On Fourier Matrices, Szegö Problem, Prolate Functions and Painlevé.

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General Prolate Functions.

The story of functions band-limited in space and/or frequency.

For a bounded subset Ω in n-dimensional space, we have a subspace R_Ω of $L_2(R^n)$ of functions $f(x)$ whose frequency is band-limited to Ω:

$$f(x) \;=\; (\frac{1}{2\pi})^{n/2} \cdot \int \cdots \int_\Omega e^{-i(x.u)} \cdot s(u)\, du$$

Let P_Ω be the projection from $L_2(R^n)$ to R_Ω - i.e. P_Ω is the operator of band-limiting to Ω.

"Space-limited" functions and projections.
If M is a subset of R^n, the operator D_M restricts functions to M:

$$D_M f(x) = f(x) \cdot \chi_M(x),$$

There is a natural band-limited basis $\{\psi_k\}$ of R_Ω of eigenfunctions of the "double projection" operator $P_\Omega D_M$:

$$P_\Omega D_M \psi_k = \lambda_k \cdot \psi_k.$$

The "double projection" operator $P_\Omega D_M$ is:

$$P_\Omega D_M(f(x)) \;=\; \int \cdots \int_M K_\Omega(x-y) \cdot f(y)\, dy,$$

with kernel $K_\Omega(x-y)$ depending on the spectral support Ω:

$$K_\Omega(x-y) \;=\; (\frac{1}{2\pi})^{n/2} \cdot \int \cdots \int_\Omega e^{-i((x-y).u)}\, du.$$

This integral equation is very complex as it ties together space and frequency.

In the incredibly **”lucky accident”** in cases of the Ω and M being balls (intervals for $n = 1$) the problem turned out to be reducible to a well studied classical one.

D. Slepian, H. Landau, and H. Pollak (1961-1983).

They called this theory a theory of prolate functions because the eigenfunctions ψ_k are actually eigenfunctions of a classical prolate spheroidal wave equation.

When $\Omega = [-W, W]$ and $M = [-T, T]$, if we put $x = Tz$, and $c = W \cdot T$, $P_\Omega D_M$ is an integral operator:

$$P_\Omega D_M(\psi(z)) = \int_{-1}^{1} \frac{\sin c(z-u)}{\pi(z-u)} \cdot \psi(u)\, du.$$

It commutes with the prolate spheroidal linear differential operator:

$$P_z = \frac{d}{dz}(1-z^2)\frac{d}{dz} - c^2 z^2.$$

Eigenfunctions $\psi_k(z)$ are eigenfunctions of P_z:

$$(1-z^2) \cdot \psi_k^{''}(z) - 2z \cdot \psi_k^{'}(z) + (\chi_k - c^2 z^2) \cdot \psi_k(z) = 0.$$

Practical problems: ill-conditioning of the integral operator $P_\Omega D_M$ with almost all eigenvalues λ_k very close to degenerate - almost all of them cluster at $\lambda = 0$ and $\lambda = 1$.

Second eigenvalues χ_k are very well separated.

General Prolate Functions and Commuting Differential Operators.

For theoretical and applied development of the theory of space/frequency limited functions one needs the second, (differential?), well-conditioned, linear problem that defines the same eigenfunctions.

Slepian, Landau, Pollak generalized prolate functions to n-dimensional balls, where the original sinc $K_\Omega(x-y)$ kernel

$$\frac{\sin\left(c \cdot (x-y)\right)}{x-y}$$

is replaced by the Bessel-like kernel

$$J_N(cxy)\sqrt{xy}.$$

There were various attempts in 60s-80s to extend this commuting miracle to other cases. The only new found kernel was Airy $Ai(x + y)$.

Morrison in 60s and Grunbaum in 80s:

the cases of commuting differential operators (or sparse matrices in the discrete cases) are basically reduced to the known ones.

Are there other completely integrable cases?

Szegö Problem and Concentrated Polynomials.

In fact, all this business started with Szegö's problem:

Szegö. *On some hermitian forms associated with two given curves of the complex plane, 1936*; see Szegö's Collected Works, v II, pp. 666-678.

Let C_1, C_2 be Jordan curves in the complex plane. What is the maximum value of

$$M_n(C_1, C_2) = \frac{\int\limits_{C_1} |P(z)|^2 \, dz}{\int\limits_{C_2} |P(z)|^2 \, dz}$$

among all polynomials $P(z)$ of degree n?

