On the qualitative effect of volatility and duration on prices of Asian options

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February 6, 2008

Abstract

We show that under the Black Scholes assumption the price of an arithmetic average Asian call option with fixed strike increases with the level of volatility . This statement is not trivial to prove and for other models in general wrong. In fact we demonstrate that in a simple binomial model no such relationship holds. Under the Black-Scholes assumption however, we give a proof based on the maximum principle for parabolic partial differential equations. Furthermore we show that an increase in the length of duration over which the average is sampled also increases the price of an arithmetic average Asian call option, if the discounting effect is taken out. To show this, we use the result on volatility and the fact that a reparametrization in time corresponds to a change in volatility in the Black-Scholes model. Both results are extremely important for the risk management and risk assessment of portfolios that include Asian options.

Keywords: Asian options, volatility, vega, duration, qualitative risk-management

Mathematics Subject Classification: 91B28, 60H30, 65H05

JEL Subject Classification: C63, G11, G31, G39

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1 Introduction

Asian options are options where the payoff depends on the average of the underlying asset during at least some part of the life time of the option. The average can be taken in several ways, each leading to a different type of Asian option. According to the Handbook of Exotic Options [6] the name Asian option was coined by employees of Bankers Trust, which sold this type of options to Japanese firms that wanted to hedge their foreign currency exposure. These firms used these options because their annual reports were also based on average exchange rates over the year. Average type options are particularly suited to hedge risk at foreign exchange markets and by reason of the averaging effect, significantly cheaper than plain vanilla options. Effectively such options are traded since the mid 1980's and first appeared in the form of commodity linked bonds. Specific examples are the Mexican Petro Bond and the Delaware Gold Index Bond. Asian options are OTC traded, however market and trading volume appear to grow very fast. A recent study of CIBC world markets revealed that Asian style options are the most commonly traded exotic options. Similar statements can be found in the Handbook of Exotic Options [6]. In the Black-Scholes model, the technically easiest case to consider is where the average is a geometric average. Since the product of log-normal distributed random variables is again log-normal distributed, explicit analytical expressions are available for the price of such options and everything appears to be well understood. On the other side, for the more natural version of an arithmetic average Asian option, no explicit pricing formulas are available and it is hard to say, how changes in parameters of the model are reflected in price changes. Our paper focuses on the case of arithmetic Asian options and how changes in volatility and duration effect its price. The average can be taken either continuously or discretely. An option of continuous average type is represented by the following example

$$\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+ \qquad \text{"continuous average price call".}$$

The continuous average type is of particular importance, since the Black-Scholes partial differential equation can be easily modified in order to obtain a Black-Scholes like partial differential equation for the prices, see section

3. However, in contrast to the standard Black-Scholes partial differential equation, it is not possible to find a closed form solution for this PDE. On the other side it has to be said, that the continuous average type is neither traded at any financial market nor between any two financial institutions and in a way only represents an approximation of the discrete average type, which is of the form

$$\left(\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - K\right)^{+} \qquad \text{"discrete average price call"}$$

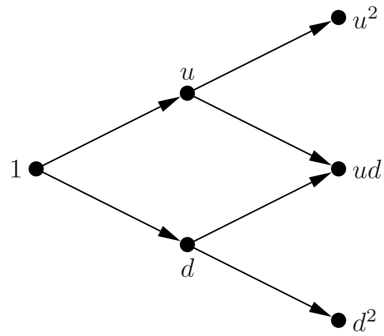
where $0 = t_0 < t_1 < t_2 < ... < t_n = T$. PDE's for a discretely sampled average Asian option are available, see for example Vecer [9], and the discussion presented in the following section can be adapted to this case. For the reason of clarity however we restrict our investigations in this article to the case of continuously sampled arithmetic Asian options, but we allow the payoff structure to be slightly more general than in the classical examples, that is of type

