Dominoes and Tetrominoes Tiling for Particular Domains
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General Definitions

Let Γ be a finite connected (undirected) graph, and let $V(\Gamma)$ and $E(\Gamma)$ be, respectively, the vertices and edges of Γ. A perfect matching on Γ is a set of edges of Γ such that every vertex is adjacent to exactly one of the edges in the matching. Of course, to have a perfect matching on Γ the number of vertices must be even.

If there is no perfect matching on Γ one uses a maximum matching as a largest in cardinality set of edges having no common vertices.

In the language of statistical mechanics the perfect matching is called a dimer configuration (dimer covering), and the edges of the matching set are called dimers. In the case of non-perfect matching the edges of that matching (say, maximum matching) set are still called dimers, and vertices that do not belong to any of the edges in a matching set are called monomers.

The existence of a perfect matching, as well as finding a maximum matching in a graph Γ can be determined in a polynomial time. However, the problem of finding the number of perfect matching is $\#P$ for general graphs Γ. It is exactly this problem, of finding the number of dimer coverings as $\mathcal{D}(\Gamma)$ of a graph Γ (or finding its asymptotical properties when Γ approaches a stable infinite configuration), that is important in many fields, particularly in statistical mechanics.

Particularly interesting in applications are cases when graphs Γ are build on subsets of lattices (or similar repeatable structures). One of the most common cases is the case of graphs on a square lattice (or its dual, if it more convenient for description). In that case one can think of vertices as 1×1 cells on a chessboard which are connected if their share a common

edge. Considering the white/black coloring of the chessboard one can see that dimers are always "dominoes" – 2×1 or 1×2 rectangles. In such case one can reduce the perfect matching to a "domino" covering problem on a bipartite graph, since only cells of opposite colors can be matched. In this case one can talk about the "domino covering problem".

In the case of planar lattices and, in fact, in the case of any planar graph Γ the counting of dimer coverings $\mathcal{D}(\Gamma)$ can be effectively computed using wonderful techniques developed by physicists in early 1960s. To present their results we need a bit more definitions.

If $A = (a_{ij})$ is a skew-symmetric matrix, then its determinant, $\det(A)$ is a square of a polynomial in a_{ij} which is called a Pfaffian, $\mathrm{Pf}(A)$, of A. The Pfaffian is not-trivial only when the size of the matrix A is even. Since we are interested in perfect matching (or subgraph resulting from a maximum matching set), we assume now that the number of vertices of a graph Γ is even.

FKT Formula and Kasteleyn Ordering of Planar Graphs

Fisher, Kasteleyn, and Temperley (thus FKT) in a series of papers, published in 1961, established an efficient algorithm and combinatorial description of a method of computing the number of dimer coverings for any planar graph Γ using Pfaffians and special orientations of edges of Γ.

To define the corresponding matrices, let us look at an orientation K of edges Γ. For simplicity (though it is by no means necessary) assume that Γ is a simple graph (not more than 1 edges between 2 vertices) with uniform weights (equal to 1). Then the skew-adjacency matrix $A^K = (a^K_{ij})$ is defined as follows for vertices i, j of Γ:

$a^K_{ij} = 0$ if there is no edge connecting i, j

$a^K_{ij} = 1$ if edge connecting i, j is oriented according to K

$a^K_{ij} = -1$ if edge connecting j, i is oriented according to K.

Kasteleyn proved that for every planar graph Γ there is an orientation K such that the Pfaffian of A^K produces (up to a sign) the number $\mathcal{D}(\Gamma)$ or dimer coverings, or:

$$\mathcal{D}(\Gamma) = \sqrt{\det A^K}.$$

The construction of this K (Kasteleyn) orientation is nearly-linear in the edge count $|E(\Gamma)|$ and typically uses the dual graph of Γ (the graph of faces

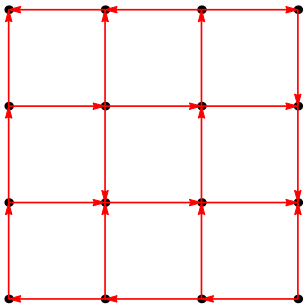

Figure 1: 4 X 4 Grid Graph

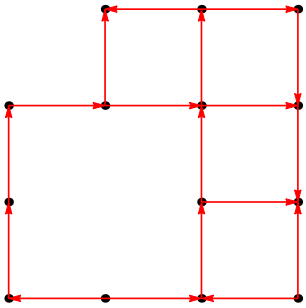

Figure 2: Holes in a Grid Graph

of Γ). The calculation of $\mathcal{D}(\Gamma)$ following this approach (when the analytic expression cannot be derived) requires big-num arithmetic since generically it is a number growing exponentially with the size of $|V(\Gamma)|$.

We wrote a K-orientation package in *Mathematica* that includes big-num evaluation of Pfaffians.

In some interesting case the analytic expression can be explicitly obtained. The case of $G(n,m)$ Grid Graph of a rectangle $n \times m$ was a particular target of FKT study. Since this Grid Graph is a product graph, the corresponding weight-adjacency matrix has analytic expression of its eigenvalues, resulting in an explicit expression of $\mathcal{D}(G(n,m))$ as a product for even n:

$$\mathcal{D}(G(n,m)) \;=\; \prod_{j=1}^{m}\prod_{k=1}^{n/2}(4\cos^2\frac{\pi j}{m+1}+4\cos^2\frac{\pi k}{n+1})$$

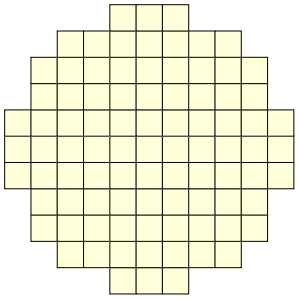

Figure 3: Grid of Large Cells

The limit of $n, m \to \infty$ defines the entropy of the Domino coverage of rectangles. In particular,

$$\lim_{n\to\infty} \frac{\log \mathcal{D}(G(n,n))}{n^2} \;=\; \frac{G}{\pi},$$

where G is Catalan's constant.

Similar results were obtained for some other planar lattice graphs.

Kasteleyn result cannot be generalized to arbitrary non-planar graphs (only to those that are Pfaffian), but he stated a result on the representation of the number of dimer coverings of a graph on a surface of genus g in terms of 4^g Pfaffians. This was used for the study of grids on a torus, important for statistical mechanics and for the development of the dimer coverage in the continuous, SLE, limit.

One can derive many interesting corollaries of FKT results. As one practical example, consider a sample grid of large square cells (super-cells) inscribed in a circle.

Take now each of these super-cells and subdivide it into an $n \times n$ cell grid.

What follows rather easily from the FKT result is that for any simply connected figure $\mathcal{F}$ of super-cells on the plane, the subdivision of super-cells into $n \times n$ squares results in a simply connected grid subgraph $\Gamma(\mathcal{F}_n)$ whose dimer coverage number converges asymptotically as follows:

$$\lim_{n\to\infty} \frac{\log \mathcal{D}(\Gamma(\mathcal{F}_n))}{|\mathcal{F}| \cdot n^2} \;=\; \frac{G}{\pi},$$

(Proof: the fact that it is a lower bound follows directly from the rectangular case, applied to an individual super-cell. The fact that it is asymp-

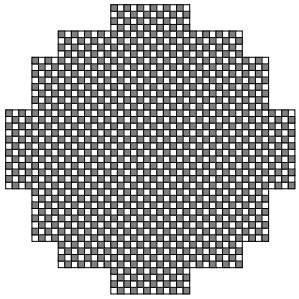

Figure 4: Subdivision into 4 x 4 cells

totically an upper bound is a consequence of, essentially, the same bound. If the domino coverings for the figure $\mathcal{F}$ is asymptotically large, then completing the figure $\mathcal{F}$ for a full square $N \times N$ of super-cells would contradict the asymptotic for that square, made of $n^2 \cdot N^2$ smaller cells.)

Using the set $\mathcal{F}$ from the Figure 3 (our **model set**), we get for $n = 6, 8, 10, 12, 14$ the following values for $|\mathcal{D}(\Gamma(\mathcal{F}_n))|$:

$$5.61702 \cdot 10^{396}, 2.33667 \cdot 10^{709}, 1.44289 \cdot 10^{1112}, 1.30638 \cdot 10^{1605}, 1.72217 \cdot 10^{2188}$$

The most general result in this direction belongs to R. Kenyon (2008), and describes the asymptotic of the number of domino covering of a simply connected rectilinear domain in the plane, when the size of a grid element is ϵ for $\epsilon \to \infty$. This result provides with necessary logarithmic corrections to FKT formula, and assumes that a building block of a covered area is a "Temperleyan polyomino" – roughly a finite (with respect to ϵ) collection of rectangles of an even size in a given grid.

It is essential to consider the simply connected domains made this way. A number of examples of simply connected domains that are coverable by a unique dimer configuration with a more complex boundary can be produced. These examples are useful because they can be tied together to produce equivalence between SAT problems and domino/polyomino covering to show high complexity of these problems.

