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A note on sufficient conditions for no arbitrage

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Abstract

It is shown that the absence of call spread, butterfly spread and calendar spread arbitrages is sufficient to exclude all static arbitrages from a set of option price quotes across strikes and maturities on a single underlier.

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1. Introduction

The absence of arbitrage opportunities is a fundamental principle underlying the modern theory of financial asset pricing. In particular, this concept is intrinsic to the statement of the first fundamental theorem due to Harrison and Kreps (1979), Harrison and Pliska (1981), and Delbaen and Schachermayer (1994). These authors establish conditions and exact definitions under which the absence of arbitrage opportunities is equivalent to the existence of an equivalent martingale measure.

In a continuous time setting, it can actually be quite difficult to confidently establish whether or not the given prices of a given set of assets are arbitrage-free. The reason is that

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the definition of an arbitrage opportunity depends critically on the nature of the information set which can be used in developing trading strategies in the specified assets. Likewise, the associated continuous time martingales are defined through conditional expectations which rely on the same informational basis. Typically, this informational basis includes the entire path of past underlying asset prices at a minimum. Hence, to certify that a given set of asset prices are arbitrage-free in continuous time, the structure of the possible price paths must be specified a priori. For example, one must specify whether the possible price paths are purely continuous, pure jump, or a combination of the two. One must also specify whether the possible price paths display finite or infinite variation over time. Since one can only observe past prices in practice and then only discretely, it is difficult to have any confidence in any particular structure which is imposed on future paths. Nonetheless, a certification that a given set of asset prices is arbitrage-free in the traditional sense does require this specification of the nature of the possible price paths.

One way out of this difficulty is to alter the technical definition of an arbitrage opportunity. By reducing the size of the information set upon which trading strategies can rely, it becomes easier to certify that a given set of asset prices are free of these restricted arbitrage opportunities. A concrete example of such a restricted set of trading strategies is given in Carr, Géman, Madan and Yor (2003) (henceforth CGMY). These authors introduce the phrase "static arbitrage" to describe a much simpler concept of an arbitrage opportunity. Parsing this phrase, the arbitrage part refers as usual to a costless trading strategy which at some future time provides a positive profit with positive probability, but has no possibility of a loss. The static part of the phrase signifies that the position taken in the underlying stock at a particular time can only depend on the time and on the contemporaneous stock price. Positions cannot depend on past prices or their path properties. CGMY establish the equivalence of no static arbitrage with the existence of a Markov martingale in some information filtration. This martingale is used to price the assets just at the initial time by discounting the expected cash flow under this process. Assuming that the initial market prices of a given set of European calls are free of static arbitrage, Madan and Yor (2002) go beyond the CGMY existence result by providing several concrete constructions of Markov martingales which are consistent with these prices.

The CGMY results suggest a related problem which is of considerable practical interest. This problem is to determine whether or not a given rectangular grid of European option prices quoted across several strikes and maturities is in fact free of static arbitrage. Many specific arbitrages are known due to the seminal results of Merton (1973) and others, but it is still unclear at this relatively late date whether the elimination of these known arbitrages is sufficient to exclude all static arbitrages. The limited but useful contribution of this short note is to establish the structure of a finite set of tests that do in fact imply that all of the option quotes are free of static arbitrage. If the given option quotes pass our set of tests, then no further tests need to be conducted in order to certify that the quotes are free of static arbitrage.

A critical step in the formation of our tests requires the extraction of risk-neutral probability measures at each maturity from quoted option prices. This fundamental idea is essentially due to Breeden and Litzenberger (1978). The extraction of these measures is followed by establishing sufficient conditions for these measures to be increasing in the convex order as described in Shaked and Shantikumar (1994). We next appeal to Kellerer

(1972) who establishes that these conditions are equivalent to the existence of a Markov martingale with the same marginals as those extracted from option prices. The filtration for this martingale includes price paths and hence the current price. Standard results then imply that there are no static arbitrage opportunities. We note that Kellerer's results are also implicit in the earlier works in economics on increasing risk published in Rothschild and Stiglitz (1970, 1971).

