The Variance Gamma Process and Option Pricing

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Abstract. A three parameter stochastic process, termed the variance gamma process, that generalizes Brownian motion is developed as a model for the dynamics of log stock prices. The process is obtained by evaluating Brownian motion with drift at a random time given by a gamma process. The two additional parameters are the drift of the Brownian motion and the volatility of the time change. These additional parameters provide control over the skewness and kurtosis of the return distribution. Closed forms are obtained for the return density and the prices of European options. The statistical and risk neutral densities are estimated for data on the S&P500 Index and the prices of options on this Index. It is observed that the statistical density is symmetric with some kurtosis, while the risk neutral density is negatively skewed with a larger kurtosis. The additional parameters also correct for pricing biases of the Black Scholes model that is a parametric special case of the option pricing model developed here.

1. Introduction

This article proposes a three parameter generalization of Brownian motion as a model for the dynamics of the logarithm of the stock price. The new process, termed the variance gamma (VG) process, is obtained by evaluating Brownian motion (with constant drift and volatility) at a *random time change* given by a gamma process.[1] Each unit of calendar time may be viewed as having an economically relevant time length given by an independent random variable that has a gamma

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[1] Earlier related work by Madan and Seneta (1990) considered a time change of Brownian motion without drift by a gamma process and this process is here termed the symmetric variance gamma process. Madan and Milne (1991) considered equilibrium option pricing for the symmetric variance gamma process in a representative agent model, under a constant relative risk aversion utility function. The resulting risk neutral process is identical with the more general variance gamma process proposed here, with the drift in the time changed Brownian motion being negative for positive risk aversion. This paper theoretically extends Madan and Milne (1991), by providing closed forms for the return density and the prices of European options on the stock.

density with unit mean and positive variance. Under the VG process, the unit period continuously compounded return is normally distributed, conditional on the realization of a random time. This random time has a gamma density. The resulting stochastic process and associated option pricing model provide us with a robust three parameter model. In addition to the volatility of the Brownian motion there are parameters that control for (i) kurtosis (a symmetric increase in the left and right tail probabilities of the return distribution) and (ii) skewness that allows for asymmetry of the left and right tails of the return density. An additional attractive feature of the model is that it nests the lognormal density and the Black-Scholes formula as a parametric special case.[2]

The risk neutral approach, first introduced by Black and Scholes (1973), to valuing derivatives is a standard paradigm in finance. While the Black-Scholes formula remains the most widely used model by practitioners, it has known biases. Two well documented biases are volatility smiles and skewness premia. Rubinstein (1985, 1994) among others, documents evidence that implied volatilities tend to rise for options that are deeply in- or out-of-the-money. Bates (1995) presents evidence that, relative to call options, put options are underpriced by the Black-Scholes formula which, in turn, suggests that the implied volatility curve is downward sloping in the strike price. On the one hand, the presence of a volatility smile suggests a risk neutral density with a kurtosis above that of a normal density, on the other, the existence of skewness premia further suggests that the left tail of the return distribution is fatter than the right tail.

Contrary to much of the literature on option pricing, the proposed VG process for log stock prices has no continuous martingale component.[3] In contrast, it is a pure jump process that accounts for high activity[4] (as in Brownian motion) by having an infinite number of jumps in any interval of time. The importance of introducing a jump component in modeling stock price dynamics has recently been noted in Bakshi, Chen and Cao (1996), who argue that pure diffusion based models have difficulties in explaining smile effects in, in particular, short-dated option prices. Poisson type jump components in jump diffusion models are designed to address these concerns. For the VG process, however, as the Black Scholes model

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[2] The resulting stochastic process is a one dimensional time homogeneous Markov process. As such we do not have a stochastic volatility model that allows for changes in the conditional volatility, as for example in GARCH models or the Heston stochastic volatility model. Higher dimensional Markov models may be obtained by for example, time changing the Heston stochastic volatility model by a gamma time process. In this paper we abstract from issues of time inhomogeneity.

