Sparse Sets in Time and Frequency with Diophantine Problems and Integrable Systems

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Overview of sparseness in space and frequency

- Space and frequency domains - setting and motivation
- Reconstruction and uncertainty in sparse signals
- Ill conditioning–effects on reconstruction
- Diophantine meaning of ill conditioning
- Band limiting and related integrable systems
- Combinatorial description of singular sparse sets

The spaces over which we operate

In this introduction, we give examples of the duality between conventional spatial and frequency representations of ‘sparse objects’, signals sparse in both space and frequency representations.
We have a space of signals, and two basis representing signals. One basis consists of confined functions (delta functions),representing atoms of space and a second basis,consisiting of widely distributed functions (sines and cosines), having unbounded support, and the operator, $\mathcal{F}$, mapping between the basis.
Then we consider the behavior of the operator, $\mathcal{F}$, and the consequences of projecting signals onto subspaces simultaneously in space and in frequency.
The examples that follow will focus on Fourier representations; however other basis sharing the property of confined/unconfined support can be used as well

For example, the standard basis can be paired with a polynomial basis.

The Fourier matrix

$$\mathcal{F}_n = \frac{1}{\sqrt{n}} \begin{pmatrix} \zeta_n^{00} & \zeta_n^{10} & \dots & \zeta_n^{(n-1)0} \\ \zeta_n^{01} & \zeta_n^{11} & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \zeta_n^{0(n-1)} & \zeta_n^{1(n-1)} & \dots & \zeta_n^{(n-1)(n-1)} \end{pmatrix}$$

$$\frac{1}{\sqrt{7}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{\frac{2i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{-\frac{2i\pi}{7}} \\ 1 & e^{\frac{4i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{-\frac{4i\pi}{7}} \\ 1 & e^{\frac{6i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{-\frac{6i\pi}{7}} \\ 1 & e^{-\frac{6i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{\frac{6i\pi}{7}} \\ 1 & e^{-\frac{4i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{\frac{2i\pi}{7}} & e^{-\frac{2i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{\frac{4i\pi}{7}} \\ 1 & e^{-\frac{2i\pi}{7}} & e^{-\frac{4i\pi}{7}} & e^{-\frac{6i\pi}{7}} & e^{\frac{6i\pi}{7}} & e^{\frac{4i\pi}{7}} & e^{\frac{2i\pi}{7}} \end{pmatrix}$$

The world (of human kind) is full of band limited signals

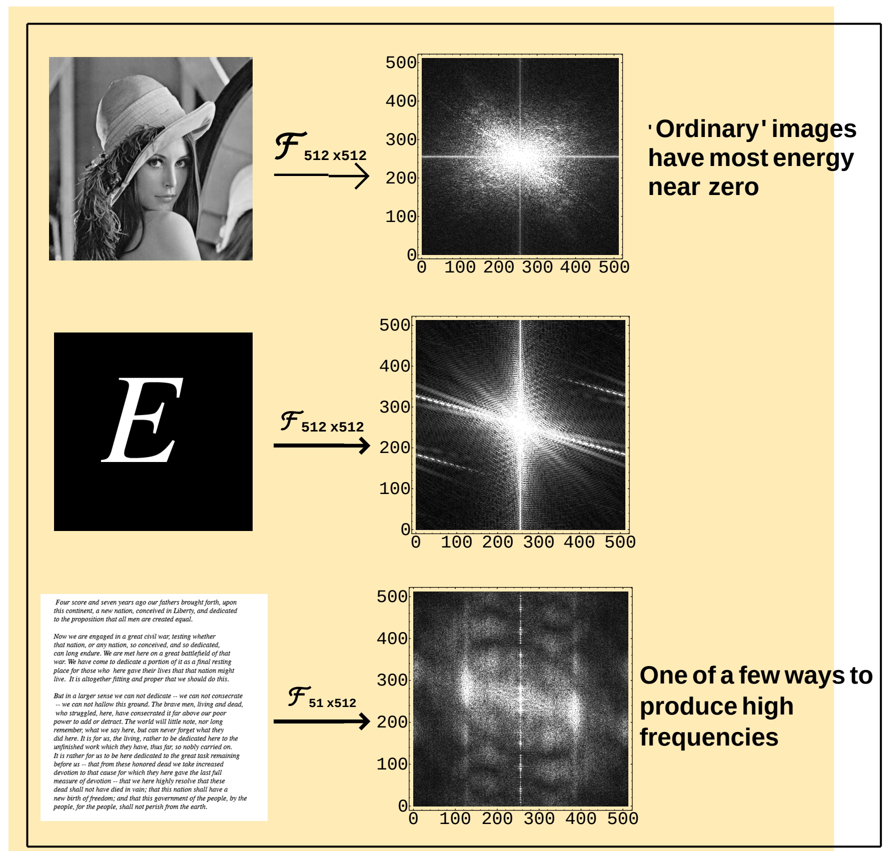
$$\begin{array}{ccc} S_F & \stackrel{\mathcal{F}}{\longleftrightarrow} & S_\delta \\ R_W \;\downarrow & & \downarrow\; P_T \\ W & \stackrel{?}{\leftrightarrow} & T \end{array}$$

is in here:

In the block diagram of a system, it’s not at all unusual for an early on block to be a band pass filter

Further, there’s an enormous ’filter’ at any system’s outputs – namely, what’s interesting to the observer?

Some examples of signals and their representations



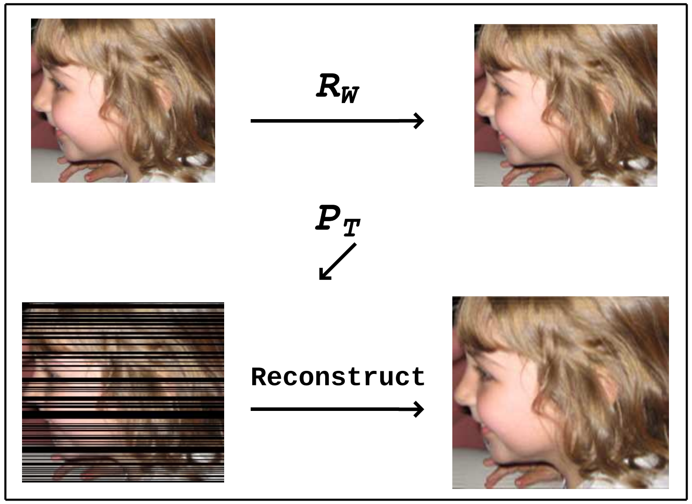
Reconstruction/concentration

$$
\begin{array}{ccccc}
 & S_F & \stackrel{\mathcal{F}}{\longleftrightarrow} & S_\delta & \\
R_W & \downarrow & & \downarrow & P_T \\
 & W & \stackrel{?}{\leftrightarrow} & T &
\end{array}
$$

- If the Fourier operator limited to the W, T subspaces is invertible, then signals can be reconstructed from partial frequency data.
- If the Fourier operator limited to the W, T subspaces is *not* invertible, then it has a null space
  Vectors in this null space are interesting, since they exhibit concentration in both space (T) and frequency $(S_F \setminus W)$ These are curious vectors, ones which are running up against uncertainty principle limits.

A visual example of reconstruction

$$
\begin{array}{ccc}
S_F & \overset{\mathcal{F}}{\longleftrightarrow} & S_\delta \\
R_W \downarrow & & \downarrow P_T \\
W & \overset{?}{\leftrightarrow} & T
\end{array}
$$



The notation for the operators of interest, and what are their actions

$$\begin{array}{ccccc} & S_F & \stackrel{\mathcal{F}}{\longleftrightarrow} & S_\delta & \\ R_W & \downarrow & & \downarrow & P_T \\ & W & \stackrel{?}{\leftrightarrow} & T & \end{array}$$

$$\begin{array}{ll} W & -\,a\ subspace\ of\ frequencies \\ T & -\,a\ subspace\ of\ spatial\ points \end{array}$$

$$\begin{array}{ll} R_W & operator\ which\ projects\ to\ W \\ P_T & operator\ which\ projects\ to\ T \\ R_{WT} & the\ composition\ of\ R_W P_T \end{array}$$

The action of R_{WT}, concretely, is straighforward:

$$\mathcal{F}_n = \frac{1}{\sqrt{n}} \begin{pmatrix} \zeta_n^{00} & \zeta_n^{10} & \dots & \zeta_n^{(n-1)0} \\ \zeta_n^{01} & \zeta_n^{11} & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \zeta_n^{0(n-1)} & \zeta_n^{1(n-1)} & \dots & \zeta_n^{(n-1)(n-1)} \end{pmatrix}$$

