Hedging Variance Options on Continuous Semimartingales

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Abstract

We find robust model-free hedges and price bounds for options on the realized variance of [the returns on] an underlying price process. Assuming only that the underlying process is a positive continuous semimartingale, we superreplicate and subreplicate variance options and forward-starting variance options, by dynamically trading the underlying asset, and statically holding European options. We thereby derive upper and lower bounds on values of variance options, in terms of Europeans.

1 Introduction

Variance swaps, which pay the realized variance of [the returns on] an underlying price process, have become a leading tool for managing exposure to volatility risk. As reported in the Financial Times, [19],

> Volatility is becoming an asset class in its own right. A range of structured derivative products, particularly those known as variance swaps, are now the preferred route for many hedge fund managers and proprietary traders to make bets on market volatility.

Dealers have met the demand for variance swaps with the help of the model-free log contract methodology which replicates realized variance, and which became in 2003 the basis for the CBOE's calculation of the VIX index. Extending that methodology, we replicate the forward-starting weighted variance of general functions of continuous semimartingales – but mainly we focus on variance *options*.

Variance options – calls and puts on realized variance – allow portfolio managers greater control over volatility risk exposure, offering them the ability to go long or short variance while limiting the

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downside to the premium paid for the option. However, they present greater hedging difficulties to the dealer. According to one practitioner [1] in 2007,

> The industry is taking a big risk writing such products [options on variance] and at some point that will be a risk that you can't assess. This industry has to fulfill investors' needs, but at the same time I don't want to write a ticking bomb.

We take a robust model-free approach to this hedging problem. Assuming only that the underlying process is a positive continuous semimartingale, we superreplicate and subreplicate variance options and forward-starting variance options, by dynamically trading the underlying asset, and statically holding European options. We thereby derive upper and lower bounds on the values of variance options, in terms of Europeans.

1.1 Related Work

In [7], Carr-Geman-Madan-Yor priced options on realized variance, assuming returns follow pure jump dynamics with independent increments; whereas we work with arbitrary continuous dynamics – without assuming independent increments. They did not address hedging, whereas we develop both subhedges and superhedges. They found pricing formulas in terms of the characteristics/parameters of the underlying process; whereas we derive bounds directly in terms of European-style option prices – *without imposing a model on the underlying dynamics*, hence without bearing the risks of misspecification and miscalibration associated with any specific model.

Indeed, we regard our results as part of a broad program which aims to use European options – which pay functions of the time-T underlying Y_T – to extract information model-independently about risks dependent on the entire *path* of Y, and to hedge or replicate those risks robustly. Three prominent examples of such path-dependent risks are: first, the maximum of a price process, robustly hedged in Hobson [20] by holding a call option to subreplicate, and by gradually selling off a portfolio of calls to superreplicate (also see Hobson-Pedersen [21] for subreplication of a forward-starting digital on the maximum); second, barrier-contingent call and put payoffs, robustly hedged in Brown-Hobson-Rogers [6] using European options together with a transaction in the underlying at the barrier passage time; and third, the variance swap payoff, robustly replicated in Neuberger [23], Dupire [16], Carr-Madan [9], Derman et al [14], and Britten-Jones/Neuberger [5], using a log contract together with dynamic trading of the underlying. This paper includes extension and unification of the replication strategies for the various flavors of variance swaps (including gamma swaps, corridor variance, and variance of transformed prices), but our main contribution to this program is to extend the management of path-dependent risks to include sub/superreplication of variance *options*.

In [8], Carr-Lee took a model-free approach to the *exact* pricing and replication of general functions of realized variance, but that paper made an independence assumption on the volatility process (while carefully immunizing its pricing and trading methodology, to first order, against violations of the independence assumption). Here we do *not* assume independence; instead we work in the very general setting of an arbitrary continuous semimartingale price process. Such minimal assumptions do not determine uniquely the prices of variance options; but we will show that they do imply bounds on those prices, enforceable by superreplicating and subreplicating portfolio strategies.

In [17], Dupire found lower bounds and subhedges for spot-starting variance options. Here we extend those bounds and subhedges to *forward*-starting options. Moreover, we find *upper* bounds and *super*hedges for spot-starting and forward-starting options.

1.2 Assumptions

Let Y denote the price of a share of the underlying asset, together with all reinvested dividends. Assume that Y is a positive continuous semimartingale on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfying the usual conditions. We interpret P as physical probability measure, so Y is not necessarily a local P-martingale. Except in Section 5, our proofs will have no need of risk-neutral measure.

Fix some $T > 0$. If we say that [a claim on] some $\mathcal{F}_T$-measurable payoff A is *tradeable*, we mean that at times $t \leq T$, it may be bought and sold frictionlessly at some finite price, denoted by $\mathbb{V}_t A$. Assume the absence of arbitrage in the class of predictable self-financing semi-static strategies in the tradeable payoffs. By semi-static we mean strategies which trade at most once in $(0, T)$.

We assume the existence of the following tradeables: the underlying asset, with payoff Y_T and price $\mathbb{V}_t Y_T = Y_t$; and a bond, with payoff 1 and price $\mathbb{V}_t 1 = 1$. Depending on the context, we may add other tradeables. We view the tradeables as the "basic assets" from which we will synthesize contracts on realized variance.

Let $X_t := \varphi(Y_t)$ where φ is the difference of convex functions, for example $\varphi(y) = \log y$. Let $\langle X \rangle$ and $\langle Y \rangle$ denote the quadratic variation of X and Y respectively, with the convention that quadratic variation at time 0 is zero.

We will model-independently (super/sub)-*replicate* claims written on $\langle X \rangle$ – including options on forward-starting variance $\langle X \rangle_T - \langle X \rangle_\theta$ for any constant $\theta \in [0, T)$ – using predictable self-financing strategies which dynamically trade Y and statically hold European-style claims on Y_T and Y_θ.

Superreplication implies upper bounds on variance option prices, by the standard logic that shorting an option bid above the upper bound, and going long the superreplicating strategy, produces an arbitrage. However, our notion of a superreplicating strategy does not promise any notion of tameness or admissibility, so to be careful and complete, we show moreover that our strategies satisfy natural margin constraints at all times $[0, T]$. We do likewise for the subreplication strategies which give lower bounds.

To summarize, we impose consistency among the prices of the tradeable basic assets by assuming the absence of semi-static arbitrage among them. We impose consistency between each variance option and its super/sub-replicating portfolios of tradeable basic assets (including Y, which we trade fully dynamically), by assuming, moreover, the absence of dynamic arbitrage satisfying natural margin constraints.

Remark 1.1. The constant bond price assumption does not restrict us to zero interest rates, because we regard all prices in this paper (except Y' and Z' in this Remark) to be denominated in units of the bond. If in practice we wish to use a different unit of denomination – let us say the "dollar" – then we have the following conversions. Letting Y' denote the dollar-denominated share price and Z' denote the dollar-denominated bond price (for a bond that pays 1 dollar at maturity T), we have the bond-denominated share price $Y_t = Y'_t/Z'_t$, and the bond-denominated bond price $Z_t = Z'_t/Z'_t = 1$.

In practice, variance contracts are written on dollar-denominated logarithmic variance, not on bond-denominated logarithmic variance; but under arbitrary deterministic (including non-constant) interest rates given by a short rate process r_t, the two notions of variance are identical. Indeed, we have $Z'_t = \exp(-\int_t^T r_t \mathrm{d}t)$, hence the bond-denominated share price $Y = Y'/Z'$ has logarithmic quadratic variation

$$\Big\langle \log Y_\cdot \Big\rangle = \Big\langle \log(Y'_\cdot/Z'_\cdot) \Big\rangle = \Big\langle \log Y'_\cdot + \int_\cdot^T r_t \mathrm{d}t \Big\rangle = \Big\langle \log Y'_\cdot \Big\rangle \tag{1.1}$$

because the dt integral has finite variation. Therefore a T-maturity contract on any function of $\langle \log Y' \rangle_T$ is identical to a T-maturity contract on that function of $\langle \log Y \rangle_T$. This holds true even if the former contract pays in dollars while the latter contract pays in bonds, because 1 time-T dollar equals 1 time-T bond. In conclusion, our constant bond price assumption entails no loss of generality relative to arbitrary non-random interest rates.

Note that the irrelevance of interest rates shown in (1.1) contrasts to the cases of lookback and barrier options [6, 20] where more care was required, because $\max_t(Y'_t/Z'_t)$ does not equal $\max_t(Y'_t)$.

2 Model-free replication

By Meyer-Itô we have

$$\mathrm{d}X_t = \varphi_y(Y_t)\mathrm{d}Y_t + \frac{1}{2}\int_{\mathbb{R}} L_t^a \varphi_{yy}(\mathrm{d}a), \tag{2.1}$$

where φ_y denotes the left-hand derivative of φ, and φ_{yy} denotes the second derivative in the sense of distributions, and L^a denotes the local time of X at a. Since φ_{yy} is the difference of two positive measures and L^a is increasing, the local time term has finite variation. Therefore

$$\mathrm{d}\langle X \rangle_t = \varphi_y^2(Y_t)\mathrm{d}\langle Y \rangle_t. \tag{2.2}$$

Let $h(y, q)$ be $C^{2,1}$. Then for all t, by Itô's rule,

$$\begin{aligned} h(Y_t, \langle X \rangle_t) &= h(Y_0, 0) + \int_0^t h_y \mathrm{d}Y_s + \int_0^t \frac{1}{2} h_{yy} \mathrm{d}\langle Y \rangle_s + \int_0^t h_q \mathrm{d}\langle X \rangle_s \\ &= h(Y_0, 0) + \int_0^t h_y \mathrm{d}Y_s + \int_0^t \Big(\frac{1}{2} h_{yy} + \varphi_y^2 h_q\Big)\mathrm{d}\langle Y \rangle_s \end{aligned}$$

where subscripts on h denote partial differentiation.

More generally, we will need Propositions 2.1 and 2.2 which are slight extensions of Bick [3] to a larger class of stopping times.

Proposition 2.1. *Let U be an open set with $(Y_0, 0) \in U \subseteq \mathbb{R}^2$. Let h be $C^{2,1}$ on U and continuous on $\bar{U}$. Then for all T and all stopping times $\tau \leq \inf\{t : (Y_t, \langle X \rangle_t) \notin U\}$,*

$$h(Y_{T\wedge\tau}, \langle X \rangle_{T\wedge\tau}) = h(Y_0, 0) + \int_0^{T\wedge\tau} h_y \mathrm{d}Y_s + \int_0^{T\wedge\tau} \left(\frac{1}{2} h_{yy} + \varphi_y^2 h_q\right) \mathrm{d}\langle Y \rangle_s. \tag{2.3}$$

If moreover $\varphi_y(y) > 0$ for all y then

$$h(Y_{T\wedge\tau}, \langle X \rangle_{T\wedge\tau}) = h(Y_0, 0) + \int_0^{T\wedge\tau} h_y \mathrm{d}Y_s + \int_0^{T\wedge\tau} \left(\frac{1}{2} \frac{h_{yy}}{\varphi_y^2} + h_q\right) \mathrm{d}\langle X \rangle_s. \tag{2.4}$$

In the integrands, the the h_y, h_{yy}, and h_q are evaluated at $(Y_s, \langle X \rangle_s)$, and φ_y is evaluated at Y_s.

