On Fourier Matrices, Szegö Problem, Prolate Functions and Painlevé.

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General Prolate Functions.

The story of functions band-limited in space and/or frequency.

For a bounded subset $\Omega$ in $n$-dimensional space, we have a subspace $R_\Omega$ of $L_2(R^n)$ of functions $f(x)$ whose frequency is band-limited to $\Omega$:

$$f(x) = \left(\frac{1}{2\pi}\right)^{n/2} \cdot \int \cdots \int_\Omega e^{-i(x.u)} \cdot s(u) \, du$$

Let $P_\Omega$ be the projection from $L_2(R^n)$ to $R_\Omega$ - i.e. $P_\Omega$ is the operator of band-limiting to $\Omega$.

"Space-limited" functions and projections. If $M$ is a subset of $R^n$, the operator $D_M$ restricts functions to $M$:

$$D_M f(x) = f(x) \cdot \chi_M(x),$$
There is a natural band-limited basis \( \{ \psi_k \} \) of \( R_\Omega \) of eigenfunctions of the "double projection" operator \( P_\Omega D_M \):

\[
P_\Omega D_M \psi_k = \lambda_k \cdot \psi_k.
\]

The "double projection" operator \( P_\Omega D_M \) is:

\[
P_\Omega D_M (f(x)) = \int \cdots \int_M K_\Omega (x-y) \cdot f(y) \, dy,
\]

with kernel \( K_\Omega (x-y) \) depending on the spectral support \( \Omega \):

\[
K_\Omega (x-y) = \left( \frac{1}{2\pi} \right)^{n/2} \cdot \int \cdots \int_\Omega e^{-i((x-y) \cdot u)} \, du.
\]

This integral equation is very complex as it ties together space and frequency.
In the incredibly "lucky accident" in cases of the $\Omega$ and $M$ being balls (intervals for $n = 1$) the problem turned out to be reducible to a well studied classical one.


They called this theory a theory of prolate functions because the eigenfunctions $\psi_k$ are actually eigenfunctions of a classical prolate spheroidal wave equation.
When $\Omega = [-W, W]$ and $M = [-T, T]$, if we put $x = Tz$, and $c = W \cdot T$, $P_\Omega D_M$ is an integral operator:

$$P_\Omega D_M(\psi(z)) = \int_{-1}^{1} \frac{\sin c(z - u)}{\pi(z - u)} \cdot \psi(u) \, du.$$  

It commutes with the prolate spheroidal linear differential operator:

$$P_z = \frac{d}{dz} (1 - z^2) \frac{d}{dz} - c^2 z^2.$$  

Eigenfunctions $\psi_k(z)$ are eigenfunctions of $P_z$:

$$(1 - z^2) \cdot \psi_k''(z) - 2z \cdot \psi_k'(z) + (\chi_k - c^2 z^2) \cdot \psi_k(z) = 0.$$  

Practical problems: ill-conditioning of the integral operator $P_\Omega D_M$ with almost all eigenvalues $\lambda_k$ very close to degenerate - almost all of them cluster at $\lambda = 0$ and $\lambda = 1$.

Second eigenvalues $\chi_k$ are very well separated.
General Prolate Functions and Commuting Differential Operators.

For theoretical and applied development of the theory of space/frequency limited functions one needs the second, (differential?), well-conditioned, linear problem that defines the same eigenfunctions.

Slepian, Landau, Pollak generalized prolate functions to $n$-dimensional balls, where the original sinc $K_{\Omega}(x - y)$ kernel

$$\frac{\sin (c \cdot (x - y))}{x - y}$$

is replaced by the Bessel-like kernel

$$J_N(cxy)\sqrt{xy}.$$
There were various attempts in 60s-80s to extend this commuting miracle to other cases. The only new found kernel was Airy $Ai(x + y)$.

Morrison in 60s and Grunbaum in 80s:

the cases of commuting differential operators (or sparse matrices in the discrete cases) are basically reduced to the known ones.

Are there other completely integrable cases?
Szegö Problem and Concentrated Polynomials.

In fact, all this business started with Szegö’s problem:

Szegö. *On some hermitian forms associated with two given curves of the complex plane, 1936*; see Szegö’s Collected Works, v II, pp. 666-678.

Let $C_1, C_2$ be Jordan curves in the complex plane. What is the maximum value of

$$M_n(C_1, C_2) = \frac{\int_{C_1} |P(z)|^2 \, dz}{\int_{C_2} |P(z)|^2 \, dz}$$

among all polynomials $P(z)$ of degree $n$?