Non zero deteriminant of certain sub-matrices of the Fourier matrix

A little more about the case with invertibility

This opens an interesting possibility. For example, you might be able to quickly recover spatial data, if you know that the spatial data was of limited support (but not where it was)

For this to work, more than ’non - zero’ is required to be known about the determinant of the the Fourier submatrices

Some estimates of the conditioning of the Fourier submatrices are needed

Statement of Chebotarev's Theorem

Chebotarev found a modular reduction of the value of the determinant of matrices of the form:

$$\det\begin{pmatrix} \zeta_p^{a_1 x_1} & \zeta_p^{a_1 x_2} & \dots & \zeta_p^{a_1 x_k} \\ \zeta_p^{a_2 x_1} & \zeta_p^{a_2 x_2} & \dots & \zeta_p^{a_2 x_k} \\ \dots & \dots & \dots & \dots \\ \zeta_p^{a_k x_1} & \zeta_p^{a_k x_2} & \dots & \zeta_p^{a_k x_k} \end{pmatrix} =$$

$$\frac{\prod(x_i - x_j)\prod(a_i - a_j)}{1!2!\ldots(k-1)!}\pi^{\frac{k(k-1)}{2}} \left(mod\ \pi^{\frac{k(k-1)}{2}+1}\right)$$

Where $\pi = (\zeta_p - 1)$, p is a prime and a_i and x_j are integers

In outline the proof proceeds by expanding the matrix entries in a series for the formal variable π , gathering the coefficients of the like terms of π and finding values for the coefficients for various powers of π in the expansion of the determinant.

More about the reconstruction operation

$$\begin{array}{ccc} S_F & \stackrel{\mathcal{F}}{\longleftrightarrow} & S_\delta \\ R_W \downarrow & & \downarrow P_T \\ W & \stackrel{?}{\leftrightarrow} & T \end{array}$$

Reconstruction consists of inverting the operator R_{WT} on the subspace W We look at a signal, s, that is band limited to W (that is,the Fourier coefficients of S are zero outside of W). We want to reconstruct the signal, s everywhere, from it's values on a subset T.

For this to succeed, |T| must be at least as great as |W|.

As a practical matter, the more observations points available in T, the better the reconstruction.

For convenience, instead of reconstructing s, we reconstruct f= $\mathcal{F}$(s) ,knowing that f is non zero only on W and recover s from f.

Writing explicitly the identity: $s = \mathcal{F}^{-1}$(s)

we have: $\frac{1}{\sqrt{n}} \sum_{u \in W} f(u) \cdot \zeta_n^{(-u \cdot x)} = s(x)\ x\ \epsilon\ T$

the sum may be restricted to W because $f = \mathcal{F}(s)$ is zero outside W. In matrix notation, this expression becomes: $R_{WT} f = s$

The pseudo-inverse and it’s application to recontruction

Given a matrix,the Moore-Penrose generalized matrix inverse is a unique matrix pseudoinverse.This matrix was independently defined by Moore in 1920 and Penrose (1955),and variously known as the generalized inverse, pseudoinverse,or Moore-Penrose inverse.

The psuedo inverse for a system A x = y : apply $A^{\mathcal{H}}$ to both sides:

$A^{\mathcal{H}}$A x = $A^{\mathcal{H}}$ y

$A^{\mathcal{H}}$A is square; invert it to find:

x = $\left(A^{\mathcal{H}}\ A\right)^{-1}\ A^{\mathcal{H}}$ y

The psuedo inverse produces a useful result, even with inconsistent data; it provides a minimal L_2 fit to the system of equations. For our system of equations, reconstruction depends on the conditioning of the matrix |W|x|W| matrix, Q = $R_{WT}R_{WT}^{\mathcal{H}}$

$(\ Q)_{u,v}$= $\sum_{t\epsilon T}\ \zeta_n^{t(u-v)}$ for (u,v) in W

The larger |T|, the better conditioned the matrix Q. In particular, for |T| = n, Q is the identity matrix of order |W|.

The engineers point of view, part 1

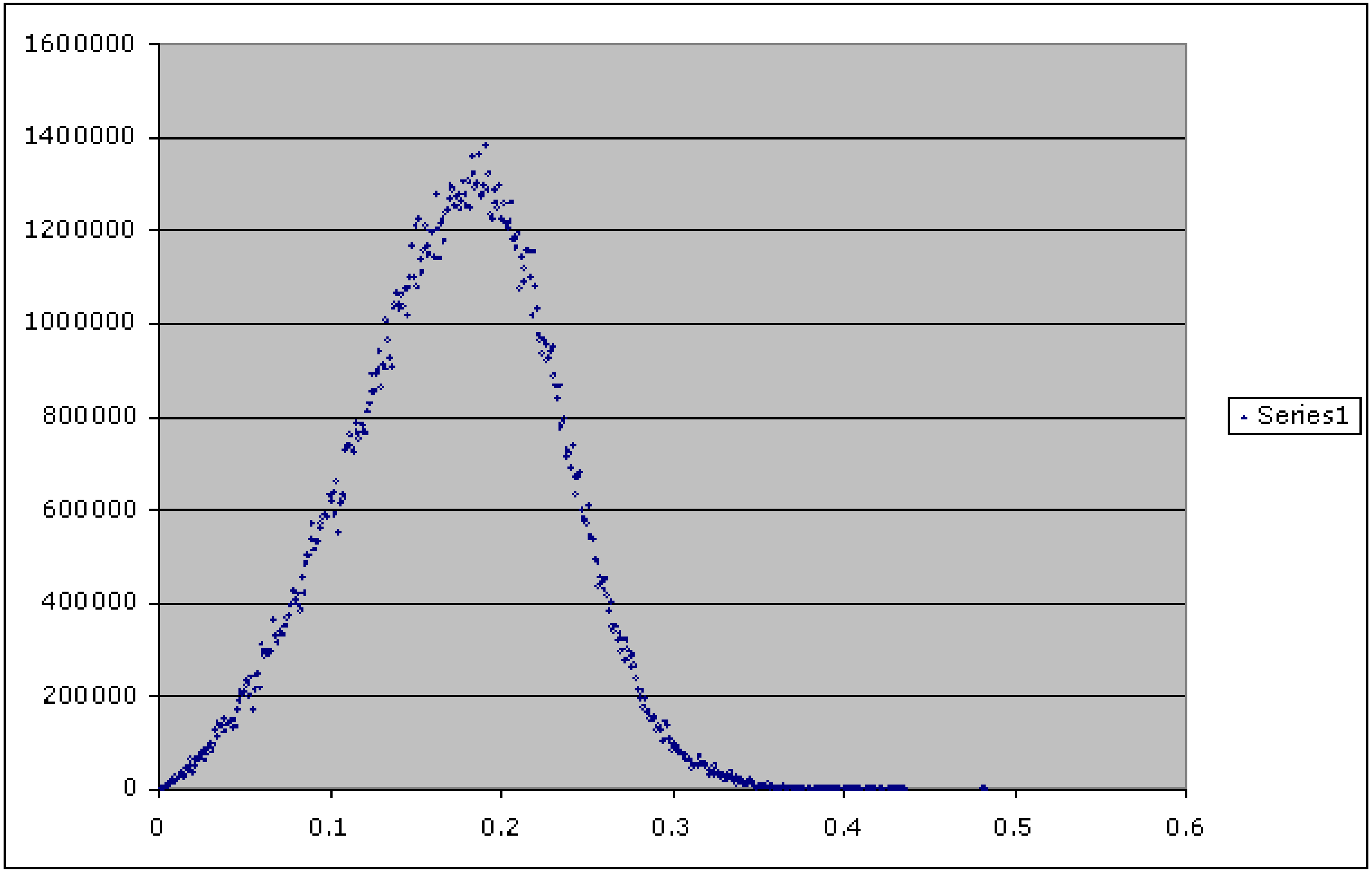
The engineer imagines ’n’ as the discretization scale. The extent to which the number theoretic properties of ’n’ are involved is unsatisfactory, and surprising. Further ’non-singularity’ isn’t good enough. For practical reconstruction, well conditioning of the matrix Q is necessary. For these purposes, matrix conditioning is measured in a variety of ways

1. The size of the determinant
2. The size of the smallest eigenvalue
3. Ratio of the largest to the smallest eigenvalue
4. The size of the Frobenius norm of the inverted matrix

