

IIE Transactions



ISSN: 0740-817X (Print) 1545-8830 (Online) Journal homepage: http://www.tandfonline.com/loi/uiie20

Joint modeling of VIX and SPX options at a single and common maturity with risk management applications

Peter Carr & Dilip B. Madan

To cite this article: Peter Carr & Dilip B. Madan (2014) Joint modeling of VIX and SPX options at a single and common maturity with risk management applications, IIE Transactions, 46:11, 1125-1131, DOI: 10.1080/0740817X.2013.857063

To link to this article: http://dx.doi.org/10.1080/0740817X.2013.857063

	Accepted author version posted online: 10 Mar 2014. Published online: 10 Mar 2014.
	Submit your article to this journal $oldsymbol{\mathbb{Z}}$
ılıl	Article views: 114
Q ^L	View related articles ☑
CrossMark	View Crossmark data ☑
4	Citing articles: 1 View citing articles 🗹

Joint modeling of VIX and SPX options at a single and common maturity with risk management applications

PETER CARR¹ and DILIP B. MADAN^{2,*}

¹Courant Institute, New York University, 251 Mercer Street, New York, NY 10012, USA ²Robert H. Smith School of Business, University of Maryland, College Park, MD 20742, USA E-mail: dbm@rhsmith.umd.edu

Received January 2013 and accepted October 2013

A double gamma model is proposed for the VIX. The VIX is modeled as gamma distributed with a mean and variance that respond to a gamma-distributed realized variance over the preceding month. Conditional on VIX and the realized variance, the logarithm of the stock is variance gamma distributed with affine conditional drift and quadratic variation. The joint density for the triple realized variance, VIX, and the SPX is in closed form. Maximum likelihood estimation on time series data addresses model adequacy. A joint calibration of the model to SPX and VIX options is employed to illustrate a risk management application hedging realized volatility options.

Keywords: Ask price minimizing hedge, distorted expectation, variance gamma model

1. Introduction

The S&P 500 index along with options written on the index are now coupled with the VIX index and options on this index. The two are not unrelated as the square of the VIX index is the price of a 1-month variance swap paying the annualized 1-month realized variance of returns on the S&P 500 index. The variance swap rate itself is typically synthesized from S&P 500 options using procedures described, for example, in Carr and Lee (2009). In fact, this is how the VIX is now computed. There is then a demand and an interest in jointly and consistently modeling options on these two indices. For recent work in this direction we cite Bergomi (2004, 2005, 2008), Buehler (2006), Broadie and Jain (2008), Gatheral (2008), and Cont and Kokholm (2013).

Although the VIX may be and is computed every day, it is by construction eventually the price of a different asset, as the variance of the S&P 500 index over disjoint months along with the price of such a payout are potentially unrelated. Of course, they may be linked by assuming a model that relates them. There are various levels of consistency that may be asked for and modeled. For example, one may ask for a consistent modeling of both sets of options across

Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/uiie.

all strikes and maturities through calendar time. Alternatively, one may seek to model the calibration date consistently across strike and maturity at a single point of time. Restricting further one may jointly model across both sets of strikes at a single maturity. The focus of this article is on the last of these alternatives.

In a partial defense of such a modeling strategy we note that Lévy processes such as the variance gamma model (Madan and Seneta (1990); Madan *et al.* (1998)) synthesized option prices just across a single maturity; an effective parsimonious synthesis across maturities using a one-dimensional Markov model came later in the Sato process of Carr *et al.* (2007). An effective synthesis across calendar time has not yet been attained for just the S&P 500 index itself, as most models are recalibrated continuously. Hence, we focus attention here on just two smiles at a common maturity.

With a view to addressing model adequacy we estimate the model on time series data for the equity return, the lagged realized quadratic variation, and the level of the square of the VIX. The estimation is conducted using maximum likelihood based on the derived density. The results are generally supportive of the model structure.