Kenyon Coverage Theorem. Let $U \subset \mathbf{R}^2$ be a rectilinear polygon with V vertices. For sufficiently small $\epsilon > 0$, let P_ϵ be a Temperleyan polyomino in $\epsilon\mathbf{Z}^2$ approximating U in the natural sense (the corners of P_ϵ are converging to the corners of U). Let A_ϵ be the area and $Perim_\epsilon$ be the perimeter of P_ϵ. Then the logarithm of the number of domino tiling of P_ϵ is

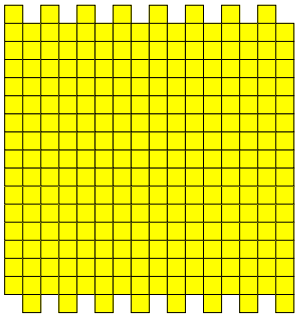

Figure 5: Single Domino tiling case

$$\frac{c_0 \cdot A_\epsilon}{\epsilon^2} + \frac{c_1 \cdot Perim_\epsilon}{\epsilon} - \frac{\pi}{48} \cdot (c_2(\epsilon) \log \frac{1}{\epsilon} + c_3(U)) + c_4 + o(1)$$

Here $c_0 = \frac{G}{\pi}$; G is Catalan's constant as above; $c_1 = \frac{G}{2\pi} + \frac{\log \sqrt{2}-1}{4}$; c_4 is independent of U, and the term

$$c_2(\epsilon) \log \frac{1}{\epsilon} + c_3(U)$$

represents the ϵ-normalized Dirichlet energy $E_\epsilon(U)$ of the limiting average height function on U. This continuous limit of the "height" of the domain is computed in terms of Weierstrass elliptic functions. It can be computed using the partitioning of the domain into its rectangular parts, with a formula for the rectangle R of the size $\alpha \times \beta$ of the form:

$$E_\epsilon(R) = \frac{24}{\pi} \cdot \log \frac{2\alpha}{\epsilon} - \frac{2}{\pi} \cdot \log(\frac{1}{2}(2\pi)^{12} \eta(e^{-2\pi \frac{\alpha}{\beta}})^{24})$$

The classical Dedekind η function used is defined as:

$$\eta(q) = \prod_{n=1}^{\infty} (1 - q^n)$$

Chessboard, Gomory's Theorem and Hamiltonian Paths

8×8 chessboard can be easily covered fully by non-overlapping dominoes. The puzzle, traced to Max Black (1946), and popularized by Martin Gardner

is the following one: if diagonally opposed corners of the chessboard are removed, will it still be tiled by dominoes?

The answer is no. The reason is rather simple: since the chessboard like any other grid subgraph is bipartite, every domino connects one black with one white vertices. Thus any dominoes tiled domain has equal number of white and black vertices. Since opposite sides of the 8×8 chessboard have cells of the same color, this mutilated domain cannot be dominoes tiled.

John McCarthy proposed it famously as a hard problem for AI automated proof system.

There is a very nice positive result here for other "mutilated chessboards" known as Gomory's theorem. R.E. Gomory was the head of IBM Research for many years and later was the President of Alfred P. Sloan Foundation.

Gomory's theorem states that if you remove two cells of opposite color from 8×8 chessboard, then the resulting board (graph) is fully domino tiled.

The method of proof is much more far-reaching: the same is true for any rectangular domain in the square grid that has even number of cells. For the rectangles whose sides are both odd the domino tiling is impossible, but if one removes any corner from such a domain, the theorem still holds.

The proof is very elegant. One takes a Hamiltonian cycle that exists in such a graph (in fact there are many such cycles). The cycle is broken in 2 places, but each part is even in length – because is start and end are of opposite color. They are obviously partitioned into dominoes since every consecutive pair represents axially aligned dominoes.

Hamiltonian paths (not only cycles) in grid graphs produce another impressive feat: for every $n \geq 2$ they provide a covering of these graphs with a maximal number of n-polyominoes possible (essentially the same argument as above). So for those lucky graphs you get the best triomino, tetromino, ..., octomino, ... coverings. Sadly, this is not really a common occurrence.

In fact, even for grid graphs (finite subgraphs of a square lattice) determining whether the graph is Hamiltonian is still **NP**. Only for grid graph representing simply connected domain (i.e., no holes) Hamiltonian paths can be identified in polynomial time.

There is a remarkably interesting approach to finding large families of nonintersecting cycles in planar graphs, even if there are no Hamiltonian paths. This is based on another set of lattice problems created in solid state physics – "double-dimers". Take two different domino tilings of the same graph (domain in the square grid). Superimposing them one gets covering of a graph by sets of two kinds: loops of even length and "double edges" – the latter are just cases of same dominoes in both coverings.

To obtain the two "non-intersecting" domino coverings of the same graph

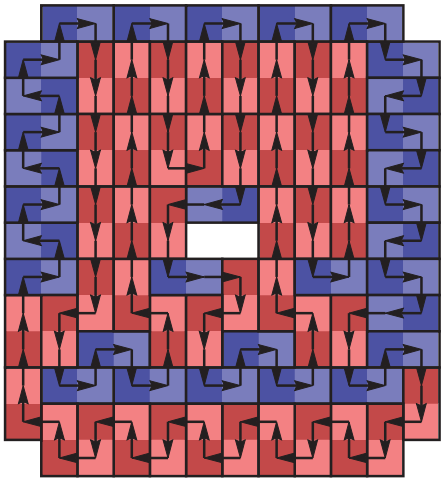

Figure 6: 12×13 grid graph with 6 holes and the domino tiling from a Hamiltonian cycle

(domain), one runs the flipping of domino pairs in two "opposite" directions – by increasing and by decreasing the height of domino coverings, since any coverings are ultimately related through the sequence of flips (easy to prove for simply connected domain and require some graph modifications otherwise).

We show one example of a Hamiltonian cycle for a particularly interesting grid graph with holes.

Note that because there is Hamiltonian path through this graph, making 2 extra defects with opposite colors creates new domino coverings of a depopulated graph because each of 2 parts of the path represent a set of dominoes, covering the newly depopulated graph (each part is even in the length because it starts and end at sites of different colors).

Simple Unknowns; a defect

The problem of an odd rectangle $(2 \cdot m+1) \times (2 \cdot n+1)$ with one (white) cell removed is the simplest example of an "unsolved" problem on the monomer-dimer coverings. Only in a rather trivial case of a "defect" on the boundary of the rectangle the explicit "simple" expression on the number of dimer coverings in the resulting grid is known (F. Wu; 2007) through a reduction to Kasteleyn determinant (and Temperley's correspondence with the number of spanning trees in the rectangle with a corner missing).

Even in this simplest setup interesting questions arise: If we are allowed to move the defect, just as we are allowed to flip double dominoes (when they are next to and parallel to each other), what kind of lattice structure we will get. It turned out that there are configurations (in particular with a defect right in the center of an odd rectangle), when there is no equivalence with a generic single monomer-dimers tiling. In a majority of case though the equivalence is there, when one combine sliding of a monomer in the direction of adjacent domino while flipping the dominoes in the dimer coverage.

There is another popular version of the same problem, originated from a "15-puzzle", when one is trying the sliding approach with a domino covering to move the monomer, but without the flips of the dominoes. Only recently (Ruelle et al. 2008) the probability that the monomer is jammed – at $\frac{57}{4} - 10\sqrt{2} = 0.1078\ldots$, and a probability that a monomer can make only one move (by 2 grid positions) – at $\frac{72817\sqrt{2}-102977}{32} = 0.0559\ldots$ were computed.

In general, for arbitrary defect locations next to nothing is known in the monomers-dimers problem; even precise conjectures. The only glimpse of the information is based on old Fisher problem about the "interaction" between pairs of monomers based on the free energy computations for the partition function with these "defects". Fisher conjectures, largely unproved yet, describe the "interactions" in terms of the electrostatic force between the monomer pairs. In the conformal limit rare cases of these conjectures were proved – for diagonal pairs on diamond shaped domains.

Fisher Conjectures on Monomer (Defect) Interaction

It seems that physicists in early 60s knew Gomory theorem, but not its proof. At about the same time as FKT papers were published, Fisher and Stephenson tried to understand what will happen in the monmer-dimer case, when there are (individual) defects – monomers, but the rest of the domain

is still domino tiled.

Using the FKT approach and extensive matrix computations they put forward in 1963 a conjecture that in a large grid graph $G(n,m)$ (with even $n \cdot m$) the removal of two sites of opposite color, say $(0,0)$ and (p,q) (with $p+q = 1 \mod 2$), leads to a light decrease in the number of dimer covering of the subgraph $G_m = G(n,m) \setminus \{(0,0),(p,q)\}$ – $\mathcal{D}(G_m)$, asymptotically as:

$$\mathcal{D}(G_m) \sim \frac{c}{\sqrt{d((p,q),(0,0))}} \cdot \mathcal{D}(G(m,n))$$

for a constant c.

