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The Valuation of Executive Stock Options in an Intensity-Based Framework ⋆

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Abstract. This paper presents a general intensity-based framework to value executive stock options (ESOs). It builds upon the recent advances in the credit risk modeling arena. The early exercise or forfeiture due to voluntary or involuntary employment termination and the early exercise due to the executive's desire for liquidity or diversification are modeled as an exogenous point process with random intensity dependent on the stock price. Two analytically tractable specifications are given where the ESO value, expected time of exercise or forfeiture, and the expected stock price at the time of exercise or forfeiture are calculated in closed-form.

Key words: Brownian area, early exercise, executive stock options, Feynman-Kac formula, forfeiture, Laplace transform, occupation time, point processes with random intensity.

JEL classification: G13, G39, M41.

1. Introduction

Executive stock options (ESOs) currently constitute a sizable fraction of many firms' total compensation expense. It is important to accurately assess the cost of these options to shareholders both for accounting purposes and from a managerial control perspective (see Carpenter, 1998; Foster et al., 1991; Jennergren and Naslund, 1993). Since 1995, the Financial Accounting Standards Board (FASB) SFAS 123 has mandated that an estimate of the cost of ESO grants be disclosed in a footnote. Although it is not required, the recommended valuation method is to use the Black Scholes European call pricing formula. The suggested maturity used in this formula is the expected life, although the maximum life (typically 10 years at grant) can also be used. Rubinstein (1995) argues on theoretical grounds that either method will tend to cause overvaluation. Similarly, Marquardt (1999) empir-

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ically determines that both methods overvalue the economic cost to shareholders of issuing ESOs.

ESOs are typically long dated American calls which differ from standard options in that they have an initial vesting period during which exercise is proscribed. Although it is straightforward to numerically determine the value and the optimal exercise policy for ESOs in a frictionless market, certain institutional frictions complicate the determination of the optimal exercise policy for ESOs. First, the holder of an ESO can not sell or transfer his option. Furthermore, the holder cannot hedge his call since short positions in the company's stock are prohibited. In contrast, the issuer is allowed to transfer their liability or hedge their obligation. In general, this asymmetry drives a wedge between the value to the recipient and the value to the issuer. Both values are affected by the exercise policy used by executives, which is in general determined both by publicly available information such as stock prices and by executive-specific information such as personal portfolio composition, risk aversion, and the executive's demand for liquidity. The optimal exercise policy employed by the executive need not match the optimal exercise policy prevailing in the absence of these frictions since early exercise may be optimal for diversification or liquidity reasons even if the underlying stock does not pay any dividends. A second reason why the executive's optimal exercise policy may deviate from the perfect markets policy is that the executive may leave the firm either voluntarily or involuntarily while the option is alive. In this case, the executive forfeits his options if they are out-of-the-money, and will have to exercise early if they are in-the-money.

Two general approaches have been adopted to modeling executive exercise decisions and valuing the cost of ESOs to the firm. In the first approach, one assumes that the executive exercises the option according to a policy that maximizes his expected utility subject to hedging restrictions (Huddart, 1994; Marcus and Kulatilaka, 1994; Detemple and Sundaresan, 1998). In this approach, one must explicitly model such unobservable variables as the executive's risk aversion, his outside wealth, and the potential gain from changing his employment. In the alternative approach, one models early exercise as an exogenous stopping time, e.g., the first jump time of some exogenous Poisson process, as in Jennergren and Naslund (1993). The Poisson process serves as a proxy for anything that causes the executive to exercise the option early, including the desire for diversification or liquidity, and voluntary or involuntary employment termination. In contrast to the utility maximation approach, the hazard rate or intensity of this exogenous Poisson process is the only parameter in the model that needs to be estimated from empirical data. In an interesting recent paper, Carpenter (1998) shows that this second reduced form intensity-based model performs as well or better than the more complicated structural model in empirical tests of the two competing ESO valuation models in predicting actual exercise patterns for a sample of 40 firms.

This dichotomy in modeling the executive's exercise decision parallels the modeling of default events required in the valuation of credit risky corporate debt. The literature on pricing credit risky debt can be subdivided into two classes: structural models and reduced-form intensity-based models. The first class of models, dating back to Black and Scholes (1973) and Merton (1974), models the default event structurally as a utility maximization decision by the equity holders (see Leland (1994) and Leland and Toft (1996)). The second class of models are reduced-form models that exogenously specify default as occuring at the first jump time of a point process with random intensity (default hazard rate) (see Duffie et al., 1996; Duffie and Singleton, 1998; Jarrow and Turnbull, 1995; Jarrow et al., 1996; Lando, 1998; Madan and Unal, 1996, 1998). Davydov et al. (1998) value credit risky debt in the intensity-based framework uisng an approach similar to ours. In all such models, the intensity of the point process is calibrated to empirical data. Due to the relative simplicity of calibration and empirical testing, the reduced-form modeling philosophy is gaining considerable popularity in the credit markets.

The contribution of this paper is two-fold. First, we develop a general stochastic intensity-based framework for the valuation of ESOs in which the early exercise or forfeiture intensity $h_t = h(S_t, t)$ depends on the underlying stock price and time. Second, we suggest two simple analytically tractable specifications of hazard rate-based models of ESOs. In the first example, the intensity is specified as follows (assuming the ESO is vested):

$$h_t = \lambda_f + \lambda_e \mathbf{1}_{\{S_t > K\}}, \tag{1}$$

where S_t is the underlying stock price, K is the ESO's strike price, λ_f is the constant intensity of early exercise or forfeiture due to the exogenous voluntary or involuntary employment termination (assumed independent of the stock price), and $\lambda_e \mathbf{1}_{\{S_t > K\}}$ is the constant intensity of the early exercise due to the executive's exogenous desire for liquidity or diversification assumed positive and constant if the ESO is in-the-money and zero otherwise ($\mathbf{1}_A$ is the indicator function of the event A; e in λ_e stands for "exercise"). Thus, the intensity of forfeiture when the stock is out-of-the-money is λ_f (f stands for "forfeiture"), while the total intensity of early exercise when the option is in-the-money is $\lambda_f + \lambda_e$. The integrated hazard linearly depends on the *occupation time* of the underlying stock above the strike K (i.e., when the ESO is in-the-money) and the corresponding ESO valuation model draws on some recent results on *occupation time derivatives* (see Akahori, 1995; Chesney et al., 1997; Dassios, 1995; Davydov and Linetsky, 1998; Embrechts et al., 1995; Hugonnier, 1998; Linetsky, 1998, 1999; Pechtl, 1995, 1998).