$$g \quad \frac{1}{T} \, {}_{0}^{T} \, S_t dt \tag{1}$$

with $g(x)$ being a convex function. The objective of this article is to understand the qualitative behavior of the price of this option with regards to changes in the Black-Scholes volatility parameter σ and the duration time T of the averaging process. We show the price is increasing in both parameters, at least if the discounting effect is neglected. From an intuitive point of view, the first result might be expected, but it is far from trivial. Carr [1] and Jagannathan [5] clearly point out that the idea of options prices rising with increasing volatility, even if these options have convex payoff profiles, is a widely held fallacy. Jagannathan clearly brings it to the point: Quote "It is commonly believed that the value of a call option is a non-decreasing function of the riskiness of its underlying security. Such a belief is not correct." Carr provides an example in which the price of a European call decreases with an increase in volatility. We adapt Carr's example to the case of an arithmetic Asian call and show that the situation here is quite similar and increases in volatility may result in a decrease in price. Whether the price of an option is increasing in volatility or not, crucially depends on the distribution of the underlying. In the case of an arithmetic Asian option the distribution of the underlying is only partly understood, see for example Carr and Schröder [2], Geman and Yor [4] or Yor [10], and it appears to be impossible from the knowledge of this distribution alone to directly infer about the volatility and duration effect on prices. We also found that it appears to be impossible to directly infer about the volatility effect from the Fourier transform of an Asian option, see for example Carr and Schröder [2]. As with regards to the effect of duration time T of the averaging process, intuition may in fact suggest that since more samples for the average are taken into account, volatility of the underlying decreases and therefore the option price does so as well. In this case not only the argument is wrong, but also its conclusion, as long as the Black-Scholes model is concerned. In fact we show that if the discounting effect is taken out, the price of an arithmetic Asian option with convex payoff profile is increasing in duration of the averaging process and that the answers for volatility effect and duration effect are strongly related by a rescaling property of Brownian motion. The remainder of this article is a follows. In section 2 we adapt the example of Carr [1] to the case of an arithmetic Asian call, while in section 3 we focus on the Black-Schole theory of Asian options and derive certain Lemma's which address symmetry properties of the price function as well as some structural relationships between the Greeks of an Asian option. In section 4 we derive the result for the volatility effect, while in section 5 we address the duration effect. The main conclusions are summarized in section 6.

2 Negative vega's for Asian calls in a Binomial model

In this section we adapt the example in Carr [1] to the case of an Asian call and demonstrate in this way that the price of an arithmetic Asian call can in general be decreasing in volatility. Let us consider a twice averaged Asian option $\left(\frac{1}{2}\sum_{i=1}^{2} S_{t_i} - K\right)^+$ with maturity 2 years in a two-period binomial model under zero interest rate. We assume that the initial price of the stock is $S_0 = 1$ and the strike price is $K = 1$, which means the option is at the money. Denote the upstate and downstate parameter pair of the binomial model by (u, d) with $u > 1 > d > 0$. Then the stock prices are provided by the following tree.



A no arbitrage assumptions enables us to calculate the risk neutral probability $p = \frac{1-d}{u-d}$ for upwards and $q = \frac{u-1}{u-d}$ for downwards, which in turn yields the option's value

$$C_0 = p^2 \left(\frac{1}{2}\left(u^2 + u\right) - 1 \right)^+ + pq \left(\frac{1}{2}(ud + u) - 1 \right)^+ + pq \left(\frac{1}{2}(ud + d) - 1 \right)^+ . \tag{2}$$

We will later refer to the volatility parameter σ in the Black-Scholes model and for this reason, we must identify its analogue in the binomial model above. Under the Black-Scholes assumption $\ln(S(t)) = \left(r - \frac{1}{2}\sigma^2\right) t + \sigma W(t)$ and therefore $\sigma^2 = var(\ln(S(t)))/t$. The volatility parameter σ in the Black-Schole model therefore corresponds to the risk-neutral log-normal variance of every single period of the binomial model, when length of each period is normalized to one. For the risk-neutral log-normal variance we obtain

$$p(\ln u - (p \ln u + q \ln d))^2 + q(\ln d - (p \ln u + q \ln d))^2 = pq \ln^2(\frac{u}{d})$$

and therefore

$$pq \ln^2 \left(\frac{u}{d}\right) = \sigma^2 \triangle t = \sigma^2,$$

or alternatively

$$\sigma = \sqrt{pq \ln^2(\frac{u}{d})}. \tag{3}$$

Note that in the setup here $\triangle t = 1$ year and σ is the annual volatility. Equations (2) and (3) connect option price with the volatility of stock price through the tree's parameters u and d. This enables us to draw the following graph showing a negative relationship between option price and volatility, once the volatility passes a critical value, in this example $\frac{1}{2}$.