Numerically this conjecture is true for a variety of domains and lattices. It is proved only in one case – for $(p,q) = (d,d-1)$ (the diagonal case) by Hartwig (1966). The constant c in that case is

$$\frac{\sqrt{e}}{2^{\frac{7}{12}} A^6},$$

where A is the Gla1sher constant, as in:

$$\zeta(-1) = \frac{1}{12} - \ln A.$$

After decades of numerical experiments, Fisher conjecture was generalized in many cases to a general set of monomers of different colors (interpreted as opposite charges) involving the electrostatic potential of their interaction as a correction factor to the number of domino coverings of domain with fixed monomers (defects). This set of conjectures assumes that the remaining sub-domain if fully domino tileable and deals only with generic cases.

The only cases that were dealt with analytically so far are of particular configurations of monomers that are collinear on the diagonal of the domain, or are well-separated on the domain's boundary (Ciucu, 2007-2014).

The original Fisher conjecture of two monomers of opposite color is still wide open.

Many defects – Percolation Problems

The first percolation problem that was encountered in physics is now known as a phase transition. P. Curie discovered that iron exposed to a magnetic field retained its magnetism below a "critical temperature" T_c and lost it at the temperature above T_c. Ising model in two dimensions turned this

into a classical "solvable" model, and lead to generalizations that we know now under the general percolation name, well-defined by Flory (1941) and Broadbent and Hammersley (1954-1957).

The simplest percolation model is the "bond/site percolation" problem. In that model you have a porous material, and water flows to one side of it. Will the water reach the "other side" of it? Model the porous material as a lattice of vertices, called sites, connected by edges of that lattices, called bonds. The bonds are open for the water with a probability p or closed with a probability $1-p$. In the bond percolation problem one wants to know for what p the water will flow from one side to another, assuming a "random" nature of connections and large size of the material block.

In practical applications this problem arises in many fields: resistance (disordered electrical) networks, ferromagnetism (as in Ising model), epidemology, ecology, etc. This problem was briefly studied in early 80s in connection with defects on chips. It re-appeared in a set of problems that we are studying now in an interesting conjunction with a problem of polyomino tiling of two-dimensional domains on a square grid.

The most remarkable feature of the general percolaton problem that, assuming a "random" nature of the bond/site defects and a large size of a domain, there is always a "critical" $p = p_c$ such that the behavior is different for $p < p_c$ and $p > p_c$. Assuming an infinite size of the domain (for any lattice) Kolmogorov's 0-1 theorem for tail events implies that the probability of having an infinite fully connected cluster (so the water will always flow from one side to another) for any given p is either 0 of 1. Combined with a monotonicity arguments this probability can be shown to be an increasing function of p. Thus there is a critical p, p_c, below which the probability of infinite cluster is always 0, and above is always 1.

The determination of the critical probability for different lattices and different models of "defects" is a subject of intensive and hard work in solid state physics, statistical mechanics, integrable systems and conformal analysis. Much of the rigorous progress was made in the last 30 years, but is confined only to special two-dimensional lattices. The most famous is Kesten (1982) theorem that $p_c = 1/2$ for a two-dimensional square lattice and a bond percolation problem.

However, already in 1936 (Sir) Rudolph Peierls showed how to get simple analytical lower and upper bounds on p_c. We sill use this argument for the site (not bond) percolation, where there is no analytic result even for the two-dimensional square lattice, let alone in the high-dimensional case.

The non-percolation case. In the case of Cartesian grid in d-dimensional lattice, the number of paths of length n starting at the origin (or any site)

is at most $(3^d-1)^n$. Thus a probability of having n distinct sites open simultaneously is smaller than p for some p, $0 < p < (3^d-1)^{-1}$. Thus the expected number of open paths starting at the origin (at most $(p\cdot(3^d-1))^n$) tends to 0 as $n \to \infty$. Then there is no percolation (or an infinite path).

The percolation case. The number of contours surrounding the origin in Z^d is bounded by $(n\cdot(3^d-2))^{n-1}$. The probability to have n distinct sites simultaneously closed is smaller than p^n some p, $0 < p < (3^d-2)^{-1}$ and large n. The number of closed circuits around the origin then is finite since

$$\sum_{n\geq 1} n(3^d-2)^{n-1}p^n \ < \ \infty$$

and there will be a path to infinity.

In two dimensions for square lattice and other special lattices (Potts, Bethe) the most exciting developments in the last 12 years was the progress in study of critical behavior at p asymptotically close to p_c and the conformal invariance found there (the "SLE" models). Among the results are the following:

There is no infinite Light (open) or Dark (closed) infinite clusters at the critical point p_c.

The probability that there is a path from the origin to a distance r decreases polynomially as r^α for a constant α

The shape of the large cluster is conformally invariant.

Sadly, nothing like that is useful in our case when coming too close to p_c would be disastrous.

Percolation Rates

The best consistent current estimate of the "critical p site percolation rate" for a square lattice with uniform distribution of defects is $p_S = 0.5927460$. If the probability p of site occupancy is $p < p_S$, there will be no "infinite" cluster among the occupied sites in the infinite square lattice.

However, in many practical cases the percolation problem arising in cases of non-uniform distribution of defects in the lattice domain. In our case of nearly "circular" domains there is clearly a non-Poisson Point Process responsible for random defects. The main feature of defect distribution here, as in many other examples studying for the last 30 years is the radial dependency of defects, where defects crowds towards the edge of the disk (with the defect probability at 1.0 and beyond the rim).

To easily parameterize the distribution shape that depends only on a radial distance in the "circular" domain we use the following curve:

$$L_c(r) \;=\; e^{u\cdot(1-|r|^c)}$$

for normalized radius of the circular domain: $0 \leq r \leq 1$ and a constant $u = u(c)$ normalizing the distribution for $0 < c$.

The main differences in the choices of this L_c Distributions for c varying from 0 to ∞, are that at $c \to \infty$ the distribution converges to a uniform, and at small c the distribution strongly hugs the edges of the "circular domain".

This is how the distribution looks like for various c.

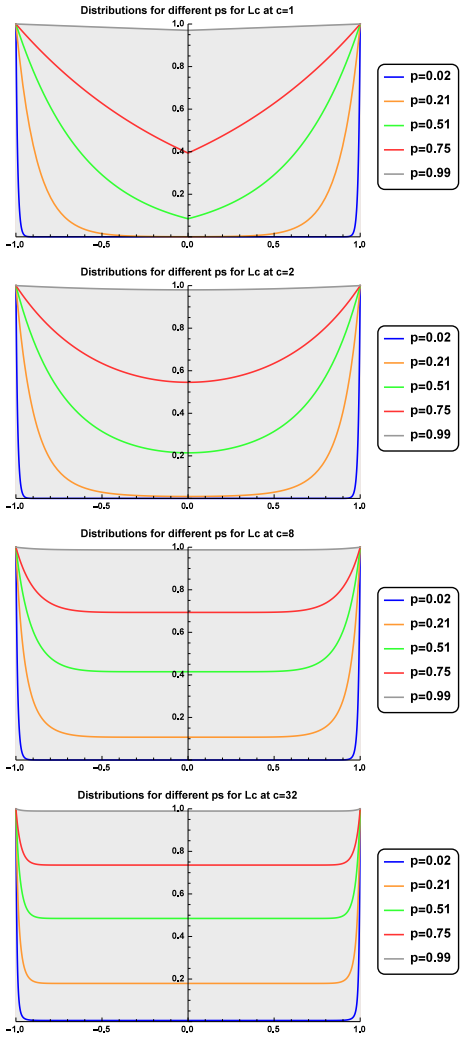

Figure 7: Radial Lc Distribution for $c = 1, 2, 8, 32$ for various p levels, $0 < p < 1$

We present a series of images showing the percolation maps for relatively low probabilities p of bad sites in the previously defined "circular domain" for different c in L_c distribution of defects.

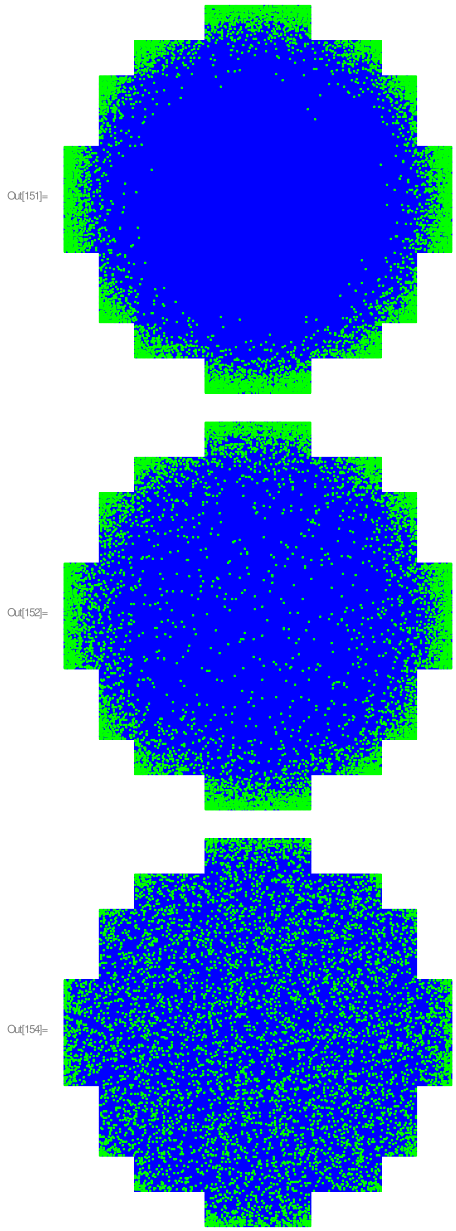

Figure 8: Percolation Map for a Probability of Bad Site $p = 0.1$ for L_c Distribution with $c = 1, 4, 16$