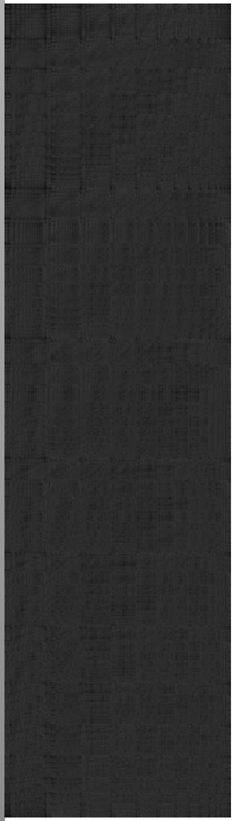
3 is a standard measure of matrix conditioning; 4 has an immediate consequence for the processing of quantized values

Related by inequalities, any of these means provides an indication of reconstruction problems

The distribution of the eigenvalues of the matrix Q shows extraordinary outliers



Numerical evidence - subset pairs for n=23,d_1=5 ,d_2=6



Sidebar - on programming to gather numerical evidence

- Problems amount to exhaustive search of all (most) subsets
- Processing of equivalence classes to extend the search space
- Optimizing programming execution time vs total time to result
- Compiled versus interpretitive environments + very basic parallelism

And then – How do you look at a billion numbers?

A bound on det Q by means of Chebotarev's theorem

$$\det\begin{pmatrix} \zeta_p^{a_1x_1} & \zeta_p^{a_1x_2} & \dots & \zeta_p^{a_1x_k} \\ \zeta_p^{a_2x_1} & \zeta_p^{a_2x_2} & \dots & \zeta_p^{a_2x_k} \\ \dots & \dots & \dots & \dots \\ \zeta_p^{a_kx_1} & \zeta_p^{a_kx_2} & \dots & \zeta_p^{a_kx_k} \end{pmatrix} =$$

$$\frac{\prod(x_i-x_j)\prod(a_i-a_j)}{1!2!\dots(k-1)!}\pi^{\frac{k(k-1)}{2}}\left(mod\pi^{\frac{k(k-1)}{2}+1}\right)$$

By Chebotarev's theorm,the determinant of

$C_{W,T}$=$\left(\zeta_n^{w\cdot t}\right)_{w\in W,t\in T}$

is non zero for n a prime with $|$W$|$= $|$T$|$. We have: $|$det$Q_{W,T}$$|$=$|detC_{W,T}|^2$

How small can this determinant be for a given W and T and n a prime?

It can be shown that: $\left|\det\left(C_{W,T}\right)\right| \geq n^{d(d-1)/4}\cdot\left(d^{d/2}\right)^{-(n-3)/2}$

The engineers point of view, part 2

$\left|\det\left(C_{W,T}\right)\right| \geq n^{d(d-1)/4}\cdot\left(d^{d/2}\right)^{-(n-3)/2}$

When d is small and n is large, this bound is exponential in n. This is once again unexpected and unsatisfying. What'd you want is something, perhaps exponential in d, but polynomial in n. Investigation so far has found for a prime n, the lowest determinant observed is for W, T 'intervals' of length d. For this case, the determinant behaves like: $n^{-d^2+o(d)}$ for n$>>$d. This is a satisfactory estimate. The hardness of the lower bound is not surprising. It is related to the classical problem in short character sums: to find for a prime p, the smallest value of the trigonometric sum of d terms:

$\sum_{i=1}^{d}\zeta_p^{x_i}$

Again the lower bounds (provable just as above but without as much congruence help) are exponential in p. The best exiting examples of small values are just as with 'intervals' of the order $p^{-d/2}$.

These problems are related, because smallness of the determinant is equivalent to smallness of d linear forms in the d roots of unity.

When Sums of Roots of Unity are Very Small?

The problem of classification of all cases when the sum of roots of unity is zero is a very old one, and it was "solved" many times. H. Mann (1964) and I. Schoenberg (1964) showed that all cases of vanishing of roots of unity can be reduced by simple algebraic operations to the primitive cyclotomic relations. It is extended to include the diophantine equations of the vanishing sums of elements from algebraic multiplicative groups. This was an "easy" problem.

The problem that seems to be much harder is the following one:

When the sum of roots of unity is non-zero but very very small?

The first formulation of this problem belongs, probably, to G. Myerson (1975). He was also the first to prove the "Liouvillean" lower bound for the absolute value of the non-vanishing sum of zeros. It marginally improved in the case of sums of pairwise distinct roots of unity.

Exponential Lower bound

Liouville Lemma. Let ξ be a non-zero sum of d distinct n-th roots of unity:

$$\xi \;=\; \sum_{s\in S} \zeta_n^s,$$

with $|S|$ = d. Then

$$|\xi| \;\geq (\frac{d\cdot(n-d)}{\varphi(n)-2})^{-\frac{\varphi(n)-2}{4}}.$$

First, the conjugates to the algebraic number ξ all have the form.

$$\xi^{(\sigma)} \;=\; \sum_{s\in S} \zeta_n^{s\cdot\sigma},$$

for some σ in $\{0,\ldots,n-1\}$.
Next, ξ is non-zero and an algebraic integer, we have

$$1 \leq |N(\xi)|^2 \leq |\xi|^4 \cdot \prod_{\sigma\neq\pm 1} |\xi^{(\sigma)}|^2.$$

Parseval's identity

Now we use Parseval's identity:

$$\sum_{\sigma=0}^{n-1} |\sum_{s\in S} \zeta_n^{s\cdot\sigma}|^2 = n \cdot d.$$

Now using the arithmetic-geometric means inequality and the Parseval's identity, we get

$$1 \leq |\xi|^4 \cdot (\frac{d\cdot(n-d)}{\varphi(n)-2})^{\varphi(n)-2}.$$

Roughly speaking, particularly for the prime n (when ξ is non-zero for any proper subset S of $\{0\ldots n-1\}$), one has a lower bound (Konyagin-Lev) on $|\xi|$ of the form

$$d^{-\frac{n-3}{4}}.$$

This lower bound on the smallest sum of roots of unity is exponential in n independently of d, typical for all Liouville-type lower bounds. Unfortunately in general we have no better bound for the sums of roots of unity.

Upper Bound on the Sum of Roots of Unity

It is rather hard even to find good upper bound for the smallest sum of roots of unity. This is another popular problem. Graham and Sloane (1984) asked:
What is the magnitude $\lambda_c(n)$ of the smallest eigenvalue of a non-singular $(0,1)$ circulant matrix of order n? This question is equivalent to the problem of determining the smallest magnitude $\lambda_c(n)$ of a non-zero sum of distinct n-th roots of unity.
Graham and Sloan proved the upper bound

$$\lambda_c(n) \leq \frac{c \cdot n}{2^{n/8}}$$

This bound holds for $n = 2p$ for a prime p.

Case of $n = 2p$

In general Dirichlet principle shows that for $n = 2p$ (and a prime p) one can get a lower bound for the absolute value of the non-vanishing sum of d of n-th roots of unity:

$$\frac{c \cdot d}{(n/2)^{\lfloor d/2 \rfloor /2}}.$$

This bound for the absolute value of the non-vanishing sum of d of n-th roots of unity is more realistic. It is exponential in d, not in n, like other "normal" linear diophantine approximation bounds are expected to be. The main difference here is the significant non-linearity of the diophantine approximations that creates difficulties of constructing even these bounds - requiring conditions like $n = 2p$.

Case of $n = p$

For a prime $n = p$ the known lower bounds are much cruder.
Relatively weak bound belongs to Konyagin-Lev (2000) and it shows that there are sums of d distinct n-th roots of unity that are smaller in magnitude than

$$d^{-(1-\epsilon)\log_2 n}$$

Hint: Take $n = p$ for a prime p and $d = 2^k < c_\epsilon n$ and define the set S as the set of sums of all subsets of

$$\{p' + 1, p' + 2, \ldots, p' + 2^{k-1}\} \text{ in } \mathbf{Z/Zp}$$

for $p' = (p-1)/2$.

How Small are Small Sums of Roots of Unity?

