2D Tiling: Domino, Tetromino and Percolation

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General Definitions

Let Γ be a finite connected (undirected) graph, and let $V(\Gamma)$ and $E(\Gamma)$ be, respectively, the vertices and edges of Γ. A perfect matching on Γ is a set of edges of Γ such that every vertex is adjacent to exactly one of the edges in the matching.
If there is no perfect matching on Γ one uses a maximum matching as a largest in cardinality.

The existence of a perfect matching, as well as finding a maximum matching in a graph Γ can be determined in a polynomial time. However, the problem of finding the number of perfect matching is $\#P$ for general graphs Γ.

One of the most common cases is the case of graphs on a square lattice. One gets a large chessboard and "domino covering problem".
In the case of planar graphs the counting of dimer coverings $\mathcal{D}(\Gamma)$ can be effectively computed using techniques developed by physicists in early 1960s.
If $A = (a_{ij})$ is a skew-symmetric matrix, then its determinant, $\det(A)$ is a square of a polynomial in a_{ij} which is called a Pfaffian, $\mathrm{Pf}(A)$, of A.

Fisher, Kasteleyn, and Temperley (FKT) in 1961, established an efficient algorithm of computing the number of dimer coverings for any planar graph Γ using Pfaffians and special orientations of edges of Γ.

To define the corresponding matrices, let us look at an orientation K of edges Γ. The skew-adjacency matrix $A^K = (a_{ij}^K)$ is defined as follows for vertices i, j of Γ:

$a_{ij}^K = 0$ if there is no edge connecting i, j

$a_{ij}^K = 1$ if edge connecting i, j is oriented according to K

$a_{ij}^K = -1$ if edge connecting j, i is oriented according to K.

Kasteleyn proved that for every planar graph Γ there is an orientation K such that the Pfaffian of A^K produces the number $\mathcal{D}(\Gamma)$ or dimer coverings, or:

$$\mathcal{D}(\Gamma) = \sqrt{\det A^K}.$$

Rectangle

In some interesting case the analytic expression can be explicitly obtained. The case of $G(n,m)$ Grid Graph of a rectangle $n \times m$ was a particular target of FKT study:

$$\mathcal{D}(G(n,m)) = \prod_{j=1}^{m} \prod_{k=1}^{n/2} \left(4\cos^2 \frac{\pi j}{m+1} + 4\cos^2 \frac{\pi k}{n+1}\right)$$
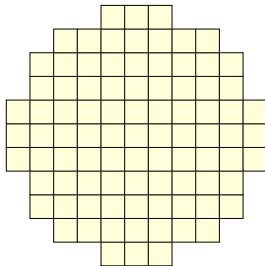
The limit of $n, m \to \infty$ defines the entropy of the Domino coverage of rectangles. In particular,

$$\lim_{n \to \infty} \frac{\log \mathcal{D}(G(n,n))}{n^2} = \frac{G}{\pi},$$
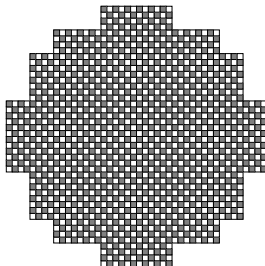
where G is Catalan's constant.

As one practical example, consider a sample grid of large square cells (super-cells) inscribed in a circle. This will be from now on our "model circular domain".

Favorite Case

Take now each of these super-cells and subdivide it into an $n \times n$ cell grid. For the "model circular domain" n will be as large as 28.