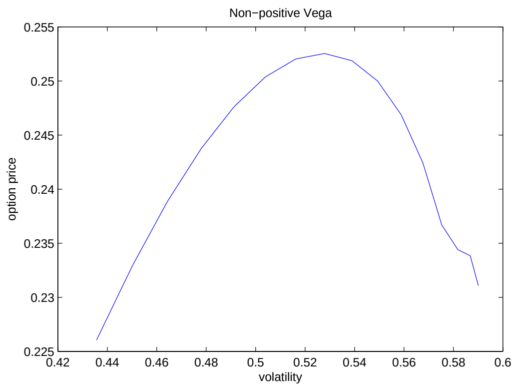


Figure 1: Non-positive Vega

To document the numerical result some pairs of volatility and option prices in the picture are listed in the table below.

volatility	price
0.4357	0.2261
0.4644	0.2389
0.5279	0.2525
0.5676	0.2424
0.5900	0.2311

3 Arithmetic Asian options in the Black-Scholes framework

In this section we briefly review some classical facts on Arithmetic Asian options in the Black-Scholes framework and in addition derive some symmetry properties of the price function. We work under the risk neutral measure and assume that stock price and bond price follow the dynamics

$$\begin{aligned} dB_t &= B_t r dt \\ dS_t &= S_t(rdt + \sigma dW_t). \end{aligned} \tag{4}$$

The arbitrage free price of an option with payoff (1) at time t is then given by

$$v(t,T,x,y,\sigma) = e^{-r(T-t)}\mathbb{E}\left[g\left(\frac{1}{T}\int_0^T S_u du\right) \middle| S_t = x, \int_0^t S_u du = y\right]. \tag{5}$$

In difference to the standard notation, we include the parameters T and σ explicitly as arguments as we will later study the price function as a function of these arguments. A similar argument as in Black-Scholes (1973) leads to the following PDE

$$v_t - rv + xv_y + \frac{1}{2}\sigma^2 x^2 v_{xx} = 0 \tag{6}$$

with boundary conditions

$$\begin{aligned} v(t,T,0,y,\sigma) &= e^{-r(T-t)}g(y), & 0 \leq t < T, y \in \mathbb{R}, \sigma > 0 \\ \lim_{y \to -\infty} v(t,T,x,y,\sigma) &= 0, & 0 \leq t < T, x \geq 0, \sigma > 0 \\ v(T,T,x,y,\sigma) &= g(y), & x \geq 0, y \in \mathbb{R}, \sigma > 0 \end{aligned} \tag{7}$$

As indicated before, in this article we are primarily interested in qualitative aspects and in particular monotonicity of the option price $v(0,T,x,0,\sigma)$ as function of T and σ. In the remainder we use the following notation for the Greeks of an arithmetic average Asian option:

$$\begin{aligned} v_t &= \text{tau} = \tau \\ v_T &= \text{chronos} = \chi \\ v_x &= \text{delta} = \Delta \\ v_y &= \text{iota} = \iota \\ v_\sigma &= \text{vega} = \mathcal{V} \\ v_{xx} &= \text{gamma} = \Gamma \end{aligned}$$