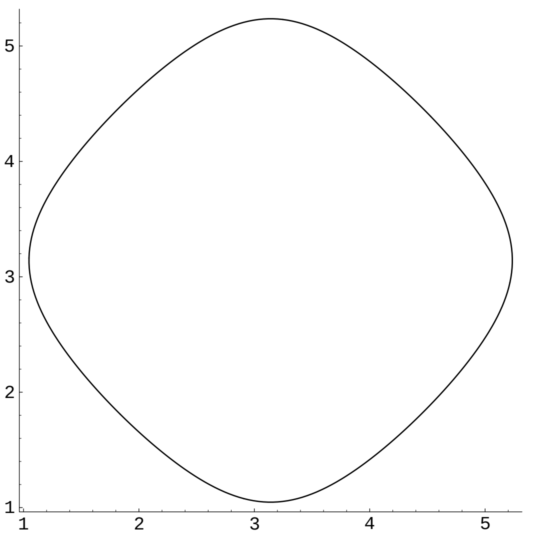
Unfortunately, the existing methods of effective (like bound on linear forms in logarithms) or ineffective diophantine geometry (like Subspace Theorem) that are well-used for the solutions of traditional exponential diophantine equations, are not applicable here, mainly due to an extreme non-linearity of the approximation problem.
Interestingly enough it is not trivially obvious what to expect here, either.
We will concentrate on the most interesting case of n sufficiently large with respect to d.
First, there is no non-trivial small sums of roots of unity for $d \leq 4$.
Efficient way of looking at the problem by looking at the zero-manifold

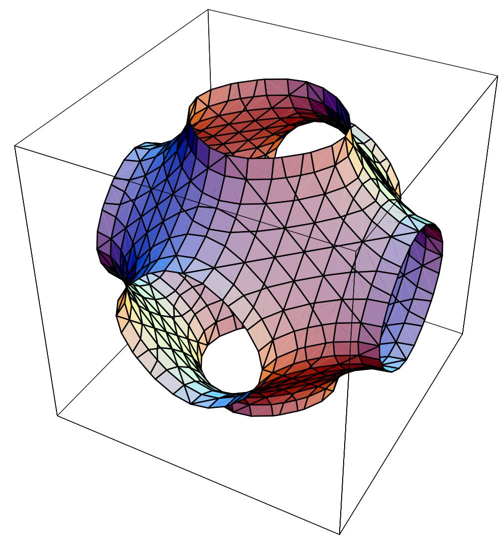
$$\sum_{i=1}^{d} e^{2\pi \cdot z_i} \;=\; 0 \;(S)$$

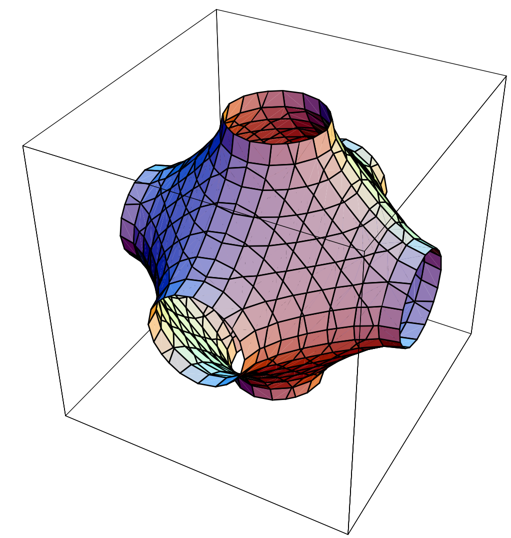
and asking how close the grid points $(\frac{a_1}{n}, \ldots, \frac{a_d}{n})$ with integers a_i can come to this zero-manifold.
Its dimension is $\lfloor d/2 \rfloor - 1$.

How the Manifold looks like

These are the images of this manifold for $d = 5$, $d = 6$ and $d = 7$.







These images of the surfaces (and their periodicity translations to cover the whole space) are illustrative of what to expect from the "singularity manifold", We hope to expect something similar from the "near-singular" or ill-conditioned minors of the Fourier matrix.

Expected Polynomial Bounds

We do not expect to see the sum of d of n-th roots of unity to be exponentially small with respect to n but non-zero when $n \gg d$; just as we do not expect anything similar for conventional diophantine problems.
What we should expect is the lower bound that is polynomial in n:

$$|\xi| = |\sum_{s \in S} \zeta_n^s| > n^{-K} \ (H)$$

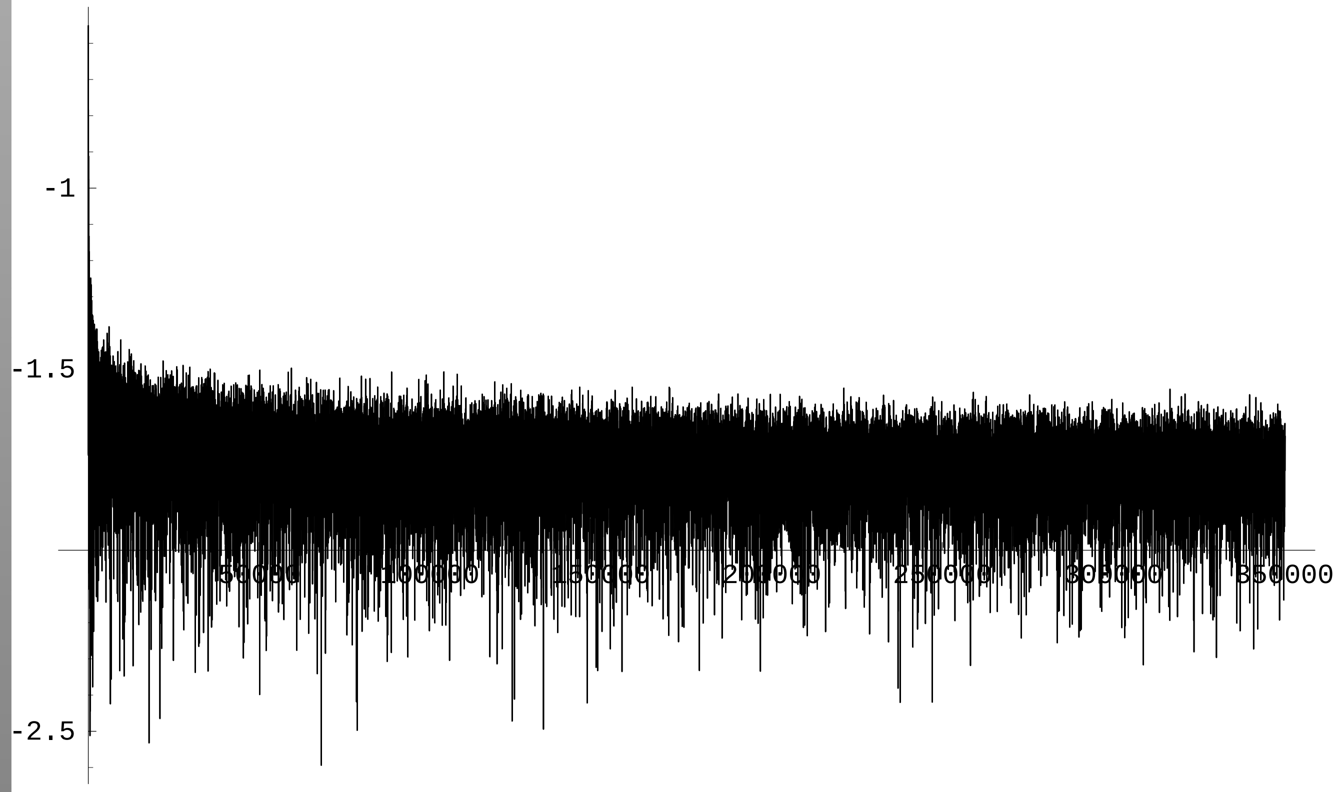
when $\xi \neq 0$, $|S| = n$ and $n \gg d$.
The precise expected exponent K is a bit difficult to predict. We already know that there is a lower bound on K:

$$K \geq \frac{\lfloor d/2 \rfloor}{2}.$$

Unfortunately the complexity of running numerical experiments with finding the closest grid point to the manifold (S) is of the order of $n^{\lfloor d/2 \rfloor - 1}$ for each n.