It follows from FKT result is that for any simply connected figure $\mathcal{F}$ of super-cells on the plane, the subdivision of super-cells into $n \times n$ squares results in a simply connected grid subgraph $\Gamma(\mathcal{F}_n)$ whose dimer coverage number converges asymptotically as follows:

$$\lim_{n \to \infty} \frac{\log \mathcal{D}(\Gamma(\mathcal{F}_n))}{|\mathcal{F}| \cdot n^2} = \frac{G}{\pi},$$

For the set $\mathcal{F}$ of the previous we get for $n = 6, 8, 10, 12, 14$ the following values for $|\mathcal{D}(\Gamma(\mathcal{F}_n))|$:

$5.61702 \cdot 10^{396}, 2.33667 \cdot 10^{709}, 1.44289 \cdot 10^{1112}, 1.30638 \cdot 10^{1605}, 1.72217 \cdot 10^{2188}$

The most general result in this direction belongs to R. Kenyon (2008), and describes the asymptotic of the number of domino covering of a simply connected rectilinear domain in the plane, when the size of a grid element is ϵ for $\epsilon \to \infty$. Assume that it is a finite (with respect to ϵ) collection of rectangles of an even size in a given grid.

Kenyon Coverage Theorem. Let $U \subset \mathbf{R}^2$ be a rectilinear polygon with V vertices. For sufficiently small $\epsilon > 0$, let P_ϵ be a Temperleyan polyomino in $\epsilon \mathbf{Z}^2$ approximating U in the natural sense (the corners of P_ϵ are converging to the corners of U). Let A_ϵ be the area and $Perim_\epsilon$ be the perimeter of P_ϵ. Then the logarithm of the number of domino tiling of P_ϵ is

$$\frac{c_0 \cdot A_\epsilon}{\epsilon^2} + \frac{c_1 \cdot Perim_\epsilon}{\epsilon} - \frac{\pi}{48} \cdot (c_2(\epsilon) \log \frac{1}{\epsilon} + c_3(U)) + c_4 + o(1)$$
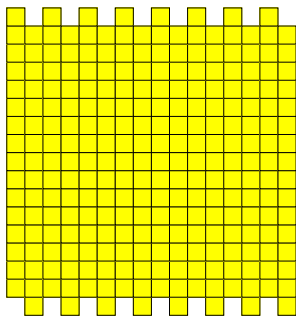
Here $c_0 = \frac{G}{\pi}$; G is Catalan's constant; $c_1 = \frac{G}{2\pi} + \frac{\log \sqrt{2} - 1}{4}$; c_4 is independent of U, and the term

$$c_2(\epsilon) \log \frac{1}{\epsilon} + c_3(U)$$

is the ϵ-normalized Dirichlet energy $E_\epsilon(U)$ (expressed in terms of elliptic functions).

But if a boundary is complex..

If instead the boundary is complex one can get simply connected domains that are coverable by a unique dimer configuration.

Chessboard, Gomory's Theorem and Hamiltonian Paths

8×8 chessboard is covered fully by non-overlapping dominoes. The puzzle (Max Black, 1946): if diagonally opposed corners of the chessboard are removed, will it still be tiled by dominoes?
The answer is no. John McCarthy proposed it famously as a hard problem for AI automated proof system.

Positive result for other "mutilated chessboards" is known as Gomory's theorem: if you remove two cells of opposite color from 8×8 chessboard, then the resulting board (graph) is fully domino tiled.

The same is true for any rectangular domain in the square grid that has even number of cells. For the rectangles whose sides are both odd the domino tiling is impossible, but if one removes any corner from such a domain, the theorem holds.

The proof is very elegant. One takes a Hamiltonian cycle. The cycle is broken in 2 places, but each part is even in length – because is start and end are of opposite color. They are obviously partitioned into dominoes since every consecutive pair represents axially aligned dominoes.

More on Hamiltonian Paths

Hamiltonian paths in grid graphs produce another impressive feat: for every $n \geq 2$ they provide a covering of these graphs with a maximal number of n-polyominoes possible. So for those lucky graphs you get the best triomino, tetromino, ..., octomino, ... coverings.