From a risk management perspective one may seek to determine a static portfolio of relatively short maturity options on the S&P 500 index and the VIX with a view to covering a comparable maturity risk exposure in a realized volatility swap. For such an application one seeks a joint

^{*}Corresponding author

1126 Carr and Madan

and consistent modeling of the two smiles. We apply the model developed here to hedge such an exposure.

The outline of the rest of this article is as follows. Section 2 sets out the joint model for the two smiles. Section 3 describes the calibration procedure followed for a single calibration date. Section 4 addresses model adequacy on time series data. Section 5 illustrates with a risk management application. Section 6 concludes.

2. The joint model

We formulate a joint model for the logarithm of the index at a fixed and, say, unit maturity that we denote by s and the square of the VIX at the same time denoted by v. A third variable of interest is the realized variance on the index that we shall take to be proportional to s. Options trade on the exponential of s and the square root of s, while s will be a hidden variable whose conditional law given $s \times v$ will be inferred for a risk management exercise. We shall in fact formulate a joint law for the triple (s, s, v).

Various models were attempted for the law of v or its square root and we mention in passing the gamma distribution or a general power of a gamma distributed variable. It was found that these models failed to match out of the money call option prices. In line with Gatheral's (2008) suggestion of a double log normal we develop here a double gamma given the relative tractability of the associated Laplace transforms. Since the variable x is scaled in all its occurences we take it to be our first gamma variable with unit mean and variance $1/\gamma$. The marginal density for x is then

$$f(x) = \frac{\gamma^{\gamma}}{\Gamma(\gamma)} x^{\gamma - 1} e^{-\gamma x}, \quad x > 0,$$

where $\Gamma(x)$ is the gamma function.

Next we specify the conditional density of v given x and we take this to be gamma distributed with scale coefficient α and shape coefficient $\beta + \delta x$. The mean and variance of v respond linearly to x and this specification models the response of the square of the VIX at unit time to the proxy for realized variance x to that time. The joint density for x, v is then

$$g(x, v) = \frac{\gamma^{\gamma} x^{\gamma - 1} e^{-\gamma x}}{\Gamma(\gamma)} \frac{\alpha^{\beta + \delta x} v^{\beta + \delta x - 1} e^{-\alpha v}}{\Gamma(\beta + \delta x)}, \quad x, v > 0.$$

The marginal density for v would require an integration over x and is not available in closed form but it may be accessed by transform methods from the joint characteristic

function that is easily seen to be

$$\Lambda(\kappa, \lambda) = E\left[\exp\left(i\kappa x + i\lambda v\right)\right]$$

$$= E\left[\exp\left(i\kappa x + (\beta + \delta x)\ln\left(\frac{\alpha}{\alpha - i\lambda}\right)\right)\right]$$

$$= \left(\frac{\alpha}{\alpha - i\lambda}\right)^{\beta} E\left[\exp\left(i\left(\kappa - i\delta\ln\left(\frac{\alpha}{\alpha - i\lambda}\right)\right)x\right)\right]$$

$$= \left(\frac{\alpha}{\alpha - i\lambda}\right)^{\beta} \left(\frac{\gamma}{\gamma - i\left(\kappa - i\delta\ln\alpha/(\alpha - i\lambda)\right)}\right)^{\gamma}$$

$$= \left(\frac{\alpha}{\alpha - i\lambda}\right)^{\beta} \left(\frac{c}{c - i\kappa - \delta\ln\alpha/(\alpha - i\lambda)}\right)^{\gamma}.$$

Conditional on x, v we suppose that the logarithm of the stock is variance gamma distributed X_{CGM} with parameters CGM. The characteristic function for X_{CGM} is

$$E\left[\exp\left(iu\,X_{CGM}\right)\right] = \exp\left(C\ln\left(\frac{GM}{(M-iu)(G+iu)}\right)\right).$$

The density for X_{CGM} is from Madan *et al.* (1998) on transformation to the CGM parameterization as per Carr *et al.* (2002):

$$f_{CGM}(x) = \frac{(GM)^{C}}{2^{C-1}\Gamma(C)\sqrt{2\pi} (G+M)/2^{C-1/2}} \times \exp\left(\frac{G-M}{2}x\right)|x|^{C-1/2}K_{C-1/2}\left(\frac{G+M}{2}|x|\right),$$

where $K_{\nu}(x)$ is the modified Bessel function.