The Greeks chronos and iota do not appear in the literature and have been named here with chronos and the Greek letter χ to indicate "chronos" which is Greek for time and iota for the Greek letter ι, as the average process $I_t = \int_0^t S_u du$, which the variable y refers to is classically denoted with an I. We will find it convenient in the following to work with the un-discounted price function

$$\tilde{v}(t,T,x,y,\sigma) = e^{r(T-t)}v(t,T,x,y,\sigma). \tag{8}$$

It is then easy to see that $\tilde{v}$ satisfies the following partial differential equation

$$\tilde{v}_t + x\tilde{v}_y + \frac{1}{2}\sigma^2 x^2 \tilde{v}_{xx} = 0 \tag{9}$$

with boundary conditions

$$\begin{aligned} \tilde{v}(t,T,0,y,\sigma) &= g(y), \quad && 0 \le t < T, y \in \mathbb{R}, \sigma > 0 \\ \lim_{y \to -\infty} \tilde{v}(t,T,x,y,\sigma) &= 0, && 0 \le t < T, x \ge 0, \sigma > 0 \\ \tilde{v}(T,T,x,y,\sigma) &= g(y), && x \ge 0, y \in \mathbb{R}, \sigma > 0. \end{aligned} \tag{10}$$

The following relationships hold for the Greeks tau and chronos

$$\begin{aligned} v_t &= rv + e^{-r(T-t)}\tilde{v}_t \\ v_T &= -rv + e^{-r(T-t)}\tilde{v}_T, \end{aligned}$$

while all other Greeks are simply the discounted partial derivatives of the un-discounted value function. We will later need a symmetry property of the un-discounted price function with regards to the variables t and T which is stated in the following Lemma.

Lemma 3.1. *The un-discounted price function $\tilde{v}(t,T,x,y,\sigma)$ of an arithmetic Asian call (1) satisfies*

$$\tilde{v}(t,T,x,0,\sigma) = \tilde{v}\left(0, T-t, \frac{T-t}{T}x, 0, \sigma\right). \tag{11}$$

Proof. For $y > 0$ we have that

$$\begin{aligned} \tilde{v}(t,T,x,y,\sigma) &= \mathbb{E}\left[g\left(\frac{1}{T}\int_0^T S_u du \right) \middle| S_t = x, \int_0^t S_u du = y \right] \\ &= \mathbb{E}\left[g\left(\frac{y}{T} + \frac{1}{T}\int_t^T S_u du \right) \middle| S_t = x \right] \\ &= \mathbb{E}\left[g\left(\frac{y}{T} + \frac{1}{T-t}\int_t^T S_u du \right) \middle| S_t = \frac{T-t}{T}x \right]. \end{aligned}$$

Now we take the limit for $y \to 0$ and obtain from the Markov property of the stock price dynamic that

$$\begin{aligned}
\tilde{v}(t,T,x,0,\sigma) &= \mathbb{E}\left[g\left(\frac{1}{T-t}\int_t^T S_u du\right)\middle| S_t = \frac{T-t}{T}x\right] \\
&= \mathbb{E}\left[g\left(\frac{1}{T-t}\int_0^{T-t} S_u du\right)\middle| S_0 = \frac{T-t}{T}x\right] \\
&= \tilde{v}\left(0, T-t, \frac{T-t}{T}x, 0, \sigma\right).
\end{aligned}$$

□

The following Lemma relates the Greeks tau, chronos and delta of the un-discounted value function.

Lemma 3.2. *The following relationship for un-discounted Greeks holds at time $t = 0$:*

$$\tilde{v}_t + \tilde{v}_T + \frac{\tilde{v}_x}{T} = 0 \tag{12}$$

Proof. It follows from the previous Lemma that for all $\epsilon > 0$

$$\tilde{v}\left(\epsilon, T+\epsilon, \left(\frac{T+\epsilon}{T}\right)x, 0, \sigma\right) = \tilde{v}\left(0, T, x, 0, \sigma\right).$$

Differentiating with respect to ϵ and evaluation at $\epsilon = 0$ gives

$$\begin{aligned}
0 &= \frac{d}{d\epsilon}\tilde{v}\left(\epsilon, T+\epsilon, \left(\frac{T+\epsilon}{T}\right)x, 0, \sigma\right)\Bigg|_{\epsilon=0} \\
&= \tilde{v}_t + \tilde{v}_T + \frac{\tilde{v}_x}{T}.
\end{aligned}$$

□

4 The volatility effect on arithmetic Asian calls

We have seen in section 2 that it is a priori not clear whether or not an increase in the volatility parameter σ leads to an increase in the price of an arithmetic average Asian option. On the other side the answer of this question is of fundamental importance for risk managers dealing with portfolios which contain Asian options. In this section we apply the maximum principle for parabolic PDE's as it can be found for example in Stroock and

Varadhan [8] in order to prove that under the Black-Scholes assumption an increase in volatility indeed leads to an increase in the option price.