However, even for grid graphs determining whether the graph is Hamiltonian is still **NP**. Only for grid graph representing simply connected domain (no holes) Hamiltonian paths can be identified in polynomial time.
We show one example of the Hamiltonian cycle for a particularly interesting grid graph with holes.
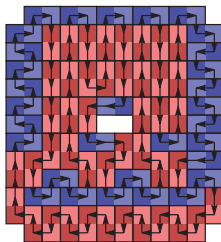
Fisher Conjectures on Monomer (Defect) Interaction

Physicists in early 60s knew Gomory theorem, but not its proof. Fisher and Stephenson: what happens in the monmer-dimer case, when there are (individual) defects – monomers, but the rest of the domain is still domino tiled.

After extensive matrix computations they put forward in 1963 a conjecture that in a large grid graph $G(n, m)$ (with even $n \cdot m$) the removal of two sites of opposite color, say $(0,0)$ and (p, q) (with $p + q = 1 \mod 2$), leads to a light decrease in the number of dimer covering of the subgraph $G_m = G(n, m) \setminus \{(0,0), (p, q)\} - \mathcal{D}(G_m)$, asymptotically as:

$$\mathcal{D}(G_m) \sim \frac{c}{\sqrt{d((p, q), (0, 0))}} \cdot \mathcal{D}(G(m, n))$$

for a constant c.

It is proved only in **one** case – for $(p, q) = (d, d - 1)$ (the diagonal case) by Hartwig (1966).

Many defects – Percolation Problems

The simplest percolation model is the "bond/site percolation" problem. In that model you have a porous material, and water flows to one side of it. Will the water reach the "other side" of it? Model the porous material as a lattice of vertices, called sites, connected by edges of that lattices, called bonds. The bonds are open for the water with a probability p or closed with a probability $1 - p$. In the bond percolation problem one wants to know for what p the water will flow from one side to another, assuming a "random" nature of connections and large size of the material block.

Practical applications: resistance networks, ferromagnetism, epidemology, ecology, etc, and particularly fracking. This problem was briefly studied in early 80s in connection with defects on chips. It re-appeared in a set of problems that we are studying now in an interesting conjunction with a problem of polyomino tiling of two-dimensional domains on a square grid.

The critical percolation probability

Assuming a "random" nature of the bond/site defects and a large size of a domain, there is always a "critical" $p = p_c$ such that the behavior is different for $p < p_c$ and $p > p_c$. Assuming an infinite size of the domain Kolmogorov's 0-1 theorem for tail events implies that the probability of having an infinite fully connected cluster (so the water will always flow from one side to another) for any given p is either 0 of 1. Combined with a monotonicity arguments this probability can be shown to be an increasing function of p. Thus there is a critical p, p_c, below which the probability of infinite cluster is always 0, and above is always 1.

The determination of the critical probability for different lattices and different models of "defects" is a subject of intensive and hard work in solid state physics, statistical mechanics, integrable systems and conformal analysis. Much of the rigorous progress was made in the last 30 years for special two-dimensional lattices.
The most famous is Kesten (1982) theorem that $p_c = 1/2$ for a two-dimensional square lattice and a bond percolation problem.

Percolation Rates in the non-uniform case

The best consistent current estimate of the "critical p site percolation rate" for a square lattice with uniform distribution of defects is $p_S = 0.5927460$. If the probability p of site occupancy is $p < p_S$, there will be no "infinite" cluster among the occupied sites in the infinite square lattice.

However, the percolation problem often arises from non-uniform distribution of defects. In our case of nearly "circular" domains there is a non-Poisson Point Process responsible for random defects. The main feature of defect distribution here, as in many other examples studied for the last 30 years, is the radial dependency of defects, where defects crowds towards the edge of the disk.

To parameterize the distribution shape that depends only on a radial distance in the "circular" domain we use the following curve:

$$L_c(r) \;=\; e^{u \cdot (1 - |r|^c)}$$

for scaled radius of the circular domain: $0 \le r \le 1$ and a constant $u = u(c)$ normalizing the distribution for $0 < c$.