The structure of this note is as follows. Section 2 establishes sufficient conditions for the existence of a risk-neutral probability measure at each maturity. Section 3 lists sufficient conditions for these measures to be increasing in the convex order. Section 3 also shows that our set of tests are jointly sufficient to establish that the provided option quotes are free of static arbitrage. Section 4 concludes.

2. The price grid and marginal probabilities

We suppose that at some fixed time, we have been given several market quotes for European call options arranged on a rectangular grid. More specifically, the quotes are for a countably infinite collection of discrete strikes and for each strike, the quotes are for a common set of maturities which are finite in number. We recognize that the assumed regularity of the provided data may fail to hold in practice, but we regard the issue of missing data as an alternative problem.

Let C_{ij} denote the given quote for a call of strike K_i, $i = 1, \dots, \infty$, and maturity T_j, $j = 1, \dots, M$. We suppose that the strikes K_i form an increasing and positive sequence as do the maturities T_j. We also suppose that the strikes tend to infinity as i tends to infinity. We require that the given call prices tend to zero as the strike becomes infinite. Furthermore, we suppose that interest rates and dividends are zero over the period ending at the longest maturity. We augment the provided call quotes with quotes for calls of strikes $K_0 \equiv 0$. For each maturity, these additional quotes are taken to be S_0, the current spot price. We also take the prices at maturity $T_0 = 0$ to be $(S_0 - K_i)^+$, $i = 1, \dots, \infty$. This gives us the augmented matrix of prices C_{ij}, with now $i = 0, \dots, \infty$ and $j = 0, \dots, M$.

Suppose that for each $j > 0$, we define the following quantities:

$$\overline{Q}_{i,j} \equiv \frac{C_{i-1,j} - C_{i,j}}{K_i - K_{i-1}}, \quad i > 0, \qquad \overline{Q}_{0,j} \equiv 1. \tag{1}$$

For each $i > 0$, $\overline{Q}_{i,j}$ is clearly the cost of a vertical spread which by definition is long $1/(K_i - K_{i-1})$ calls of strike K_{i-1} and short $1/(K_i - K_{i-1})$ calls of strike K_i. A graph of the payoff from this position against the terminal stock price indicates that this payoff is bounded below by zero and above by one. We therefore require for our first test that $\overline{Q}_{i,j} \in [0, 1]$ for all i, j.

Next, for each $j > 0$, we define the following quantities:

$$BS_{i,j} \equiv C_{i-1,j} - \frac{K_{i+1} - K_{i-1}}{K_{i+1} - K_i} C_{i,j} + \frac{K_i - K_{i-1}}{K_{i+1} - K_i} C_{i+1,j}, \quad i > 0.$$

For each $i > 0$, $BS_{i,j}$ is clearly the cost of a butterfly spread which by definition is long the call struck at K_{i-1}, short $(K_{i+1} - K_{i-1})/(K_{i+1} - K_i)$ calls struck at K_i, and long

$(K_i - K_{i-1})/(K_{i+1} - K_i)$ calls struck at K_{i+1}. A graph quickly indicates that the butterfly spread payoff is nonnegative and hence our second test requires that

$$C_{i-1,j} - \frac{K_{i+1} - K_{i-1}}{K_{i+1} - K_i} C_{i,j} + \frac{K_i - K_{i-1}}{K_{i+1} - K_i} C_{i+1,j} \geqslant 0.$$

Equivalently, we require that

$$C_{i-1,j} - C_{i,j} \geqslant \frac{K_i - K_{i-1}}{K_{i+1} - K_i} (C_{i,j} - C_{i+1,j}). \tag{2}$$