Proposition 4.1. *Assume that stock and bond follow the Black-Scholes dynamic (4) and assume that $g(x)$ is a continuous convex function, which is piecewise C^1 with bounded derivatives and at most finitely many singularities. Then the price of a continuous type arithmetic Asian option* $g \;\; \frac{1}{T}\int_0^T S_t dt$ *is a strictly increasing function of the volatility parameter* $\sigma > 0$.

Proof. As the vega is just the discounted vega of the un-discounted value function $\tilde{v}$ we assume w.l.o.g. that the interest rate satisfies $r = 0$. In this case the two value functions coincide and the partial differential equation for the value function $v(t, x, y, \sigma)$ is given by

$$v_t + xv_y + \frac{1}{2}\sigma^2 x^2 v_{xx} = 0 \tag{13}$$

with boundary conditions (10). Differentiating (13) with respect to σ gives

$$v_{t\sigma} + xv_{y\sigma} + \frac{1}{2}\sigma^2 x^2 v_{xx\sigma} + \sigma x^2 v_{xx} = 0 \tag{14}$$

Denoting the vega $v_\sigma(t, x, y, \sigma)$ of the option with $\mathcal{V}$ we find that

$$\mathcal{V}_t + L_t^{AS}\mathcal{V} = -\sigma x^2 v_{xx} \tag{15}$$

where L_t^{AS} is the differential operator

$$L_t^{AS} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y}$$

associated to the Asian option PDE (6) ($r = 0$). Differentiating the boundary conditions (10) with respect to σ we obtain that $\mathcal{V}$ vanishes on the whole of the boundary. As σ was assumed to be positive, the maximum principle for parabolic PDE's (see for example Theorem 3.11, page 66 in [8]) therefore implies that $\mathcal{V}$ is positive, given that the right hand side of (15) is negative. The latter is true if v_{xx} is positive. It therefore suffices to show that the gamma v_{xx} of the Asian option is positive. In order to do this, it suffices to show that

$$\frac{\partial^2}{\partial x^2}\mathbb{E}\left[g\left(\frac{x}{T}\int_0^T \tilde{S}_t dt\right)\right] > 0 \tag{16}$$

where $x = S_0$ and $\tilde{S}_t := \exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right)$ denotes the normalized stock price. Let us first assume that g is two times continuously differentiable. As $\tilde{S}_t$ does not depend on x taking the second derivative in (16) gives

$$\frac{\partial^2}{\partial x^2}\mathbb{E}\left[g\left(\frac{x}{T}\int_0^T \tilde{S}_t dt\right)\right] = \mathbb{E}\left[g''\left(\frac{x}{T}\int_0^T \tilde{S}_t dt\right)\left(\frac{1}{T}\int_0^T \tilde{S}_t dt\right)^2\right]. \tag{17}$$

It follows from the convexity of g that the expression on the right hand side of (17) is positive. If g is not two times continuously differentiable, an approximation such as in [3] (proof of Proposition 3.2. page 23) enables us to get the same result. $\square$

The result from Proposition 4.1. also holds in the case of a discrete type arithmetic Asian option $g\left(\frac{1}{n}\sum_{i=1}^n S_{t_i}\right)$ with convex payoff profile. This can be shown using results of Vecer (2005) who derives a single partial differential equation for the price of a discrete type arithmetic Asian option (equation (3.9) in [9] with q_t chosen in (3.5)). Positivity of the coefficient in front of the second order term in Vecer's equation (3.9) guarantees that the methodology presented above using the maximum principle works in this case as well, positivity of the gamma follows in exactly the same way as before. We omit the details as in this article we focus on the continuous type.