In the second analytically tractable example, the intensity is specified as follows (assuming the ESO is vested):

$$h_t = \lambda_f + \lambda_e (\ln S_t - \ln K)^+. \tag{2}$$

In this case, the first term due to termination is still independent of the stock price,[1] but the second term due to the desire for liquidity or diversification is now a monotonically increasing function of the underlying stock price if the ESO is in-the-money and zero otherwise ($x^+ := x\mathbf{1}_{\{x>0\}}$ denotes the positive part of x). The integrated hazard linearly depends on the so-called *Brownian area* and the corresponding ESO valuation model draws on the results of Davydov, Linetsky and Lotz (1998) on *area options*.

The remainder of this paper is organized as follows. In Section 2, we consider a general stochastic intensity-based framework for the valuation of ESOs. In Section 3, we solve the model with the intensity specification given in (1). In Section 4, we solve the model with the intensity specification (2). Numerical examples are given in Section 5. Section 6 concludes the paper.

2. A General Intensity-Based Formulation

We assume frictionless markets, no dividends, a constant riskfree rate r, and that the underlying stock price obeys the following diffusion process under the risk-neutral probability measure Q:

$$dS_t = rS_t dt + \sigma(S_t, t) S_t dW_t^Q, \quad t > 0, \quad S_0 = S,$$

where W_t^Q is a standard Brownian motion, the process is starting at $S_0 = S$ at time $t = 0$, and the local volatility function $\sigma(S, t)$ is assumed continuous and strictly positive for all $S \in [0, \infty)$ and bounded as $S \to \infty$ (for all $t \geq 0$).

The time of early exercise or forfeiture $\mathcal{T}$ can be thought of as the first jump time of a point process with random intensity (hazard rate) h_t, which is generally a function of time and the underlying stock price, $h_t = h(S_t, t)$. Then the probability under Q of no early exercise up to time t for a given stock price path $\{S_u, 0 \leq u \leq t\}$ is (see Bremaud (1980) and Lando (1998) for details on point processes with random intensity):

$$Q(\mathcal{T} > t | \{S_u, 0 \leq u \leq t\}) = e^{-\int_0^t h(S_u, u) du}, \tag{3}$$

and

$$Q(\mathcal{T} > t) = E_{0,S}^Q \left[e^{-\int_0^t h(S_u, u) du} \right],$$

where the expectation is with respect to the risk-neutral measure Q.

Letting $t = 0$ be the ESO grant date and $t_v \in [0, T]$ be the ESO vesting date, the value at $t \in [0, T]$ of an unexercised ESO with strike price K and maturity T is given by the risk-neutral expectation:

[1] In general, one could also make the forfeiture intensity λ_f a function of the stock price arguing that the executive is more likely to leave the firm when the stock price is low relative to the strike price of his or her ESOs. For simplicity we assume that λ_f is constant.

$$C(S,t;K,T) = e^{-r(T-t)}E^{Q}_{t,S}[\mathbf{1}_{\{\mathcal{T}\geq T\}}(S_T-K)^+] + E^{Q}_{t,S}[e^{-r(\mathcal{T}-t)}\mathbf{1}_{\{\max(t_v,t)\leq\mathcal{T}<T\}}(S_{\mathcal{T}}-K)^+], \quad (4)$$

where $\mathcal{T}$ is a stopping time assumed to be the first jump time of the point process with intensity h_t, and the subscript t, S in the expectation operator $E_{t,S}$ signifies that the stock price is S at time t. Note that, following Jennergren and Naslund (1993), we assume that the jump risk is non-priced, i.e., that it can be diversified away by issuing a diversified portfolio of ESOs. Since many firms issue multiple ESOs[2], we regard this as a reasonable assumption in practice. The first term on the right hand side of Equation (4) is the present value of the option payoff at maturity given no early exercise. The second term is the present value of the payoff at the time of exercise, given that the option is exercised early. This decomposition of value is analogous to a decomposition of value arising for defaultable securities. The first term in (4) is analogous to the present value of the promised payment conditional on no default, while the second term is the present value of the recovery payment paid at the time of default if default occurs prior to maturity.

Due to the key relationship (3), the expectation can be re-written in the form:

$$C(S,t;K,T) = e^{-r(T-t)}E^{Q}_{t,S}\left[e^{-\int_t^T h_u du}(S_T-K)^+\right] + \int_{\max(t_v,t)}^{T} e^{-r(u-t)}E^{Q}_{t,S}\left[e^{-\int_t^u h_s ds}h_u(S_u-K)^+\right]du.$$

By the Feynman-Kac theorem (see, e.g., Karatzas and Shreve (1992)), the ESO value $C(S,t;K,T)$ at time t, $0 \leq t < T$, is the unique solution to the Cauchy problem for the PDE:

$$\frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC + h(S,t)[\mathbf{1}_{\{t>t_v\}}(S-K)^+ - C] + \frac{\partial C}{\partial t} = 0, \quad (5)$$

subject to the terminal condition

$$C(S,T;K,T) = (S-K)^+. \quad (6)$$

The financial meaning of the second last term on the left-hand- side of Equation (5) is that over an infinitesimal time period dt, there is a probability $h_t dt$ of the executive exercising his option and receiving $(S_t - K)^+$ in exchange if the ESO is vested ($t > t_v$) and nothing otherwise (the option is forfeited).