It follows from (1) and (2) that the risk-neutral complementary distribution functions are declining at each maturity:

$$\overline{Q}_{i,j} \geqslant \overline{Q}_{i+1,j}, \quad i, j \geqslant 0.$$

Given that the call quotes pass both tests, suppose we now define

$$q_{i,j} \equiv \overline{Q}_{i,j} - \overline{Q}_{i+1,j} = \frac{C_{i-1,j} - C_{i,j}}{K_i - K_{i-1}} - \frac{C_{i,j} - C_{i+1,j}}{K_{i+1} - K_i}, \quad i = 1, \ldots, \infty. \tag{3}$$

The sequence of nonnegative numbers q_i, $i = 1, \ldots, \infty$, defines a probability mass function (PMF) since $\sum_{i=1}^{\infty} q_{i,j} = 1$. We may interpret each $q_{i,j}$ as the marginal risk-neutral probability that the stock price at maturity T_j equals K_i for $i = 1, \ldots, \infty$. Furthermore, (3) implies that the risk-neutral complementary distribution function can be recovered from the risk-neutral PMF by

$$\overline{Q}_{i,j} = \sum_{k=i}^{\infty} q_{k,j}, \quad i, j > 0.$$

Summarizing to this point, we have imposed two tests on the provided call quotes. Assuming that the quotes pass both tests, the quotes have been manipulated in a standard way to yield the marginal risk-neutral probabilities of the stock price being at level K_i at time T_j for all $i, j \geqslant 0$. We may now define the call pricing function on the continuum of strikes $K \geqslant 0$ for each discrete maturity T_j by

$$C_j(K) \equiv \sum_l (K_l - K)^+ q_{l,j}.$$

The call prices obtained in this way linearly interpolate the provided call quotes. For future use, we also associate with each maturity a risk-neutral probability measure defined by

$$Q_j(K) = \sum_{K_l \leqslant K} q_{l,j}.$$

As K increases from 0, this measure will be piecewise constant and increasing from 0 to 1.

3. The convex order

As a third and final test on the provided call quotes, we require that for each discrete strike K_i, $i \quad 0$, and each discrete maturity T_j, $j \quad 0$:

$$C_{i,j+1} - C_{i,j} \quad 0, \quad i, j \quad 0. \tag{4}$$

The left-hand side of (4) is clearly the cost of a calendar spread consisting of long one call of maturity T_{j+1} and short one call of maturity T_j, with both calls struck at K_i. Hence, our third test requires that calendar spreads comprised of adjacent maturity calls are not negatively priced at each maturity.

As the call pricing functions are linear interpolations of the provided quotes, we have that at each maturity T_j, calendar spreads are not negatively priced for the continuum of strikes $K > 0$. Since all convex payoffs may be represented as portfolios of calls with nonnegative weights, it follows that all convex functions $\phi(S)$ are priced higher when promised at T_{j+1} than when they are promised at T_j. In turn, this ordering implies that the risk-neutral probability measures $Q_j(K)$ constructed in the last section are increasing in the convex order with respect to the index j.

By the results of Kellerer (1972), it follows that there exists a discrete time Markov martingale M_j with

$$E[M_{j+1} \mid M_j] = M_j,$$

such that all of the call prices satisfy

$$C_j(K) = E\big[(M_j - K)^+\big], \quad K > 0,\ j = 0, 1, \ldots, m.$$

This further implies that there exists a martingale measure which is consistent with the call quotes and which is defined on some filtration that includes at least the stock price and time. Finally, it follows that the provided call quotes are free of static arbitrage by standard results in arbitrage pricing theory.

4. Conclusion

This short note has shown that a rectangular grid of European call quotes are free of static arbitrage if all adjacent vertical spreads, butterfly spreads, and calendar spreads are nonnegatively priced. The result relies on deep results on constructing Markov martingales which match given marginals when the latter are found to be increasing in the convex order. These results go back to the work of Rothschild and Stiglitz (1970, 1971) in economics and Kellerer (1972) in probability theory. Although both of these works are widely cited in their respective fields, their relevance for option pricing has not been previously highlighted to our knowledge. The current note rectifies this oversight by drawing attention to these papers and linking them to other well-known results in the option pricing literature.

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