The quadratic variation of a variance gamma process with density in *CGM* parameterization is

$$C\left(\frac{1}{M^2}+\frac{1}{G^2}\right).$$

We suppose the conditional density for the logarithm of the stock has a conditional drift adapted to x, v as

$$r + \eta x + \zeta v$$
,

with quadratic variation modeled to respond to the level of x, v as

$$C = k + ax + bv$$
.

The parameters G, M are constants.

The conditional drift for the stock is organized by writing:

$$s = r + \eta x + \zeta v + (k + ax + bv)$$

$$\times \ln \left(\frac{(M-1)(G+1)}{GM} \right) + X_{CGM} + \omega,$$

where the constant ω is chosen to set the unconditional drift to be the interest rate.

The conditional expectation of the exponential of s given x, v is

$$E[e^{s}|x,v] = \exp(r + \eta x + \zeta v + \omega).$$

VIX and SPX options 1127

Hence we set

$$\omega = -\ln E \left[\exp \left(\eta x + \zeta v \right) \right],$$

and on evaluation of the joint characteristic function of x, v at $-i\eta$, $-i\zeta$ we infer that

$$\omega = \beta \ln \left(\frac{\alpha - \zeta}{\alpha} \right) + \gamma \ln \left(\frac{\gamma - \eta - \delta \ln \alpha / (\alpha - \zeta)}{\gamma} \right).$$

The joint characteristic function of the triple s, x, v is then

$$E\left[\exp\left(ius + i\kappa x + i\lambda v\right)\right]$$

$$= \exp\left(iu\left(r + \omega + k\ln\left(\frac{(M-1)(G+1))}{GM}\right)\right)$$

$$\times E\left[\begin{array}{c} iu\left(\eta + a\ln\left(\frac{(M-1)(G+1))}{GM}\right)\right)x \\ + iu\left(\zeta + b\ln\left(\frac{(M-1)(G+1))}{GM}\right)\right)v \\ + i\kappa x + i\lambda v \\ + (k + ax + bv)\ln\left(\frac{GM}{(M-iu)(G+iu)}\right) \end{array}\right]$$

$$= \exp\left(iu\left(r + \omega + k\ln\left(\frac{(M-1)(G+1))}{GM}\right)\right)$$

$$+ k\ln\left(\frac{GM}{(M-iu)(G+iu)}\right)\right)$$

$$\times \Lambda \begin{pmatrix} \kappa + u\left(\eta + a\ln\left(\frac{(M-1)(G+1))}{GM}\right)\right) \\ -ia\ln\left(\frac{GM}{(M-iu)(G+iu)}\right), \\ \lambda + u\left(\zeta + b\ln\left(\frac{(M-1)(G+1))}{GM}\right)\right) \\ -ib\ln\left(\frac{GM}{(M-iu)(G+iu)}\right). \end{pmatrix}$$

There are 11 parameters in the model and they are γ , α , β , δ , k, a, b, η , ζ , G, and M.

The joint density for the triple (s, x, v) is given by

$$\begin{split} h(s,x,v) &= g(x,v) \times \frac{(GM)^C}{2^{C-1}\sqrt{2\pi}}\Gamma(C)(G+M)/2^{C-.5} \\ &\times \exp\left(\frac{G-M}{2}w\right)|w|^{C-.5}K_{C-.5}\left(\frac{G+M}{2}|w|\right) \\ w &= s-r-\left(\eta+a\ln\left(\frac{(M-1)(G+1))}{GM}\right)\right)x \\ &-\left(\zeta+b\ln\left(\frac{(M-1)(G+1))}{GM}\right)\right)v-\omega \\ &-k\ln\left(\frac{(M-1)(G+1))}{GM}\right) \\ C &= k+ax+bv \\ g(x,v) &= \frac{1}{\Gamma(\gamma)}c^{\gamma}x^{\gamma-1}e^{-cx}\frac{\alpha^{\beta+\delta x}}{\Gamma(\beta+\delta x)}v^{\beta+\delta x-1}e^{-\alpha v} \\ \omega &= \beta\ln\left(\frac{\alpha-\zeta}{\alpha}\right)+\gamma\ln\left(\frac{c-\eta-\delta\ln\alpha/(\alpha-\zeta)}{c}\right). \end{split}$$