The main differences in the choices of this L_c Distributions for c varying from 0 to ∞, are that at $c \to \infty$ the distribution converges to a uniform, and at small c the distribution hugs the edges of the "circular domain".

Percolation Map for a Probability of Bad Site $p = 0.1$ for L_c Distribution with $c = 1, 4, 16$

We use the same "model circular domain" for different c in L_c distribution of defects.
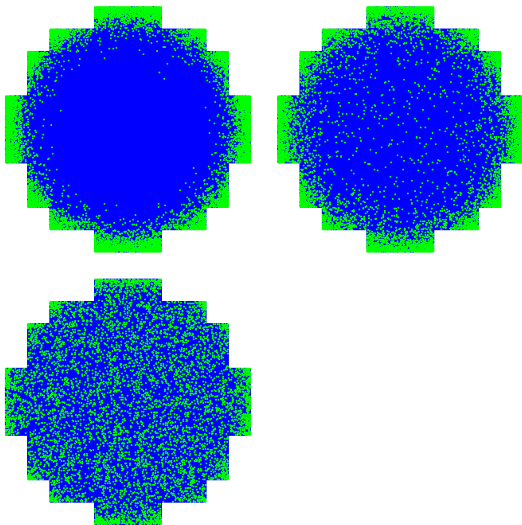
Light Clusters and Domino Tilings in them.

In the case of uniform percolation, where the large cluster will be gone by the time the probability p of a site defect will be close to 0.4, we need to know not only the expected size of the connected "good sites" cluster in the case of a finite domain, but more importantly we need to know the largest connected domino-tiled cluster in that big cluster.

The finite case is crucial for us. Thus take a particular shape and subdivide its sites further. As above we take a polygonal, nearly circular "model domain" and scale each of the big squares in this domain to small sites (using 28×28 sub-grids). This gives 69776 total sites before site defects.

The dependence of the size of the clusters and the domino coverings of defect rate currently can be determined only experimentally, with low and upper bounds provided by extensive Monte-Carlo testing.

We are interested in expected sizes of:
Largest Light (open) connected site cluster
Largest Dark (closed) connected site cluster
Largest size of dominoes on the domain (not necessarily connected)
Largest size of connected domino-tiled cluster in the Light Cluster.
It is the last size that is most important in practical applications.

Poisson Process Percolation till critical p_c

Clusters and Matching Set Sizes for
Uniform Poisson percolation rate p

Legend:
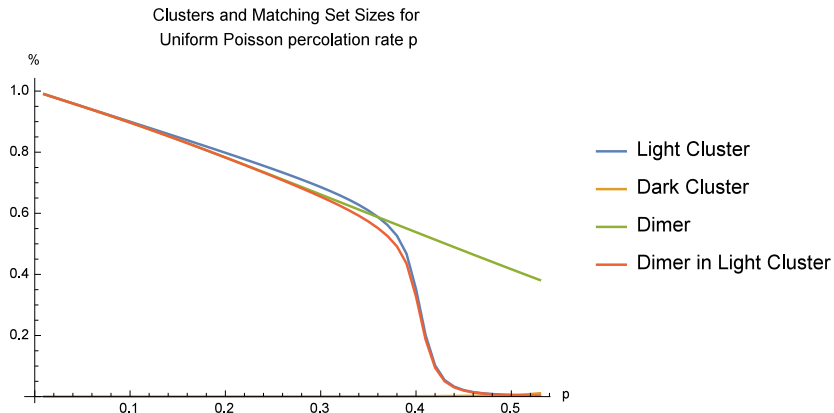- Light Cluster
- Dark Cluster
- Dimer
- Dimer in Light Cluster

Figure : Uniform Percolation till critical p_c: ; Dimensions are in fractions of the Domain cardinality

One can see clearly the effect of the collapse of the Light cluster and the rise of the Dark cluster despite the finiteness of the domain (the number of Monte-Carlo tests here is about 1.2 million).