Proof. Let $\tau_n := \inf\{t :$ there exist $(y, q) \in (\mathbb{R} \times \mathbb{R}^+) \setminus U$ such that $|Y_t - y| + |\langle X \rangle_t - q| < 1/n\}$. Itô's rule applies to the stopped process $(Y_{t\wedge\tau_n}, X_{t\wedge\tau_n})$, so for all T

$$h(Y_{T\wedge\tau_n}, \langle X \rangle_{T\wedge\tau_n}) = h(Y_0, 0) + \int_0^{T\wedge\tau_n} h_y \mathrm{d}Y_s + \int_0^{T\wedge\tau_n} \left(\frac{1}{2} h_{yy} + \varphi_y^2 h_q\right) \mathrm{d}\langle Y \rangle_s$$

Now let $n \to \infty$. By continuity of Y and h, we have (2.3). By (2.2) we have (2.4). $\square$

2.1 Vanishing $\langle X \rangle$ integral

To proceed from (2.4), we can choose h to make the $\mathrm{d}\langle X \rangle$ integral vanish. Then the $\langle X \rangle$-dependent LHS can be created, using the trading strategy in bonds and shares given by the two remaining terms on the RHS.

Proposition 2.2. *Under the hypotheses of Proposition 2.1, we assume, moreover, that*

$$\frac{1}{2} \frac{h_{yy}}{\varphi_y^2} + h_q = 0 \qquad \text{for } (y, q) \in U. \tag{2.5}$$

Then for any T the payoff $h(Y_{T\wedge\tau}, \langle X \rangle_{T\wedge\tau})$ can be replicated by holding at each time $t \leq T \wedge \tau$

$$\begin{aligned} h_y(Y_t, \langle X \rangle_t) &\quad \text{shares} \\ h(Y_t, \langle X \rangle_t) - Y_t h_y(Y_t, \langle X \rangle_t) &\quad \text{bonds.} \end{aligned} \tag{2.6}$$

The replicating portfolio has time-0 value $h(Y_0, 0)$.

Proof. With $Z_t := 1$ denoting the bond price, the portfolio's value at any time $t \leq \tau \wedge T$ is

$$\begin{aligned} V_t &:= (h(Y_t, \langle X \rangle_t) - Y_t h_y(Y_t, \langle X \rangle_t)) \times Z_t + h_y(Y_t, \langle X \rangle_t) \times Y_t \\ &= h(Y_t, \langle X \rangle_t) \\ &= h(Y_0, 0) + \int_0^t h_y(Y_s, \langle X \rangle_s) \mathrm{d}Y_s + 0, \end{aligned}$$

by (2.4) and (2.5). Therefore

$$\mathrm{d}V_t = h_y(Y_t, \langle X\rangle_t)\mathrm{d}Y_t + (h(Y_t, \langle X\rangle_t) - Y_t h_y(Y_t, \langle X\rangle_t))\mathrm{d}Z_t$$

which is by definition the self-financing condition. □

Remark 2.3. Equation (2.5) is a backward Kolmogorov PDE, with quadratic variation playing the role of time. We return to this point in Remark 2.12.

We will need the following slight extension of Bick [3], who has the case that f is a put payoff. Let $\mathbb{R}_+ = (0, \infty)$ denote the positive reals.

Proposition 2.4 (Claims on price when variance reaches a barrier)**.** *Let $X_t = \log(Y_t/Y_0)$.*

Let τ be the first passage time of $\langle X\rangle$ to level Q.

For any $y > 0$, any $v \geq 0$, and any continuous $f : \mathbb{R}_+ \to \mathbb{R}$ such that $|f(e^z)| \leq F(e^{|z|})$ for some polynomial F and all $z \in \mathbb{R}$, let

$$BS(y, v; f) := \begin{cases} \int_{-\infty}^{\infty} f(ye^z)\, \frac{1}{\sqrt{2\pi v}} \exp\left[-\frac{(z+v/2)^2}{2v}\right] \mathrm{d}z & \text{if } v > 0 \\ f(y) & \text{if } v = 0 \end{cases} \tag{2.7}$$

and let BS_y denote its y-derivative. Then the strategy of holding at each time $t \leq T \wedge \tau$

$$\begin{array}{rl} BS_y(Y_t, Q - \langle X\rangle_t; f) & \textit{shares} \\ BS(Y_t, Q - \langle X\rangle_t; f) - Y_t BS_y(Y_t, Q - \langle X\rangle_t; f) & \textit{bonds} \end{array} \tag{2.8}$$

replicates the time-$(T \wedge \tau)$ payoff

$$f(Y_\tau)\mathbb{I}_{\tau\leq T} + BS(Y_T, Q - \langle X\rangle_T; f)\mathbb{I}_{\tau>T} \tag{2.9}$$

The replicating portfolio has time-0 value $BS(Y_0, Q; f)$.

Proof. Let $h(y, q) := BS(y, Q - q; f)$. Directly verify that $\frac{1}{2}y^2 h_{yy} + h_q = 0$ on $U = \mathbb{R}_+ \times (-\infty, Q)$ and h is continuous on $\bar{U}$; then apply Proposition 2.2. □

Remark 2.5. No longer purely theoretical, similar contracts, of perpetual type, have been traded by Société Générale [2], and described as "timer" options.

Remark 2.6. Intuitively the $BS(y, v; f)$ function gives the value of the payoff $f(Y_\tau)$, which is computed by the *Black-Scholes formula* with dimensionless volatility parameter $Q - q$. We say *dimensionless* to emphasize that this parameter represents a total "unannualized" variance until expiration, not variance per unit time. Proposition 2.4 shows that starting with bonds and shares of total value $BS(Y_0, 0)$, and at each time t "delta-hedging at dimensionless BS volatility $Q - \langle X\rangle_t$" will produce $f(Y_\tau)$ if and when $\langle X\rangle$ reaches Q.

Corollary 2.7 (How to make profit/loss if realized volatility $\leq / \geq$ implied BS volatility)**.** *Under the conditions of Proposition 2.4, further assume convexity of f, which therefore has a left derivative f'. Then strategy* (2.8)*, extended to times $t > \tau$ by holding at all $t \in (T \wedge \tau, T]$ the* static *portfolio*

$$\begin{array}{rl} f'(Y_\tau) & \textit{shares} \\ f(Y_\tau) - f'(Y_\tau)Y_\tau & \textit{bonds} \end{array} \tag{2.10}$$

subreplicates (resp. superreplicates) $f(Y_T)$ if $\tau \leq T$ (resp. $\tau \geq T$).

Proof. If $\tau \geq T$, then by (2.9) the portfolio has time-T value $BS(Y_T, Q - \langle X \rangle_T; f) \geq f(Y_T)$ by convexity. If $\tau \leq T$, then the portfolio has time-T value $f(Y_\tau) + f'(Y_\tau)(Y_T - Y_\tau) \leq f(Y_T)$. $\square$

Remark 2.8. Suppose a contract paying $f(Y_T)$ has time-0 value $BS(Y_0, Q, f)$; for example, this holds if f is a call, and Q is its BS implied volatility. Then Corollary 2.7 implies immediately that going long the $f(Y_T)$ contract and short the portfolio (2.8,2.10) is a zero-initial-cost strategy whose time-T value is nonnegative if $\tau \leq T$, nonpositive if $\tau \geq T$.

In addition to variance-barrier contracts, we also replicate price-barrier contracts.

Proposition 2.9 (Claims on variance until price reaches a down-barrier)**.** *Let $X_t = \log(Y_t/Y_0)$.*

Let τ be the first passage time of Y to a barrier $b \in (0, Y_0)$.

For any continuous $g : \mathbb{R} \to \mathbb{R}$ such that $|g| \leq G$ for some polynomial G, let

$$BP(y, q; b, g) := \begin{cases} \int_0^\infty g(q+z) \, \frac{|\log(b/y)|}{\sqrt{2\pi z^3}} \exp\left[-\frac{(\log(b/y)+z/2)^2}{2z} \right] dz & \textit{if } y \neq b \\ g(q) & \textit{if } y = b \end{cases} \tag{2.11}$$

and let BP_y denote its y-derivative. Then the strategy of holding at each time $t \leq T \wedge \tau$

$$\begin{array}{rl} BP_y(Y_t, \langle X \rangle_t; b, g) & \textit{shares} \\ BP(Y_t, \langle X \rangle_t; b, g) - Y_t BP_y(Y_t, \langle X \rangle_t; b, g) & \textit{bonds} \end{array} \tag{2.12}$$

replicates the time-$(T \wedge \tau)$ payoff

$$g(\langle X \rangle_\tau)\mathbb{I}_{\tau \leq T} + BP(Y_T, \langle X \rangle_T; b, g)\mathbb{I}_{\tau > T}. \tag{2.13}$$

If g is monotonically increasing, then the strategy superreplicates $g(\langle X \rangle_{\tau \wedge T})$.

*The replicating portfolio has time-*0 *value $BP(Y_0, 0; b, g)$.*

Proof. Let $h(y, q) := BP(y, q; b, g)$. Then directly verify that $\frac{1}{2}y^2 h_{yy} + h_q = 0$ on $U = (b, \infty) \times \mathbb{R}$, and that h is continuous on $\bar{U}$.

Proposition 2.2 implies replication of (2.13), and superreplication of $g(\langle X \rangle_{\tau \wedge T})$ follows because $BP(y, q; b, g) \geq g(q)$ for increasing g. $\square$

We chose the notation BP for "Brownian passage," to be explained in Remark 2.11; but first we give the analogue of Proposition 2.9 for a double barrier – more precisely, for claims on the variance from time 0 until the price's exit time from a finite interval.

Proposition 2.10 (Claims on variance to an exit time)**.** *Let* $X_t = \log(Y_t/Y_0)$.

Let $0 < b_d < Y_0 < b_u$, *and let* τ *be the exit time of* Y *from the interval* (b_d, b_u).

For any continuous $g : \mathbb{R} \to \mathbb{R}$ *such that* $|g| \le G$ *for some polynomial* G, *let*

$$BP(y, q; b_d, b_u, g) := \begin{cases} \int_0^\infty g(q+z)p(\log(y/b_d), \log(y/b_u), z)\mathrm{d}z & \text{if } b_d < y < b_u \\ g(q) & \text{otherwise} \end{cases} \tag{2.14}$$

$$p(\beta_d, \beta_u, z) := e^{-z/8}[e^{-\beta_d/2}\,\psi(\beta_u, \beta_u - \beta_d, z) + e^{-\beta_u/2}\,\psi(-\beta_d, \beta_u - \beta_d, z)] \tag{2.15}$$

$$\psi(r, R, z) := \sum_{k=-\infty}^{\infty} \frac{R - r + 2kR}{\sqrt{2\pi} z^{3/2}} e^{-(R-r+2kR)^2/(2z)}. \tag{2.16}$$

Then the strategy of holding at each time $t \le T \wedge \tau$

$$\begin{aligned} BP_y(Y_t, \langle X\rangle_t; b_d, b_u, g) & \quad \textit{shares} \\ BP(Y_t, \langle X\rangle_t; b_d, b_u, g) - Y_t BP_y(Y_t, \langle X\rangle_t; b_d, b_u, g) & \quad \textit{bonds} \end{aligned} \tag{2.17}$$

replicates the time-$(T \wedge \tau)$ *payoff*

$$g(\langle X\rangle_\tau)\mathbb{I}_{\tau \le T} + BP(Y_T, \langle X\rangle_T; b_d, b_u, g)\mathbb{I}_{\tau > T}. \tag{2.18}$$

If g *is monotonically increasing, then the strategy superreplicates* $g(\langle X\rangle_{\tau\wedge T})$.

*The replicating portfolio has time-*0 *value* $BP(Y_0, 0; b_d, b_u, g)$.

Proof. Let $h(y, q) := BP(y, q; b_u, b_d, g)$. Then directly verify that $\frac{1}{2}y^2 h_{yy} + h_q = 0$ on $U = (b_d, b_u) \times \mathbb{R}$, and that h is continuous on $\bar{U}$.