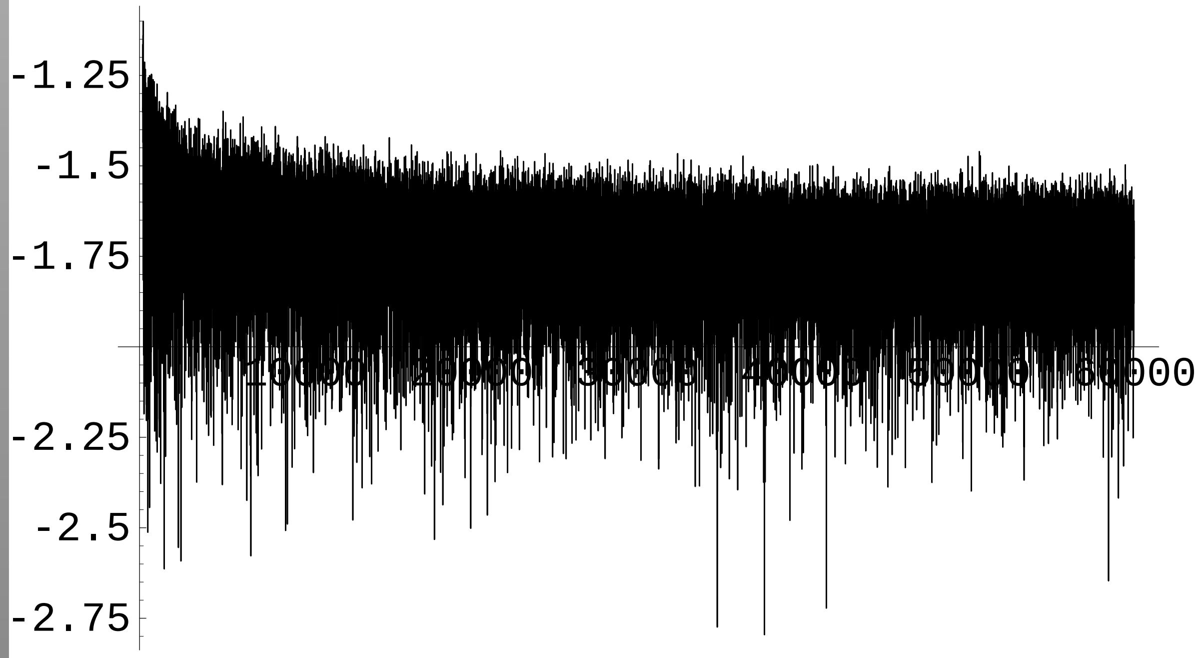
Graphs of the polynomial exponent

In the simplest case of $d = 5$ one gets the following image of the exponent K in (H) as a function of n for prime n only.



Graphs of the polynomial exponent, cont.

If we look at all n, not divisible by 5 or 6 (when there are vanishing sums of 5 of n-th roots of unity), we get the following image of the exponent K in (H) as a function of n.



Experimental Evidence

From a limited amount of numerical experiments one can come back with a reasonable guess that the exponent K (as a function of d only for $n \gg d$) should satisfy

$$K \leq \lfloor \frac{d}{2} \rfloor .$$

In fact for a positive density of n (even among the primes) it seems that K is much smaller than that; only of order $d/4$; close to the analytic upper bounds. Certainly the exponent is not a simple monotonic function of n (for a given d); a numerical answer to G. Myerson's problem.

It is (NP) Hard to deal with Sparse Sums of Roots of Unity.

The computational cost of operating on Fourier transforms of sparse sets and determining their global properties is huge not only in practice but also in theory.

Sparse-Poly-Divis Given an integer N and a set $\{p_1(x), \ldots, p_k(x)\}$ of sparse polynomials, determine whether $x^N - 1$ is not a factor of $\prod_{i=1}^{k} p_i(x)$.

Sparse-Poly-NonRoot Given a sparse polynomial $p(x)$ with integer coefficients and an integer M, determine if $p(\omega) \neq 0$ where ω is a primitive M-th root of unity.

Sparse-Poly-Root-Modulus-1 Given a sparse polynomial $p(x)$ with integer coefficients, determine if $p(z)$ has a root on the complex unit circle.

Trig-Inequality Given integers $a_1, \ldots, a_n, b_1, \ldots, b_n, c$, determine whether there exists a real θ such that the inequality

$$c + \sum_{i=1}^{n} a_i \cos b_i\theta > 0$$

fails to hold.

Plaisted proved that Sparse-Poly-Divis is **NP** complete. and that Sparse-Poly-NonRoot, Sparse-Poly-Root-Modulus-1, and Trig-Inequality are **NP** hard.

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From Discrete to Continuous Band-Limited Signals (and Back).

We started at trigonometric sums, and problems of singular and nearly singular minors of the Fourier transform matrix. From this subject of signal reconstruction and diophantine approximation we are now led to the subjects that we like a lot: Padé approximations and isomonodromy transformations. It will provide us with a better look at those ill-conditioned matrices and their deep analytical properties.

General Prolate Functions.

The story of functions band-limited in space and/or frequency.
For a bounded subset Ω in n-dimensional space, we have a subspace R_Ω of $L_2(R^n)$ of functions $f(x)$ whose frequency is band-limited to Ω:

$$f(x) \; = \; (\frac{1}{2\pi})^{n/2} \cdot \int \cdots \int_{\Omega} e^{-i(x.u)} \cdot s(u) \, du$$

Let P_Ω be the projection from $L_2(R^n)$ to R_Ω - i.e. P_Ω is the operator of band-limiting to Ω.
"Space-limited" functions and projections.
If M is a subset of R^n, the operator D_M restricts functions to M:

$$D_M f(x) = f(x) \cdot \chi_M(x),$$

There is a natural band-limited basis $\{\psi_k\}$ of R_Ω of eigenfunctions of the "double projection" operator $P_\Omega D_M$:

$$P_\Omega D_M \psi_k = \lambda_k \cdot \psi_k.$$

The "double projection" operator $P_\Omega D_M$ is:

$$P_\Omega D_M(f(x)) \; = \; \int \cdots \int_M K_\Omega(x-y) \cdot f(y)\, dy,$$

with kernel $K_\Omega(x-y)$ depending on the spectral support Ω:

$$K_\Omega(x-y) \; = \; (\frac{1}{2\pi})^{n/2} \cdot \int \cdots \int_\Omega e^{-i((x-y).u)}\, du.$$

This integral equation is very complex as it ties together space and frequency. It is a "continuous" version of the symmetric square of the general Chebotarev matrix.

In the incredibly **"lucky accident"** in cases of the Ω and M being balls (intervals for $n = 1$) the problem turned out to be reducible to a well studied classical one.
D. Slepian, H. Landau, and H. Pollak (1961-1983).
They called this theory a theory of prolate functions because the eigenfunctions ψ_k are actually eigenfunctions of a classical prolate spheroidal wave equation.

When $\Omega = [-W, W]$ and $M = [-T, T]$, if we put $x = Tz$, and $c = W \cdot T$, $P_\Omega D_M$ is an integral operator:

$$P_\Omega D_M(\psi(z)) \;=\; \int_{-1}^{1} \frac{\sin c(z-u)}{\pi(z-u)} \cdot \psi(u)\, du.$$

It commutes with the prolate spheroidal linear differential operator:

$$P_z \;=\; \frac{d}{dz}(1-z^2)\frac{d}{dz} \;-\; c^2 z^2.$$

Eigenfunctions $\psi_k(z)$ are eigenfunctions of P_z:

$$(1-z^2)\cdot\psi_k^{''}(z) - 2z\cdot\psi_k^{'}(z) + (\chi_k - c^2 z^2)\cdot\psi_k(z) \;=\; 0.$$

Practical problems: ill-conditioning of the integral operator $P_\Omega D_M$ with almost all eigenvalues λ_k very close to degenerate - almost all of them cluster at $\lambda = 0$ and $\lambda = 1$. Second eigenvalues χ_k are very well separated.

General Prolate Functions and Commuting Differential Operators.

For theoretical and applied development of the theory of space/frequency limited functions one needs the second, (differential?), well-conditioned, linear problem that defines the same eigenfunctions.

Slepian, Landau, Pollak generalized prolate functions to n-dimensional balls, where the original sinc $K_{\Omega}(x-y)$ kernel

$$\frac{\sin\left(c \cdot (x-y)\right)}{x-y}$$

is replaced by the Bessel-like kernel

$$J_N(cxy)\sqrt{xy}.$$

There were various attempts in 60s-80s to extend this commuting miracle to other cases.
The only new found kernel was Airy $Ai(x+y)$.
Morrison in 60s and Grunbaum in 80s:
the cases of commuting differential operators (or sparse matrices in the discrete cases) are basically reduced to the known ones.
Are there other completely integrable cases?

Szegö Problem and Concentrated Polynomials.

In fact, all this business started with Szegö's problem:

Szegö. *On some hermitian forms associated with two given curves of the complex plane, 1936*; see Szegö's Collected Works, v II, pp. 666-678.

