb{E}G(S_0 e^{2\mu T - X_T}) = \mathbb{E}G \quad \frac{S_0^2 e^{2\mu T}}{S_T} \quad .$$

If $p = 0$, then let

$$Y_t := pX_t + (T - t)\Psi(-ip).$$

We have

$$\infty > \mathbb{E}e^{pX_t} = e^{t\Psi(-ip)},$$

so $\exp(Y_t)$ is a martingale. Moreover,

$$\nu(-x) = e^{px}\nu(x),$$

so Y has Lévy density $\tilde{\nu}$ with

$$\tilde{\nu}(-y) = e^y\tilde{\nu}(y).$$

Therefore PCS holds for $\exp(Y_t) = S_t^p e^{\Psi(-ip)(T-t)} = \left(S_t e^{\alpha(T-t)}\right)^p$.

Applying Theorem 2.2 to the function $x \mapsto G(x^{1/p})$, we have

$$\mathbb{E}G(S_T) = \mathbb{E}\left[\left(\frac{S_T}{S_0 e^{\alpha T}}\right)^p G\left(\frac{S_0^2 e^{2\alpha T}}{S_T}\right)\right]$$

as claimed. □

8 Conclusion

When symmetry holds, any payoff of European or single/double/sequential-barrier type has a conjugate European-style payoff with the same value, from which we construct semi-static hedges of those barrier options.

We have established necessary and sufficient conditions for symmetry to hold. When symmetry does *not* hold, we have found techniques which map the pricing and hedging results for symmetric processes into the corresponding relationships for asymmetric processes.

We view these results as part of a broad program which aims to use European options – whose values are determined by *marginal* distributions – to extract information about *path-dependent* risks (including barrier-contingent payoffs) and to hedge those risks robustly.

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