Proposition 2.2 implies replication of (2.18), and superreplication of $g(\langle X\rangle_{\tau\wedge T})$ follows because $BP(y, q; b, g) \ge g(q)$ for increasing g. □

Remark 2.11. By Borodin-Salminen [4] Formula 2.3.0.2, the p function is the density of the exit time of drift $-1/2$ Brownian motion from the interval (β_d, β_u). Intuitively, the BP function gives the expected value of g at this "Brownian Passage" time.

Remark 2.12. The formulas (2.7) and (2.11) and (2.14)-(2.16) can be understood via *time change*. We have

$$\mathrm{d}X_t = \frac{1}{Y_t}\mathrm{d}Y_t - \frac{1}{2Y_t^2}\mathrm{d}\langle Y\rangle_t = \frac{1}{Y_t}\mathrm{d}Y_t - \frac{1}{2}\mathrm{d}\langle X\rangle_t.$$

Under risk-neutral measure the underlying Y is a continuous local martingale, hence so is M where

$$M_t := \int_0^t \frac{1}{Y_s}\mathrm{d}Y_s = X_t + \frac{1}{2}\langle X\rangle_t.$$

By Dambis/Dubins-Schwarz ([12, 15]; henceforth DDS), there exists (on an enlarged probability space if needed) a Brownian motion W with $W_{\langle X\rangle_t} = M_t$ for all $t \le T$. So $X_t = W_{\langle X\rangle_t} - \frac{1}{2}\langle X\rangle_t$ and hence $Y_t = G_{\langle X\rangle_t}$ where $G_u := Y_0 \exp(W_u - u/2)$. Therefore, with respect to *business time* $\langle X\rangle_t$, the underlying Y is driftless geometric Brownian motion. So, even in our completely general continuous semimartingale setting, Black-Scholes prevails under the stochastic clock which identifies time with quadratic variation.

Forde [18] independently notes the relevance of DDS to pricing variance-to-a-barrier claims. Dupire [17] uses DDS to cast volatility derivatives into the framework of the Skorokhod embedding problem. Our hedging proofs do not rely on DDS – indeed they do not even rely on the existence of a risk-neutral measure – but the time change perspective adds insight.

In the case of a call payoff, we find an easily computable formula for BP.

Proposition 2.13 (Fourier representation for calls on variance until an exit time)**.** *For a call payoff $g(q) = (q - Q)^+$, the function BP of Proposition 2.10 for $y \in (b_d, b_u)$ has the representation*

$$BP = \int_{-\infty-\alpha i}^{\infty-\alpha i} \frac{\sqrt{y/b_u}\sinh(\log(b_d/y)\sqrt{1/4-2iz}) - \sqrt{y/b_d}\sinh(\log(b_u/y)\sqrt{1/4-2iz})}{2\pi z^2 e^{i(Q-q)z}\sinh(\log(b_u/b_d)\sqrt{1/4-2iz})}\mathrm{d}z$$

where $\alpha > 0$; any such α gives the same value for the integral.

Remark 2.14. Abusing notation, we will write $BP(y, q; b_d, b_u, Q)$ to mean $BP(y, q; b_d, b_u, g)$, where $g(q) := (q - Q)^+$.

Proof. Combine Borodin-Salminen [4] Formula 2.3.0.1, which gives the Laplace transform of p, with Lee [22] Theorem 5.1 (for the "G_2" payoff), which obtains BP from that transform. □

Proposition 2.15 (Properties of BP for a call)**.** *For any $q \geq 0$, $Q \geq 0$, $y > 0$,*

$$BP(y, q; b_d, b_u, Q) - BP(y, 0; b_d, b_u, Q) \geq (q - Q)^+. \tag{2.19}$$

For $q = Q = 0$ and $b_d < y < b_u$,

$$BP(y, 0; b_d, b_u, 0) = 2\log(y/b_u) - 2\frac{\log(b_u/b_d)}{b_u - b_d}(y - b_u). \tag{2.20}$$

Proof. For $y \notin (b_d, b_u)$, inequality (2.19) clearly holds. For $y \in (b_d, b_u)$,

$$\begin{aligned} BP(y, q; b_d, b_u, Q) &= \int_0^\infty (q + z - Q)^+ p(\log(y/b_d), \log(y/b_u), z)\, \mathrm{d}z \\ &\geq \int_0^\infty [(q - Q)^+ + (z - Q)^+] p(\log(y/b_d), \log(y/b_u), z)\, \mathrm{d}z \\ &= (q - Q)^+ + BP(y, 0; b_d, b_u, Q), \end{aligned}$$

and (2.19) again holds.

Equation (2.20) holds because each side equals the expectation of the exit time of drift $-1/2$ Brownian motion from the interval $(\log(b_d/y), \log(b_u/y))$: the LHS by definition of BP and Remark 2.11, and the RHS by the usual method of extracting an expectation from the known Laplace transform of the exit time density. □

2.2 Nonvanishing $\langle X\rangle$ integral

An alternative way to proceed from (2.4) is to generate the quadratic variation dependence in the $\mathrm{d}\langle X\rangle_s$ integral, instead of in $h(Y_T, \langle X\rangle_T)$. In particular, by making $h(x,q)$ depend on x alone, the quadratic-variation-dependent integral on the RHS of (2.3) can be created from the LHS (which has thereby become simply a European claim) minus the bonds and shares terms on the RHS.

Proposition 2.16 ((Sub)replication of forward-starting weighted variance of $\varphi(Y)$)**.** *Let the* weight $w : \mathbb{R}_+ \to [0,\infty)$ *be a Borel function and let* τ *be a stopping time. Let* $\lambda : \mathbb{R}_+ \to \mathbb{R}$ *be a difference of convex functions, let* λ_y *denote its left-hand derivative, and assume that its second derivative in the distributional sense has a (signed) density, denoted* λ_{yy}*, which satisfies for all* $y \in \mathbb{R}_+$

$$\lambda_{yy}(y) \le 2\varphi_y^2(y)w(y). \tag{2.21}$$

If claims on $\lambda(Y_T)$ *and* $\lambda(Y_{\tau\wedge T})$ *are tradeable, then the strategy of holding at each time* $t \in (0, \tau\wedge T]$

$$\begin{array}{ll} 1 & \textit{claim on } \lambda(Y_T) \\ 1 & \textit{claim on } -\lambda(Y_{\tau\wedge T}) \end{array} \tag{2.22a}$$

and holding at each time $t \in (\tau\wedge T, T]$

$$\begin{array}{rl} 1 & \textit{claim on } \lambda(Y_T) \\ -\lambda_y(Y_t) & \textit{shares} \\ -\lambda(Y_{\tau\wedge T}) - \int_{\tau\wedge T}^{t} \lambda_y(Y_s)\mathrm{d}Y_s + Y_t\lambda_y(Y_t) & \textit{bonds,} \end{array} \tag{2.22b}$$

subreplicates the forward-starting weighted variance *of* $X = \varphi(Y)$*, defined by*

$$\langle X\rangle^w_{\tau,T} := \int_{\tau\wedge T}^{T} w(Y_s)\ \mathrm{d}\langle X\rangle_s.$$

*The subreplicating portfolio has time-*0 *value* $\mathbb{V}_0\lambda(Y_T) - \mathbb{V}_0\lambda(Y_{\tau\wedge T})$*.*

If equality holds in (2.21) *then the strategy replicates* $\langle X\rangle^w_{\tau,T}$ *exactly.*

Proof. The strategy clearly self-finances and has the claimed time-0 value. By Meyer-Itô

$$\lambda(Y_T) = \lambda(Y_0) + \int_0^T \lambda_y(Y_s)\mathrm{d}Y_s + \int_0^T \frac{1}{2}\lambda_{yy}(Y_s)\ \mathrm{d}\langle Y\rangle_s$$

and, by Meyer-Itô applied to the stopped process $Y_{t\wedge\tau}$,

$$\lambda(Y_{\tau\wedge T}) = \lambda(Y_0) + \int_0^{\tau\wedge T} \lambda_y(Y_s)\mathrm{d}Y_s + \int_0^{\tau\wedge T} \frac{1}{2}\lambda_{yy}(Y_s)\ \mathrm{d}\langle Y\rangle_s.$$

Taking the difference,

$$\lambda(Y_T) = \lambda(Y_{\tau\wedge T}) + \int_{\tau\wedge T}^{T} \lambda_y(Y_s)\mathrm{d}Y_s + \int_{\tau\wedge T}^{T} \frac{1}{2}\lambda_{yy}(Y_s)\;\mathrm{d}\langle Y\rangle_s \tag{2.23}$$

$$\leq \lambda(Y_{\tau\wedge T}) + \int_{\tau\wedge T}^{T} \lambda_y(Y_s)\mathrm{d}Y_s + \int_{\tau\wedge T}^{T} \varphi_y^2(Y_s)w(Y_s)\;\mathrm{d}\langle Y\rangle_s \tag{2.24}$$

$$= \lambda(Y_{\tau\wedge T}) + \int_{\tau\wedge T}^{T} \lambda_y(Y_s)\mathrm{d}Y_s + \int_{\tau\wedge T}^{T} w(Y_s)\;\mathrm{d}\langle X\rangle_s, \tag{2.25}$$

hence

$$\lambda(Y_T) - \lambda(Y_{\tau\wedge T}) - \int_{\tau\wedge T}^{T} \lambda_y(Y_s)\mathrm{d}Y_s \leq \langle X\rangle_{\tau,T}^{w}, \tag{2.26}$$

which proves subreplication of $\langle X\rangle_{\tau,T}^{w}$. If equality holds in (2.21), then it holds in (2.24) and (2.26), which proves exact replication. $\square$

Remark 2.17. The strategy (2.22) can be described as delta-hedging the λ claim "at zero vol," because its share holding $-\lambda_y(Y_t)$ is identical to $-BS_y(Y_t, v; \lambda)\big|_{v=0}$.

Proposition 2.16 includes as special cases the classical results on replication of various flavors of variance swaps.

Example 2.18 (Replication of forward-starting variance of $\log Y$). Consider the weight function $w(y) := 1$. If

$$\lambda(y) = A_1 y + A_0 - 2\log y \tag{2.27}$$

where A_0, A_1 are arbitrary constants, then (2.21) holds with equality, so if claims on $\lambda(Y_T)$ and $\lambda(Y_{\tau\wedge T})$ are tradeable, then the strategy (2.22) replicates $\langle X\rangle_T - \langle X\rangle_{\tau\wedge T}$, where $X = \log Y$. This recovers the known strategy (Neuberger [23], Dupire [16], Carr-Madan [9], Derman et al [14]) of using a log contract to replicate logarithmic quadratic variation.

Example 2.19 (Replication of forward-starting corridor variance of $\varphi(Y)$). Let the *corridor* C be a Borel set and let the weight function be the indicator $w(y) := \mathbb{I}(y \in C)$. If λ is convex and $\lambda_{yy} = 2\varphi_y^2$ in C and $\lambda_{yy} = 0$ outside of C, then (2.22) replicates *corridor variance* [9]

$$\int_{\tau\wedge T}^{T} \mathbb{I}(Y_s \in C)\;\mathrm{d}\langle X\rangle_s.$$

The replicating portfolio has time-0 value $\mathbb{V}_0\lambda(Y_T) - \mathbb{V}_0\lambda(Y_{\tau\wedge T})$.

Taking $C = \mathbb{R}^+$ produces the full forward-starting variance of $\varphi(Y)$.