Let C_1, C_2 be Jordan curves in the complex plane. What is the maximum value of

$$M_n(C_1, C_2) = \frac{\int\limits_{C_1} |P(z)|^2\, dz}{\int\limits_{C_2} |P(z)|^2\, dz}$$

among all polynomials $P(z)$ of degree n?

Then $M_n(C_1, C_2)$ can be interpreted as the energy ratio, and $P(z)$ as a polynomial having its energy most concentrated in C_1 at the expense of its energy in C_2.

A particular case that was treated in great detail is that of the interval C_1 and a circle C_2: Let C_1 is an interval $(0, 1)$, and C_2 is the unit circle. One gets then the quadratic form:

$$\int_0^1 (x_0 + x_1 t + x_2 t^2 + \cdots + x_n t^n)^2 dt = \sum_{\mu,\nu=0}^{n} \frac{x_\mu x_\nu}{\mu + \nu + 1}$$

and Szegö obtain for its smallest characteristic value the asymptotics

$$\lambda_n \cong 2^{15/4} \pi^{3/2} n^{1/2} \cdot (2^{1/2} - 1)^{4n+4}.$$

This matrix is the Hilbert matrix. Since its largest eigenvalue is very close to π, this matrix is notoriously ill-conditioned. For example, for $n = 100$, the smallest eigenvalue of Hilbert matrix is

$$1.71 \cdot 10^{-152}$$

While the inverse of Hilbert matrix is analytically known, no analytic properties of its eigenfunction/values were known.

1

Hilbert Matrix and a Commuting Differential Operator.

The eigenvectors of the Hilbert matrix

$$H[n] = \left(\frac{1}{i-j+1}\right)_{i,j,=0}^{n}$$

can be described using a "very natural" 4-th order Fuchsian linear differential operator:

$$\begin{aligned}
&L_n^{\{4\}} = \\
&x^3 \cdot (x-1)^2 \cdot \frac{d^4}{dx^4} + \\
&2 \cdot x^2 \cdot (5x-3) \cdot (x-1) \cdot \frac{d^3}{dx^3} + \\
&x \cdot (6-(n^2+2n)(x-1)^2+4x(6x-7)) \cdot \frac{d^2}{dx^2} + \\
&(-n(n+2)+4(-2+n(n+2))x-3(-4+n(n+2))x^2) \cdot \frac{d^1}{dx^1} + \\
&(C-n(n+2)x) \cdot \frac{d^0}{dx^0}
\end{aligned}$$

The equation

$$L_n^{\{4\}} Q \;=\; 0$$

has a polynomial solution $Q(x)$ of degree n when and only when the vector of coefficients of $Q(x)$ is the eigenvector of $H[n]$:

$$H[n] \cdot v \;=\; \lambda \cdot v;$$

$Q(x) \;=\; \sum_{i=0}^{n} v_i \cdot x^i$.
The relationship between the commuting eigenvalues - C (the accessory parameter of the Fuchsian l.d.e.) and λ (the matrix eigenvalue) shows the role of Padé approximations.

The Hilbert eigenvalue problem becomes an over-convergence Padé- approximation problem:

Find a polynomial $Q_n(x)$ of degree n such that the linear form

$$\log\left(1-\frac{1}{x}\right)\cdot Q_n(x) - P_{n-1}(x) - \lambda\cdot Q_n\left(\frac{1}{x}\right)x^{2n} \;=\; O(x^{-n-2})$$

at $x\to\infty$.

The differential equation formulation replaces an ill-conditioned problem with the equivalent well-conditioned, and a dense matrix with a commuting sparse one.

Szegö Problem and Arbitrary Unions of Intervals.

The most interesting case of the Szegö problem is that of sets C_1, C_2 the unions of intervals.
In 1977 Slepian and Gilbert tried to find differential operators commuting with the concentration problem. They found that there were **exactly** 2 such cases: C_1, C_2 are single intervals and
C_2 centrally positioned inside C_1,
or C_1, C_2 adjacent.
There is **no** commuting differential operators in the multiple interval cases neither in Szegö problem, nor in the space/frequency bandlimiting problem.
Nevertheless these problems can be solved using classical isomonodromy deformation using methods we studied 27 years ago.
Start with Padé approximation.

The rational function $P_n(x)/Q_n(x)$ is a Padé approximation to $f(x)$ of order n if

$$Q_n(x) \cdot f(x) - P_n(x) = O(x^{2n+1}).$$

If the order is exactly x^{2n+1}, the Padé approximation is called **normal.**
Szegö problem is equivalent to finding cases of non- normality of Padé approximation to the general logarithmic function

$$f(x) = \sum_{i=1}^{m} w_i \log(1 - a_i x).$$

for a fixed set $\{a_i\}_{i=1}^m$ of singularities, made from the ends of intervals comprising C_1 and C_2.

Namely, when

$$C_1 \ = \bigcup_{i_1=1}^{d_1} (b_{i_1}, c_{i_1}); \ C_2 \ = \bigcup_{i_2=1}^{d_2} (d_{i_2}, e_{i_2}),$$

we will associate weight $w(.)$ with the ends of these intervals:

$$w(b_{i_1}) = 1, w(c_{i_1}) = -1, w(d_{i_2}) = \lambda, w(e_{i_2}) = -\lambda.$$

λ is an eigenvalue in $M_n(C_1, C_2)$ problem **if and only if** the Padé approximation $P_n(x)/Q_n(x)$ of order n to

$$f(x) \ = \sum_{i=1}^{m} w(a_i) \log{(1 - a_i x)},$$

is **non-normal**, and then polynomial $Q_n(x)$ is the most concentrated polynomial.

This immediately associates with the $M_n(C_1, C_2)$ problem a Fuchsian linear differential equation of the second order satisfied by

$$Q_n(x) \text{ and } Q_n(x) \cdot f(x) - P_n(x).$$

Garnier Isomonodromy Deformation Equations.

The Fuchsian equations for the Szegö problem start with the set $\{a_i\}_{i=1}^m$ of ends of intervals.
We have m regular (logarithmic) singularities. We also have $m-2$ apparent singularities $\{b_i\}_{i=1}^{m-2}$ - these are spurious zeros of $Q_n(x)$ outside of C_1, C_2.

$$\frac{d^2}{dx^2}Y + \left(\sum_{i=1}^{m} \frac{1}{x-a_i} - \sum_{i=1}^{m-2} \frac{1}{x-b_i}\right)\frac{d}{dx}Y + \frac{p_{2m-4}(x)}{\prod(x-a_i)\prod(x-b_i)}Y = 0$$

There are exactly $m-2$ free (accessory) parameters $\{c_j\}$ in $p_{2m-4}(x)$:

$$c_j = Res_{x=b_j} \frac{p_{2m-4}(x)}{\prod(x-a_i)\prod(x-b_i)}$$

2

The monodromy group depends only on the number m of logarithmic singularities, and thus the dependence of this equation on position of singularities $\{a_i\}$ defines isomonodromy deformation equations, known as Garnier system (Garnier, 1912-1919). Hamiltonian form of Garnier system:

$$\frac{\partial b_k}{\partial a_j} = \frac{\partial K_j}{\partial c_k};$$

$$\frac{\partial c_k}{\partial a_j} = -\frac{\partial K_j}{\partial b_k}.$$

The Hamiltonians K_j:

$$T(x) = \prod(x - a_i);\ \ L(x) = \prod(x - b_i)$$

$$K_j = -\frac{L(a_j)}{T'(a_j)}(\sum_{l=1}^{m-2} \frac{T(b_l)}{L'(b_l)(b_l - a_j)}(c_l^2 + c_l \sum_{i=1}^{m-2} \frac{\delta_{ij}}{b_l - a_i}) + n(n+1)).$$

In the case of $m = 3$ the Garnier system is Painlevé VI.
Isomonodromy deformation systems posses birational Darboux-Schelsinger transformations. These are explicit nonlinear algebraic transformations that relate parameters a_i, b_j, c_k for a given n to parameters for $n+1$ (or $n-1$).
Zeros $\{z_j\}_{j=1}^n$ of $Q_n(z)$ on the complex plane are governed by Heune-Stiltjes theory of electrostatic particles minimizing the energy

$$\prod_{i \neq j} |z_i - z_j| \cdot \prod_{i,k} |z_i - a_k|^2 \cdot \prod_{i,l} |z_i - b_l|^{-2}.$$