Then $M_n(C_1, C_2)$ can be interpreted as the energy ratio, and $P(z)$ as a polynomial having its energy most concentrated in C_1 at the expense of its energy in C_2.

A particular case that was treated in great detail is that of the interval C_1 and a circle C_2:

Let C_1 is an interval $(0,1)$, and C_2 is the unit circle. One gets then the quadratic form:

$$\int_0^1 (x_0 + x_1 t + x_2 t^2 + \ldots + x_n t^n)^2 dt \; = \sum_{\mu,\nu=0}^{n} \frac{x_\mu x_\nu}{\mu + \nu + 1}$$

and Szegö obtain for its smallest characteristic value the asymptotics

$$\lambda_n \cong 2^{15/4} \pi^{3/2} n^{1/2} \cdot (2^{1/2} - 1)^{4n+4}.$$

This matrix is the Hilbert matrix. Since its largest eigenvalue is very close to π, this matrix is notoriously ill-conditioned. For example, for $n = 100$, the smallest eigenvalue of Hilbert matrix is

$$1.71 \cdot 10^{-152}$$

While the inverse of Hilbert matrix is analytically known, no analytic properties of its eigenfunction/values were known.

Hilbert Matrix and a Commuting Differential Operator.

The eigenvectors of the Hilbert matrix

$$H[n] = \left(\frac{1}{i-j+1}\right)_{i,j,=0}^{n}$$

can be described using a "very natural" 4-th order Fuchsian linear differential operator:

$$L_n^{\{4\}} = x^3 \cdot (x-1)^2 \cdot \tfrac{d^4}{dx^4} +$$
$$2 \cdot x^2 \cdot (5x-3) \cdot (x-1) \cdot \tfrac{d^3}{dx^3} +$$
$$x \cdot (6-(n^2+2n)(x-1)^2+4x(6x-7)) \cdot \tfrac{d^2}{dx^2} +$$
$$(-n(n+2)+4(-2+n(n+2))x-3(-4+n(n+2))x^2) \cdot \tfrac{d^1}{dx^1} +$$
$$(C-n(n+2)x) \cdot \tfrac{d^0}{dx^0}$$

The equation

$$L_n^{\{4\}} Q = 0$$

has a polynomial solution $Q(x)$ of degree n when and only when the vector of coefficients of $Q(x)$ is the eigenvector of $H[n]$:

$$H[n] \cdot v = \lambda \cdot v;$$

$Q(x) = \sum_{i=0}^{n} v_i \cdot x^i$.

The relationship between the commuting eigenvalues - C (the accessory parameter of the Fuchsian l.d.e.) and λ (the matrix eigenvalue) shows the role of Padé approximations.

The Hilbert eigenvalue problem becomes an over-convergence Padé-approximation problem:

Find a polynomial $Q_n(x)$ of degree n such that the linear form

$$\log(1 - \frac{1}{x}) \cdot Q_n(x) - P_{n-1}(x) - \lambda \cdot Q_n(\frac{1}{x})x^{2n} = O(x^{-n-2})$$

at $x \to \infty$.

The differential equation formulation replaces an ill-conditioned problem with the equivalent well-conditioned, and a dense matrix with a commuting sparse one.

Szegö Problem and Arbitrary Unions of Intervals.

Sets C_1, C_2 are unions of intervals. In 1977 Slepian and Gilbert tried to find differential operators commuting with the concentration problem. They found that there were **exactly** 2 such cases: C_1, C_2 are single intervals and

C_2 centrally positioned inside C_1,

or C_1, C_2 adjacent.

There is **no** commuting differential operators in the multiple interval cases neither in Szegö problem, nor in the space/frequency bandlimiting problem.

Nevertheless these problems can be solved using classical isomonodromy deformation using methods we studied 27 years ago.

Start with Padé approximation.

The rational function $P_n(x)/Q_n(x)$ is a Padé approximation to $f(x)$ of order n if

$$Q_n(x) \cdot f(x) - P_n(x) = O(x^{2n+1}).$$

If the order is exactly x^{2n+1}, the Padé approximation is called **normal**.

Szegö problem is equivalent to finding cases of non- normality of Padé approximation to the general logarithmic function

$$f(x) = \sum_{i=1}^{m} w_i \log(1 - a_i x).$$

for a fixed set $\{a_i\}_{i=1}^{m}$ of singularities, made from the ends of intervals comprising C_1 and C_2.