[3] In this regard the VG process is a departure from existing option pricing literature, where the main mode of analysis is a diffusion, that has a martingale component with sample paths that are continuous functions of calendar time. The Black Scholes (1973) option pricing model makes this assumption, as do most other diffusion and jump diffusion models of Praetz (1972), Merton (1976), Cox and Ross (1976), Jones (1984), Hull and White (1987), Scott (1987), Wiggins (1987), Mellino and Turnbull (1990), Naik and Lee (1990), Heston (1993a,b) and Bates (1991, 1995).

[4] The level of activity may here loosely be measured by the volume or number of transactions, or the associated number of price changes.

is a parametric special case already, and high activity is already accounted for, it is not necessary to introduce a diffusion component in addition: hence the absence of a continuous martingale component.

Unlike Brownian motion, the sum of the absolute log price changes is finite for the VG process.[5] Since the VG process is one of finite variation, it can be written as the difference of two increasing processes, the first of which accounts for the price increases, while the second explains the price decreases. In the case of the VG process, the two increasing processes that are differenced to obtain the VG process are themselves gamma processes.[6]

The statistical and risk neutral processes are postulated to be VG processes for the dynamics of the S&P500 index and estimates are obtained using data on the index and on a large cross-section of option prices for S&P 500 futures. First, the VG model successfully corrects strike and maturity biases in Black Scholes pricing. Second, estimation of the statistical process shows that the hypothesis of zero skewness in the statistical return distribution cannot be rejected, while the hypothesis of zero excess kurtosis over the normal distribution can be rejected. Third, estimates of the risk neutral process show that the hypotheses of zero skewness and zero kurtosis, can both be rejected. Thus, we reject the Black Scholes and symmetric VG special cases of our general option pricing formula associated with the asymmetric VG process. Furthermore, excess kurtosis estimates are substantially larger for the risk neutral process than they are for the statistical process.

The outline of the paper is as follows. The VG process is defined in Section 1 and its properties are presented and discussed. Closed forms for the statistical density and the prices of European options, when the stock price follows the VG process, are presented in Section 2. The data is described in Section 3. Empirical findings and the analysis of pricing errors are presented in Sections 4 and 5. Section 6 concludes. All proofs are to be found in the Appendix.

2. The VG Process for Statistical and Risk Neutral Log Stock Prices

This section defines the VG process, generalizing the two parameter stochastic process studied in Madan and Seneta (1990) and Madan and Milne (1991), that controlled for volatility and kurtosis, to a three parameter process that now addresses skewness as well.[7] Consider a continuous time economy, over the interval $[0, \Upsilon]$, in which are traded a stock, a money market account, and options on the stock for all strikes and maturities $0 < T \leq \Upsilon$. We suppose a constant continu-

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[5] Brownian motion is a process of infinite variation but finite quadratic variation and the log price changes must be squared before they are summed, to get a finite result.

[6] Such a process has also been considered by Heston (1993c) in the context of interest rate modeling.

[7] The statistical process in Madan and Milne (1991) was symmetric and did not allow for skewness, but the risk neutral process had skewness obtained via representative agent model with constant relative risk aversion utility. This paper provides an alternative derivation of this process, seen here as Brownian motion with drift, time changed by a gamma process.

ously compounded interest rate of r with money market account value of $\exp(rt)$, stock prices of $S(t)$ and European call option prices of $c(t; K, T)$ for strike K and maturity $T > t$, at time t.

The VG process is obtained by evaluating Brownian motion with drift at a random time given by a gamma process. Let

$$b(t; \theta, \sigma) = \theta t + \sigma W(t) \tag{1}$$

where $W(t)$ is a standard Brownian motion. The process $b(t; \theta, \sigma)$ is a Brownian motion with drift θ and volatility σ.