In addition to the ESO value, we are also interested in the expected time of exercise or forfeiture (the *expected ESO maturity*):

$$\bar{\mathcal{T}} = T E^{P}_{0,S}[\mathbf{1}_{\{\mathcal{T}\geq T\}}] + E^{P}_{0,S}[\mathbf{1}_{\{\mathcal{T}<T\}}\mathcal{T}], \quad (7)$$

[2] For example, Marquardt (1999) examines 58 Fortune 100 firms over a 21 year period and finds an average of 17 grants per firm.

and the expected stock price at the time of exercise or forfeiture:

$$\bar{S}_{\mathcal{T}} = E^P_{0,S}[\mathbf{1}_{\{\mathcal{T}\geq T\}} S_T] + E^P_{0,S}[\mathbf{1}_{\{\mathcal{T}<T\}} S_{\mathcal{T}}]. \tag{8}$$

Note that, in contrast to the ESO value calculation which is carried out under the risk-neutral measure Q, these quantities are calculated under the statistical measure P where:

$$dS_t = mS_t dt + \sigma(S_t, t) S_t dW^P_t, \quad S_0 = S,$$

and m is the expected annualized percentage rate of return on the stock in the real world (m is assumed constant). Using the key relationship (3) (considered under P), it is easy to see that Equations (7)- (8) reduce to:

$$\begin{aligned}\bar{\mathcal{T}} &= TP(\mathcal{T} \geq T) - \int_0^T t \frac{\partial P(\mathcal{T} > t)}{\partial t} dt = \int_0^T P(\mathcal{T} > t) dt \\ &= \int_0^T E^P_{0,S}\left[e^{-\int_0^t h(S_u,u)du}\right] dt, \end{aligned} \tag{9}$$

and

$$\bar{S}_{\mathcal{T}} = E^P_{0,S}\left[e^{-\int_0^T h(S_t,t)dt} S_T\right] + \int_0^T E^P_{0,S}\left[e^{-\int_0^t h(S_u,u)du} h(S_t, t) S_t\right] dt. \tag{10}$$

Carpenter (1998), Huddart and Lang (1996), and Marquardt (1999) all give empirical expected times of exercise and average stock prices at the time of exercise for their samples. Given the values of parameters m, σ, S, t_v, and T, one can calibrate the exercise or forfeiture intensity h_t to the empirical data using Equations (9) and (10).

3. The Occupation Time Specification: A Step Option Model for Valuing ESOs

In this section, we restrict the setup discussed in the previous section with a view towards obtaining explicit solutions for the quantities of interest. We assume constant volatility, i.e. $\sigma(S, t) = \sigma$, and that the option is vested, i.e., $t_v = 0$ (we extend to the case of options that are not yet vested at the end of this Section). We also consider a particularly simple specification for the exercise or forfeiture intensity:

$$h_t = \lambda_f + \lambda_e \mathbf{1}_{\{S_t > K\}}, \tag{11}$$

where S_t is the underlying stock price, K is the ESO's strike price, λ_f is the constant intensity of the early exercise or forfeiture due to the exogenous voluntary or involuntary employment termination (assumed independent of the stock price), and $\lambda_e \mathbf{1}_{\{S_t > K\}}$ is the constant intensity of the early exercise due to the executive's exogenous desire for liquidity or diversification assumed positive and constant if the ESO is in-the-money and zero otherwise.

Under these assumptions, the initial (i.e., $t = 0$) ESO value (4) simplifies to[3]:

$$C(S; K, T; \lambda_f, \lambda_e) = e^{-(r+\lambda_f)T} E^{Q}_{0,S}\left[e^{-\lambda_e \tau^+_K(T)}(S_T - K)^+\right] + (\lambda_f + \lambda_e)\int_0^T e^{-(r+\lambda_f)t} E^{Q}_{0,S}\left[e^{-\lambda_e \tau^+_K(t)}(S_t - K)^+\right] dt, \quad (12)$$

where $\tau^+_K(t) = \int_0^t \mathbf{1}_{\{S_u > K\}} du$ is the *occupation time* of the in-the-money region $\{S > K\}$ up to time t. This expectation can be expressed as a portfolio of *up-and-out geometric step options* with knock-out rate λ_e and knock-out barrier equal to the strike:

$$C(S; K, T; \lambda_f, \lambda_e) = e^{-\lambda_f T} C^+_{\lambda_e}(S; T, K, K) + (\lambda_f + \lambda_e)\int_0^T e^{-\lambda_f t} C^+_{\lambda_e}(S; t, K, K) dt, \quad (13)$$

where $C^+_{\lambda_e}(S; t, K, K)$ is the value of an up-and-out geometric step call with strike price K, knock-out rate λ_e, knock-out barrier level K, and maturity t (see Linetsky (1998, 1999)):

$$C^+_{\lambda_e}(S; t, K, K) = e^{-rt} E^{Q}_{0,S}[e^{-\lambda_e \tau^+_K(t)}(S_t - K)^+]. \quad (14)$$

The payoff at maturity t of a geometric step call can be interpreted as that of a standard call, except that the underlying share notional is path-dependent in that it depends on the occupation time above the strike: $e^{-\lambda \tau^+_K(t)}$. In other words, a geometric step call loses a given fraction of its notional per unit time above the barrier.

Introduce the following notation:

$$x := \frac{1}{\sigma}\ln\left(\frac{S}{K}\right), \quad \nu := \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right), \quad \xi := r + \frac{\nu^2}{2}. \quad (15)$$

Then the expectation in Equation (14) reduces to:

$$C^+_{\lambda_e}(S; t, K, K) = e^{-(\xi+\lambda_e)t - \nu x} K[\Psi_{-\lambda_e}(\nu + \sigma; 0, x, t) - \Psi_{-\lambda_e}(\nu; 0, x, t)], \quad (16)$$

[3] Note that the constant forfeiture intensity λ_f is added to the discount rate in Equation (12). Intuitively, the possibility of forfeiture lowers the value of the ESO in the same fashion as the possibility of default lowers the value of a defaultable bond, and the intensity of forfeiture is added to the risk-free rate as a "credit spread".

where the function Ψ is defined as:

$$\Psi_\rho(\nu; k, x, t) := E_{0,x}\left[e^{\nu W_t - \rho\Gamma_0^-(t)}\mathbf{1}_{\{W_t \geq k\}}\right], \tag{17}$$

where the expectation $E_{0,x}$ is conditional on the Brownian motion W_t starting at x at $t = 0$ and $\Gamma_0^-(t) = \int_0^t \mathbf{1}_{\{W_u<0\}}du$ is the occupation time of the negative half-line $(-\infty, 0)$ up to time t.[4] This expectation is computed in closed form in Linetsky (1999). For the reader's convenience, the explicit analytical form of the function Ψ is given in Appendix A. Thus, Equations (13) and (16) provide a simple analytical solution for the ESO value under the specification (11) for the exercise and forfeiture intensity.