Darboux Transformation for m=3 case

These are the simplest formulas for $m = 3$ (Painleve VI) with one singularity $a_3 = t$; one apparent singularity λ and an accessory parameter μ.
It shows the complexity of the transformation from n to $n-1$.

$$\mu' = \frac{(\lambda^3\mu^3-\lambda^2(t+1)\mu^3+n^2(t\mu+\mu+2n)+\lambda(\mu^3 t-3\mu n^2))(\mu^4\lambda^6-2\mu^4(t+1)\lambda^5+}{(\mu^2\lambda^3-\mu(t\mu+\mu+2n)\lambda^2+(t\mu^2+2n\mu+n^2)\lambda+}$$
$$\frac{(\mu^4(t^2+4t+1)-6\mu^2n^2)\lambda^4-2\mu(t(t+1)\mu^3-4n^2(t+1)\mu-4n^3)\lambda^3+}{n^2(t-1))(\mu^2\lambda^3-\mu(t\mu+\mu+2n)\lambda^2+(t\mu^2+2nt\mu+n^2)\lambda-}$$
$$\frac{(t^2\mu^4-2n^2(t^2+5t+1)\mu^2-8n^3(t+1)\mu-3n^4)\lambda^2+2n^2(t(t+1)\mu^2+4nt\mu+n^2(t+1))\lambda+n^4(t-1)^2)}{n^2(t-1))(\mu^2\lambda^3-\mu(t\mu+\mu+2n)\lambda^2+(t\mu^2+2n(t+1)\mu+n^2)\lambda-n(tn+n+2\mu t))}$$

$$\lambda' = \frac{\lambda((\lambda-1)(n-\lambda\mu)^2-((\lambda-1)\lambda\mu^2-2(\lambda-1)n\mu+n^2)t)^2}{(\lambda-1)((\lambda-1)\lambda^2\mu^2+2(\lambda-1)\lambda n\mu-(3\lambda+1)n^2)(n-\lambda\mu)^2+}$$
$$\frac{}{(n^2-(\lambda-1)\lambda\mu^2)^2t^2-2(\lambda-1)((\lambda-1)\lambda^3\mu^4-\lambda(4\lambda-1)n^2\mu^2+4\lambda n^3\mu-n^4)t}$$

Poles of λ and μ give the location of eigenvalues for 2-interval sets.

3

Generalized Prolate Functions and another Isomonodromy Problem.

Expression of generalized prolate functions for sets Ω and M that are unions of intervals also can be reduced to the Padé approximations and isomonodromy deformation (Garnier) equations.
The difference in the approximating function:

$$f(x) = \sum w_i \cdot \log \frac{x - a_i}{x - \frac{1}{a_i}}$$

for a_i on the unit circle.
The function $f(x)$ is approximated both at $x = 0$ and $x = \infty$:

$$Q(x) \cdot f(x) - P(x) = O(x^{-1});\ x \to \infty$$

$$Q(x) \cdot f(x) - P(x) + \lambda \cdot Q(x) = O(x^n);\ x \to 0.$$

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Fourier Matrices in 2D.

Singular set are rather trivial (only coinciding elements) for the case $n = p$ - prime. What about the case $n = pq$? What are the singular sets for the case reduced Fourier matrix for $n = pq$?

For composite $n = pq$, the Fourier transform may be found either by directly by operation in $\mathbb{Z}/\mathbb{Z}_n$ or by successive operation in $\mathbb{Z}/\mathbb{Z}_p$ and in $\mathbb{Z}/\mathbb{Z}_q$ separately. This is a consequence of the group isomorphism between $\mathbb{Z}/\mathbb{Z}_{pq}$ and the direct sum of the $\mathbb{Z}/\mathbb{Z}_p$ $\oplus$ $\mathbb{Z}/\mathbb{Z}_q$groups, with index renumbering given by the Chinese Remainder Theorem.

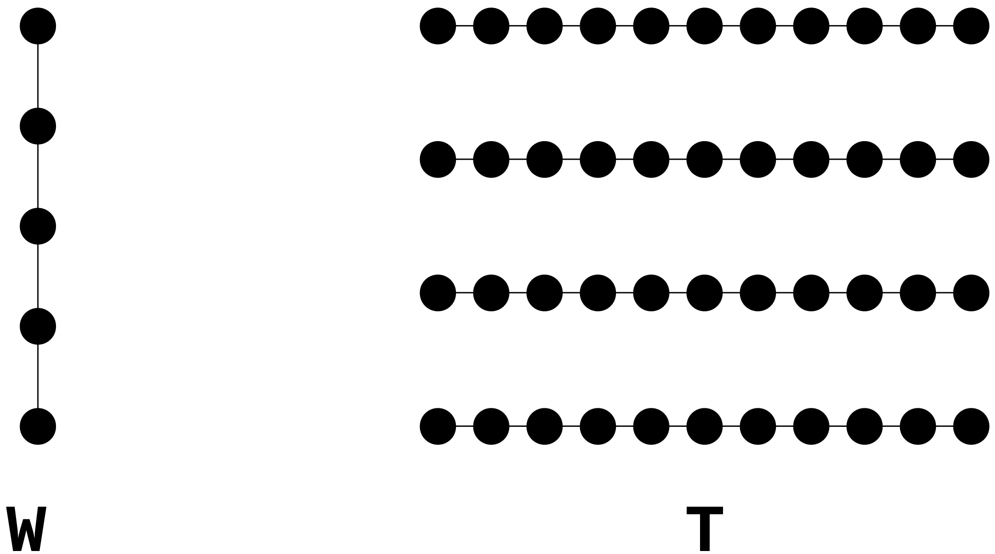
Thus, the 1-dimensional Fourier matrix for $n = pq$ can be represented as a 2-dimensional matrices over $\mathbb{Z}/\mathbb{Z}_p \oplus \mathbb{Z}/\mathbb{Z}_q$ separately by re-arrangement of rows and columns:

$$F_{j_p\, q^{-1}\, q + j_q\, p^{-1}\, p} = \sum_{k_q=0}^{q-1} \left(\sum_{k_p=0}^{p-1} x_{k_q\, p + k_p\, q} e^{\left(\frac{-2\pi i}{p} j_p\, k_p\right)} \right) e^{\left(\frac{-2\pi i}{q} j_q\, k_q\right)}$$

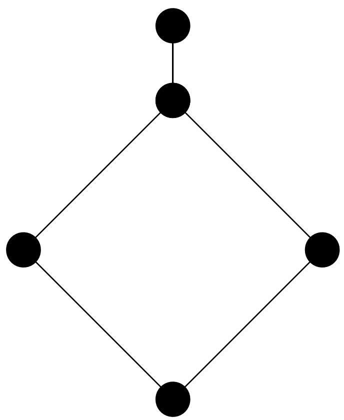
where q^{-1} is the inverse of q mod p ; p^{-1} is the inverse of p mod q and j_p and j_q range over the mixed radix representation of $0 \ldots (n-1)$.

Some examples of a singular configurations in p, q coordinates.

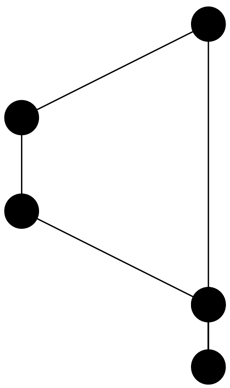
For singular configurations in the $n = pq$, the two dimensional representations clarifies the nature of the singular configurations (this the example of Donoho and Stark).



Some examples of a singular configurations in p, q coordinates - continued.



W



T

These configurations of W, T persist for varying n.

Composite case of $n = p \cdot q$*.*

When one tries to find all cases of singular pairs W, T for a composite $n = p \cdot q$ one finds that the singular cases seem to be valid "independently" of particular p and q and arise from the two-dimensional nature of the corresponding minors of the 2-dimensional Fourier matrix.
Namely, the numerical evidence that we accumulated seems to suggest the following. If the 2-dimensional representation of the $R_{W,T}$ matrix for $n = p \cdot q$

$$C2_{W,T} = (\zeta_p^{x_i \cdot u_j} \cdot \zeta_q^{y_i \cdot v_j})_{i,j=1}^d$$

is singular, then the sets $T = \{(x_i, y_i)\}_{i=1}^d$ and $W = \{(u_j, v_j)\}_{j=1}^d$ are specializations of the case when a full "functional" matrix

$$R_{W,T} = (z_1^{x_i \cdot u_j} \cdot z_2^{y_i \cdot v_j})_{i,j=1}^d$$

is singular (has a zero determinants) as a function of z_1 and z_2.