The gamma process $\gamma(t; \mu, \nu)$ with mean rate μ and variance rate ν is the process of independent gamma increments over non-overlapping intervals of time $(t, t + h)$. The density, $f_h(g)$, of the increment $g = \gamma(t + h; \mu, \nu) - \gamma(t; \mu, \nu)$ is given by the gamma density function with mean μh and variance νh. Specifically,

$$f_h(g) = \left(\frac{\mu}{\nu}\right)^{\frac{\mu^2 h}{\nu}} \frac{g^{\frac{\mu^2 h}{\nu} - 1} \exp\left(-\frac{\mu}{\nu} g\right)}{\Gamma(\frac{\mu^2 h}{\nu})}, \, g > 0, \tag{2}$$

where $\Gamma(x)$ is the gamma function. The gamma density has a characteristic function, $\phi_{\gamma(t)}(u) = E[\exp(iu\gamma(t; \mu, \nu))]$, given by,

$$\phi_{\gamma(t)}(u) = \left(\frac{1}{1 - iu\frac{\nu}{\mu}}\right)^{\frac{\mu^2 t}{\nu}}. \tag{3}$$

The dynamics of the continuous time gamma process is best explained by describing a simulation of the process. As the process is an infinitely divisible one, of independent and identically distributed increments over non-overlapping intervals of equal length, the simulation may be described in terms of the Lévy measure (Revuz and Yor (1991, p. 110)), $k_\gamma(x)dx$ explicitly given by

$$k_\gamma(x)dx = \frac{\mu^2 \exp\left(-\frac{\mu}{\nu} x\right)}{\nu x} dx, \text{ for } x > 0 \text{ and } 0 \text{ otherwise.} \tag{4}$$

Since the Lévy measure has an infinite integral, we see that the gamma process has an infinite arrival rate of jumps, most of which are small, as is indicated by the concentration of the Lévy measure at the origin. The process is pure jump and may be approximated as a compound Poisson process. To simulate the compound Poisson approximation, we truncate the Lévy measure near the origin, thereby ignoring jumps of a size below ε. We then use the area under the truncated Lévy measure as

the Poisson arrival rate of jumps. The normalized truncated Lévy measure provides the conditional density of jump magnitudes, given the arrival of a jump.

The VG process $X(t; \sigma, \nu, \theta)$, is defined in terms of the Brownian motion with drift $b(t; \theta, \sigma)$ and the gamma process with unit mean rate, $\gamma(t; 1, \nu)$ as

$$X(t; \sigma, \nu, \theta) = b(\gamma(t; 1, \nu); \theta, \sigma). \tag{5}$$

The VG process is obtained on evaluating Brownian motion at a time given by the gamma process.[8] The VG process has three parameters: (i) σ the volatility of the Brownian motion, (ii) ν the variance rate of the gamma time change and (iii) θ the drift in the Brownian motion with drift. The process therefore provides two dimensions of control on the distribution over and above that of the volatility. We observe below that control is attained over the skew via θ and over kurtosis with ν.

The density function for the VG process at time t can be expressed conditional on the realization of the gamma time change g as a normal density function. The unconditional density may then be obtained on integrating out g employing the density (2) for the time change g. This gives us the density for, $X(t)$, $f_{X(t)}(X)$, as

$$f_{X(t)}(X) = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi g}} \exp\left(-\frac{(X-\theta g)^2}{2\sigma^2 g}\right) \frac{g^{\frac{t}{\nu}-1} \exp\left(-\frac{g}{\nu}\right)}{\nu^{\frac{t}{\nu}} \Gamma\left(\frac{t}{\nu}\right)} dg. \tag{6}$$

The characteristic function for the VG process, $\phi_{X(t)}(u) = E[\exp(iuX(t)]$, is[9]

$$\phi_{X(t)}(u) = \left(\frac{1}{1 - i\theta\nu u + (\sigma^2\nu/2)u^2}\right)^{t/\nu}. \tag{7}$$

The VG process may also be expressed as the difference of two independent increasing gamma processes, specifically (see the appendix for deatils)[10]

$$X(t; \sigma, \nu, \theta) = \gamma_p(t; \mu_p, \nu_p) - \gamma_n(t; \mu_n, \nu_n). \tag{8}$$

The explicit relation between the parameters of the gamma processes differenced in (8) and the original parameters of the VG process (5) is given by (see the Appendix for details),

$$\mu_p = \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} + \frac{\theta}{2} \tag{9}$$

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[8] The VG process is therefore in the class of subordinated processes. Such processes were first considered for stock prices by Clark (1973). Recent investigations using such processes in a financial context include Ané and Geman (1995), Bossaerts, Ghysels and Gourieroux (1996) and Geman and Ané (1996).