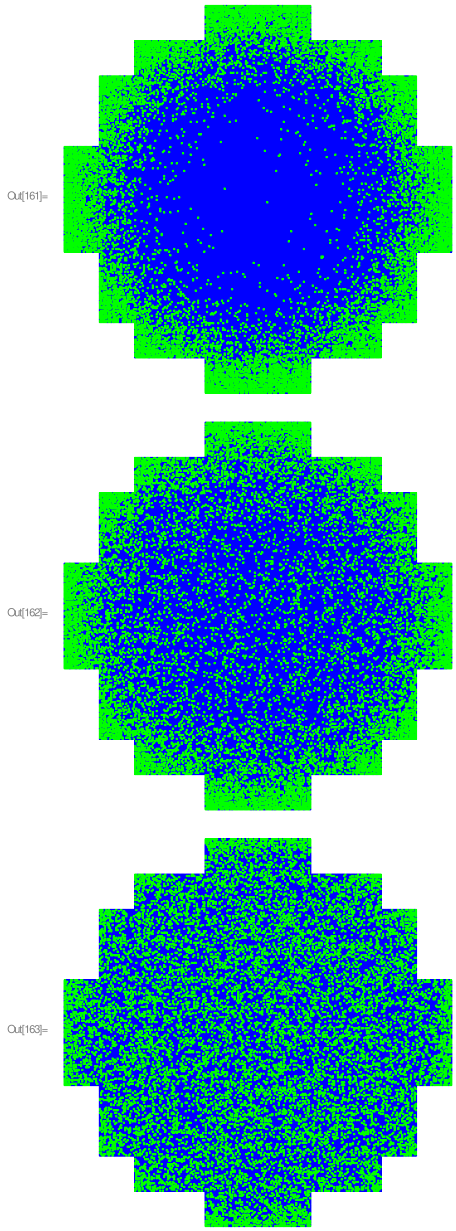

Figure 9: Percolation Map for a Probability of Bad Site $p = 0.2$ for L_c Distribution with $c = 1, 4, 16$

Light Clusters and Domino Tilings in them.

In the case of uniform percolation, where the large cluster will be gone by the time the probability p of a site defect will be close to 0.4, we need to know not only the expected size of the connected "good sites" cluster in the case of a finite domain, but more importantly we need to know the largest connected domino-tiled cluster in that big cluster.

The finite, but relatively large, case is crucial for us. We use the same approach as for the domino tiling of a general rectangle: take a particular shape and scale the square sites to the number of sites become large. Just as before, we take a polygonal, nearly circular, domain and scale each of the big squares in this domain to very small sites (to be precise made of 28×28 sub-grids). This bring us to 69776 total sites before defects of broken sites are introduced.

The dependence of the size of the clusters and the domino coverings can be determined so far in 2015 only experimentally, with good low and upper bounds provided by extensive Monte-Carlo testing.

The provable bounds for domains of arbitrary sizes are quite off the experimentally determined by numerical testing. In most cases nobody knows yet what tools to use, though one would expect the help of Lowener equations and Brownian curves appearing in both percolation and spanning trees study. Another promising approach that started to emerge in the last couple of years is based on the fact that Hirota and discrete Painleve equations on a lattice seem to describe the tiling phenomena. A real problem is that the tiling with holes break a simple nature of possible solutions requiring use of high dimensional Abelian varieties.

So let us focus on the numerical experiments for now. If all site defects are random, as in the original definition of the percolation problem, we get the following results for the expected sizes of the:

Largest Light (open) connected site cluster

Largest Dark (closed) connected site cluster

Largest size of dominoes on the domain (not necessarily connected)

Largest size of connected domino-tiled cluster in the Light Cluster.

It is the last size that is most important in practical applications.

One can see clearly the effect of the collapse of the Light cluster and the rise of the Dark cluster despite the finiteness of the domain (the number of Monte-Carlo tests here is about 1.2 million). What Monte-Carlo experiments also show is a relatively simple formula for the Expected size of the Dimer in the Light Cluster in terms of the total number of the elements in the domain (in the case of uniform distribution of defects):

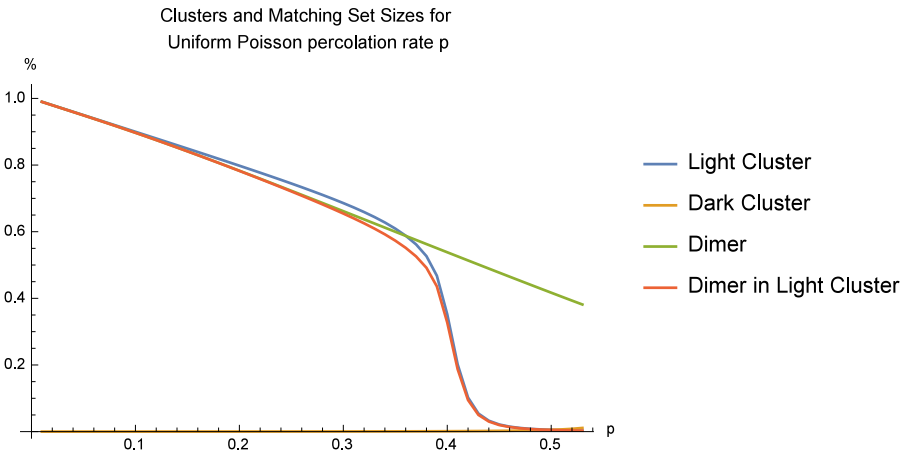

Figure 10: Uniform Percolation till critical p_c: ; Dimensions are in fractions of the Domain cardinality

$$|\text{LightDimer}| \sim (1 - p - \frac{p^2}{2} - \frac{p^4}{4} + O(p^6)) \cdot |\text{Domain}|$$

where p is a probability of a defect site in a domain of a cardinality of $|\text{Domain}| \to \infty$, with the Dimer in the connected Light Cluster in that domain of cardinality $|\text{LightDimer}|$ and

$$p < 0.35.$$

It is not known whether this numerical result can be actually proved, even as a weaker statement:

$$|\text{LightDimer}| < (1 - p - O(p^2)) \cdot |\text{Domain}|,\ p = o(1),\ |\text{Domain}| \to \infty$$

The best we can prove so far is that the upper bound for a small p is of the order of $1 - \frac{3}{4} \cdot p$ of the domain's cardinality for polygonal scaled domain.

Main Conjecture on Defects

Current techniques are insufficient to make further rigorous progress in the proof of the expected sizes of dimers in the presence of monomers (defects),

but the conjectured story (supported so far only by numerical evidence) is very interesting.

Monomer-Dimer Conjecture. For any large rectilinear (Temperleyan) polygon U on a square grid, and for n sufficiently small compared to the number of sites in U, and for any n white and n black monomers (defects) in a generic position in U, the remaining subgraph of U is domino tiled.

For $n = 1$ one can get this from Gomory theorem. Generally speaking there seems to be no approach to the proof of this conjecture, as well as its natural generalization along Fisher's conjecture lines.

However, in the cases of $n = 2$ and $n = 3$ the exceptional configurations where dimer covering promised by the Monomer-Dimer Conjecture fails can be explicitly determined for even size rectangular domains. These exceptional configurations are in corners of the domain for $n = 2$ and at edges for $n = 3$.

Some slight modification of the conjecture with random n defects, and extra n monomers to create domino tiling, can be proved for large rectilinear domains. The method still uses creation of Hamiltonian paths in the subgraphs, and thus can be used for arbitrary polyomino shapes (see below).

The proof of the full conjecture would justify Monte-Carlo experiments showing the dimers indeed can cover almost all Light Cluster in the set of percolation problems with the site failure rate p not too far from the critical one.

Relief from Percolation Limit in Non-Poisson Percolation

The critical defect rate of $p \sim 0.41$ is relieved somewhat when one has clustering and/or radial dependence of defects, especially with defects crowding towards the rim of the domain.

We look at L_c radial distribution of random site defects at $c = 4$ (somewhat close to at least one practical case). In this case the expected size of the cluster as percentages of the domain cardinality vs the probability p of defects look more advantageous (as far the moment of the breakout of the largest cluster is concerned).