The expected time of exercise or forfeiture (9) under this specification is:

$$\bar{\mathcal{T}} = \int_0^T e^{-(\lambda_f+\lambda_e+\nu_P^2/2)t-\nu_P x}\Psi_{-\lambda_e}(\nu_P; -\infty, x, t)dt, \tag{18}$$

where (recall that $\bar{\mathcal{T}}$ and $\bar{S}_{\mathcal{T}}$ are computed under the statistical measure P):

$$\nu_P := \frac{1}{\sigma}\left(m - \frac{\sigma^2}{2}\right). \tag{19}$$

The expected stock price at the time of exercise or forfeiture is:

$$\begin{aligned}\bar{S}_{\mathcal{T}} &= e^{-(\lambda_f+\lambda_e+\nu_P^2/2)T-\nu_P x}K\Psi_{-\lambda_e}(\nu_P+\sigma; -\infty, x, T)\\ &\quad +K\int_0^T e^{-(\lambda_f+\lambda_e+\nu_P^2/2)t-\nu_P x}[\lambda_f\Psi_{-\lambda_e}(\nu_P+\sigma; -\infty, x, t)\\ &\quad +\lambda_e\Psi_{-\lambda_e}(\nu_P+\sigma; 0, x, t)]dt.\end{aligned} \tag{20}$$

Now consider the case $t_v > 0$, i.e., the option is not yet vested. Suppose $S_v = S(t_v)$ is the stock price on the vesting date. The ESO value on the vesting date t_v is given by $C(S_v; K, T - t_v; \lambda_f, \lambda_e)$ defined by Equation (13) (note that the time to maturity is now equal to $T - t_v$, so we need to substitute $T \to T - t_v$ in Equation (13)). Then the ESO value at time $t = 0$ is computed by taking the expectation:

$$\begin{aligned}&C(S, 0; K, t_v, T; \lambda_f, \lambda_e)\\ &\quad = e^{-(r+\lambda_f)t_v}\int_0^\infty C(S_v; K, T-t_v; \lambda_f, \lambda_e)p^Q(S_v, t_v|S, 0)dS_v,\end{aligned} \tag{21}$$

[4] For the background on occupation times and other functionals of Brownian motion and diffusion processes, as well as Feynman-Kac-type calculations of their laws, see Karatzas and Shreve (1992), Borodin and Salminen (1996) and Revuz and Yor (1994).

where p^Q is the (lognormal) probability density of the stock price on the vesting date, given the known stock price today (at time $t = 0$):

$$p^Q(S_v, t_v|S, 0) = \frac{1}{S_v\sqrt{2\pi\sigma^2 t_v}} \exp\left\{-\frac{\left[\ln\left(\frac{S_v}{S}\right) - \mu t_v\right]^2}{2\sigma^2 t_v}\right\}, \quad \mu = r - \frac{\sigma^2}{2}. \tag{22}$$

4. The Brownian Area Specification: An Area Option Model for Valuing ESOs

As in the previous section, we first assume that the option is already vested, i.e., $t_v = 0$. Under the occupation time specification, the exercise or forfeiture intensity is constant above the strike. An analytically tractable alternative is:

$$h_t = \lambda_f + \lambda_e(\ln S_t - \ln K)^+ = \lambda_f + \lambda_e \quad \ln \quad \frac{S_t}{K} \quad ^+. \tag{23}$$

In this case, the first term due to voluntary or involuntary employment termination is still independent of the stock price, but the second term due to the desire for liquidity or diversification is now an increasing function of the moneyness S_t/K if the ESO is in-the-money and zero otherwise (x^+ denotes the positive part of x). A similar specification for the default hazard rate was used by Davydov, Linetsky, and Lotz (1998) to model credit risky corporate debt.

The vested ESO value (4) under this specification takes the form:

$$\begin{aligned} C(S; K, T; \lambda_f, \lambda_e) = {} & e^{-(r+\lambda_f)T} E^Q_{0,S}\left[\exp \quad -\lambda_e \int_0^T (\ln S_t - \ln K)^+ dt \quad (S_T - K)^+\right] \\ & + \int_0^T e^{-(r+\lambda_f)t} E^Q_{0,S}\left[\exp \quad -\lambda_e \int_0^t (\ln S_u - \ln K)^+ du \right. \\ & \left. \times \left[\lambda_f + \lambda_e \ln \quad \frac{S_t}{K}\right)\right] (S_t - K)^+\right] dt. \end{aligned} \tag{24}$$

To calculate this expectation, we first note that the stock price process can be represented as:

$$S_t = K e^{\sigma(\nu t + W_t)}, \tag{25}$$

where W_t is a Brownian motion starting at x (defined in Equation (15)) at time $t = 0$. Then due to Girsanov's theorem:

$$\begin{aligned} C(S;K,T;\lambda_f,\lambda_e) = {} & e^{-(r+\lambda_f)T} E_{0,x}\left[e^{\nu(W_T - x) - \frac{\nu^2}{2}T - \sigma\lambda_e \int_0^T W_t^+ dt} (K e^{\sigma W_T} - K)^+\right] \\ & + \int_0^T e^{-(r+\lambda_f)t} E_{0,x}\left[e^{\nu(W_t - x) - \frac{\nu^2}{2}t - \sigma\lambda_e \int_0^t W_u^+ du} [\lambda_f + \sigma\lambda_e W_t]\right. \end{aligned}$$

$$
\begin{aligned}
&\times (Ke^{\sigma W_t} - K)^+ \ dt \\
= \ & e^{-(\xi+\lambda_f)T-\nu x} K[\Phi_{\sigma\lambda_e}(\nu+\sigma; 0, x, T) - \Phi_{\sigma\lambda_e}(\nu; 0, x, T)] \\
&+ e^{-\nu x} K \int_0^T e^{-(\xi+\lambda_f)t} \ \lambda_f \Phi_{\sigma\lambda_e}(\nu+\sigma; 0, x, t) \\
&-\lambda_f \Phi_{\sigma\lambda_e}(\nu; 0, x, t) + \sigma\lambda_e \frac{\partial \Phi_{\sigma\lambda_e}(\nu+\sigma; 0, x, t)}{\partial \nu} \\
&-\sigma\lambda_e \frac{\partial \Phi_{\sigma\lambda_e}(\nu; 0, x, t)}{\partial \nu} \ \ dt, \qquad (26)
\end{aligned}
$$

where we introduced the following notation:

$$\Phi_\alpha(\nu; k, x, t) := E_{0,x}\left[e^{\nu W_t - \alpha A_t^+} \mathbf{1}_{\{W_t \geq k\}}\right], \qquad (27)$$

$$A_t^+ := \int_0^t W_u^+ du. \qquad (28)$$

The functional A_t^+ is called *Brownian area* until time t (see Perman and Wellner, 1996). It is equal to the (random) *area under the positive part of a Brownian sample path from zero to time* t. The expectation in Equation (27) is calculated by Davydov, Linetsky, and Lotz (1998) via the Feynman-Kac theorem:

$$
\begin{aligned}
\Phi_\alpha(\nu; k, x, t) &= \int_k^\infty e^{\nu y} E_{0,x}\left[e^{-\alpha A_t^+}; W_t \in dy\right] \\
&= \int_k^\infty e^{\nu y} \mathcal{L}_t^{-1}\{G_\alpha(x, y; s)\} dy, \qquad (29)
\end{aligned}
$$

where the expectation inside the integral is expressed as the inverse Laplace transform in s of the resolvent kernel $G_\alpha(x, y; s)$. Its analytical form is given in Appendix B.[5]

The expected time of exercise or forfeiture under this specification is:

$$\bar{\mathcal{T}} = \int_0^T e^{-(\lambda_f + \nu_P^2/2)t - \nu_P x} \Phi_{\sigma\lambda_e}(\nu_P; -\infty, x, t) dt, \qquad (30)$$

where ν_P is given in Equation (19). The expected stock price at the time of exercise or forfeiture is:

[5] The calculation of this functional is close in spirit to the calculations of Geman and Yor (1993) for Asian options and Geman and Yor (1996) for double-barrier options and relies on the Feynman-Kac formula.

$$\begin{aligned}\bar{S}_{\mathcal{T}} &= e^{-(\lambda_f+\nu_P^2/2)T-\nu_P x} K \Phi_{\sigma\lambda_e}(\nu_P+\sigma;-\infty,x,T) \\ &\quad + K\int_0^T e^{-(\lambda_f+\nu_P^2/2)t-\nu_P x} \ \lambda_f \Phi_{\sigma\lambda_e}(\nu_P+\sigma;-\infty,x,t) \\ &\quad + \sigma\lambda_e \frac{\partial\Phi_{\sigma\lambda_e}(\nu_P+\sigma;0,x,t)}{\partial\nu_P} \ dt. \end{aligned} \tag{31}$$

The case $t_v > 0$, i.e., the option is not yet vested, is treated similarly to Equation (21).

5. Numerical Examples

To illustrate our models, consider a ten year ESO granted at-the- money[6] ($S = K = 100$) and vested immediately ($t_v = 0$). We assume that the underlying stock has volatility of 30% per annum, pays no dividends, the riskfree rate is 5% per annum, and the expected annualized percentage rate of return on the stock under the statistical measure P is $m = 15\%$ per annum (recall that the expected time of exercise or forfeiture and the expected stock price at the time of exercise or forfeiture are calculated under the statistical measure). Tables I and II give the ESO value at the grant date, the expected time of exercise or forfeiture, and the expected stock price at the time of exercise or forfeiture as functions of the parameters of the point process λ_f and λ_e under the occupation time specification (11) and the Brownian area specification (23), respectively. For $\lambda_f = \lambda_e = 0$, the ESO value is equal to the ten-year Black-Scholes value, the expected exercise time is equal to the ESO maturity (ten years), and the expected stock price at the time of exercise is equal to $e^{10m}S$ (no early exercise or forfeiture). As the rates λ_f and λ_e increase, the ESO value, expected exercise or forfeiture time and the expected stock price at the time of exercise or forfeiture all decrease. Given $\bar{\mathcal{T}}$ and $\bar{S}_{\mathcal{T}}$, one can calibrate our models by backing out the intensity parameters λ_f and λ_e, and value ESOs with these parameter values. Carpenter (1998) reports that average exercise times for 10 year ESOs in her sample are about 5.8 years, with the average stock price at the time of exercise of about 2.8 times the ESO strike price. Marquardt (1999), who studies a different sample of ESO granting firms, reports that average exercise times for 10 year ESOs in her sample are about 5.06 years, with the average stock price at the time of exercise of about 2.02 times the ESO strike price. Thus, empirically, typical exercise times are in the five to six year range, with the stock price at the time of exercise of two to three times the ESO strike.

Consider an example of the occupation time model with $\lambda_f = 8\%$ per annum and $\lambda_e = 12\%$ per annum. The expected exercise time for these intensities is 4.99 years, with the expected stock price at the time of exercise of 2.31 times the ESO

6 Marquardt (1999) found that 85% of the 987 ESOs in her sample were issued with ten years to maturity. She states that most are issued with strike equal to stock price at grant.

Table I. **Occupation Time Model.** ESO values, expected times of exercise or forfeiture and expected stock prices at the time of exercise or forfeiture as functions of the intensity parameters λ_f and λ_e. Parameters: $K = 100$, $S_0 = 100$, $T = 10$ years, $\sigma = 0.30$, $r = 0.05$, $m = 0.15$, $t_v = 0$, no dividends