We term this model sxvvgwadqv for modeling the triple s, x, v and specifying s as vg conditional on x, v with affine drift and quadratic variation.

3. Calibration of the joint model for the triple s, x, v

We may organize the parameters into two groups, γ , α , β , δ of the double gamma model for the square of the VIX, followed by k, a, b, η , ζ , G, M for the VG in CGM format. Conditional on x, v the parameter C is modeled by the affine function k + ax + bv. Similarly, conditional on x, v, the stock drift is $\eta x + \zeta v$. The first step is to find the double gamma parameters for the square of the VIX. Unfortunately, there are no quoted options on the square of the VIX but just options on the VIX. One could build options on the square of the VIX from VIX options but the range of traded strikes on the VIX may be too narrow for evaluating the price of the tail of the square. Instead, we proceed by first fitting the double gamma model to the VIX even though it is a model for the square.

We next use these parameters to generate prices for calls on VIX over a wide range of strikes and we then use these prices to determine prices on calls for the square of the VIX. For a call option on the square of the VIX with strike k the call price C(k) is obtained in terms of call prices c(x) on the VIX for strike x and density f(x) by

$$C(k) = e^{-rt} \int_{\sqrt{k}}^{\infty} (x^2 - k) f(x) dx$$
$$= e^{-rt} \int_{\sqrt{k}}^{\infty} 2x (1 - F(x)) dx$$
$$= \int_{\sqrt{k}}^{\infty} 2x (-c'(x)) dx. \tag{1}$$

The procedure may also be reversed with density g(k) for the squared VIX by writing

$$c(x) = e^{-rt} \int_{x^2}^{\infty} (\sqrt{k} - x)g(k)dk$$

$$= e^{-rt} \int_{x^2}^{\infty} \frac{1}{2\sqrt{k}} (1 - G(k))dk$$

$$= \int_{x^2}^{\infty} \frac{1}{2\sqrt{k}} \left(-C'(k)\right) dk. \tag{2}$$

Equation (1) is employed to generate option prices on the squared VIX to get starting values for the double gamma model for the squared VIX after fitting this model to the VIX. Equation (2) is then used in calibrating VIX options for a model on the squared VIX.

We illustrate the procedure for data on October 20, 2008, a month after the Lehman collapse. Fitting the double gamma model to VIX options yielded the following

1128 Carr and Madan

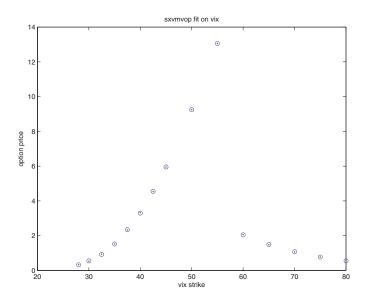


Fig. 1. Fit of double gamma model to VIX option directly on October 20, 2008. Market prices are represented by circles and model prices are shown as dots.

parameter values:

$$\gamma = 0.6617$$
 $\alpha = 0.5474$
 $\beta = 18.7080$
 $\delta = 5.8241.$

Figure 1 presents a graph of the fit of the double gamma model to VIX options.

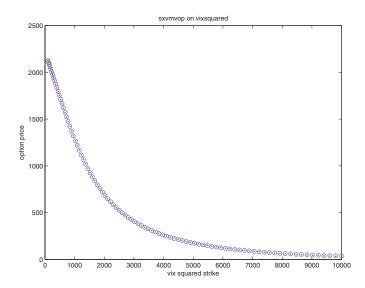


Fig. 2. Fit of the double gamma model to call option prices for the squared VIX. Circles represent the call prices generated from the double gamma fit to the VIX. The double gamma model for the VIX squared is shown as dots.