One can see that the collapse of the Light cluster is delayed till about $p = 0.5$ in this radial defect distribution. Again, one can see only a slight loss of the size of the LightDimer (the Dimer in the Light Cluster) all the way till

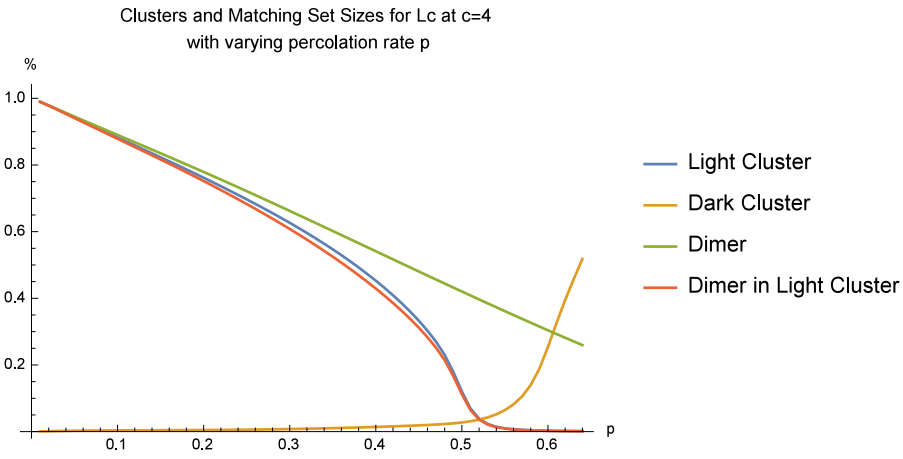

Figure 11: Percolation graphs for L_c distribution for $c = 4$ till critical point; Dimensions are in fractions of the Domain cardinality

$$p < 0.45.$$

Numerically (for the L_c radial distribution at $c = 4$) the expected size of the LightDimer in this range of p has an excellent behavior:

$$|\text{LightDimer}| \sim (1 - p - p^2 - O(p^4)) \cdot |\text{Domain}|.$$

This behavior is only numerical, but it was computed for other types of circular domain, and is sufficiently stable to vouch for one leading digit after the period for the range of p in $p < 0.45$.

We do not expect this bound to be proved soon (see conjectures on the domino covering of domains with many monomers) especially because the non-Poisson Point Process percolation problems are rarely studied so far. These processes are occurring mostly in the practical applications (particularly in biology and chemistry), and not in theoretical solid state physics and statistical mechanics driving the percolation study.

However, we cannot resist the temptation to point out that Potts q-model and Baxter's eight vertex models with their elliptic function parameterization might are somewhat connected with the non-Poisson defect Point Processes similar to those we consider.

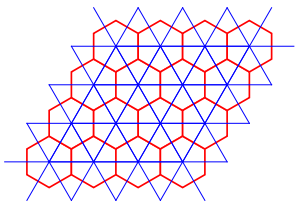

Figure 12: 4×4 Square Mesh as a Hexagonal Grid with Main Diagonal connections

Hexagonal Lattice and Bibones

Main constants of energy (entropy) and percolation depend on geometry of the site connectivity. In the planar case different lattices give rise to different tiling problems.

In all cases we are considering the underlying lattice is a square lattice with sites that can be viewed as unit size squares anchored at integer grid location, with basic mesh (axial) connectivity. Any subgraph of that mesh graph is a planar graph, making it possible to have a FKT detailed analysis, and conformal invariance statements for refined sub-meshes.

Allowing arbitrary diagonal connections between square lattice sites creates a non-planar graph, but allowing the diagonal connections along only the main (or only along the minor) diagonals, still creates planar graphs. We chose the main diagonal (from bottom right to upper left) as another axis of connection on the square grid.

Allowing for such connectivity effectively turns the square lattice (with 6 points of contact) into a hexagonal lattice. It is easy to visualize by turning the unit squares into regular hexagons.

In terms of tiling (and matching problems) the pairs of primitive objects used for matching have different names depending on the lattice type. In the case of the square lattice we have our dominoes. For the triangular lattice the name of the 2 adjacent cells is a *lozenge*. For the hexagonal lattice the name that stuck is that of a *bibone* (two regular hexagons having a common edge). We color them according to their orientations: horizontal are Red; vertical are Blue, and diagonal are Green.

It is harder to study bibone tiling compared to the domino tiling because three techniques that were important for detailed domino tiling analysis are missing here:

bipartite structure of the planar graph (white and black fields of the

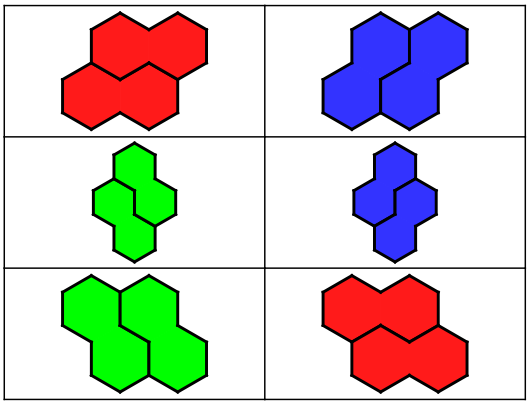

Figure 13: 2 way local bibone flips

dominoes);

height function on the dominoes tiling (and a similar height function for lozenge tiling);

bijection between tilings and spanning trees.

Still, the FKT formalism works, and shows that the number of bibone tilings for rectangular domains rather expectantly significantly exceeds the number of dominoes tilings. Instead of per-dimer molecular freedom value of $\frac{G}{\pi} = 0.291561$ for dominoes one gets for bibones the value of 0.428595.

Moreover, nearly linear (in area) algorithm of determining the covering with bibones of a simply connected domain exists (Kenyon-Remila). It is based on one important property that bibone coverings share with domino coverings: **local flip connectivity**.

In the case of dominoes coverings of a simply connected domain in a square lattice, a sequence of domino local flips in 2×2 arrangement of 2 dominoes can turn any domino tiling into any other.

In the case of the bibone tiling of a simply connected domain in the hexagonal case, a more complex set of flips does the same thing: turn any bibone tiling of a simply connected domain with N sites into any other bibone tiling in at most $5N$ iterations of flips.

The set of local flips in bibone tiling that allow such local flip connectivity is rather small:

3 of 2-bibone flips; 2 of 3-bibone flips, and 3 of 4-bibone flips.

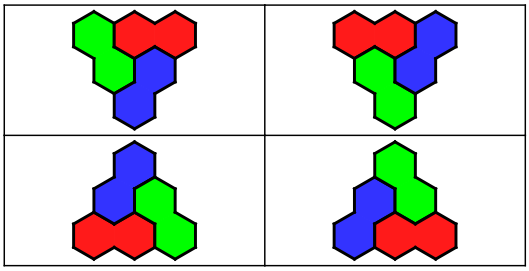

Figure 14: 3 way local bibone flips

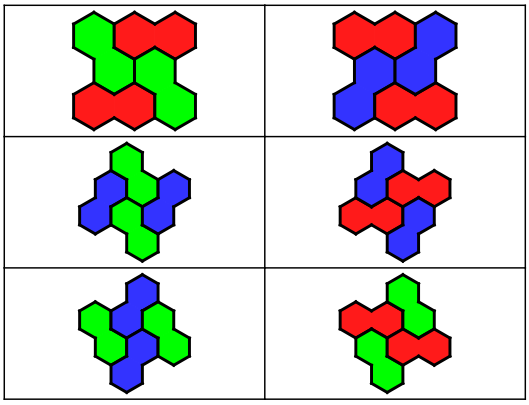

Figure 15: 4 way local bibone flips

Polyomino Tilings: Tetrominoes, Octominoes

Polyomino is any axially connected finite subgraph on the square grid; they are named by their sizes. General problems to determine whether a finite domain is tileable by a particular set of polyominoes are **NP**. The dominoes are exceptions here as well as easy generalizations of them to $a \times b$ and $b \times a$ rectangle tilings.

It turned out that we still need the generalizations of domino tilings with bigger polyominoes. Among them the tetrominoes (of size 4) allow wonderful theoretical and numerical opportunities. These are most widely known from the Tetris game of the old.

The good news is that the tetromino tilings are not the end of the road. We can double the size of building blocks once more and go to octominoes.

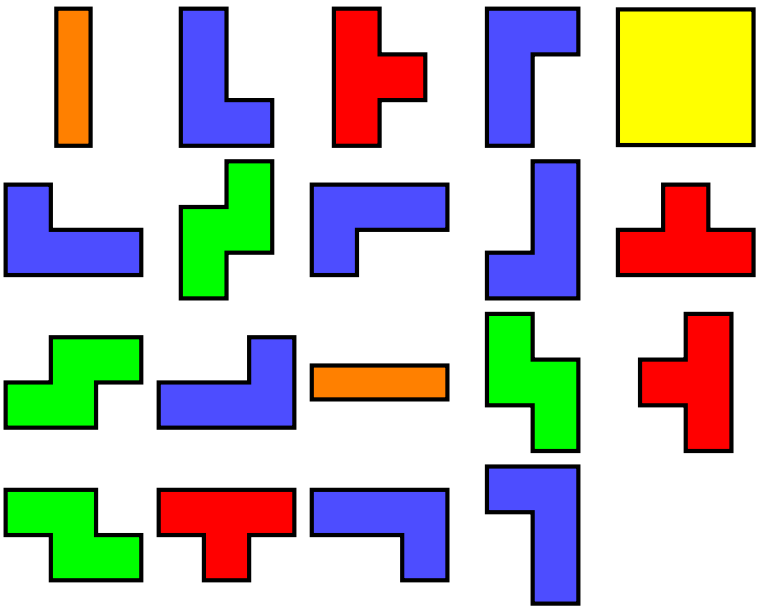

Figure 16: 19 Tetrominoes

Figure 17: 2725 Octominoes

The reason why the path from dominoes to tetrominoes to octominoes works is because we look at every step as a matching problems, even though graphs are no longer planar (but in most cases still Pfaffian).