[9] This expression is obtained by first conditioning on the gamma time and employing the characteristic function of the normal to obtain the conditional characteristic function. One next employs equation (2) to integrate out the gamma time with respect to its density.

[10] Such gamma processes have recently been used to model order queues by Gourieroux, LeFol and Meyer (1996).

$$\mu_n = \frac{1}{2}\overline{\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}}} - \frac{\theta}{2} \tag{10}$$

$$\nu_p = \left(\frac{1}{2}\overline{\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}}} + \frac{\theta}{2}\right)^2 \nu \tag{11}$$

$$\nu_n = \left(\frac{1}{2}\overline{\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}}} - \frac{\theta}{2}\right)^2 \nu \tag{12}$$

The Lévy measure for the VG process has three representations, two in terms of the parameterizations introduced above, as time changed Brownian motion, and the difference of two gamma processes and the third in terms of a symmetric VG process subjected to a measure change induced by a constant relative risk aversion utility function as in Madan and Milne (1991). When viewed as the difference of two gamma processes as in (8) we may write the Lévy measure for $X(t)$, employing (4) as

$$k_X(x)dx = \begin{cases} \dfrac{\mu_n^2}{\nu_n}\dfrac{\exp\left(-\dfrac{\mu_n}{\nu_n}|x|\right)}{|x|}dx & \text{for } x < 0 \\[3ex] \dfrac{\mu_p^2}{\nu_p}\dfrac{\exp\left(-\dfrac{\mu_p}{\nu_p}x\right)}{x}dx & \text{for } x > 0 \end{cases} \tag{13}$$

We observe from (13) that the VG process inherits the property of an infinite arrival rate of price jumps, from the gamma process. The role of the original parameters is more easily observed when we write the Lévy measure directly in terms of these parameters. In terms of (σ, ν, θ) one may write the Lévy measure as

$$k_X(x)dx = \frac{\exp(\theta x/\sigma^2)}{\nu|x|}\exp\left(-\frac{\sqrt{\frac{2}{\nu}+\frac{\theta^2}{\sigma^2}}}{\sigma}|x|\right)dx \tag{14}$$

The special case of $\theta = 0$ in (14) yields a Lévy measure that is symmetric about zero. This yields the symmetric VG process employed by Madan and Seneta (1990) and Madan and Milne (1991) for describing the statistical process of continuously compounded returns. We also observe from (14), that when $\theta < 0$, negative values of x receive a higher relative probability than the corresponding positive value. Hence, negative values of θ give rise to a negative skewness. We note further that

large values of v, lower the exponential decay rate of the Lévy measure symmetrically around zero, and hence raise the likelihood of large jumps, thereby raising tail probabilities and kurtosis. The VG process can therefore be expected to flatten the volatility smiles at the low end of the maturity spectrum.

The third form of the Lévy measure for the VG process is in terms of the representation employed in Madan and Milne (1991). The risk neutral VG process for the stock price can be derived from a Lucas-type general equilibrium economy in which the representative agent has a constant relative risk aversion utility function with relative risk aversion ζ, and in which the statistical process for the log price dynamics is given by the symmetric VG process ($\theta = 0$), with volatility s and time change volatility v. The risk neutral Lévy measure (see Madan and Milne (1991)) is given by[11]

$$k_X(x)dx = \frac{\exp(-\zeta x)}{v\,|x|} \exp\left(-\frac{\overline{2}}{s}\,\frac{}{\overline{v}}\,|x|\right).$$ (15)

This is in agreement with the definition of equation (14) on defining

$$\zeta = -\frac{\theta}{\sigma^2}$$ (16)

and

$$s = \frac{\sigma}{\sqrt{1 + \left(\frac{\theta}{\sigma}\right)^2 \frac{v}{2}}}.$$ (17)