Functional Determinants in $2D$.

In fact it is true at least for the case of d sufficiently small with respect to p and q. For this we use results of H. Mann (1964) and I. Schoenberg (1964) on vanishing sums of roots of unity.
The condition for this is

$$d! \ \ll \ \min(p,q).$$

The result is also true for $d \leq 5$ due to our algebraic and numerical analysis.
It seems reasonable to conjecture that this reduction to the Functional Case is true in general; at least for

$$d \ < \ \min(p,q).$$

Two-Dimensional Singular Pairs of Sets

To find the "Singularity Manifold" in the 2-dimensional moduli space we have to solve the following problem:

2D Singularity Problem. For a given d find all configurations of d- sets $T = \{(x_i, y_i)\}_{i=1}^d$ and $W = \{(u_j, v_j)\}_{j=1}^d$ such that the Z-transform matrix

$$R_{W,T} = (z_1^{x_i \cdot u_j} \cdot z_2^{y_i \cdot v_j})_{i,j=1}^d$$

is singular (has a zero determinants) as a function of z_1 and z_2. If $R_{W,T}$ is singular (for all z_1, z_2), we call a pair W, T singular.

An obvious singularity case is that 2 rows of columns of $R_{W,T}$ being the same. This would mean that T or W would have the cardinality less than d.

The two-dimensional nature of W and T brings new non-trivial classes of singularity, hard to describe in a simple 1-dimensional language.

The most general class of singular sets arises from the classical theory of partitions and Young tableau. This class is particularly crucial in application to $n = p^2$, where it characterizes all singular cases.

Singularity Lemma

Singularity Permutations Lemma. Two sets $T = (x_i, y_i)_{i=1}^d$ and $W = (u_j, v_j)_{j=1}^d$ are singular in the sense above if and only if for every permutation τ from of $\{1 \ldots d\}$ there is a permutation τ' such that signs of permutations τ and τ' are different and

$$\sum_{i=1}^{d} x_i \cdot u_{\tau(i)} = \sum_{i=1}^{d} x_i \cdot u_{\tau'(i)}$$

$$\sum_{i=1}^{d} y_i \cdot v_{\tau(i)} = \sum_{i=1}^{d} y_i \cdot v_{\tau'(i)}$$

Partitions Notations

Singularity of 2-dimensional sets can occur only for some particular degeneracies among x- and y-projections of W, T. To describe them we need to introduce the partitions of d for these projections.
To fix the notations, let S by (a 1-dimensional) list of d elements, not necessarily distinct. We partition the list S into the sublists of equal elements:

$$S = \bigcup_{l=1}^{k} S_k$$

where each S_k has exactly one distinct element, but N_k repeated copies of that element of S. This naturally creates a partition of d:

$$d = N_1 + \cdots + N_k$$

which we order in a natural fashion: $N_1 \geq \cdots \geq N_k \geq 1$.

This partition will be denoted by

$$p(S) = (N_1, \cdots, N_k)$$

with k being the length of the partition of S, denoted by

$$k = Length(p(S))$$

and N_1 the maximal element of this partition of S, denoted by

$$N_1 = Max(p(S)).$$

Partitions Relations

Let's define a classical order on the set of all partitions. For two partitions

$$\alpha = (\alpha_1, \ldots, \alpha_i, \ldots)$$

and

$$\beta = (\beta_1, \ldots, \beta_i, \ldots)$$

we define

$$\alpha \leq \beta \text{ iff } \sum_{j=1}^{i} \alpha_j \leq \sum_{j=1}^{i} \beta_j, i = 1, \ldots$$

We also need a definition of a partition α' associated with the partition α (also known as a transposed partition):

$$\alpha'_i = \sum_{j, \alpha_j \geq i} 1$$

Note: the associated partition is easy to understand from the view of Young tableau $[\alpha]$ of α. Just interchange the rows and columns of the tableau (reflect $[\alpha]$ in its main diagonal).

Partitions Order and Singularity

Theorem. Let $\alpha = p(T_x)$ and $\beta = p(W_y)$ be partitions of the x-projection of T and the y-projection of W. Let the following condition is satisfied:

$$\alpha \leq \beta' \text{ is false } (R)$$

Then a pair W, T is singular. Similarly, if

$$p(T_y) \leq p(W_x)' \text{ is false}$$

or

$$p(W_x) \leq p(T_y)' \text{ is false}$$

or

$$p(W_y) \leq p(T_x)' \text{ is false}$$

then a pair W, T is singular.

The condition (R) has a rather simple combinatorial meaning that shows that it is a symmetric condition in α and β alone. The condition (R) is satisfied if and only if the coefficient of the monomial

$$x_1^{\alpha_1} \cdots x_i^{\alpha_i} \cdots \cdot y_1^{\beta_1} \cdots y_i^{\beta_i} \cdots$$

in the expansion of

$$\prod_{i,j}(1 + x_i y_j)$$

is zero.

Young Subgroups

The proof of this theorem uses another condition equivalent to (R) in terms of the induced representations of the subgroups S_α and S_β.
For an arbitrary partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of n and subsets $\bar{n}_i^\lambda$ of $\bar{n} = \{1, \ldots, n\}$ that are pairwise disjoint and which satisfy:

$$|\bar{n}_i^\lambda| = \lambda_i;\ \bar{n} = \bigcup_{i=1} \bar{n}_i^\lambda,$$

we define a subgroup S_λ as the product

$$S_\lambda = \prod_{i=1} S_i^\lambda,$$

where S_i^λ is the subgroup of S_n consisting of the $\lambda_i!$ elements that leave each point in $\bar{n} \backslash \bar{n}_i^\lambda$ fixed.

The subgroup S_λ is isomorphic to the direct product

$$S_{\lambda_1} \times S_{\lambda_2} \times \ldots .$$

S_λ is called a Young subgroup corresponding to λ (λ can be an improper partition - i.e. its parts do not have to be non-increasing).

Ruch and Schönhofer Theorem

The main auxiliary result we need here is

Theorem of Ruch and Schönhofer. For two partitions α and β of n and two Young subgroups S_α and S_β we have the condition (R) satisfied (i.e. $\alpha \leq \beta'$ not true) if and only if for every double coset $S_\alpha \pi S_\beta$ there is a nontrivial intersection

$$S_\alpha \bigcap \pi S_\beta \pi^{-1} \neq \{1\}$$

(in fact if this is true, there exists an odd sign permutation in $S_\alpha \bigcup \pi S_\beta \pi^{-1}$).
This theorem actually describes the intertwining number of the representation of S_n induced by identity and alternating representations $IS_\alpha \uparrow S_n$ and $AS_\beta \uparrow S_n$.
Using this theorem we only need a singularity lemma above to complete the proof.

Conditions of the form (R) as applied to the x-projections of W, T in the discrete case of $n = p^2$ are not only sufficient but also necessary for the singularity of the pair W, T. However for $d \geq 4$ the conditions (R) are not the only ones that define the singular pairs W, T. There are some other classes of exceptional cases defined by systems of linear equations on coordinates of the points from W and T separately.

Powers of a Prime.

The case of $n = p^k$ for a prime p is easier than that of a general factorable number. First of all since there is no prime factors of n other than p, this is not a truly multi-dimensional case. It is neither a one-dimensional case, so we call it "1.5-dimensional case". We have only "x-projections" of W, T in the form of $(\bmod p)$ for W, T as d-subsets of $\mathbf{Z/Zp^k}$.
The first non-trivial non-singularity result is

Theorem. Let $n = p^k$ and let elements of T be distinct $\bmod p$ and both T and W have d distinct elements from $\mathbf{Z/Zn}$. Then the pair W, T is non-singular, i.e. the matrix

$$C_{W,T} = (\zeta_n^{x \cdot u})_{x \in T,\, u \in W}$$

is non-singular. Similarly, if elements of W are distinct $\bmod p$, $C_{W,T}$ is non-singular.

The singularity statement is very similar to the two-dimensional case.

Theorem. Let $n = p^k$ and both T and W have d distinct elements from $\mathbf{Z}/\mathbf{Zn}$. $\alpha = p(T(\mathrm{mod} p))$ and $\beta = p(W(\mathrm{mod} p))$ be partitions of T and $W \mod p$. If this condition is false

$$\alpha \leq \beta' \quad (R)$$

Then a pair W, T is singular.

Only in the case $n = p^2$ the converse of this result is true.