5 The duration effect on arithmetic Asian calls

In this section we study the price effect of duration T over which the arithmetic Asian option (1) is averaged. Intuitively one may think that if the discounting effect is taken aside, since the average over a longer period is taken, the variance of the average decreases and so does the option price. In the discussion of the volatility effect we have already seen that this kind of intuitive thinking is very misleading, in fact in this case it leads to the wrong conclusion. Increasing the duration in fact has two effects, one is that an average over a larger sample is taken, but the other one is that the variance of the samples taken at a later time increases, as the asset is assumed to follow a geometric Brownian motion. While the first effect gives the option price a tendency to decrease, the second effect will lead to an increase in the option price. We will show that if the discount effect e^{-rT} is taken out, the option price indeed increases in T, i.e. the chronos of the un-discounted value function is positive. This means that the first effect mentioned above dominates the second. We will prove this result by establishing a symmetry between the vega and the chronos of an Asian option. Before we do this however we indicate that the result is not trivial and that it can not be proved in analogy to the case of a plain vanilla call, where a sub-martingale argument applies. In order to do this note that under the risk neutral measure, which in this article is identified with the subjective probability measure $\mathbb{P}$, the discounted Black-Scholes stock price represents a martingale. Assuming for simplicity that the interest rate is equal to zero the Jensen inequality for the conditional expectation, see for example [7], page 70, implies that for a monotonic increasing and convex payoff function $g(\cdot)$ we have

$$\mathbb{E}\left(g(S_{T+\epsilon})|\,\mathcal{F}_T\right) \geq g\left(\mathbb{E}\left(S_{T+\epsilon}|\,\mathcal{F}_T\right)\right) \geq g(S_T).$$

Taking expectation and using the tower property of the conditional expectation shows that the option price is increasing with T. In the case of a plain vanilla call we also find a strong relation ship between the tau and the chronos. In the case of an arithmetic Asian option the relationship between tau and chronos is more complicated and depends on delta and T as equation (12) shows. Furthermore the underlying process $\tilde{I}_t = \frac{1}{t} I_t = \frac{1}{t}\int_0^t S_u du$ is not a sub-martingale. This follows from the fact that it has bounded variation, and as such could only be a submartingale if it would be increasing a.s., which it is obviously not. There is hence no direct conclusive martingale based argument which proves positivity of the chronos of the un-discounted value-function of an arithmetic Asian option of type (1). In the following we will establish a symmetry between the chronos and the vega and then use the positivity result obtained for the vega in Proposition 4.1. In order to do this we use a scaling property of Brownian motion which allows us to represent a change in volatility as an increase in speed. For $\epsilon > 0$ denote

$$S_t^\epsilon = \exp\left((\sigma+\epsilon)W_t - \frac{1}{2}(\sigma+\epsilon)^2 t\right)$$

$$\tilde{S}_t^\epsilon = \exp\left(\sigma W_{\left(\frac{\sigma+\epsilon}{\sigma}\right)^2 t} - \frac{1}{2}(\sigma+\epsilon)^2 t\right).$$

Lemma 5.1. *The processes $S_{(\cdot)}^\epsilon$ and $\tilde{S}_{(\cdot)}^\epsilon$ have the same distribution.*

Proof. It follows directly from the scaling property of Brownian motion that

$$\frac{1}{\left(\frac{\sigma+\epsilon}{\sigma}\right)} W_{\left(\frac{\sigma+\epsilon}{\sigma}\right)^2 (\cdot)} \sim W_{(\cdot)}$$

which in turn implies the statement of the lemma. □

Corollary 5.1. *For $\epsilon > 0$ we have*

$$\frac{1}{T}\int_0^T S_t^\epsilon dt \sim \frac{1}{\left(\frac{\sigma+\epsilon}{\sigma}\right)^2 T}\int_0^{\left(\frac{\sigma+\epsilon}{\sigma}\right)^2 T} S_u du, \tag{18}$$

i.e. the two sides have the same distribution.