Example 2.20 (Replication of forward-starting *gamma swap* on $\varphi(Y)$). Let the weight function be $w(y) := ay$ where a is a constant, typically $a = 1/Y_0$. If λ is convex and $\lambda_{yy}(y) = 2a\varphi_y^2(y)y$ then (2.22) replicates the *gamma swap* payout

$$\int_{\tau\wedge T}^{T} aY_s\;\mathrm{d}\langle X\rangle_s.$$

In particular, for the usual logarithmic case $\varphi(y) = \log(y)$, the ODE $\lambda_{yy}(y) = 2a/y$ is solved by

$$\lambda(y) = ay \log y + A_1 y + A_0$$

for arbitrary constants A_0 and A_1. The replicating portfolio has time-0 value $\mathbb{V}_0\lambda(Y_T) - \mathbb{V}_0\lambda(Y_{\tau\wedge T})$.

The final example we designate as a Corollary, due to its relevance to one of our main goals – subreplicating a forward-starting variance call.

Corollary 2.21 (Subreplication of forward-starting variance of $\log Y$)**.** *Let $\lambda : \mathbb{R}_+ \to \mathbb{R}$ be a difference of convex functions, let λ_y denote its left-hand derivative, and assume that its second derivative in the distributional sense has a density, denoted λ_{yy}, which satisfies for all $y \in \mathbb{R}_+$*

$$\lambda_{yy}(y) \leq 2/y^2. \tag{2.28}$$

Let τ be a stopping time. If claims on $\lambda(Y_{\tau\wedge T})$ and $\lambda(Y_T)$ are tradeable, then

$$\lambda(Y_T) - \lambda(Y_{\tau\wedge T}) - \int_{\tau\wedge T}^{T} \lambda_y(Y_s)\mathrm{d}Y_s \leq \langle X\rangle_{\tau,T} = \langle X\rangle_T - \langle X\rangle_{\tau\wedge T} \tag{2.29}$$

and the strategy (2.22) *subreplicates $\langle X\rangle_T - \langle X\rangle_{\tau\wedge T}$, where $X = \log Y$.*

Proof. Take $w = 1$ and $\varphi(y) = \log y$ in Proposition 2.16. □

3 Variance call: Lower bound

In this section let $X_t := \log(Y_t/Y_0)$. Let $Q \geq 0$ and $T > 0$.

3.1 Spot-starting variance call: Dupire's subreplication

Consider a variance call with strike Q and expiry T.

Dupire's [17] subreplication strategy has the following intuition. Let λ be convex and satisfy the hypotheses of Corollary 2.21.

If and when $\langle X\rangle$ hits Q prior to time T, we need to subreplicate a variance swap, so we want to have a claim on $\lambda(Y_T)$ plus a claim on $-\lambda(Y_{\tau_Q})$. The former is a European claim, and the latter is synthesized by a bond-and-shares strategy, according to Proposition 2.4.

If $\langle X\rangle$ does not hit Q prior to time T, then our time-T portfolio is $\lambda(Y_T)$ minus a claim on $\lambda(Y_{\tau_Q})$. By convexity of λ, the latter has greater value than the former. So the portfolio value is negative, as desired.

Proposition 3.1 (Dupire [17])**.** *Consider a variance call which pays*

$$(\langle X\rangle_T - Q)^+.$$

Assume λ is convex and satisfies the hypotheses of Corollary 2.21. Define

$$N_t := \begin{cases} -BS_y(Y_t, Q - \langle X\rangle_t; \lambda) & \text{if } t \le \tau_Q \\ -\lambda_y(Y_t) & \text{if } t > \tau_Q. \end{cases}$$

Then for any T the following strategy subreplicates the variance call: at each time $t < T$ hold

$$\begin{aligned} 1 \quad & \textit{claim on } \lambda(Y_T) \\ N_t \quad & \textit{shares} \\ -BS(Y_0, Q; \lambda) + \int_0^t N_s \mathrm{d}Y_s - N_t Y_t \quad & \textit{bonds.} \end{aligned} \tag{3.1}$$

*The subreplicating portfolio has time-*0 *value $-BS(Y_0, Q; \lambda) + \mathbb{V}_0\lambda(Y_T)$.*

Proof. The strategy clearly self-finances and has the claimed time-0 value.

If $\tau_Q \le T$, then the time-T portfolio value is

$$\begin{aligned} -BS(Y_0, Q; \lambda) + \int_0^{\tau_Q} N_s \mathrm{d}Y_s + \int_{\tau_Q}^T N_s \mathrm{d}Y_s + \lambda(Y_T) &= -\lambda(Y_{\tau_Q}) + \int_{\tau_Q}^T N_s \mathrm{d}Y_s + \lambda(Y_T) \\ &\le \langle X\rangle_T - \langle X\rangle_{\tau_Q} = (\langle X\rangle_T - Q)^+ \end{aligned}$$

by Proposition 2.4 and Corollary 2.21. If $\tau_Q > T$, then the time-T portfolio value is

$$-BS(Y_0, Q; \lambda) + \int_0^T N_s \mathrm{d}Y_s + \lambda(Y_T) = -BS(Y_T, Q - \langle X\rangle_T; \lambda) + BS(Y_T, 0; \lambda) \tag{3.2}$$

$$\le 0 = (\langle X\rangle_T - Q)^+. \tag{3.3}$$

Equality (3.2) holds by Proposition 2.4. Inequality (3.3) holds because the convexity of λ implies that BS is increasing in its second argument. □

Remark 3.2. Dupire chooses λ to maximize the lower bound, as follows. Let

$$\mathrm{van}_K(y) := \begin{cases} (y - K)^+ & \text{if } K \ge Y_0 \\ (K - y)^+ & \text{if } K < Y_0 \end{cases} \tag{3.4}$$

denote the payoff function of the OTM vanilla option at strike K, and assume tradeability of $\mathrm{van}_K(Y_T)$ for all K.

Define the time-0 dimensionless Black-Scholes *implied volatility* for an underlying Y, strike K, and expiry T, to be the unique $I_0(K, T)$ such that

$$BS(Y_0, I_0(K, T); \mathrm{van}_K) = \mathbb{V}_0 \mathrm{van}_K(Y_T). \tag{3.5}$$

Then we may rewrite the lower bound as

$$\begin{aligned}\mathbb{V}_0\lambda(Y_T) - BS(Y_0, Q; \lambda) &= \int_0^\infty \lambda_{yy}(K)[\mathbb{V}_0\mathrm{van}_K - BS(Y_0, Q; \mathrm{van}_K)]\mathrm{d}K \\ &= \int_0^\infty \lambda_{yy}(K)[BS(Y_0, I_0(K,T); \mathrm{van}_K) - BS(Y_0, Q; \mathrm{van}_K)]\mathrm{d}K.\end{aligned} \tag{3.6}$$

Under the constraint $0 \leq y^2\lambda_{yy}(y) \leq 2$, the optimal λ consists of $2/K^2\mathrm{d}K$ OTM vanilla payoffs at all K where the dimensionless BS implied volatility $I_0(K,T)$ exceeds Q:

$$\lambda(y) = \int_{\{K:I_0(K,T)>Q\}} \frac{2}{K^2}\mathrm{van}_K(y)\ \mathrm{d}K.$$

If a variance call is offered below its lower bound, then short the $\lambda(Y_T)$ claim and borrow $BS(Y_0, Q; \lambda)$, the λ claim's Black-Scholes valuation using dimensionless volatility Q. Use the proceeds to buy the variance call, for a net credit. Then dynamically trade shares to lock in this credit.

3.2 Forward-starting variance call: Subreplication

Let the forward-start date be a constant $\theta \in [0, T)$.

Proposition 3.3. *Consider a forward-starting variance call which pays*

$$(\langle X\rangle_T - \langle X\rangle_\theta - Q)^+.$$

Assume that λ is convex and satisfies the hypotheses of Corollary 2.21.

Let $\tau_Q := \inf\{t : \langle X\rangle_t - \langle X\rangle_\theta \geq Q\}$ and

$$N_t := \begin{cases} -BS_y(Y_t, Q - (\langle X\rangle_t - \langle X\rangle_\theta); \lambda) & \text{if } \theta \leq t \leq \tau_Q \\ -\lambda_y(Y_t) & \text{if } t > \tau_Q. \end{cases}$$

If claims on $BS(Y_\theta, Q; \lambda)$ and $\lambda(Y_T)$ are tradeable, then the following strategy subreplicates the forward-starting variance call. At each time $t \in [0, \theta]$ hold

$$\begin{aligned} 1 &\quad \textit{claim on } \lambda(Y_T) \\ 1 &\quad \textit{claim on } -BS(Y_\theta, Q; \lambda). \end{aligned} \tag{3.7a}$$

At each time $t \in (\theta, T)$ hold

$$\begin{aligned} 1 &\quad \textit{claim on } \lambda(Y_T) \\ N_t &\quad \textit{shares} \\ -BS(Y_\theta, Q; \lambda) + \int_\theta^t N_s\mathrm{d}Y_s - N_tY_t &\quad \textit{bonds.} \end{aligned} \tag{3.7b}$$

The subreplicating portfolio has time-0 value $\mathbb{V}_0\lambda(Y_T) - \mathbb{V}_0BS(Y_\theta, Q; \lambda)$.

Proof. The strategy clearly self-finances and has the claimed time-0 value.

If $\tau_Q \le T$, then the time-T portfolio value is

$$\begin{aligned}-BS(Y_\theta, Q; \lambda) + \int_\theta^{\tau_Q} N_s \mathrm{d}Y_s + \int_{\tau_Q}^{T} N_s \mathrm{d}Y_s + \lambda(Y_T) &= -\lambda(Y_{\tau_Q}) + \int_{\tau_Q}^{T} N_s \mathrm{d}Y_s + \lambda(Y_T) \\ &\le \langle X\rangle_T - \langle X\rangle_{\tau_Q} = (\langle X\rangle_T - Q)^+\end{aligned}$$

by Proposition 2.4 and Corollary 2.21. If $\tau_Q > T$, then the time-T portfolio value is

$$-BS(Y_\theta, Q; \lambda) + \int_\theta^{T} N_s \mathrm{d}Y_s + \lambda(Y_T) = -BS(Y_T, Q - (\langle X\rangle_T - \langle X\rangle_\theta); \lambda) + BS(Y_T, 0; \lambda) \tag{3.8}$$

$$\le 0 = (\langle X\rangle_T - Q)^+. \tag{3.9}$$

Equality (3.8) holds by Proposition 2.4. Inequality (3.9) holds because the convexity of λ implies that BS is increasing in its second argument. □

3.3 Subreplication under a margin constraint

The value V^{sub} of the subreplicating portfolio is a lower bound on the price of the variance call, in the sense that if the variance call is offered at a price below V^{sub}, then buying the variance call and shorting the portfolio produces an arbitrage that is well-behaved in the following way.

We prove that the subreplication strategy (3.7) satisfies a natural margin constraint on $[0, T]$. Specifically, we show that V_t^{sub} is at all times $t \le T$ dominated by the market's time-t "expectation of" $\langle X\rangle_T - \langle X\rangle_{(t\vee\theta)\wedge\tau_Q}$, by which we mean the RHS of (3.10).

This constraint prevents the magnitude of our short position in the subreplicating portfolio from becoming too large, relative to the collateral that we own, having gone long the variance call.