	λ_e									
λ_f	0	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16	0.18
	ESO value									
0	52.56	51.74	50.97	50.24	49.57	48.93	48.33	47.76	47.22	46.71
0.02	49.03	48.31	47.63	47.00	46.41	45.85	45.32	44.82	44.34	43.89
0.04	45.89	45.26	44.67	44.11	43.59	43.09	42.63	42.18	41.76	41.36
0.06	43.09	42.54	42.02	41.53	41.07	40.63	40.22	39.83	39.45	39.10
0.08	40.59	40.10	39.64	39.21	38.80	38.42	38.05	37.71	37.37	37.06
0.1	38.35	37.92	37.51	37.13	36.77	36.43	36.10	35.79	35.50	35.22
0.12	36.33	35.95	35.60	35.26	34.94	34.63	34.34	34.07	33.80	33.55
0.14	34.52	34.18	33.86	33.56	33.28	33.01	32.75	32.50	32.27	32.04
0.16	32.88	32.58	32.30	32.03	31.77	31.53	31.30	31.08	30.87	30.67
0.18	31.39	31.12	30.87	30.63	30.41	30.19	29.98	29.78	29.59	29.41
	Expected exercise or forfeiture time (years)									
0	10.00	9.60	9.24	8.91	8.62	8.36	8.11	7.89	7.69	7.50
0.02	9.06	8.72	8.40	8.12	7.87	7.64	7.43	7.23	7.05	6.89
0.04	8.24	7.94	7.67	7.43	7.20	7.00	6.82	6.65	6.49	6.35
0.06	7.52	7.26	7.02	6.81	6.62	6.44	6.28	6.13	5.99	5.87
0.08	6.88	6.66	6.45	6.27	6.10	5.94	5.80	5.67	5.55	5.44
0.1	6.32	6.12	5.95	5.78	5.63	5.50	5.37	5.26	5.15	5.05
0.12	5.82	5.65	5.50	5.35	5.22	5.10	4.99	4.89	4.80	4.71
0.14	5.38	5.23	5.09	4.97	4.86	4.75	4.65	4.56	4.48	4.40
0.16	4.99	4.86	4.74	4.63	4.53	4.43	4.35	4.27	4.19	4.12
0.18	4.64	4.52	4.42	4.32	4.23	4.15	4.07	4.00	3.94	3.87
	Expected stock price at time of exercise or forfeiture relative to strike									
0	4.48	4.18	3.93	3.71	3.53	3.37	3.23	3.10	2.99	2.90
0.02	4.08	3.82	3.61	3.42	3.26	3.12	3.00	2.90	2.80	2.72
0.04	3.73	3.51	3.33	3.17	3.03	2.91	2.81	2.72	2.63	2.56
0.06	3.43	3.25	3.09	2.95	2.83	2.73	2.64	2.56	2.49	2.42
0.08	3.17	3.01	2.87	2.76	2.65	2.57	2.49	2.42	2.35	2.30
0.10	2.95	2.81	2.69	2.59	2.50	2.42	2.35	2.29	2.24	2.19
0.12	2.75	2.63	2.53	2.44	2.36	2.30	2.24	2.19	2.14	2.10
0.14	2.58	2.48	2.39	2.31	2.25	2.19	2.14	2.09	2.05	2.01
0.16	2.43	2.34	2.26	2.20	2.14	2.09	2.04	2.00	1.97	1.93
0.18	2.30	2.22	2.15	2.10	2.05	2.00	1.96	1.93	1.90	1.87

Table II. **Area Model.** ESO values, expected times of exercise or forfeiture and expected stock prices at the time of exercise or forfeiture as functions of the intensity parameters λ_f and λ_e. Parameters: $K = 100$, $S_0 = 100$, $T = 10$ years, $\sigma = 0.30$, $r = 0.05$, $m = 0.15$, $t_v = 0$, no dividends

	λ_e									
λ_f	0	0.02	0.04	0.06	0.08	0.1	0.12	0.14	0.16	0.18
	ESO value									
0	52.56	50.33	48.29	46.42	44.71	43.13	41.68	40.35	39.11	37.96
0.02	49.03	47.05	45.23	43.57	42.04	40.63	39.33	38.13	37.02	35.98
0.04	45.89	44.13	42.51	41.02	39.66	38.39	37.23	36.14	35.14	34.20
0.06	43.09	41.52	40.07	38.74	37.51	36.38	35.33	34.35	33.44	32.59
0.08	40.59	39.19	37.89	36.69	35.59	34.56	33.61	32.73	31.90	31.13
0.10	38.35	37.09	35.92	34.85	33.85	32.92	32.06	31.26	30.51	29.80
0.12	36.33	35.20	34.15	33.18	32.28	31.44	30.65	29.92	29.24	28.59
0.14	34.52	33.50	32.55	31.67	30.85	30.08	29.37	28.70	28.08	27.49
0.16	32.88	31.96	31.10	30.30	29.55	28.85	28.20	27.59	27.01	26.47
0.18	31.39	30.56	29.77	29.04	28.36	27.73	27.13	26.57	26.04	25.54
	Expected exercise or forfeiture time (years)									
0	10.00	9.32	8.72	8.18	7.69	7.26	6.86	6.51	6.19	5.89
0.02	9.06	8.47	7.94	7.47	7.04	6.65	6.31	5.99	5.71	5.45
0.04	8.24	7.72	7.26	6.84	6.46	6.12	5.82	5.54	5.28	5.05
0.06	7.52	7.06	6.65	6.29	5.95	5.65	5.38	5.13	4.90	4.70
0.08	6.88	6.48	6.12	5.79	5.50	5.23	4.99	4.77	4.57	4.38
0.10	6.32	5.97	5.65	5.36	5.10	4.86	4.64	4.44	4.26	4.10
0.12	5.82	5.51	5.23	4.97	4.74	4.52	4.33	4.15	3.99	3.84
0.14	5.38	5.10	4.85	4.62	4.41	4.22	4.05	3.89	3.74	3.61
0.16	4.99	4.74	4.52	4.31	4.13	3.96	3.80	3.66	3.52	3.40
0.18	4.64	4.42	4.22	4.03	3.87	3.71	3.57	3.44	3.32	3.21
	Expected stock price at time of exercise or forfeiture relative to strike									
0	4.48	4.14	3.83	3.57	3.34	3.13	2.96	2.80	2.66	2.53
0.02	4.08	3.78	3.52	3.29	3.09	2.92	2.76	2.62	2.50	2.39
0.04	3.73	3.48	3.25	3.05	2.88	2.72	2.59	2.47	2.36	2.27
0.06	3.43	3.21	3.02	2.84	2.69	2.56	2.44	2.33	2.24	2.16
0.08	3.17	2.98	2.81	2.66	2.53	2.41	2.31	2.22	2.13	2.06
0.10	2.95	2.78	2.63	2.50	2.39	2.28	2.19	2.11	2.04	1.97
0.12	2.75	2.60	2.48	2.36	2.26	2.17	2.09	2.02	1.95	1.90
0.14	2.58	2.45	2.34	2.24	2.15	2.07	2.00	1.94	1.88	1.83
0.16	2.43	2.32	2.22	2.13	2.05	1.98	1.92	1.87	1.81	1.77
0.18	2.30	2.20	2.11	2.04	1.97	1.91	1.85	1.80	1.76	1.72

strike. The ESO value corresponding to these parameters is \$33.61. In contrast, the FASB-recommended valuation method is to use the Black Scholes European call pricing formula. The maturity used in this formula can be either the maturity date (ten years in this case) or an estimate of the expected life (4.99 years in this case). The corresponding Black- Scholes value of a ten year call is \$52.56. It is 56.38% higher than the value predicted by our model. The Black-Scholes value of a 4.99 year call is \$35.92, 6.87% higher than the value predicted by our model. Thus, the ESO values computed according to the intensity-based model are significantly lower than the corresponding Black-Scholes values, accounting for the suboptimal behavior of the executive. This has significant accounting implications. If one were to value ESOs for accounting purposes using the Black-Scholes model as recommended by FASB, one would significantly overstate their true costs to shareholders and unfairly penalize companies granting ESOs.