From Bibones to Tetrominoes

If one looks at an arbitrary tetromino, one can see that T-tetromino cannot be broken into two kissing dominoes, while all other tetromino can be.

However with bibones on the hex lattice all tetrominoes can be nicely split into kissing bibones. Moreover, the graph of bibones is a planar graphs. This allows us to use a full power of FKT techniques, including the counting the number of possible tetromino tilings.

An important thing here is that the quality of the tetromino tiling depends on the starting bibone graph. This is altered **only** by the bibone local flips without changing the covered domain.

The main purpose of the local flips of bibones for us is to build better tetrominoes, and ultimately octominoes, as a maximal matching of tetrominoes graph.

We took one simple random example of a 28×28 grid with corners removed and a (double) hole in the middle. Random bibone coverage does not produce enough tetrominoes to cover the whole area (one bibone have to be left open because the number of occupied sites in not divisible by 4).

After the local bibone flips one finally gets the peak (maximum) tetromino coverage.

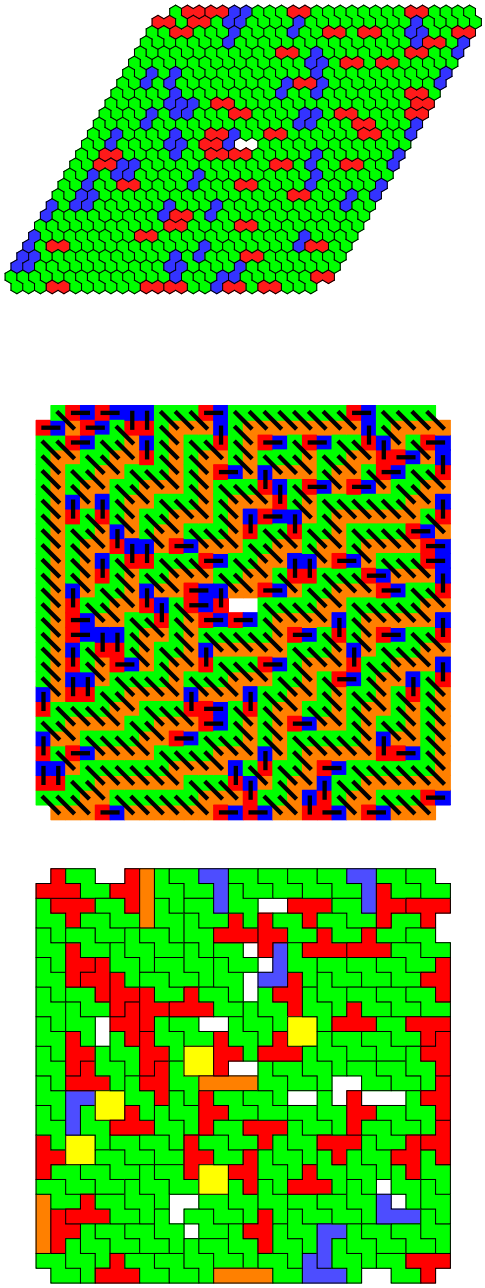

Figure 18: Random bibone full coverage; resulting tetrominoes have holes

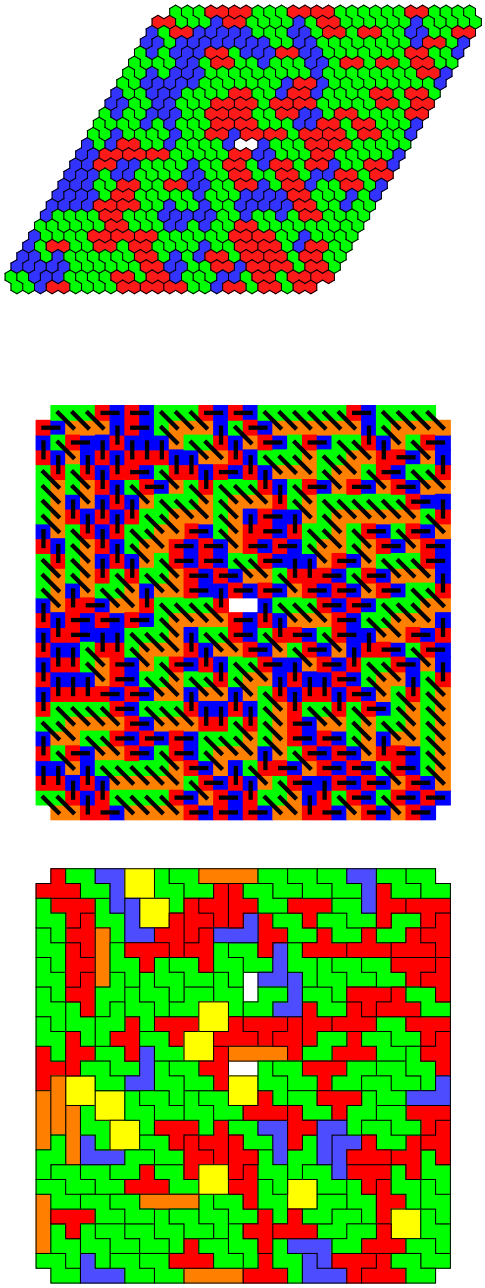

Figure 19: After local bibone flips get peak tetrominoes coverage

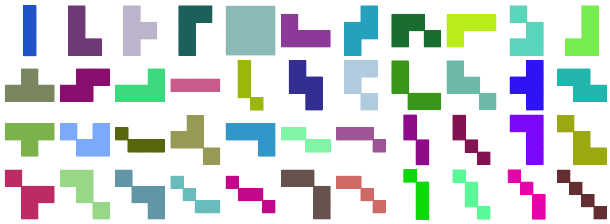

Figure 20: The full set of Hex-Tetrominoes

From Bibones to Hex-Tetrominoes to Octominoes

In the hexagonal lattice underlying the bibones one can form size 4 hex-lattice connected "hex-tetrominoes".

The graph of hex-tetrominoes is formed by matching bibones from the bibone covering of the domain in the hexagonal lattice.

The matching in that graph produces the best octomino tiling of that domain, sometimes better than the domino tiling.

The next set shows the progression of the domino, tetromino, and finally octomino coverage on a 28×28 grid with corners removed and a (double) hole in the middle, together with the extra $p = 0.1$ defects in the remaining domain.

It shows the progress of turning the maximum domino coverage (with its 5 extra defects) to the maximum tetromino coverage (with 1 more defects), and finally to the octomino coverage – with less defects than for dominoes (only 3 extra defects for the whole domain with holes).