Expected Size of the Light Dimer

What Monte-Carlo experiments also show is a relatively simple formula for the Expected size of the Dimer in the Light Cluster in terms of the total number of the elements in the domain (for the uniform distribution of defects):

$$|\text{LightDimer}| \sim (1 - p - \frac{p^2}{2} - \frac{p^4}{4} + O(p^6)) \cdot |\text{Domain}|$$

where p is a probability of a defect site in a domain of a cardinality of $|\text{Domain}| \to \infty$, with the Dimer in the connected Light Cluster in that domain of cardinality $|\text{LightDimer}|$ and

$$p < 0.35.$$

It is not known whether this numerical result can be actually proved, even as a weaker statement:

$$|\text{LightDimer}| < (1 - p - O(p^2)) \cdot |\text{Domain}|, \ \ p = o(1), \ \ |\text{Domain}| \to \infty$$

The best we can prove so far is that the upper bound for a small p is of the order of $1 - \frac{3}{4} \cdot p$ of the domain's cardinality for a polygonal scaled domain.

Main Conjecture on Defects

Nobody knows how to make further rigorous progress in determining expected sizes of dimers in the presence of monomers (defects), but the conjecture supported by numerical evidence is interesting.

Monomer-Dimer Conjecture. For any large rectilinear (Temperleyan) polygon U on a square grid, for n sufficiently small compared to the number of sites in U, and for any n white and n black monomers (defects) in a generic position in U, the remaining subgraph of U is domino tiled.
For $n = 1$ one can get this from Gomory theorem. In the cases of $n = 2$ and $n = 3$ the exceptional configurations where dimer covering promised by the Monomer-Dimer Conjecture do not exist can be explicitly determined for even size rectangular domains. These exceptional configurations are typically in corners of the domain for $n = 2$ and at edges for $n = 3$

Some slight modification of the conjecture with random n defects, and extra n monomers to create domino tiling, can be proved for large rectilinear domains. The method still uses creation of Hamiltonian paths in the subgraphs, and thus can be used for arbitrary polyomino shapes (see below).

The natural generalization of this conjecture (along Fisher's conjecture lines) would justify Monte-Carlo experiments showing the dimers can cover almost all Light Cluster when site failure rate p not too far from the critical one.
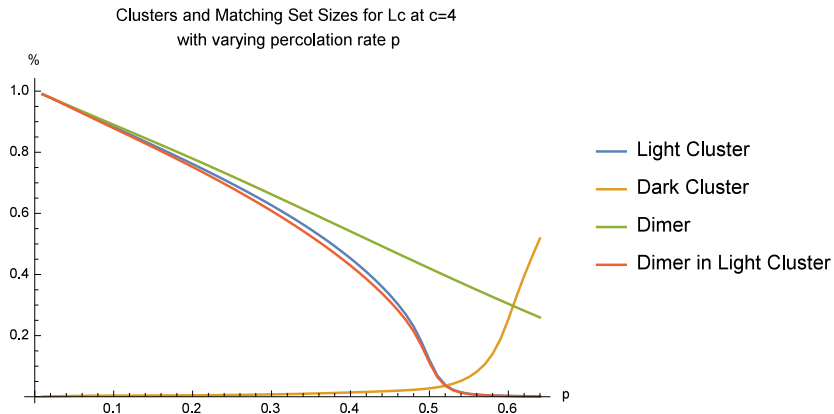
Relief from Percolation Limit in Non-Poisson Percolation

Clusters and Matching Set Sizes for Lc at c=4
with varying percolation rate p

Light Cluster
Dark Cluster
Dimer
Dimer in Light Cluster

Figure : Percolation graphs for L_c distribution for $c = 4$ till critical point; Dimensions are in fractions of the Domain cardinality

One can see that the collapse of the Light cluster is delayed till about $p = 0.5$ in this radial defect distribution. Again, one can see only a slight loss of the size of the LightDimer (the Dimer in the Light Cluster) all the way till

$$p < 0.45.$$

Numerically (for the L_c radial distribution at $c = 4$) the expected size of the LightDimer in this range of p has an excellent behavior:

$$|\text{LightDimer}| \sim (1 - p - p^2 - O(p^4)) \cdot |\text{Domain}|.$$

This behavior is only numerical, but it was computed for other types of circular domain, and is sufficiently stable to vouch for one leading digit after the period for the range of p in $p < 0.45$.