Namely, when

$$C_1 = \bigcup_{i_1=1}^{d_1} (b_{i_1}, c_{i_1}); C_2 = \bigcup_{i_2=1}^{d_2} (d_{i_2}, e_{i_2}),$$

we will associate weight $w(.)$ with the ends of these intervals:

$$w(b_{i_1}) = 1, w(c_{i_1}) = -1, w(d_{i_2}) = \lambda, w(e_{i_2}) = -\lambda.$$

λ is an eigenvalue in $M_n(C_1, C_2)$ problem **if and only if** the Padé approximation $P_n(x)/Q_n(x)$ of order n to

$$f(x) = \sum_{i=1}^{m} w(a_i) \log(1 - a_i x),$$

is **non-normal**, and then polynomial $Q_n(x)$ is the most concentrated polynomial.

This immediately associates with the $M_n(C_1, C_2)$ problem a Fuchsian linear differential equation of the second order satisfied by

$$Q_n(x) \text{ and } Q_n(x) \cdot f(x) - P_n(x).$$

Garnier Isomonodromy Deformation Equations.

The Fuchsian equations for the Szegö problem start with the set $\{a_i\}_{i=1}^m$ of ends of intervals.

We have m regular (logarithmic) singularities. We also have $m-2$ apparent singularities $\{b_i\}_{i=1}^{m-2}$ - these are spurious zeros of $Q_n(x)$ outside of C_1, C_2.

$$\frac{d^2}{dx^2}Y + \Big(\sum_{i=1}^{m}\frac{1}{x-a_i} - \sum_{i=1}^{m-2}\frac{1}{x-b_i}\Big)\frac{d}{dx}Y + \frac{p_{2m-4}(x)}{\prod(x-a_i)\prod(x-b_i)}Y = 0$$

There are exactly $m-2$ free (accessory) parameters $\{c_j\}$ in $p_{2m-4}(x)$:

$$c_j \;=\; Res_{x=b_j}\frac{p_{2m-4}(x)}{\prod(x-a_i)\prod(x-b_i)}$$

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The monodromy group depends only on the number m of logarithmic singularities, and thus the dependence of this equation on position of singularities $\{a_i\}$ defines isomonodromy deformation equations, known as Garnier system (Garnier, 1912-1919).

Hamiltonian form of Garnier system:

$$\frac{\partial b_k}{\partial a_j} = \frac{\partial K_j}{\partial c_k};$$

$$\frac{\partial c_k}{\partial a_j} = -\frac{\partial K_j}{\partial b_k}.$$

The Hamiltonians K_j:

$$T(x) = \prod(x - a_i);\; L(x) = \prod(x - b_i)$$

$$K_j = \frac{L(a_j)}{T'(a_j)}(\sum_{l=1}^{m-2} \frac{T(b_l)}{L'(b_l)(b_l - a_j)}(c_l^2 + c_l \sum_{i=1}^{m-2} \frac{\delta_{ij}}{b_l - a_i}) - n(n+1)).$$

In the case of $m = 3$ the Garnier system is Painlevé VI.

Isomonodromy deformation systems posses birational Darboux-Schelsinger transformations. These are explicit nonlinear algebraic transformations that relate parameters a_i, b_j, c_k for a given n to parameters for $n+1$ (or $n-1$).

Zeros $\{z_j\}_{j=1}^{n}$ of $Q_n(z)$ on the complex plane are governed by Heune-Stiltjes theory of electrostatic particles minimizing the energy

$$\prod_{i \neq j} |z_i - z_j| \cdot \prod_{i,k} |z_i - a_k|^2 \cdot \prod_{i,l} |z_i - b_l|^{-2}.$$

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Generalized Prolate Functions and another Isomonodromy Problem.

Expression of generalized prolate functions for sets Ω and M that are unions of intervals also can be reduced to the Padé approximations and isomonodromy deformation (Garnier) equations.

The difference in the approximating function:

$$f(x) = \sum w_i \cdot \log \frac{x - a_i}{x - \frac{1}{a_i}}$$

for a_i on the unit circle.

The function $f(x)$ is approximated both at $x = 0$ and $x = \infty$:

$$Q(x) \cdot f(x) - P(x) = O(x^{-1});\ x \to \infty$$

$$Q(x) \cdot f(x) - P(x) + \lambda \cdot Q(x) = O(x^n);\ x \to 0.$$

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