Proof. It follows from the previous Lemma that

$$\frac{1}{T}\int_0^T S_t^\epsilon dt \sim \frac{1}{T}\int_0^T \tilde{S}_t^\epsilon dt.$$

Now using the definition and substitution of $s = \left(\frac{\sigma+\epsilon}{\sigma}\right)^2 t$ we obtain

$$\begin{aligned} \frac{1}{T}\int_0^T \tilde{S}_t^\epsilon dt &= \frac{1}{T}\int_0^{\left(\frac{\sigma+\epsilon}{\sigma}\right)^2 T} \exp\left(\sigma W_s - \frac{1}{2}\sigma^2 s\right)\left(\frac{\sigma}{\sigma+\epsilon}\right)^2 ds \\ &= \frac{1}{\left(\frac{\sigma+\epsilon}{\sigma}\right)^2 T}\int_0^{\left(\frac{\sigma+\epsilon}{\sigma}\right)^2 T} S_u du. \end{aligned}$$

□

From Corollary 5.2 we immediately obtain the following symmetry for the un-discounted value function $\tilde{v}$:

Proposition 5.1. *For $\epsilon > 0$ we have*

$$\tilde{v}(0, T, x, 0, \sigma+\epsilon) = \tilde{v}\left(0, \left(\frac{\sigma+\epsilon}{\sigma}\right)^2 T, x, 0, \sigma\right).$$

Proof. Follows directly from (18) and (5). □

Using the previous result we can now easily derive the relationship between the chronos and the vega of an arithmetic Asian option.

Proposition 5.2. *The following identities hold at $t = 0$*

$$\begin{aligned} \tilde{v}_T &= \frac{\sigma}{2}\tilde{v}_\sigma & (19)\\ v_T &= \frac{\sigma}{2}v_\sigma + rv. & (20) \end{aligned}$$

Proof. We conclude from Proposition 5.1 that

$$\begin{aligned} \tilde{v}_\sigma &= \frac{d}{d\epsilon}\,\tilde{v}(0, T, x, 0, \sigma+\epsilon)\big|_{\epsilon=0} \\ &= \frac{d}{d\epsilon}\,\tilde{v}\left(0, \left(\frac{\sigma+\epsilon}{\sigma}\right)^2 T, x, 0, \sigma\right)\Bigg|_{\epsilon=0} \\ &= \frac{2}{\sigma}\tilde{v}_T, \end{aligned}$$

which shows (19). The second equality follows from taking discounting into account.

□

Finally we obtain the following result which addresses the monotonicity of the un-discounted value function with respect to the duration T over which the average is taken.

Proposition 5.3. *Assume that the payoff function $g(x)$ is convex. Then the un-discounted Black-Scholes price of an Asian option with payoff $g\left(\frac{1}{T}\int_0^T S_t dt\right)$ is increasing with duration time T.*

Proof. This follows directly from equation (19) and Proposition 4.1.

□

6 Conclusions

We have shown that the Black-Scholes price of an arithmetic Asian option with convex payoff profile is increasing in the volatility parameter σ. This result is important for the risk management of portfolios which contain Asian options. It is not a trivial result, as we indicate with an example which shows, that when the Black-Scholes assumption is disregarded, it does no longer hold. We also show, that if the discounting effect is taken away, the price of an arithmetic Asian option with convex payoff profile is increasing in the duration time. Again this is a non trivial result and quite significant, if deciding about the duration time, when an Asian option contract is setup.

Acknowledgments

Christian-Oliver Ewald acknowledges support from the "Deutsche Forschungs Gesellschaft" Grant 529817, a travel grant from the Royal Society, the grant "Dependable Adaptive Systems and Mathematical Modeling" (DASMOD) from the state Rheinland Pfalz (Germany), support from the ITWM Kaiserslautern (Germany) and the Center of Dynamic Macroeconomic Analysis at St.Andrews University. The authors are very thankful for useful suggestions, hints and remarks from Marc Yor, University Pierre et Marie Curie, Paris.

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