The parameters of the VG process, only indirectly reflect the skewness and kurtosis of the return distribution. Explicit expressions for the first four central moments of the return distribution over an interval of length t are derived in the Appendix, and are as follows:

$$E[X(t)] = \theta t,$$

$$E[(X(t) - E[X(t)])^2] = (\theta^2 v + \sigma^2)t,$$ (18)

$$E[(X(t) - E[X(t)])^3] = (2\theta^3 v^2 + 3\sigma^2\theta v)t,$$ (19)

$$E[(X(t) - E[X(t)])^4] = (3\sigma^4 v + 12\sigma^2\theta^2 v^2 + 6\theta^4 v^3)t$$
$$+ (3\sigma^4 + 6\sigma^2\theta^2 v + 3\theta^4 v^2)t^2.$$ (20)

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[11] The measure change function $\exp(-\zeta x)$ is precisely the ratio of marginal utilities for a representative investor holding stock with marginal utility function $S^{-\zeta}$. For a jump in the log price of x, the ratio of marginal utilities is $(S\exp(x))^{-\zeta}/S^{-\zeta} = \exp(-\zeta x)$.

We observe from Equation (19), that $\theta = 0$ does indeed imply that there is no skewness, and furthermore the sign of the skewness is that of θ. Furthermore, we note from Equation (20) and Equation (18), that when $\theta = 0$, the fourth central moment divided by the squared second central moment is $3(1 + \nu)$ and so ν is the percentage excess kurtosis in the distribution.

3. VG Stock Price Dynamics, Densities and Option Prices

This section describes the statistical and risk neutral dynamics of the stock price in terms of the VG process and derives closed forms for the return density and the prices of European options on the stock.[12] Such closed forms are useful in econometric estimation of the statistical and risk neutral processes by maximum likelihood methods. First the statistical and risk neutral processes are defined, followed by results on the return density function and the option price.

The new specification for the statistical stock price dynamics is obtained by replacing the role of Brownian motion in the original Black-Scholes geometric Brownian motion model by the VG process. Let the statistical process for the stock price be given by

$$S(t) = S(0) \exp(mt + X(t; \sigma_S, \nu_S, \theta_S) + \omega_S t), \tag{21}$$

where the subscript S on the VG parameters indicates that these are the statistical parameters, $\omega_S = \frac{1}{\nu_S} \ln(1 - \theta_S \nu_S - \sigma_S^2 \nu_S/2)$, and m is the mean rate of return on the stock under the statistical probability measure.[12]

Under the risk neutral process, money market account discounted stock prices are martingales and it follows that the mean rate of return on the stock under this probability measure is the continuously compounded interest rate r. Let the risk neutral process be given by

$$S(t) = S(0) \exp(rt + X(t; \sigma_{RN}, \nu_{RN}, \theta_{RN}) + \omega_{RN} t), \tag{22}$$

where the subscript RN on the VG parameters indicates that these are the risk neutral parameters, and $\omega_{RN} = \frac{1}{\nu_{RN}} \ln(1 - \theta_{RN} \nu_{RN} - \sigma_{RN}^2 \nu_{RN}/2)$.[12]

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[12] We interpret closed forms to mean reduction to the special functions of mathematics that have representations as integrals of elementary functions. This is an advance over integral representations that employ special functions in the integrand: the latter being a double integral of elementary functions. Furthermore, advances in the computation of special functions (by other methods employing, for example, functional approximation by polynomial methods) are often being made in the mathematics literature.

[12] The value for ω_S is determined by evaluating the characteristic function for $X(t)$ at $u = 1/i$, so that the expectation of $S(t) = S(0) \exp(mt)$, or equivalently the expectation of $\exp(X(t)) = \exp(-\omega_S t)$.

[12] It is important to note, in contrast to the situation for diffusion based continuous price processes, the risk neutral parameters do not have to equal their statistical counterparts, and can in principle be

The density of the log stock price relative over an interval of length t is, conditional on the realization of the gamma time change, a normal density function. The unconditional density is obtained by integrating out the gamma variate and the result is in terms of the modified bessel functions of the second kind.