We do not expect this bound to be proved soon.

Hexagonal Lattice

So far we considering only domains in a square lattice. Any subgraph of that lattice is a planar graph, making it possible to have a FKT detailed analysis.

Allowing diagonal connections between square lattice sites creates a non-planar graph, but allowing the diagonal connections along only the main diagonals, still creates planar graphs.

Such connectivity turns the square lattice into a hexagonal lattice. It is easy to visualize it by turning the unit squares into regular hexagons.
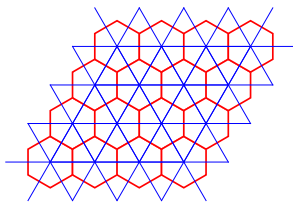

Figure : 4×4 Square Mesh as a Hexagonal Grid with Main Diagonal connections

Bibones

In terms of tiling (and matching problems) the pairs of primitive objects used for matching have different names depending on the lattice type. For a square lattice we get dominoes. For the triangular lattice the name of the 2 adjacent cells is a *lozenge*. For the hexagonal lattice the name that stuck is a *bibone*. The FKT formalism works, and shows that the number of bibone tilings for rectangular domains significantly exceeds the number of dominoes tilings. Instead of per-dimer molecular freedom value of $\frac{G}{\pi} = 0.291561$ for dominoes one gets for bibones 0.428595.

Nearly linear (in area) algorithm of determining the covering with bibones of a simply connected domain exists (Kenyon-Remila). It is based on one important property that bibone coverings share with domino coverings: **local flip connectivity**.

In the case of dominoes coverings of a simply connected domain in a square lattice, a sequence of domino local flips in 2×2 arrangement of 2 dominoes can turn any domino tiling into any other.

In the case of the bibone tiling of a simply connected domain in the hexagonal case, a more complex set of flips does the same thing: turn any bibone tiling of a simply connected domain with N sites into any other bibone tiling in at most $5N$ iterations of flips.

Local Flips

The set of local flips in bibone tiling that allow such local flip connectivity is rather small:

3 of 2-bibone flips; 2 of 3-bibone flips, and 3 of 4-bibone flips.
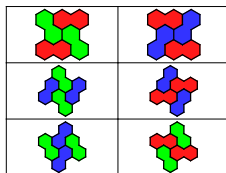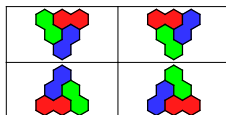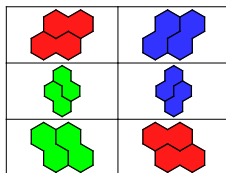
Polyomino Tilings: start with Tetrominoes

Polyomino is any axially connected finite subgraph on the square grid; they are named by their sizes. General problems to determine whether a finite domain is tileable by a particular set of polyominoes are **NP**. The dominoes are exceptions here as well as easy generalizations of them to $a \times b$ and $b \times a$ rectangle tilings. It turned out that we need the generalizations of domino tilings with bigger polyominoes. Among them the tetrominoes (of size 4) allow wonderful theoretical and numerical opportunities. These are most widely known from the Tetris game of the old.
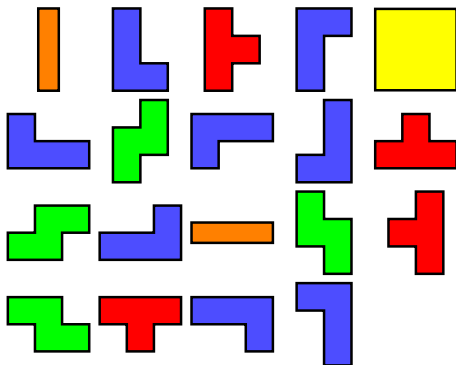