6. Conclusion and Directions for Future Research

The contribution of this paper is two-fold. First, we develop a general stochastic intensity-based framework for the valuation of executive stock options. Second, we suggest two analytically tractable specifications for the exercise and forfeiture intensity. Both specifications have the form (assuming the ESO is vested):

$$h_t = \lambda_f + \lambda_e \phi(S_t)\mathbf{1}_{\{S_t > K\}},$$

where λ_f is the constant Poisson intensity of early exercise or forfeiture due to early voluntary or involuntary employment termination, and $\lambda_e \phi(S_t)\mathbf{1}_{\{S_t > K\}}$ is the early exercise intensity due to the executive's desire for liquidity or diversification. The latter intensity is positive only when the option is in-the-money. Under the first specification, $\phi(S) = 1$. This leads to the analytically tractable occupation time model for ESOs, where the probability of early exercise due to the executive's desire for liquidity or diversification depends on the occupation time of the in-the-money region. Under the second specification, $\phi(S) = \ln S - \ln K$, leading to the analytically tractable Brownian area model. Both specifications reflect the fact that there are two distinct economic factors influencing the executive exercise decision. These are the executive's desire for liquidity or diversification which only induces exercise when the option is vested and in-the-money, and the possibility of voluntary or involuntary employment termination (this is equally likely when the option is in- or out-of-the-money and is assumed to be independent of the stock price). We argue that our specification with two separate intensity parameters provides a more complete description of the economic situation at hand than previous work[7] which modeled early exercise and forfeiture as arising from a Poisson process with a single constant intensity parameter independent of the stock price.

[7] See Shimko (1990) and Jennergen and Naslund (1993) for the special case of our model with $\lambda_e = 0$.

Our results can be further extended in several ways. First, in practice firms sometimes reset the terms of previously issued ESOs, especially when declining stock prices have moved the option deep out-of-the-money. In some interesting recent work, Brenner, Sundaram, and Yermack (1998) develop a model to value ESOs, which accounts for the possibility of *repricing*. Repricing involves specifying a new strike price when the stock price declines significantly.[8] When the option is repriced, the new strike price is specified (in practice, the new strike is often set equal to the then-current stock price, i.e. the option is re-written at-the-money). Brenner, Sundaram and Yermack (1998) note that, ignoring the possibility of early exercise or forfeiture, an ESO whose strike price K will change to K^* the first time the stock price falls below a pre-specified barrier B, can be valued as a portfolio of a down-and-out call with the strike price K (old strike) and a down-and-in call with the strike K^* (new strike). Then the standard barrier option valuation formulas are used to value the ESO (see Rubinstein and Reiner (1991) for example). Our approach to modeling early exercise and forfeiture can be extended to ESOs subject to repricing in this manner by adding a lower barrier to our analysis.

Consistent with our approach to modeling forfeiture and early exercise, an alternative approach to modelling repricing is to assume that it occurs at the first jump time of a point process, with some intensity dependent on the stock price. One possible (and analytically tractable) choice would be:

$$h_t = \lambda_r \mathbf{1}_{\{S_t < H\}},$$

where H is some barrier set at or below the strike K, and λ_r is constant. We note that the model of Brenner, Sundaram, and Yermack (1998) arises as a special case of this framework by letting λ_r approach infinity. A second possible (and analytically tractable) choice for the specification of the repricing intensity would be:

$$h_t = \lambda_r(\ln H - \ln S_t)^+,$$

where again $H \le K$, and λ_r is constant. As in the first specification, the probability of repricing in this model is zero if the option is in-the-money and positive when the option is out- of-the-money. Now, however the probability of repricing increases as the stock price declines below the barrier H.

Second, our methodology can be extended to *indexed ESOs*. Johnson and Tian (1999) design and develop a pricing model for an ESO with a strike price indexed to a benchmark index. The indexed option filters out common risks beyond the executive's control, thereby increasing the efficiency of incentive contracts by focusing them on the relative performance of the company stock relative to a benchmark. Johnson and Tian (1999) derive the ESO pricing formula based on

[8] The empirical evidence in Chance, Kumar, and Todd (1999) suggests that ESOs are usually repriced when the stock declines by about 25%

Margrabe's (1978) exchange option formula, ignoring the effects of early exercise and forfeiture. Our approach can be used to relax the latter assumption.

A third extension of this line of research would involve valuing ESOs of companies which pay sizeable dividends. Formally, this is an extension of our results to time and stock price dependent intensity which becomes infinite if the stock price is above the critical stock price at an ex-dividend date. This extension would be most relevant for firms such as utilities which typically have large dividends and low volatilities.

Finally, our methodology can be applied to value other assets. For example, it is well known that mortgages are not usually prepaid optimally and that companies often call their debt late. Potential explanations for late calling include bounded rationality, signalling phenomena, or agency costs. The latter two explanations account for the realistic possibility that the decision depends on private as well as public information. A model in which the probability of prepayment or call depends on the interest rate (and stock prices in the case of callable convertibles) might tractably capture the behavior of investors or managers more reliably than requiring that decisions be based on publicly available information. In general, the implications for asset pricing of optimizing behavior based on both public and private information is a fascinating avenue for future research.