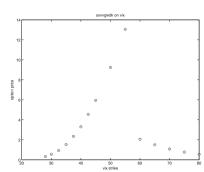


Fig. 3. Fit of *sxvvgwadqv* to VIX options on October 20, 2008. Market prices are represented by circles and model prices are shown as dots.

The double gamma parameter values fitted to VIX options are then used to generate a set of call prices on the square of the VIX and the double gamma model is then fit to the generated squared VIX call prices. The resulting parameter values for the double gamma squared VIX model are

$$\gamma = 0.198797$$

 $\alpha = 0.002392$
 $\beta = 3.583938$
 $\delta = 1.753731$.

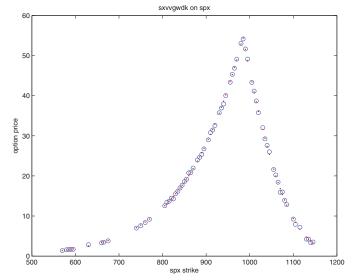


Fig. 4. Fit of *sxvvgwadqv* to SPX options on October 20, 2008. Market prices are represented by circles and model prices are shown as dots.

VIX and SPX options 1129

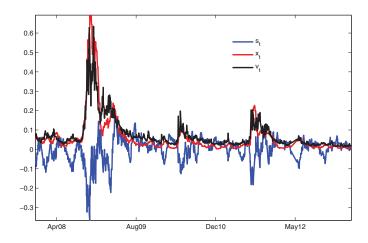


Fig. 5. Graph of the triple s, x, and v for the period December 18, 2007, to April 30, 2013.

Figure 2 shows the fit of the double gamma model to call prices on the squared VIX.

The 11-parameter model sxvvgwadqv is then fit simultaneously to options on the SPX and VIX. The resulting parameter values for October 20, 2008; were

$$\gamma = 0.1572$$
 $\alpha = 0.0024$
 21
 $\beta = 3.7288$
 42
 $\delta = 1.8025$
 46
 $k = 0.1444$
 01

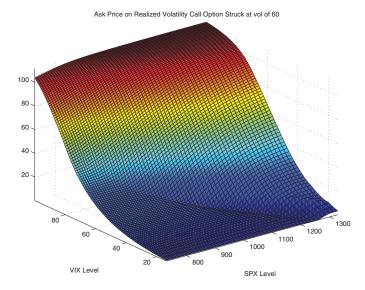


Fig. 6. Ask price for a call option on realized volatility struck at a volatility of 60 constructed from the conditional distribution of *x* given the SPX and the VIX by *minmaxvar* distortion at stress level 0.25.

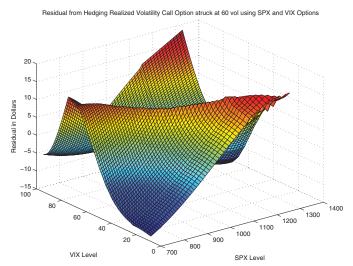


Fig. 7. Difference between the hedge cash flow and the target cash flow attained on minimizing the least squares distance between the two cash flows.

a = 0.0074 b = 0.0002 $\eta = 0.0526$ $\zeta = 0.0001$ G = 8.7586M = 20.201 96.

Figures 3 and 4 present graphs of the fit to VIX and SPX options on October 20, 2008.

4. Model adequacy assessed on time series data

With a view to addressing the adequacy of the model we take time series data on the daily close of the S&P 500 and VIX indices from October 22, 2007, to April 30, 2013. Beginning on December 18, 2007, we construct observations on realized variance x_t each day using 21-day lagged squared log price relative returns. We also construct the 21-day equity return s_t as the log price relative for current price relative to its value 21 days ago. Finally, we construct the series v_t for the square of the VIX each day. This gives us 1347 observations on the triple (s_t, x_t, v_t) . We de-mean or center the data on s_t and employ a centered unconditional variance gamma density for the law of s_t that conditionally on s_t has drift and quadratic variation affine in s_t v. Figure 5 presents a graph of the time series for the triple (s_t, x_t, v_t) .