Figure : 19 Tetrominoes

Octominoes

From Bibones to Tetrominoes

The reason why the path from dominoes to tetrominoes to octominoes works is because we look at every step as a matching problems, even though graphs are no longer planar (but in most cases still Pfaffian).

If one looks at an arbitrary tetromino, one can see that T-tetromino cannot be broken into two kissing dominoes, while all other tetromino can be.

However with bibones on the hex lattice all tetrominoes can be nicely split into kissing bibones. Moreover, the graph of bibones is planar. This allows us to use a full power of FKT techniques, including the counting the number of possible tetromino tilings (there are more of those than dominoes).

An important thing here is that the quality of the tetromino tiling depends on the starting bibone graph. This is altered **only** by the bibone local flips without changing the covered domain.

The main purpose of the local flips of bibones for us is to build better tetrominoes, and ultimately octominoes, as a maximal matching of tetrominoes graph.

The next set shows the progression of the domino, tetromino, and finally octomino coverage on a 28×28 grid with corners removed and a (double) hole in the middle, together with the extra $p = 0.1$ defects in the remaining domain.

It shows the progress of turning the maximum domino coverage (with its 5 extra monomers in addition to fixed defects) to the maximum tetromino coverage (with 1 extra defect), and finally to the octomino coverage. There there are less new defects than for dominoes: only 3 extra defects for the whole domain with holes.

After local bibone flips get peak tetrominoes coverage
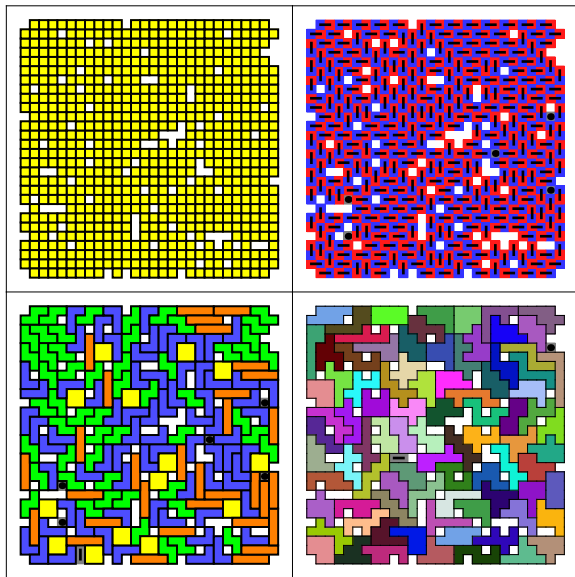

Figure : After local bibone flips get peak tetrominoes coverage. 28X28 Grid with
corner and extra 10% defects. Sites: 699; Dominoes 347; Tetrominoes 173;

Any chance for Polyominoes of arbitrary sizes?

Yes, the problem of the tiling of a connected (say, simply connected) domain in the square grid by arbitrary Polyominoes is **NP**. But one might find faster if not a perfect tiling but a "nearly perfect" coverage with $o(1)$ area missing.

The main problem becomes a "flip" problem of shaping the Polyominoes coverage, by restricting the Polyominoes of a fixed size N of particular shape. The generalized semi-group of "flips" is rather large and complex for any $N > 3$.

Driven by some practical applications we had a look at $N = 8$ (octominoes), and $N = 24$, $N = 256$.

In the next few slides we show the progression from one tiling (driven by cycles in the grid subgraphs) to tilings by corresponding Polyominoes driven by minimization of the L_2 digital diameter.