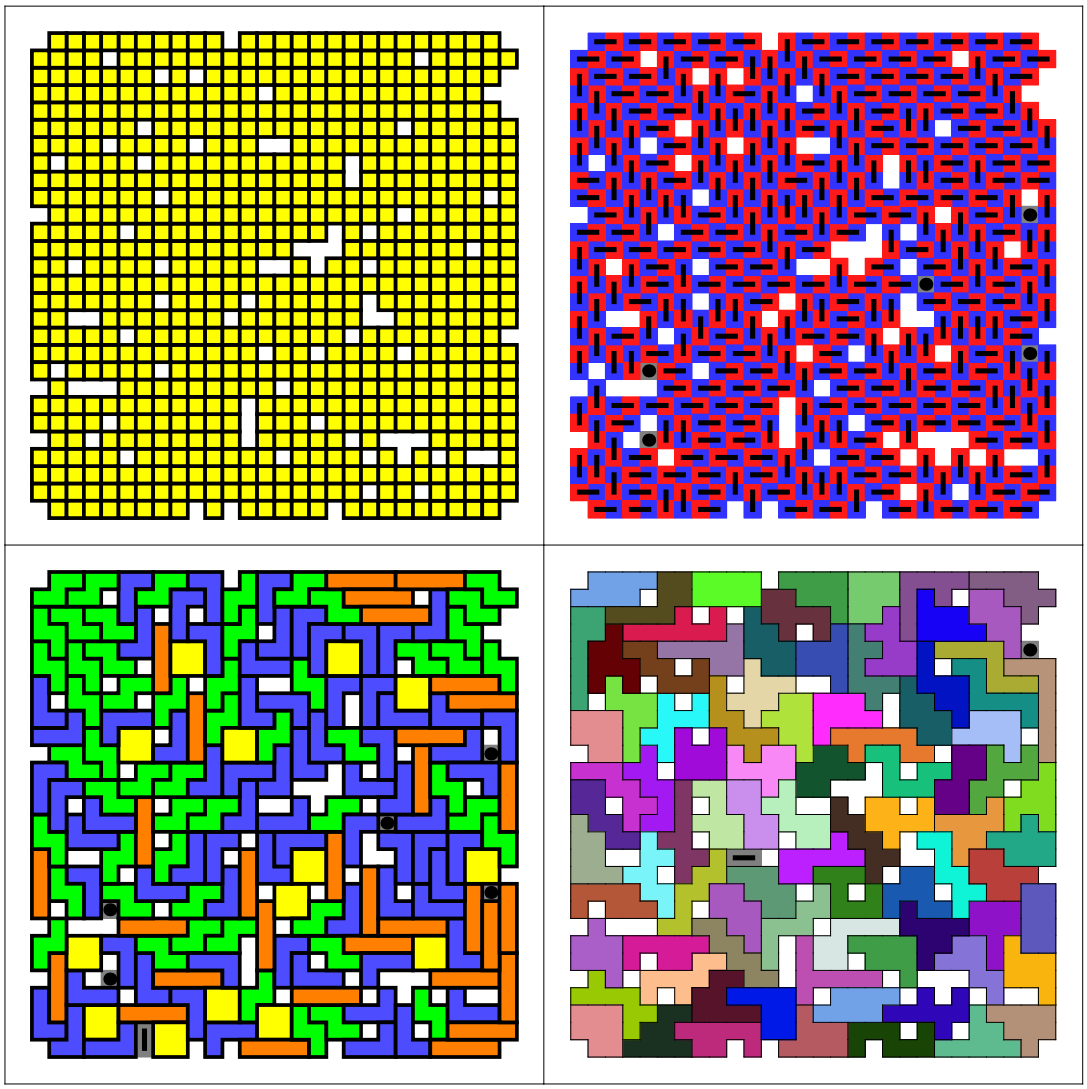

Figure 21: After local bibone flips get peak tetrominoes coverage. 28X28 Grid with corner and extra 10% defects. Sites: 699; Dominoes 347; Tetrominoes 173; Octominoes 87

Flip Animation

See attached animation for the same "model shape" with 12×12 subgrid (and 6 corner and center monomers inside). It starts with the full Tetromino coverage deduced from a set of cycles in the graph. Then local flips are used to improve the quality of the Tetrominoes, making them more confined.

That last feature is rather important in applications.

Any chance for Polyominoes of arbitrary sizes?

Yes, the problem of the tiling of a connected (say, simply connected) domain in the square grid by arbitrary Polyominoes is **NP**. It seems so, based on our initial numerical experiments, that it might not be hopeless after all, **if** one allows not a perfect tiling but a "nearly perfect" coverage with $o(1)$ area missing.

The main problem becomes a "flip" problem of shaping the Polyominoes coverage, by restricting the Polyominoes of a fixed size N of particular shape. The generalized semi-group of "flips" is rather large and complex for any $N > 3$.

Driven by some practical applications we had a look at $N = 8$ (octominoes), and $N = 24, N = 256$.

In the next few slides we show the progression from one tiling (driven by cycles in the grid subgraphs) to tilings by corresponding Polyominoes driven by minimization of the L_2 digital diameter.