Appendix

A. The expectation $E_{0,x}\left[e^{\nu W_T-\rho\Gamma_0^-(T)}\mathbf{1}_{\{W_T\geq k\}}\right]$

Let $0 \leq t < T$. Introduce the following notation:

$$d_1 = \frac{-k+x+\nu T}{\sqrt{T}}, \quad d_2 = d_1 + \sigma\sqrt{T},$$

$$d_3 = \frac{-k-x+\nu T}{\sqrt{T}}, \quad d_4 = d_3 + \sigma\sqrt{T},$$

$$d_5 = \frac{-k-x+\nu t}{\sqrt{t}}, \quad d_6 = d_5 + \sigma\sqrt{t},$$

$$d_7 = \frac{-k+\nu t}{\sqrt{t}}, \quad d_8 = d_7 + \sigma\sqrt{t},$$

$$C_1 = 1 - \frac{x^2}{T-t} - \nu x, \quad C_2 = t^{-1/2}C_1 - t^{-3/2}xk, \quad C_3 = C_1 - \sigma x.$$

Then the function $\Psi_\rho(\nu; k, x, T) \equiv E_x\left[e^{\nu W_T-\rho\Gamma_0^-(T)}\mathbf{1}_{\{W_T\geq k\}}\right]$ is given by (Linetsky, 1999):

- *Region I*: $k \geq 0$ and $x \geq 0$

$$\Psi_\rho^I(\nu; k, x, T) = e^{\nu x + \frac{\nu^2}{2}T} N(d_1) - e^{-\nu x + \frac{\nu^2}{2}T} N(d_3)$$
$$+ e^{-\nu x} \int_0^T \frac{(1-e^{-\rho(T-t)})e^{\frac{\nu^2}{2}t}}{\sqrt{2\pi}\rho(T-t)^{3/2}} \quad \nu N(d_5) + t^{-1/2} N'(d_5) \quad dt;$$

- *Region II*: $k \geq 0$ and $x \leq 0$ [9]

$$\Psi_\rho^{II}(\nu; k, x, T) = \int_0^T \frac{(1-e^{-\rho(T-t)})e^{\frac{\nu^2}{2}t}}{\sqrt{2\pi}\rho(T-t)^{3/2}} \quad \nu C_1 N(d_7) + C_2 N'(d_7)$$
$$\times e^{-\frac{x^2}{2(T-t)}} dt;$$

- *Region III*: $k \leq 0$ and $x \geq 0$

$$\Psi_\rho^{III}(\nu; k, x, T) = \Psi_\rho^I(\nu; 0, x, T) + e^{-\rho T}[\Psi_{-\rho}^{II}(-\nu; 0, -x, T)$$
$$-\Psi_{-\rho}^{II}(-\nu; -k, -x, T)];$$

- Region IV: $k \leq 0$ and $x \leq 0$

$$\Psi_\rho^{IV}(\nu; k, x, T) = \Psi_\rho^{II}(\nu; 0, x, T) + e^{-\rho T}[\Psi_{-\rho}^I(-\nu; 0, -x, T)$$
$$-\Psi_{-\rho}^I(-\nu; -k, -x, T)],$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{z^2}{2}} dz, \quad N'(x) = \frac{dN(x)}{dx} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

is the cumulative standard normal and its density.

B. The expectation $E_{0,x}\left[e^{\nu W_t - \alpha A_t^+} \mathbf{1}_{\{W_t \geq k\}}\right]$

Introduce the following notation:

$$y_1 = (2\alpha)^{-2/3}(2s + 2\alpha y), \quad y_2 = (2\alpha)^{-2/3} 2s, \quad y_3 = (2\alpha)^{-2/3}(2s + 2\alpha x),$$

$$W_\pm = \sqrt{2s} Ai(y_2) \pm (2\alpha)^{1/3} Ai'(y_2), \quad V = \sqrt{2s} Bi(y_2) - (2\alpha)^{1/3} Bi'(y_2),$$

where $Ai(z)$ and $Bi(z)$ are Airy functions defined by (Abramowitz and Stegun, 1965):

$$Ai(z) = \frac{1}{\pi} \int_0^\infty \cos \quad uz + \frac{u^3}{3} \quad du,$$

$$Bi(z) = \frac{1}{\pi} \int_0^\infty \quad \exp \quad uz - \frac{u^3}{3} \quad + \sin \quad uz + \frac{u^3}{3} \quad du.$$

[9] For $k = 0$, the function $\Psi_\rho^{II}(\nu; 0, x, T)$ is defined as a limit of the integral for $k \to 0$: $\Psi_\rho^{II}(\nu; 0, x, T) = \lim_{k \to 0} \Psi_\rho^{II}(\nu; k, x, T)$.

Then the function $G_\alpha(x, y; s)$ entering the expression (29) and defined as the Laplace transform

$$\int_0^\infty e^{-st} E_{0,x}\left[e^{-\alpha A_t^+}; W_t \in dy\right] dt = G_\alpha(x, y; s) dy$$

is given by (Davydov et al., 1998):

- *Region I*: $x \leq 0 \leq y$

$$G_\alpha^I(x, y; s) = \frac{2Ai(y_1)}{W_-} e^{\sqrt{2s}x},$$

- *Region II*: $x \leq y \leq 0$

$$G_\alpha^{II}(x, y; s) = \frac{1}{\sqrt{2s}} \quad e^{\sqrt{2s}(x-y)} + \frac{W_+}{W_-} e^{\sqrt{2s}(x+y)} \quad ,$$

- *Region III*: $y \leq x \leq 0$

$$G_\alpha^{III}(x, y; s) = G_\alpha^{II}(y, x; s),$$

- *Region IV*: $y \leq 0 \leq x$

$$G_\alpha^{IV}(x, y; s) = G_\alpha^{I}(y, x; s),$$

- *Region V*: $0 \leq y \leq x$

$$G_\alpha^{V}(x, y; s) = \frac{2\pi Ai(y_3)}{(2\alpha)^{1/3}} \quad Bi(y_1) - \frac{V}{W_-} Ai(y_1) \quad ,$$

- *Region VI*: $0 \leq x \leq y$

$$G_\alpha^{VI}(x, y; s) = G_\alpha^{V}(y, x; s).$$

The Airy functions are computed using the asymptotic expansions found in Abramowitz and Stegun (1965). To compute the inverse Laplace transform in Equation (29) numerically, we employ the Euler algorithm developed by Abate and Whitt (1995). This algorithm was previously applied to option pricing problems by Fu, Madan, and Wang (1998) and Davydov and Linetsky (1998). Then the integral in y in (29) is calculated numerically. Finally, (26) gives the ESO value under the forfeiture and early exercise intensity specification (23).

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