Then $M_n(C_1, C_2)$ can be interpreted as the energy ratio, and $P(z)$ as a polynomial having its energy most concentrated in $C_1$ at the expense of its energy in $C_2$. 
A particular case that was treated in great detail is that of the interval $C_1$ and a circle $C_2$:

Let $C_1$ is an interval $(0, 1)$, and $C_2$ is the unit circle. One gets then the quadratic form:

$$\int_0^1 (x_0 + x_1 t + x_2 t^2 + \ldots + x_n t^n)^2 dt = \sum_{\mu,\nu=0}^{n} \frac{x_\mu x_\nu}{\mu + \nu + 1}$$

and Szegö obtain for its smallest characteristic value the asymptotics

$$\lambda_n \approx 2^{15/4} \pi^{3/2} n^{1/2} \cdot (2^{1/2} - 1)^{4n+4}.$$
This matrix is the Hilbert matrix. Since its largest eigenvalue is very close to $\pi$, this matrix is notoriously ill-conditioned. For example, for $n = 100$, the smallest eigenvalue of Hilbert matrix is

$$1.71 \cdot 10^{-152}$$

While the inverse of Hilbert matrix is analytically known, no analytic properties of its eigenfunction/values were known.
Hilbert Matrix and a Commuting Differential Operator.

The eigenvectors of the Hilbert matrix

\[ H[n] = \left( \frac{1}{i - j + 1} \right)^n_{i,j=0} \]

can be described using a "very natural" 4-th order Fuchsian linear differential operator:

\[
L^{\{4\}}_n = x^3 \cdot (x - 1)^2 \cdot \frac{d^4}{dx^4} + 2 \cdot x^2 \cdot (5x - 3) \cdot (x - 1) \cdot \frac{d^3}{dx^3} + x \cdot (6 - (n^2 + 2n)(x - 1)^2 + 4x(6x - 7)) \cdot \frac{d^2}{dx^2} + (-n(n + 2) + 4(-2 + n(n + 2))x - 3(-4 + n(n + 2))x^2) \cdot \frac{d^1}{dx^1} + (C - n(n + 2)x) \cdot \frac{d^0}{dx^0}
\]
The equation

\[ L_n^{4} Q = 0 \]

has a polynomial solution \( Q(x) \) of degree \( n \) when and only when the vector of coefficients of \( Q(x) \) is the eigenvector of \( H[n] \):

\[ H[n] \cdot v = \lambda \cdot v; \]

\[ Q(x) = \sum_{i=0}^{n} v_i \cdot x^i. \]

The relationship between the commuting eigenvalues - \( C \) (the accessory parameter of the Fuchsian l.d.e.) and \( \lambda \) (the matrix eigenvalue) shows the role of Padé approximations.
The Hilbert eigenvalue problem becomes an over-convergence Padé-approximation problem:

Find a polynomial $Q_n(x)$ of degree $n$ such that the linear form

$$\log \left(1 - \frac{1}{x}\right) \cdot Q_n(x) - P_{n-1}(x) - \lambda \cdot Q_n \left(\frac{1}{x}\right)x^{2n} = O(x^{-n-2})$$

at $x \to \infty$.

The differential equation formulation replaces an ill-conditioned problem with the equivalent well-conditioned, and a dense matrix with a commuting sparse one.
Szegö Problem and Arbitrary Unions of Intervals.

Sets $C_1, C_2$ are unions of intervals. In 1977 Slepian and Gilbert tried to find differential operators commuting with the concentration problem. They found that there were exactly 2 such cases: $C_1, C_2$ are single intervals and

- $C_2$ centrally positioned inside $C_1$,
- or $C_1, C_2$ adjacent.

There is no commuting differential operators in the multiple interval cases neither in Szegö problem, nor in the space/frequency bandlimiting problem.

Nevertheless these problems can be solved using classical isomonodromy deformation using methods we studied 27 years ago.

Start with Padé approximation.
The rational function $P_n(x)/Q_n(x)$ is a Padé approximation to $f(x)$ of order $n$ if

$$Q_n(x) \cdot f(x) - P_n(x) = O(x^{2n+1}).$$

If the order is exactly $x^{2n+1}$, the Padé approximation is called \textbf{normal}.