Definition 3.4 (Call buyer's margin constraint). Assume that claims on $\log(Y_T)$ and $\log(Y_\theta)$ are tradeable. We say that a self-financing trading strategy with time-t value V_t satisfies the *call buyer's margin constraint* if for all $t \in [0, T]$,

$$V_t \le \langle X\rangle_t - \langle X\rangle_{t\wedge\tau_Q} - 2\mathbb{V}_t \log(Y_T / Y_{t\vee\theta}). \tag{3.10}$$

Proposition 3.5 (Subreplicating strategy satisfies the margin constraint)**.** *Assume that claims on* $\log(Y_T)$ *and* $\log(Y_\theta)$ *are tradeable. Then, under the hypotheses of Proposition 3.3, the subreplicating strategy* (3.7) *satisfies the call buyer's margin constraint.*

Therefore the (3.7) *strategy's value* V_t^{sub} *is a lower bound on the buyer's price of the variance call, where buyer's price is defined as the supremum of the prices of all subreplicating strategies satisfying the call buyer's margin constraint.*

Proof. For all $t \ge \tau_Q$, Corollary 2.21 implies

$$\begin{aligned}V_t^{\text{sub}} = -\lambda(Y_{\tau_Q}) + \int_{\tau_Q}^{t} N_s \mathrm{d}Y_s + \mathbb{V}_t\lambda(Y_T) &= -\lambda(Y_{\tau_Q}) + \int_{\tau_Q}^{t} N_s \mathrm{d}Y_s + \lambda(Y_t) - \lambda(Y_t) + \mathbb{V}_t\lambda(Y_T) \\ &\le \langle X\rangle_t - \langle X\rangle_{\tau_Q} + 2\log(Y_t) - 2\mathbb{V}_t \log(Y_T).\end{aligned}$$

For all $t \in (\theta, \tau_Q)$, Proposition 2.4 and the convexity of λ imply

$$\begin{aligned} V_t^{\text{sub}} &= -BS(Y_t, Q - (\langle X\rangle_t - \langle X\rangle_\theta); \lambda) + \mathbb{V}_t\lambda(Y_T) \le -BS(Y_t, 0; \lambda) + \mathbb{V}_t\lambda(Y_T) \\ &= -\lambda(Y_t) + \mathbb{V}_t\lambda(Y_T) \le 2\log(Y_t) - 2\mathbb{V}_t\log(Y_T). \end{aligned}$$

For all $t \le \theta$, the convexity of λ implies

$$V_t^{\text{sub}} = -\mathbb{V}_t BS(Y_\theta, Q; \lambda) + \mathbb{V}_t\lambda(Y_T) \le -\mathbb{V}_t\lambda(Y_\theta) + \mathbb{V}_t\lambda(Y_T) \le \mathbb{V}_t(2\log(Y_\theta) - 2\log(Y_T)),$$

as claimed. □

3.4 Forward-starting variance call: Lower bounds

For any λ satisfying the hypotheses of Corollary 2.21, we have established the lower bound

$$\mathbb{V}_0\lambda(Y_T) - \mathbb{V}_0 BS(Y_\theta, Q; \lambda)$$

on the time-0 value of the variance call.

Extending Dupire to the forward-starting case, we choose λ to maximize the lower bound, as follows. Define van_K by (3.4), and assume tradeability of $\text{van}_K(Y_\theta)$ and $\text{van}_K(Y_T)$ for all K.

Define the time-0 dimensionless Black-Scholes *forward implied volatility* for an underlying Y, a strike K, and a time interval $[\theta, T]$ to be the unique $I_0(K, [\theta, T])$ such that

$$\mathbb{V}_0 BS(Y_\theta, I_0(K, [\theta, T]); \text{van}_K) = \mathbb{V}_0\text{van}_K(Y_T). \tag{3.11}$$

Then we may rewrite the lower bound as

$$\begin{aligned} \mathbb{V}_0\lambda(Y_T) - \mathbb{V}_0 BS(Y_\theta, Q; \lambda) &= \int_0^\infty \lambda_{yy}(K)[\mathbb{V}_0\text{van}_K(Y_T) - \mathbb{V}_0 BS(Y_\theta, Q; \text{van}_K)]\mathrm{d}K \\ &= \int_0^\infty \lambda_{yy}(K)[\mathbb{V}_0 BS(Y_\theta, I_0(K, [\theta, T]); \text{van}_K) - \mathbb{V}_0 BS(Y_\theta, Q; \text{van}_K)]\mathrm{d}K. \end{aligned}$$

Under the constraint $0 \le y^2\lambda_{yy}(y) \le 2$, the optimal λ is λ^* consisting of $2/K^2\mathrm{d}K$ OTM vanilla payoffs at all K for which the dimensionless BS *forward* implied volatility exceeds Q:

$$\lambda^*(y) = \int_{\{K: I_0(K, [\theta, T]) > Q\}} \frac{2}{K^2}\text{van}_K(y)\, \mathrm{d}K$$

where forward implied volatility on $[\theta, T]$ is defined by (3.11). Note that we have shown that in this context the appropriate notion of forward implied volatility $I_0(K, [\theta, T])$ involves the *entire* market-implied distribution of Y_θ; starting from this distribution (not necessarily lognormal) at time θ, run a geometric Brownian motion with dimensionless volatility Q on $[\theta, T]$; the unique Q which recovers the time-0 price of the K-strike T-expiry option is what we mean by forward implied volatility.

The optimized lower bound is

$$V^{\text{SUB}} := \mathbb{V}_0\lambda^*(Y_T) - \mathbb{V}_0 BS(Y_\theta, Q; \lambda^*). \tag{3.12}$$

If variance call is offered below this lower bound, then short the $\lambda^*(Y_T)$ claim and go long a claim on $BS(Y_\theta, Q; \lambda^*)$, which is the λ^* claim's Black-Scholes time-θ valuation using dimensionless volatility Q; this future value is completely determined by Y_θ, so it can be synthesized at time 0 using θ-expiry Europeans. Use the proceeds to buy the variance call, for a net credit. Starting at time θ, dynamically trade shares to lock in this credit.

Remark 3.6. Intuitively, this lower bound says that a variance call dominates a τ_Q-starting *corridor* variance swap, where the corridor can be arbitrarily chosen (and need not be contiguous).

In turn, the τ_Q-starting corridor variance swap value at time 0 dominates the sum, over all K in the corridor, of $(2/K^2)\mathrm{d}K$ OTM T-expiry vanillas less those vanillas' time-0 Black-Scholes valuation using dimensionless volatility Q on $[\theta, T]$. This holds for an *arbitrary* corridor, so the optimal corridor includes exactly those K which make a positive contribution to the sum.

4 Variance call: Upper bound

In this section let $X_t := \log(Y_t/Y_0)$. Consider a variance call with strike $Q \geq 0$ and expiry $T > 0$.

Our strategy to superreplicate $(\langle X\rangle_T - Q)^+$ comes from the following intuition. Let τ_b be the exit time of Y from some fixed interval (b_d, b_u). Although we cannot perfectly replicate $(\langle X\rangle_T - Q)^+$, we can perfectly replicate $(\langle X\rangle_{\tau_b} - Q)^+$ by trading a portfolio having initial value $BP(Y_0, 0; Q)$, as shown in Proposition 2.13.

If $\tau_b \leq T$ then the shortfall is covered by creating the remaining variance $\langle X\rangle_T - \langle X\rangle_{\tau_b}$. To do so, we follow Example 2.18 and include in our holdings a claim on $L(Y_T) - L(Y_{\tau_b})$, where $L(y) := -2\log(y) + A_1 y + A_0$. By choosing (A_0, A_1) such that $L(b_d) = L(b_u) = 0$, we make the $-L(Y_{\tau_b})$ term vanish, so the claim's payoff simplifies to $L(Y_T)$.

If $\tau_b > T$ then at time T we are long a claim on $(\langle X\rangle_{\tau_b} - Q)^+$ but we also hold $L(Y_T) < 0$, a liability which we cannot always afford. We *can* always afford to accept the smaller liability $-BP(Y_T, 0; Q) \geq L(Y_T)$ and still superreplicate, because $(\langle X\rangle_{\tau_b} - Q)^+ - (\langle X\rangle_{\tau_b} - \langle X\rangle_T - Q)^+ \geq (\langle X\rangle_T - Q)^+$. So in the interval $b_d < Y_T < b_u$, let us replace the $L(Y_T)$ payoff by a $-BP(Y_T, 0; Q)$ payoff. This increase in the payoff preserves superreplication in the case $\tau_b \leq T$.

The following proof makes this argument precise, and extends it to forward-starting variance.

4.1 Forward-starting variance call: Superreplication

Let the forward-start date be a constant $\theta \in [0, T)$.

Proposition 4.1 (Forward-starting variance call superreplication)**.** *Consider a forward-starting variance call which pays*

$$(\langle X\rangle_T - \langle X\rangle_\theta - Q)^+.$$

Choose any $b_d \in (0, Y_0]$ *and any* $b_u \in [Y_0, \infty)$*. Let*

$$BP(y, q; Q) := BP(y, q; b_d, b_u, Q),$$

which has the Fourier representation given in Proposition 2.13. Define

$$L(y) := L(y;\ b_d, b_u) := \begin{cases} -2\log(y/b_u) + 2\frac{\log(b_u/b_d)}{b_u - b_d}(y - b_u) & \text{if } b_d = b_u \\ -2\log(y/Y_0) + 2y/Y_0 - 2 & \text{if } b_d = b_u = Y_0 \end{cases}$$

and

$$L^*(y) := L^*(y; b_d, b_u, Q) := \begin{cases} L(y) & \text{if } y \notin (b_d, b_u) \\ -BP(y, 0; Q) & \text{if } y \in (b_d, b_u). \end{cases} \tag{4.1}$$

Let $\tau_b := \inf\{t \geq \theta :\ Y_t \notin (b_d, b_u)\}$*. Let*

$$N_t := \begin{cases} BP_y(Y_t, \langle X\rangle_t - \langle X\rangle_\theta; Q) & \text{if } \theta \leq t \leq \tau_b \\ -L_y(Y_t) & \text{if } t > \tau_b. \end{cases} \tag{4.2}$$

Assume that claims on $-L^*(Y_\theta)$ *and* $L^*(Y_T)$ *are tradeable. Then the following strategy superreplicates the forward-starting variance call: at each time* $t \in [0, \theta]$ *hold*

$$\begin{array}{ll} 1 & \textit{claim on } L^*(Y_T) \\ 1 & \textit{claim on } -L^*(Y_\theta) \end{array} \tag{4.3a}$$

and at each time $t \in (\theta, T)$ *hold*

$$\begin{array}{rl} 1 & \textit{claim on } L^*(Y_T) \\ N_t & \textit{shares} \\ -L^*(Y_\theta) + \int_\theta^t N_s \mathrm{d}Y_s - N_t Y_t & \textit{bonds.} \end{array} \tag{4.3b}$$

*The superreplicating portfolio has time-*0 *value* $\mathbb{V}_0[L^*(Y_T) - L^*(Y_\theta)]$.

Proof. The strategy clearly self-finances and has the claimed time-0 value.