We have formulated an 11-parameter joint density for this triple and we estimate it by maximum likelihood on these data. Table 1 presents the parameter estimates and *t*-statistics based on this estimation.

1130 Carr and Madan

Table 1. Maximum likelihood estimates for *sxvvgwadqv*

	Estimate	t-Statistic
γ	0.2585	9.46
α	1.4794	16.43
β	0.0003	0.1123
δ	0.1177	30.34
k	0.0073	0.1461
a	4.9294	10.44
b	13.9980	17.30
η	0.0019	1.90
ζ	-0.4949	-41.41
\widetilde{G}	18.4096	29.56
M	27.7451	18.34

We observe a significant response of the VIX to past realized variance as reflected in the t-statistic for δ . The response of quadratic variation for the stock to both x, v is significant as reflected by the t-statistics for a, b. The drift response to the realized variance is not significant; however, the response to the forward-looking VIX is significant and reflects the leverage effect relationship between volatility and returns. The coefficients γ , α , G, and M are also significant. We conclude that the general model structure is supported by the time series data.

5. A risk management application

By way of a risk management application we construct a portfolio of options on the S&P 500 index and VIX options with a view to earning the ask price on the liability of a call option on realized volatility struck at a volatility of 60. This is a little out of the money as the variance swap quote for a

SPX Cash flow held as hedge on Realized Volatility Call

-60
-80
-100
-120
-140
-160
-800
-1800
-1800
-1800
-1800
-1800
-1800
-1800
-1800
-1800
-1800
-1800
-1800
-1800
-1800
-1800
-1800
-1800
-1800

Fig. 8. Cash flow contingent on the level of the SPX held as a hedge for the realized volatility call.

month was estimated from the S&P 500 index option data at 51.08.

The mean of the hidden variable x is unity and we let it represent realized variance by scaling it by the square of the variance swap quote. This scale factor was 2609.47. The level of the SPX was 984.41. The cash flow on a realized volatility call struck at 60 is then modeled as

$$c = \left(\sqrt{2609.47 \times x} - 60\right)^{+}.$$

The ask price for this cash flow seen as a liability is evaluated using the methods of Cherny and Madan (2010) and Carr et al. (2011) as the negative of the distorted expectation of -c, where we distort the conditional distribution p(x|s, v) of x given s, v. The conditional distribution is obtained from the joint density of the triple by the Bayes' rule. The distortion used is minmaxvar at the stress level of 0.25.

Figure 6 presents a graph of this ask price as function of the the level of the SPX and the VIX.

We construct a portfolio of positions in the money market, the SPX, the VIX and options thereon to minimize by least squares the gap between the hedge cash flow and the target cash flow given by the ask price function displayed in Fig. 6. Figure 7 presents the gap between the hedge cash flow and the target cash flow. The strikes used for the option positions are the same as those displayed in the model calibration.

The cash flows contingent on the level of the SPX and VIX held as hedges are presented in Figs. 8 and 9.

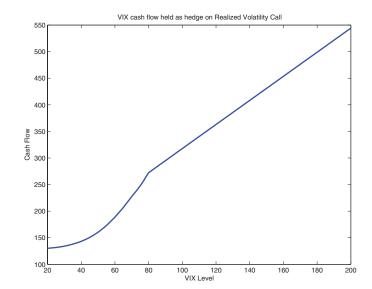


Fig. 9. Cash flow contingent on the level of the VIX held as a hedge for the realized volatility call.