Szegö problem is equivalent to finding cases of non-normality of Padé approximation to the general logarithmic function

$$f(x) = \sum_{i=1}^{m} w_i \log (1 - a_i x).$$

for a fixed set $\{a_i\}_{i=1}^{m}$ of singularities, made from the ends of intervals comprising $C_1$ and $C_2$. 
Namely, when

\[ C_1 = \bigcup_{i_1=1}^{d_1} (b_{i_1}, c_{i_1}) ; C_2 = \bigcup_{i_2=1}^{d_2} (d_{i_2}, e_{i_2}), \]

we will associate weight \( w(.) \) with the ends of these intervals:

\[ w(b_{i_1}) = 1, w(c_{i_1}) = -1, w(d_{i_2}) = \lambda, w(e_{i_2}) = -\lambda. \]

\( \lambda \) is an eigenvalue in \( M_n(C_1, C_2) \) problem if and only if the Padé approximation \( P_n(x)/Q_n(x) \) of order \( n \) to

\[ f(x) = \sum_{i=1}^{m} w(a_i) \log (1 - a_i x), \]

is non-normal, and then polynomial \( Q_n(x) \) is the most concentrated polynomial.

This immediately associates with the \( M_n(C_1, C_2) \) problem a Fuchsian linear differential equation of the second order satisfied by

\[ Q_n(x) \text{ and } Q_n(x) \cdot f(x) - P_n(x). \]
Garnier Isomonodromy Deformation Equations.

The Fuchsian equations for the Szegö problem start with the set \( \{a_i\}_{i=1}^m \) of ends of intervals.

We have \( m \) regular (logarithmic) singularities. We also have \( m - 2 \) apparent singularities \( \{b_i\}_{i=1}^{m-2} \) - these are spurious zeros of \( Q_n(x) \) outside of \( C_1, C_2 \).

\[
\frac{d^2}{dx^2} Y + \left( \sum_{i=1}^{m} \frac{1}{x-a_i} - \sum_{i=1}^{m-2} \frac{1}{x-b_i} \right) \frac{d}{dx} Y + \frac{p_{2m-4}(x)}{\prod(x-a_i)\prod(x-b_i)} Y = 0
\]

There are exactly \( m - 2 \) free (accessory) parameters \( \{c_j\} \) in \( p_{2m-4}(x) \):

\[
c_j = \text{Res}_{x=b_j} \frac{p_{2m-4}(x)}{\prod(x-a_i)\prod(x-b_i)}
\]
The monodromy group depends only on the number $m$ of logarithmic singularities, and thus the dependence of this equation on position of singularities $\{a_i\}$ defines isomonodromy deformation equations, known as Garnier system (Garnier, 1912-1919).

Hamiltonian form of Garnier system:

$$\frac{\partial b_k}{\partial a_j} = \frac{\partial K_j}{\partial c_k};$$

$$\frac{\partial c_k}{\partial a_j} = -\frac{\partial K_j}{\partial b_k}.$$

The Hamiltonians $K_j$:

$$T(x) = \prod (x - a_i); \quad L(x) = \prod (x - b_i)$$

$$K_j = \frac{L(a_j) T'(b_l)}{T'(a_j) L'(b_l)(b_l - a_j)} \left( c_l^2 + c_l \sum_{i=1}^{m-2} \frac{\delta_{ij}}{b_l - a_i} \right) - n(n + 1).$$
In the case of $m = 3$ the Garnier system is Painlevé VI.

Isomonodromy deformation systems possesses birational Darboux-Schelsinger transformations. These are explicit nonlinear algebraic transformations that relate parameters $a_i, b_j, c_k$ for a given $n$ to parameters for $n + 1$ (or $n - 1$).

Zeros $\{z_j\}_{j=1}^n$ of $Q_n(z)$ on the complex plane are governed by Heun-Stiltjes theory of electrostatic particles minimizing the energy

$$\prod_{i \neq j} |z_i - z_j| \cdot \prod_{i,k} |z_i - a_k|^2 \cdot \prod_{i,l} |z_i - b_l|^{-2}.$$
Generalized Prolate Functions and another Isomonodromy Problem.

Expression of generalized prolate functions for sets $\Omega$ and $M$ that are unions of intervals also can be reduced to the Padé approximations and isomonodromy deformation (Garnier) equations.

The difference in the approximating function:

$$ f(x) = \sum w_i \cdot \log \frac{x - a_i}{x - \frac{1}{a_i}} $$

for $a_i$ on the unit circle.

The function $f(x)$ is approximated both at $x = 0$ and $x = \infty$:

$$ Q(x) \cdot f(x) - P(x) = O(x^{-1}); \quad x \to \infty $$

$$ Q(x) \cdot f(x) - P(x) + \lambda \cdot Q(x) = O(x^n); \quad x \to 0. $$
References.

Some references:


R. Fuchs, Math. Annalen, 63, 1907, 301-321.