If $\tau_b \geq T$, then the portfolio has time-T value

$$L^*(Y_T) + BP(Y_T, \langle X\rangle_T - \langle X\rangle_\theta; Q) = -BP(Y_T, 0; Q) + BP(Y_T, \langle X\rangle_T - \langle X\rangle_\theta; Q) \tag{4.4}$$

$$\geq (\langle X\rangle_T - \langle X\rangle_\theta - Q)^+. \tag{4.5}$$

by (2.19). If $\tau_b < T$ then the portfolio has time-T value

$$\int_\theta^{\tau_b} BP_y(Y_s, \langle X\rangle_s - \langle X\rangle_\theta; Q)\mathrm{d}Y_s - L^*(Y_\theta) - \int_{\tau_b}^T L_y(Y_s)\mathrm{d}Y_s + L^*(Y_T) \tag{4.6}$$

$$= (\langle X\rangle_{\tau_b} - \langle X\rangle_\theta - Q)^+ - L^*(Y_\theta)^+ - \int_{\tau_b}^T L_y(Y_s)\mathrm{d}Y_s + L^*(Y_T) \tag{4.7}$$

$$\geq (\langle X\rangle_{\tau_b} - \langle X\rangle_\theta - Q)^+ - L(Y_{\tau_b}) - \int_{\tau_b}^T L_y(Y_s)\mathrm{d}Y_s + L(Y_T) \tag{4.8}$$

$$= (\langle X\rangle_{\tau_b} - \langle X\rangle_\theta - Q)^+ + \langle X\rangle_T - \langle X\rangle_{\tau_b} \geq (\langle X\rangle_T - \langle X\rangle_\theta - Q)^+ \tag{4.9}$$

as desired. In case $\tau_b = \theta$, lines (4.7) and (4.8) use $L^*(Y_\theta) = L(Y_{\tau_b}) \geq 0$. In case $\tau_b > \theta$, line (4.7) uses $L^*(Y_\theta) = -BP(Y_\theta, 0, Q)$ and Proposition 2.10 (applied to the semimartingale $Y_{t+\theta}$ relative to filtration $\{\mathcal{F}_{t+\theta}\}$); and line (4.8) uses $L^*(Y_\theta)^+ = L(Y_{\tau_b}) = 0$. In both cases, line (4.8) also uses

$$L^*(Y_T) = -BP(Y_T, 0, Q) \geq -BP(Y_T, 0, 0) = L(Y_T) \tag{4.10}$$

by (2.20). The equality in line (4.9) holds by Example 2.18. □

4.2 Superreplication under a margin constraint

The value V^{super} of the superreplicating portfolio is an upper bound on the price of the variance call, in the sense that if the variance call is bid at a price above V^{super}, then shorting the variance call and going long the portfolio produces an arbitrage that is well-behaved in the following way.

We prove that the superreplication strategy (4.3) satisfies a natural margin constraint on $[0, T]$. Specifically, we show that V_t^{super} at all times $t \leq T$ exceeds the "intrinsic" value of the variance call, as defined by the RHS of (4.11).

There are at least two plausible ways to define intrinsic value, so let us clarify: we prove that at all times t our portfolio value exceeds $(q - Q)^+$ evaluated not merely at $q = \langle X\rangle_t - \langle X\rangle_{t\wedge\theta}$, but indeed that it exceeds $(q - Q)^+$ evaluated at the *sum* of $\langle X\rangle_t - \langle X\rangle_{t\wedge\theta}$ *and* the market's "expectation" of the remaining variance $\langle X\rangle_T - \langle X\rangle_{t\vee\theta}$.

This constraint prevents the intrinsic value of the call (which we are short) from becoming too large, relative to the collateral that we own, having gone long the superreplicating portfolio.

Definition 4.2 (Call seller's margin constraint). Assume that claims on $\log(Y_T)$ and $\log(Y_\theta)$ are tradeable. We say that a self-financing trading strategy with time-t value V_t satisfies the *call seller's margin constraint* if for all $t \in [0, T]$,

$$V_t \geq (\langle X\rangle_t - \langle X\rangle_{t\wedge\theta} - 2\mathbb{V}_t \log(Y_T/Y_{t\vee\theta}) - Q)^+. \tag{4.11}$$

Remark 4.3. Our seller's margin/collateral constraint (4.11) is the natural analogue (for realized variance contracts) of a tameness notion (for European contracts on price) set forth in Cox-Hobson [11]. Their Definition 5.1 defined the fair seller's price of an option with payoff $H(S_T)$ and with collateral requirement function G to be the smallest initial fortune needed to construct a self-financing wealth process W_t satisfying the superreplication condition $W_T \geq H(S_T)$ and the collateral condition $W_t \geq G(S_t)$ for all $t < T$.

In particular, for the case of a European call payoff $H(s) = (s - K)^+$, it is natural to impose a collateral constraint of the call payoff function itself, thus $G(s) = H(s)$. In other words, the requirement is simply that seller of the option must, at each time t, have wealth sufficient to cover the *intrinsic value* of the option, $(S_t - K)^+$. Cox-Hobson cited practical precedent to justify this criterion, stating that (modulo notational differences):

> European call options on stocks cannot be exercised before maturity, but the terms and conditions of options on Internet stocks often included the proviso that if the firm was subject to a takeover at time $t < T$, then the option paid $(S_t - K)^+$. In order to super-replicate the call option it is necessary to have a wealth process which satisfies both a condition at maturity and this condition at intermediate times.

In our setting, with a call on variance instead of price, it is appropriate to replace the European call's intrinsic value $(S_t - K)^+$ with instead the intrinsic value of the variance call (for notational convenience in this remark, let us say a spot-starting variance call with $\theta = 0$). The variance call's intrinsic value could be defined as $(\langle X\rangle_t - Q)^+$, but we will show that indeed our strategy satisfies the stronger constraint (4.11), which defines the margin/collateral requirement to be the "forward-looking" intrinsic value which replaces $\langle X\rangle_t$ by $\langle X\rangle_t - 2\mathbb{V}_t \log(Y_T/Y_t)$.

Similar reasoning explains our definition of buyer's margin/collateral constraint (3.10).

Proposition 4.4 (Superreplicating strategy satisfies the margin constraint)**.** *Assume that claims on* $\log(Y_T)$ *and* $\log(Y_\theta)$ *are tradeable. Then, under the hypotheses of Proposition 4.1, the super-replicating strategy* (4.3) *satisfies the call seller's margin constraint.*

Therefore, the (4.3) *strategy's value* V_t^{super} *is an upper bound on the seller's price of the variance call, where seller's price is defined as the infimum of the prices of all superreplicating strategies satisfying the call seller's margin constraint.*

Proof. If $t \le \theta$ then $V_t^{\text{super}} = \mathbb{V}_t[L^*(Y_T) - L^*(Y_\theta)] \ge 0$ because L^* is convex; moreover,

$$
\begin{aligned}
V_t^{\text{super}} &= \mathbb{V}_t[L^*(Y_T) - L^*(Y_\theta)] && (4.12)\\
&\ge \mathbb{V}_t[L(Y_T) - L(Y_\theta) + L(Y_\theta) - L^*(Y_\theta)] && (4.13)\\
&= \mathbb{V}_t[L(Y_T) - L(Y_\theta) + \mathbb{I}^{(b_d,b_u)}(Y_\theta)(-BP(Y_\theta,0;0) + BP(Y_\theta,0;Q))] && (4.14)\\
&\ge \mathbb{V}_t[L(Y_T) - L(Y_\theta) - Q]. && (4.15)
\end{aligned}
$$

The remaining calculations use the results referenced in the proof of Proposition 4.1.

If $\theta < t \le \tau_b$, then

$$
\begin{aligned}
V_t^{\text{super}} &= \mathbb{V}_t L^*(Y_T) - L^*(Y_\theta) + \int_\theta^t BP_y(Y_s, \langle X\rangle_s - \langle X\rangle_\theta; Q)\mathrm{d}Y_s && (4.16)\\
&= \mathbb{V}_t L^*(Y_T) + BP(Y_t, \langle X\rangle_t - \langle X\rangle_\theta; Q) && (4.17)\\
&\ge \mathbb{V}_t L(Y_T) - L(Y_t) + L(Y_t) + BP(Y_t, \langle X\rangle_t - \langle X\rangle_\theta; Q) && (4.18)\\
&= \mathbb{V}_t L(Y_T) - L(Y_t) - BP(Y_t,0;0) + BP(Y_t,0;Q - (\langle X\rangle_t - \langle X\rangle_\theta)) && (4.19)\\
&\ge \mathbb{V}_t L(Y_T) - L(Y_t) + \langle X\rangle_t - \langle X\rangle_\theta - Q. && (4.20)
\end{aligned}
$$

If $t > \tau_b$ then V_t^{super} is

$$\int_{\theta}^{\tau_b} BP_y(Y_t, \langle X\rangle_t - \langle X\rangle_\theta; Q)\mathrm{d}Y_t - \int_{\tau_b}^{t} L_y(Y_s)\mathrm{d}Y_s + \mathbb{V}_t L^*(Y_T) - L^*(Y_\theta) \tag{4.21}$$

$$= (\langle X\rangle_{\tau_b} - \langle X\rangle_\theta - Q)^+ - \int_{\tau_b}^{t} L_y(Y_s)\mathrm{d}Y_s + \mathbb{V}_t L^*(Y_T) - L(Y_\theta)^+ \tag{4.22}$$

$$\geq (\langle X\rangle_{\tau_b} - \langle X\rangle_\theta - Q)^+ - \int_{\tau_b}^{t} L_y(Y_s)\mathrm{d}Y_s + \mathbb{V}_t L(Y_T) - L(Y_{\tau_b}) + L(Y_t) - L(Y_t) \tag{4.23}$$

$$\geq (\langle X\rangle_{\tau_b} - \langle X\rangle_\theta - Q)^+ + \langle X\rangle_t - \langle X\rangle_{\tau_b} + \mathbb{V}_t L(Y_T) - L(Y_t) \tag{4.24}$$

$$\geq (\langle X\rangle_t - \langle X\rangle_\theta - Q + \mathbb{V}_t L(Y_T) - L(Y_t))^+ \tag{4.25}$$

as claimed. □

4.3 Forward-starting variance call: Upper bounds

Each choice of (b_d, b_u) gives an upper bound $\mathbb{V}_0[L^*(Y_T; b_d, b_u, Q) - L^*(Y_\theta; b_d, b_u, Q)]$ on the time-0 price of the variance call. Hence

$$V^{\text{SUPER}} := \inf_{(b_d, b_u)} \mathbb{V}_0[L^*(Y_T; b_d, b_u, Q) - L^*(Y_\theta; b_d, b_u, Q)] \tag{4.26}$$

gives an optimized upper bound.

Remark 4.5. Because $(\langle X\rangle_T - Q)^+ \leq \langle X\rangle_T$, the spot-starting variance call has a naive upper bound, namely the value of the $\langle X\rangle_T$-replicating portfolio: $\mathbb{V}_0[-2\log(Y_T/Y_0)]$.

Taking $b_d = b_u = Y_0$ in our upper bound recovers the naive upper bound, because

$$\mathbb{V}_0 L^*(Y_T; Y_0, Y_0, Q) = \mathbb{V}_0[-2\log(Y_T/Y_0)].$$

Since our bound optimizes over all pairs (b_d, b_u), it never does worse than the naive upper bound.