VIX and SPX options 1131

6. Conclusions

A double gamma model is proposed for the VIX where the mean and variance of the VIX respond to a proxy viewed as the realized variance over the preceding month. Conditional on the realized variance proxy and the VIX the logarithm of the stock is modeled as conditionally variance gamma distributed with affine conditional drift and quadratic variation. The resulting model for the triple (i) realized variance over the month; (ii) VIX at month end; and (iii) S&P 500 index at month end is a closed-form joint density with 11 parameters. The model is generally supported by time series data for the period December 2007 to April 2013. The model is calibrated jointly to SPX and VIX options for the data as at October 20, 2008. A risk management application hedging an ask price for a call on realized volatility struck at a volatility of 60 illustrates a model application.

Acknowledgement

We are grateful to Hitendra Varsani and Abake Adenle for proposing the problem.

References

Bergomi, L. (2004) Smile dynamics. Risk, 17, 117-123.

Bergomi, L. (2005) Smile dynamics II. Risk, 18, 67-73.

Bergomi, L. (2008) Smile dynamics III. Risk, 21, 90-96.

Broadie, M. and Jain, A. (2008) Pricing and hedging volatility derivatives. *Journal of Derivatives*, **15**, 7–24.

Buehler, H. (2006) Consistent variance curve models. *Finance and Stochastics*, **10**, 178–203.

Carr, P., Geman, H., Madan, D. and Yor, M. (2002) The fine structure of asset returns: An empirical investigation. *Journal of Business*, **75**(2), 205, 232

Carr, P., Geman, H., Madan, D.B. and Yor, M. (2007) Selfdecomposability and option pricing. *Mathematical Finance*, 17, 31– 57 Carr, P. and Lee, R. (2009) Volatility derivatives. Annual Review of Financial Economics, 1, 319–339.

Carr, P., Madan, D.B. and Vicente Alvarez, J.J. (2011) Markets, profits, capital, leverage and returns. *Journal of Risk*, 14, 95–122.

Cherny, A. and Madan, D.B. (2010) Markets as a counterparty: an introduction to conic finance. *International Journal of Theoretical and Applied Finance*, 13, 1149–1177.

Cont, R. and Kokholm, T. (2013) Consistent pricing model for index options and volatility derivatives. *Mathematical Finance*, 23, 248– 274

Gatheral, J. (2008) Consistent modeling of SPX and VIX options. Presentation at Bachelier Congress, London, July 2008.

Madan, D., Carr, P. and Chang, E. (1998) The variance gamma process and option pricing. *European Finance Review*, **2**, 79–105.

Madan, D. B. and Seneta, E. (1990) The variance gamma (VG) model for share market returns. *Journal of Business*, 63, 511–524.

Biographies

Peter Carr is a Managing Director at Morgan Stanley with 15 years of experience in the derivatives industry. He was also a finance professor for 8 years at Cornell University, after obtaining his Ph.D. from UCLA in 1989. He is presently the Executive Director of the Math Finance program at NYU's Courant Institute, the Treasurer of the Bachelier Finance Society, and a trustee for the Museum of Mathematics in New York. He has over 70 publications in academic and industry-oriented journals and serves as an Associate Editor for eight journals related to mathematical finance. He was selected as Quant of the Year by *Risk Magazine* in 2003 and shared in the ISA Medal for Science in 2008. Last December, the International Association of Financial Engineers (IAFE) and Sungard jointly announced that they selected Dr. Carr as its 2010 Financial Engineer of the Year.

Dilip Madan is Professor of Finance at the Robert H. Smith School of Business. He specializes in mathematical finance. Currently he serves as a consultant to Morgan Stanley, Meru Capital, Norges Bank Investment Management, and MarketToppers. He has also consulted with Citigroup, Bloomberg, the FDIC, Wachovia Securities, and Caspian Capital. He is a founding member and Past President of the Bachelier Finance Society. He received the 2006 von Humboldt Award in Applied Mathematics, was the 2007 *Risk Magazine* Quant of the year, received the 2008 Medal for Science from the University of Bologna, and held the 2010 Eurandom Chair. He has served as the Managing Editor of *Mathematical Finance* and currently is co-editor of the *Review of Derivatives* Research and an Associate Editor for numerous other journals.