Figure 22: Initial Configuration of Tiling by 24-sized Polyominoes

Figure 23: Flip Iteration of Tiling by 24-sized Polyominoes

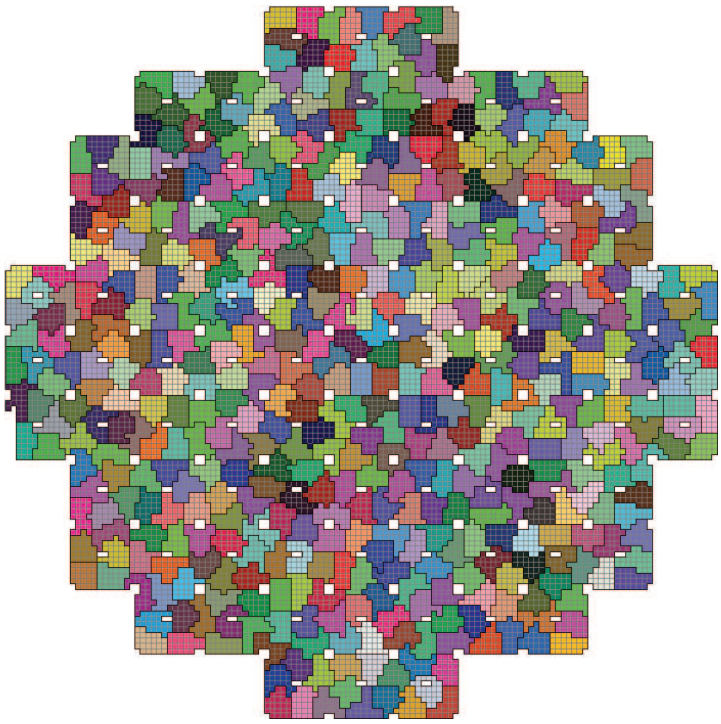

Figure 24: Flip Iteration of Tiling by 24-sized Polyominoes

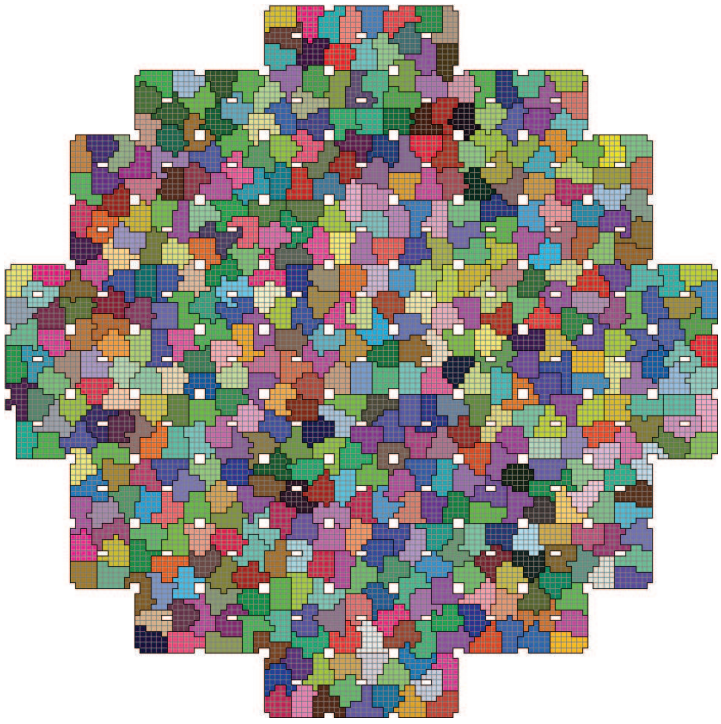

Figure 25: Flip Iteration of Tiling by 24-sized Polyominoes

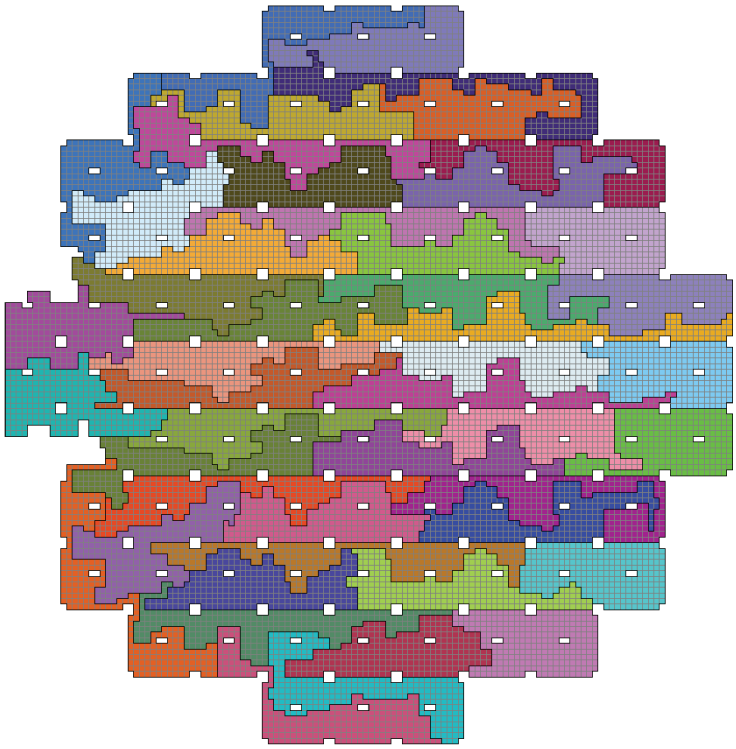

Figure 26: Initial Configuration of Tiling by 256-sized Polyominoes

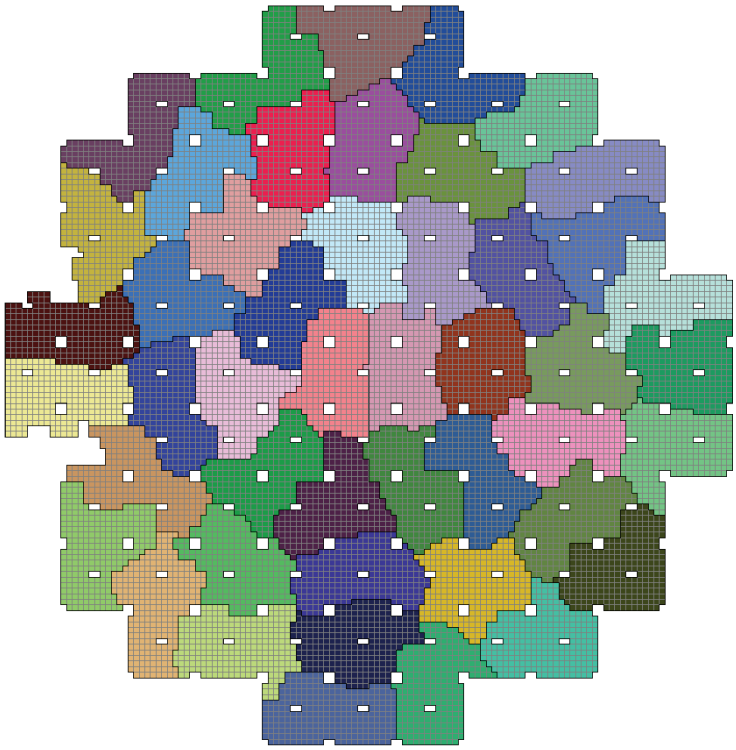

Figure 27: Flip Iteration of Tiling by 256-sized Polyominoes

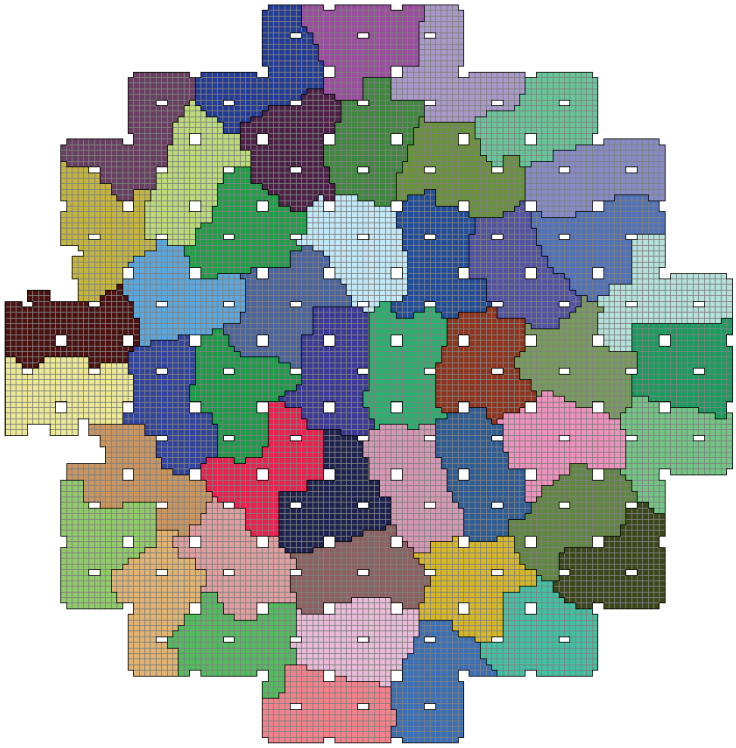

Figure 28: Flip Iteration of Tiling by 256-sized Polyominoes

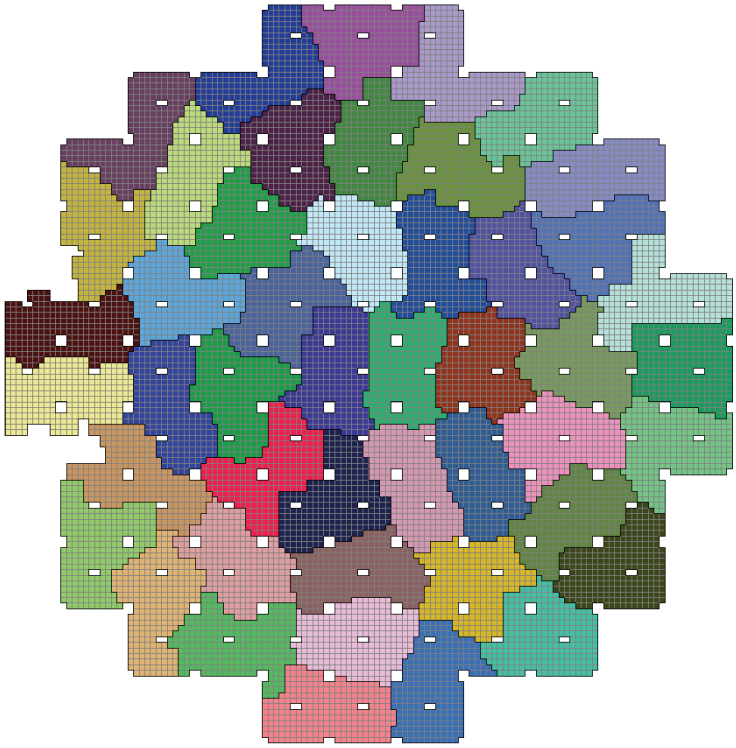

Figure 29: Flip Iteration of Tiling by 256-sized Polyominoes

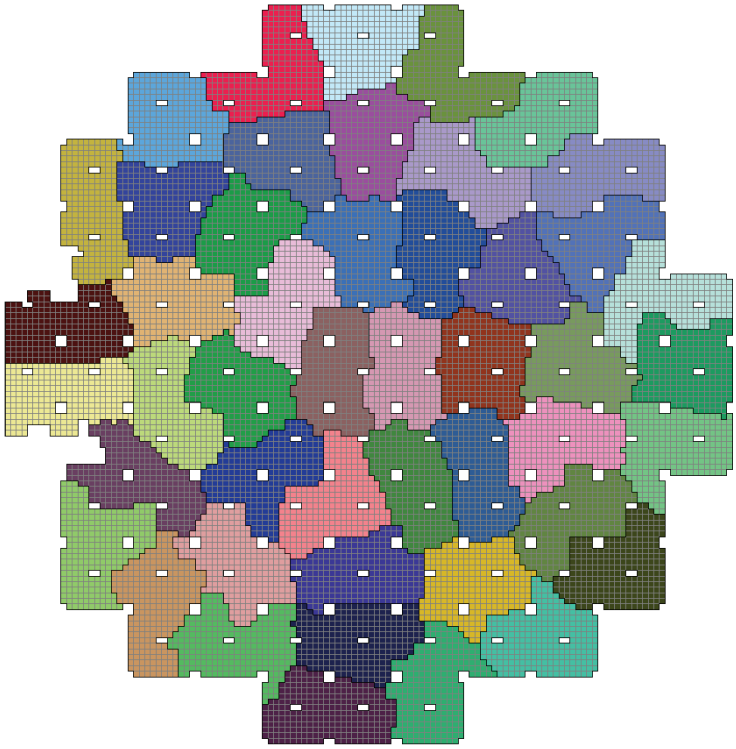

Figure 30: Flip Iteration of Tiling by 256-sized Polyominoes

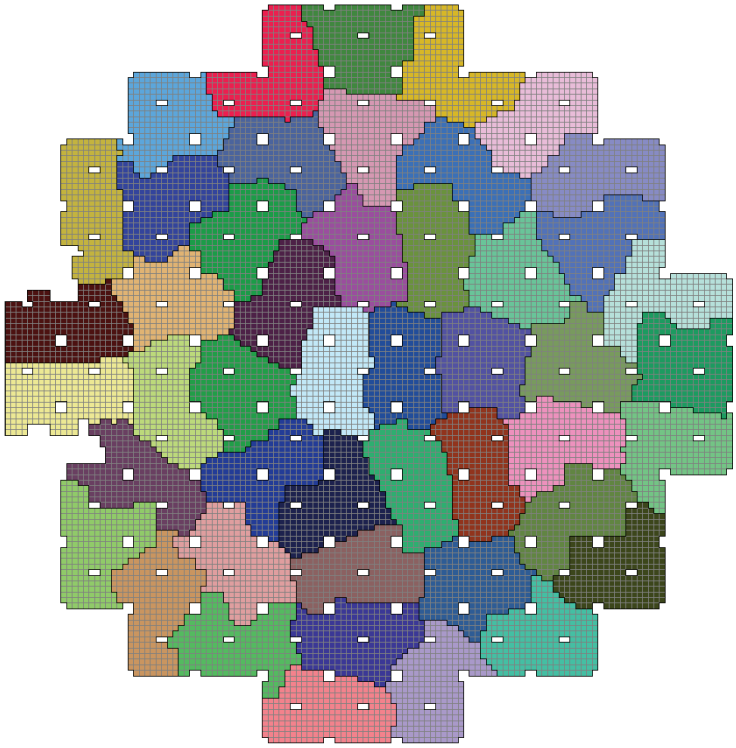

Figure 31: Flip Iteration of Tiling by 256-sized Polyominoes