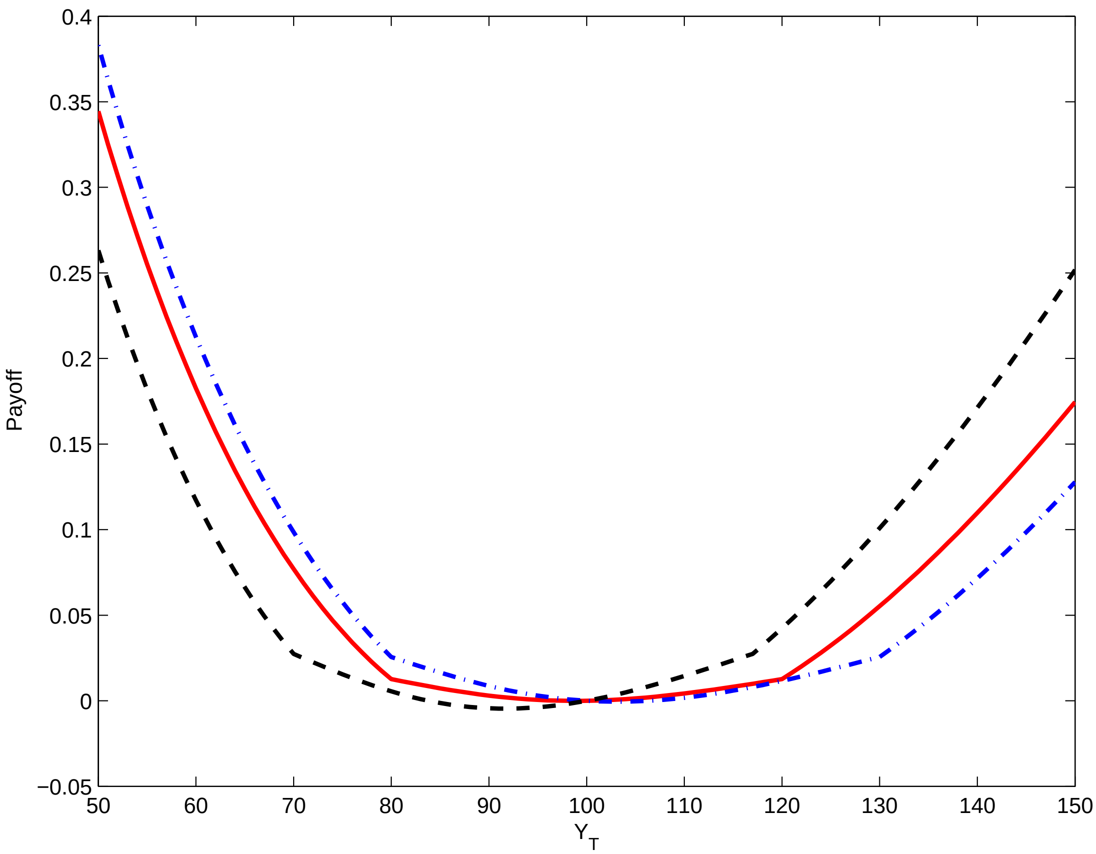
Likewise, for forward-starting variance calls, our upper bound never does worse than the naive upper bound $\mathbb{V}_0[-2\log(Y_T/Y_\theta)]$.

Remark 4.6. Figure 1 shows four examples of model-independent superreplicating portfolios for a variance call.

Remark 4.7. If barrier options are available, then we can improve the upper bound. In (4.3), replace the claim on $L^*(Y_T;\ b_d, b_u, Q)$ by a double knock-in claim on $L(Y_T;\ b_d, b_u)$ plus a double knock-out claim on $-BP(Y_T, 0, Q)$, where each claim has barriers at b_d and b_u, monitored on the time interval $[\theta, T]$.

Remark 4.8. The difference between a variance put with payoff $(Q - \langle X\rangle_T)^+$ and the variance call with payoff $(\langle X\rangle_T - Q)^+$ is a claim on $\langle X\rangle_T - Q$, which is perfectly replicable by Example 2.18. Hence subreplication and superreplication strategies for the variance put follow directly from the corresponding strategies for the variance call.

Figure 1: Superreplicating portfolios for a variance call



Let $Y_0 = 100$. A claim on any one of these time-T payoffs, together with dynamic trading of shares, model-independently superreplicates a spot-starting T-expiry variance call with strike $Q = 0.04$. Referring to (4.1), the plots show $L^*(Y_T) - L^*(Y_0)$ for three particular choices of (b_d, b_u). Each superreplicating payoff is universally valid for all continuous semimartingales with $Y_0 = 100$. The market prices of Europeans expiring at T determine which of the infinite family of superreplicating portfolios is cheapest; of course the cheapest need not be among these three examples.

Remark 4.9. We have used only the European options information available at *inception* (time 0), but an analysis from the standpoint of the European options information available at time $t > 0$ follows immediately. In particular, suppose that we have a call on $[\theta, T]$ variance, struck at $Q \geq 0$. If, at time $t \leq T$, the "running variance" $\langle X \rangle_t - \langle X \rangle_{\theta \wedge t}$ exceeds Q, then we are guaranteed to finish in-the-money, so the call reduces to a variance swap paying $\langle X \rangle_T - \langle X \rangle_\theta - Q$, which can be priced and replicated exactly, by Example 2.18. If, on the other hand, $\langle X \rangle_t - \langle X \rangle_{\theta \wedge t} \leq Q$, then the seasoned Q-strike call given running variance $\langle X \rangle_t - \langle X \rangle_{\theta \wedge t}$ is equivalent to a newly-issued call (given zero running variance) with an "effective strike" $Q - (\langle X \rangle_t - \langle X \rangle_{\theta \wedge t})$, to which our analysis applies directly.

Remark 4.10. As quadratic variation accumulates during the life of a variance call, the call's effective strike decreases. Either the call finishes out-of-the-money and pays nothing, or it finishes in-the money and the effective strike approaches zero at some time. In the latter case, our upper and lower bounds converge (to the price of a variance swap) as the effective strike approaches zero (equivalently, as running variance approaches strike). Thus, even if the observed Europeans data may generate – at inception – a significant gap between our upper and lower bounds for a particular variance contract, our results can nonetheless offer further insight for hedging and risk management at later times, because the gap approaches zero as running variance approaches the strike.

Moreover, even if a wide interval exists at inception (or any other time), our bounds additionally offer immediately usable information: the size of the interval gives an upper bound on the model risk present if one attempts to price the variance call by specifying a model and calibrating to Europeans.

5 Numerical examples

In order to specify and to compute some examples of variance call values and bounds, this section assumes the existence of a martingale measure that prices all European contracts and variance contracts. We take the European prices as given, but this will not uniquely determine the martingale measure. Each "model" – meaning each choice of martingale measure consistent with the Europeans – generates an arbitrage-free variance call valuation. We compare the valuations generated by various models against the bounds arising from our sub/superreplication strategies.

Suppose that the time-0 prices of T-expiry European contracts paying $(Y_T - K)^+$ are given by $\mathbb{E}^{\mathbb{P}_{\mathrm{Heston}}}(Y_T - K)^+$ for all K, where $T = 1$ and the expectation is with respect to a measure $\mathbb{P}_{\mathrm{Heston}}$, under which the paths of Y have distribution given by the Heston dynamics

$$
\begin{aligned}
\mathrm{d}Y_t &= \sqrt{V_t} Y_t \mathrm{d}W_{1t}, \\
\mathrm{d}V_t &= 1.15(0.04 - V_t)\mathrm{d}t + 0.39\sqrt{V_t}\mathrm{d}W_{2t}, \qquad V_0 = 0.04.
\end{aligned}
\tag{5.1}
$$

where W_1 and W_2 are independent Brownian motions. The prices of variance contracts may or may not be given by $\mathbb{P}_{\mathrm{Heston}}$-expected payoffs. Some martingale measure $\mathbb{P}$ does price, via expected payoffs, the Europeans and the variance contracts, but $\mathbb{P}$ need not be $\mathbb{P}_{\text{Heston}}$; it may agree with $\mathbb{P}_{\text{Heston}}$ on expectations of European payoffs but not variance payoffs.

In other words, the Heston dynamics (5.1) are one way to generate those particular observed prices of Europeans, but not the only way – for example, there exist local volatility models which imply, for all T-expiry Europeans, the same prices as (5.1). Therefore, path-dependent contracts, such as variance calls, admit a range of values consistent with the given European prices.

Regard the process Y as a random variable taking values in the space consisting of all positive continuous price paths on $[0,T]$, and define on this space the family $\mathcal{P}$ of probability measures $\mathbb{P}$ such that the Y is a $\mathbb{P}$-martingale satisfying the consistency condition for all K

$$\mathbb{E}^{\mathbb{P}_{\text{Heston}}}(Y_T-K)^+ = \mathbb{E}^{\mathbb{P}}(Y_T-K)^+. \tag{5.2}$$

Each of the measures $\mathbb{P}$ can be described as a "model," in the sense of Cont [10].

By (5.2) the models agree on the value of the observable Europeans, but they produce a range of different values for $\mathbb{E}^{\mathbb{P}}(\langle X\rangle_T-Q)^+$. One value in that range is the Heston-model (5.1) expectation $\mathbb{E}^{\mathbb{P}_{\text{Heston}}}(\langle X\rangle_T-Q)^+$. The middle curve in Figure 2 plots this Heston variance call price for strikes $0.0\le Q\le 0.1$.

Aside from the Heston model, we shall exhibit two other models – a "Root" model and a "Rost" model – consistent with the European values $\mathbb{E}^{\mathbb{P}_{\text{Heston}}}(Y_T-K)^+$. Equivalently, letting ν denote the $\mathbb{P}_{\text{Heston}}$-distribution of Y_T, we shall exhibit two other models under which Y_T has distribution ν. Both constructions are ideas of Dupire [17], adapted by us to the case of logarithmic quadratic variation. In both cases, suppose G is a driftless unit-volatility geometric Brownian motion with respect to some measure $\mathbb{P}_G$, and let $G_0=Y_0$.

First consider the *Root* construction. Rost [25] Theorem 1 and Corollary 3 imply that there exists a space-time "barrier" $B_{\text{Root}}\subset(0,\infty)\times[0,\infty)$ such that

(i) $\tau_{\text{root}}:=\inf\{u\ge 0:(u,G_u)\in B_{\text{Root}}\}$ is a finite stopping time that satisfies $G_{\tau_{\text{root}}}\sim\nu$.

(ii) For each $y>0$, there exists $u_{\text{root}}(y)\in[0,\infty]$ such that $(u,G_u)\in B_{\text{Root}}$ for all $u>u_{\text{root}}(y)$ and $(u,G_u)\notin B_{\text{Root}}$ for all $u<u_{\text{root}}(y)$.

Choosing any such barrier, define the *Root* model $\mathbb{P}_{\text{Root}}$ by specifying that the paths of Y have $\mathbb{P}_{\text{Root}}$-distribution identical to the $\mathbb{P}_G$-distribution of the paths of the process $t\mapsto G_{\tau_{\text{root}}\wedge(t/(T-t))}$ for $t\le T$, with the convention $t/(T-t):=\infty$ for $t=T$.

For numerical computation purposes, we obtain B_{Root} and the transition density $p_{\text{root}}(u,y)$ of the process $u\mapsto G_{u\wedge\tau_{\text{root}}}$ (a transition density in the sense that $p_{\text{root}}(u,y)=\mathbb{P}_G(G_{u\wedge\tau_{\text{root}}}\in \mathrm{d}y)$) by setting up the forward Kolmogorov equation, and solving numerically the following free boundary problem to find the "business time" $u_{\text{root}}(y)$ at which the barrier begins for each y:

$$
\begin{aligned}
p_{\text{root}}(0, y) &= \delta(y - Y_0) \\
\frac{\partial p_{\text{root}}}{\partial u} &= \begin{cases} \frac{1}{2} y^2 \frac{\partial^2 p_{\text{root}}}{\partial y^2} & u < u_{\text{root}}(y) \\ 0 & u > u_{\text{root}}(y) \end{cases} \\
p_{\text{root}}(u_{\text{root}}(y), y) &= p_\nu(y)
\end{aligned}
\tag{5.3}
$$

where p_ν denotes the density function of ν. Then

$$
\mathbb{E}^{\mathbb{P}_{\text{Root}}}(\langle X \rangle_T - Q)^+ = \mathbb{E}^{\mathbb{P}_G}(\langle \log G \rangle_{\tau_{\text{root}}} - Q)^+ = \mathbb{E}^{\mathbb{P}_G}(\tau_{\text{root}} - Q)^+ = \quad (u_{\text{root}}(y) - Q)^+ p_\nu(y) \mathrm{d}y. \tag{5.4}
$$

The lower dashed curve in Figure 2 plots the Root-model variance call price as a function of Q.

Similarly, we compute a "reversed" barrier $B_{\text{Rost}} \subset (0, \infty) \times [0, \infty)$ such that

(iii) $\tau_{\text{rost}} := \inf\{t \geq 0 : (t, G_t) \in B_{\text{Rost}}\}$ is a finite stopping time that satisfies $G_{\tau_{\text{rost}}} \sim \nu$.

(iv) For each $y > 0$, there exists $u_{\text{rost}}(y) \in [0, \infty]$ such that $(u, G_u) \in B_{\text{Rost}}$ for all $u < u_{\text{rost}}(y)$ and $(u, G_u) \notin B_{\text{Root}}$ for all $u > u_{\text{rost}}(y)$.

by solving numerically the free boundary problem

$$
\begin{aligned}
p_{\text{rost}}(0, y) &= \delta(y - Y_0) \\
\frac{\partial p_{\text{rost}}}{\partial u} &= \begin{cases} 0 & u < u_{\text{rost}}(y) \\ \frac{1}{2} y^2 \frac{\partial^2 p_{\text{rost}}}{\partial y^2} & u > u_{\text{rost}}(y) \end{cases} \\
p_{\text{rost}}(u_{\text{rost}}(y), y) &= p_\nu(y)
\end{aligned}
\tag{5.5}
$$

to find the business time $u_{\text{rost}}(y)$ at which the barrier ends for each y. Define the *Rost* model $\mathbb{P}_{\text{Rost}}$ by specifying that the paths of Y have $\mathbb{P}_{\text{Rost}}$-distribution identical to the $\mathbb{P}_G$-distribution of the paths of the process $t \mapsto G_{\tau_{\text{rost}} \wedge (t/(T-t))}$ for $t \leq T$, and compute

$$
\mathbb{E}^{\mathbb{P}_{\text{Rost}}}(\langle X \rangle_T - Q)^+ = \quad (u_{\text{rost}}(y) - Q)^+ p_\nu(y) \mathrm{d}y. \tag{5.6}
$$

The upper dashed-dotted curve in Figure 2 plots the Rost-model variance call price as a function of Q. Intuitively, the Rost model embeds the given distribution ν in the geometric Brownian motion G by stopping some G paths very "early" and stopping other G paths very "late," leading to high variance of business time (equivalently, high variance of realized variance), hence high prices for calls on realized variance. In contrast, the Root model does the embedding by stopping the G paths neither early nor late, leading to low variance of business time, hence low prices for calls on realized variance. See Dupire [16] which established the link between volatility derivative pricing and Skorokhod embedding, and Obloj [24] which surveyed the Skorokhod embedding problem.

Finally, the top and bottom solid curves in Figure 2 show, respectively, the model-free upper bound V^{SUPER} and lower bound V^{SUB}, given in (4.26) and (3.12), and enforceable by static positions in Europeans and dynamic trading of the underlying shares. The Root model's prices are almost indistinguishable from the lower bound, and the Rost model's prices are close to the upper bound.

We make the following observations regarding the quality of the bounds V^{SUB} and V^{SUPER} derived from our sub/superreplicating hedges.

Remark 5.1. Given this set of European option prices, our upper and lower price bounds are nearly "sharp" from a pricing standpoint. For example, consider the at-the-money (strike 0.04) variance call. As shown in Figure 2, there exists at least one model (Rost) for which the variance call price is within 2.6% of our upper bound, and there exists at least one model (Root) for which the variance call price is within 0.3% of our lower bound.

From a replication standpoint, see Remark 5.3.

Remark 5.2. The Root-model variance call value (5.4) is a sharp lower bound on the variance call value in the following sense: for any model $\mathbb{P}$ such that Y is a martingale and Y_T has distribution ν, we have

$$\mathbb{E}^{\mathbb{P}_{\text{Root}}}(\langle X\rangle_T - Q)^+ \leq \mathbb{E}^{\mathbb{P}}(\langle X\rangle_T - Q)^+. \tag{5.7}$$

We prove this using G, the DDS geometric $\mathbb{P}$-Brownian motion of Y, as defined in Remark 2.12. Because $G_{\tau_{\text{root}}}$ and $G_{\langle X\rangle_T}$ have $\mathbb{P}$-distribution ν, Rost [25] Definition 1 and Theorem 2 imply that $\mathbb{E}^{\mathbb{P}}(\tau_{\text{root}} - Q)^+ \leq \mathbb{E}^{\mathbb{P}}(\langle X\rangle_T - Q)^+$; then (5.4) implies the conclusion (5.7). Sharpness holds in the sense that $\mathbb{P} = \mathbb{P}_{\text{Root}}$ attains equality.

Remark 5.3. This paper's primary purpose was to *hedge*, by finding an explicit trading strategy that guarantees sub/superreplication *universally* across all models. From that standpoint, it comes as no surprise that our lower bound V^{SUB} – the initial value of a universally valid trading strategy – is slightly lower than the "sharp" lower bound (the Root valuation), which has not been shown to be universally enforceable.

In other words, if the Root model prevails, then it can be shown that the variance call admits a replicating strategy with initial value $\mathbb{E}^{\mathbb{P}_{\text{Root}}}(\langle X\rangle_T - Q)^+$. However, a Root-specific replicating strategy may fail to subreplicate if some *other* model $\mathbb{P} \in \mathcal{P}$ governs the distribution of Y paths. (By itself, (5.7) does not guarantee that the same strategy that replicates under the Root model also subreplicates under the $\mathbb{P}$ model.) The subreplicating strategy presented in this paper is universal across *all* models, and still manages to produce an initial value V^{SUB} which is only 0.3% lower than the Root valuation in Figure 2's ATM strike.

This point may be restated from an arbitrage perspective. A variance call price quote in violation of the lower bound $\mathbb{E}^{\mathbb{P}_{\text{Root}}}(\langle X\rangle_T - Q)^+$ admits arbitrage, but so far the known strategies are model-*dependent*, in the sense that they depend on which $\mathbb{P} \in \mathcal{P}$ prevails. On the other hand, we have shown that any variance call price quote in violation of our lower bound V^{SUB} admits a model-*independent* arbitrage, in the sense that going long the variance call and short our subreplication strategy generates arbitrage – regardless of which model actually prevails among all the continuous semimartingale models consistent with the observed European prices. Davis-Hobson [13] explore the subtle distinction between model-dependent and model-independent arbitrage.

The same point applies to the upper bound: under the assumption that the Y dynamics follow the Rost model, there exists a strategy that exactly replicates $(\langle X\rangle_T - Q)^+$ and has initial value $\mathbb{E}^{\mathbb{P}_{\text{Rost}}}(\langle X\rangle_T - Q)^+$. However, a Rost-specific strategy may fail to superreplicate, if some *other* model $\mathbb{P} \in \mathcal{P}$ governs the Y path distribution. The superreplicating strategy presented in this paper is *universal*, and still manages to produce, in Figure 2's ATM strike, an initial value $V^{\text{SUPER}} = 0.0274$, which is only 2.6% higher than the Rost model valuation 0.0267.

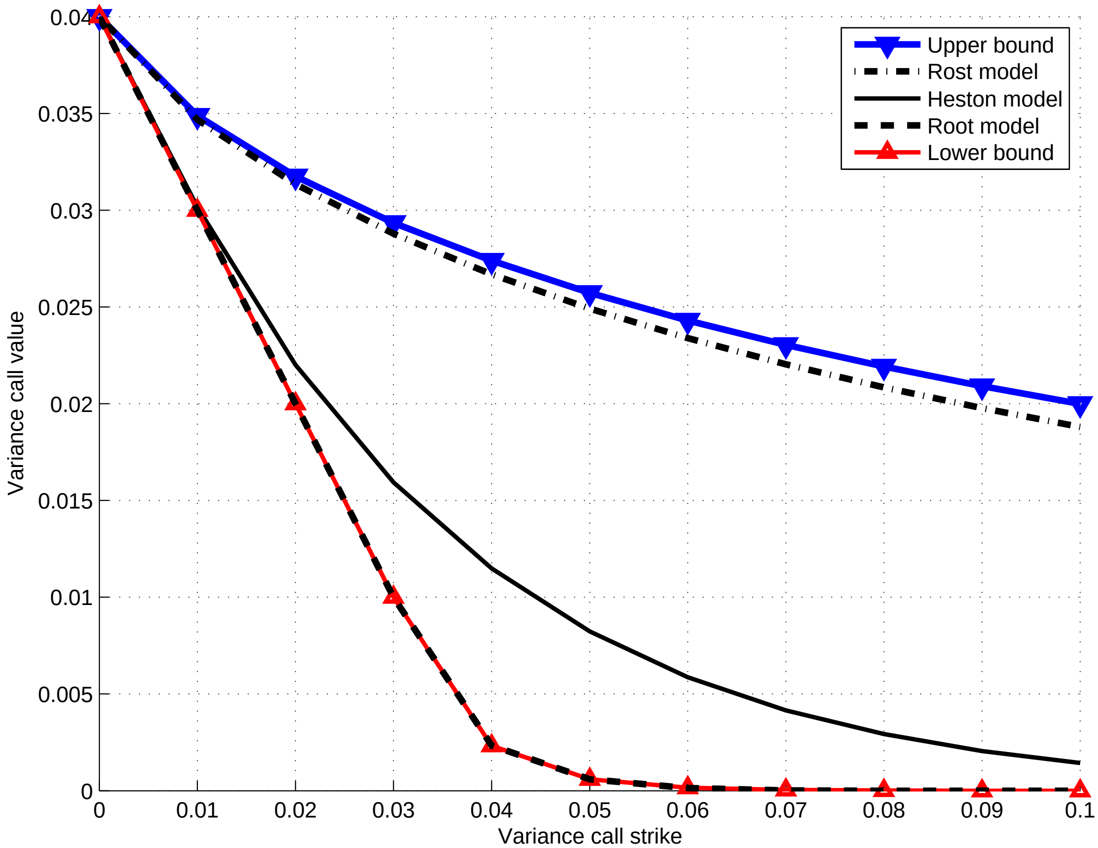
6 Conclusion

For spot-starting and forward-starting variance calls, we have found robust subreplication and superreplication strategies, hence upper and lower bounds, universally valid for all continuous semimartingales. This extends Dupire's subreplication of spot-starting variance calls. The strategies hold Europeans statically and trade the underlying asset dynamically.

From a practical standpoint, we have investigated the pricing and hedging of a contract that appeals to portfolio managers seeking to trade variance. From a methodological standpoint, we have explored the model-free replicability of general functions of price and variance, payable at general boundaries in price-variance space; we have exploited properties of geometric Brownian motion, which arises even in the general continuous semimartingale setting, due to the DDS time change by which quadratic variation becomes the "business-time" clock; and we have applied these business-time devices carefully to hedge contracts expiring at a fixed calendar time.

From a broader perspective, we have continued the ongoing investigation into extracting information about fully path-dependent risks from one-dimensional information about Y_T alone, and into hedging those path-dependent risks using Europeans.

Figure 2: Example: upper and lower bounds on a variance call



Let $T = 1$. Given T-expiry European option prices consistent with the Heston model (5.1), the dynamics of Y are not uniquely determined. Three models of Y dynamics consistent with those prices are the Heston model itself, the Rost model, and the Root model, each of which generates a different profile of variance call values. We plot all three profiles, together with the lower bound V^{SUB} and upper bound V^{SUPER} derived, in (3.12) and (4.26), from the subreplicating and super-replicating hedges. The lower bound is, to the naked eye, indistinguishable from the Root model valuation (which is in fact larger than the lower bound at strikes 0.04 and higher, specifically larger by 0.3% at the ATM strike 0.04). The upper bound is nearly (within 2.6% at the ATM strike 0.04) attained by the Rost model.

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