THEOREM. The density for the log price relative $z = \ln(S(t)/S(0))$ when prices follow the VG process dynamics of equation (21) is given by

$$
h(z) = \frac{2 \exp(\theta x/\sigma^2)}{\nu^{t/\nu} \sqrt{2\pi} \, \sigma \, \Gamma(\frac{t}{\nu})} \left(\frac{x^2}{2\sigma^2/\nu + \theta^2} \right)^{\frac{t}{2\nu} - \frac{1}{4}} X
$$

$$
K_{\frac{t}{\nu} - \frac{1}{2}} \left(\frac{1}{\sigma^2} \sqrt{x^2(2\sigma^2/\nu + \theta^2)} \right) , \tag{23}
$$

where K is the modified bessel function of the second kind,

$$
x = z - mt - \frac{t}{\nu} \ln(1 - \theta\nu - \sigma^2\nu/2)
$$

and it is understood that the VG parameters employed are the statistical ones.

The price of a European call option, $c(S(0); K, t)$, for a strike of K and maturity t, is by a standard result given by

$$
c(S(0); K, t) = e^{-rt} E[\max(S(t) - K, 0)], \tag{24}
$$

where the expectation is taken under the risk neutral process of Equation (22). The evaluation of the option price (24) proceeds by first conditioning on a knowledge of the random time change g that has an independent gamma distribution. Conditional on g, $X(t)$ is normally distributed and the option value is given by a Black-Scholes type formula. The European option price for VG risk neutral dynamics is then obtained on integrating this conditional Black-Scholes formula over g with respect the gamma density. This was the procedure followed in Madan and Milne (1991) where the price was obtained by numerical integration.[15] Here we obtain an ana-

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significantly different. The basic intuition for pure jump processes, like the VG, is that each instant is like a one period infinite state model and the risk neutral density can be $q(x)$ (where x is the log of the stock price and $-\infty < x < \infty$) while the statistical density is $p(x)$ with the change of measure density being $\lambda(x) = q(x)/p(x)$. If q is VG with parameters $\sigma_{RN}, \theta_{RN}, \nu_{RN}$ and p is VG with parameters $\sigma_S, \theta_S, \nu_S$ then λ is appropriately defined and there is no link between the statistical and risk neutral parameters. Both q and p charge the entire real line and so they are equivalent, but other than this point there is no relationship between them. Of course, for the formal measure change density process in the current context, one needs to consider the ratio of Lévy measures as opposed to densities and these considerations are beyond the scope of this paper.

[15] As noted earlier, the integrand in Madan and Milne (1991) employed special functions and was therefore a double integral of elementary functions.

lytical reduction in terms of the special functions of mathematics, or a closed form expression for the price.

THEOREM 2. The European call option price on a stock, when the risk neutral dynamics of the stock price is given by the VG process (22) is (for risk neutral parameters σ, ν, θ),

$$
\begin{aligned}
c(S(0); K, t) \;=\; & S(0)\Psi\left(d\sqrt{\frac{1-c_1}{\nu}}, (\alpha+s)\sqrt{\frac{\nu}{1-c_1}}, \frac{t}{\nu}\right) \\
& -K\exp(-rt)\Psi\left(d\sqrt{\frac{1-c_2}{\nu}}, \alpha s\sqrt{\frac{\nu}{1-c_2}}, \frac{t}{\nu}\right)
\end{aligned}
\tag{25}
$$

where

$$
d = \frac{1}{s}\left[\ln\left(\frac{S(0)}{K}\right) + rt + \frac{t}{\nu}\ln\left(\frac{1-c_1}{1-c_2}\right)\right],
\tag{26}
$$

$\alpha = \zeta s$ and ζ, s are as defined in (16) and (17),

$$
c_1 = \frac{\nu(\alpha+s)^2}{2},
\tag{27}
$$

$$
c_2 = \frac{\nu\alpha^2}{2},
\tag{28}
$$

and the function Ψ is defined in terms of the modified bessel function of the second kind and the degenerate hypergeometric function of two variables by equation (A11) of the Appendix.

The European call option pricing formula (25) has the usual form of the stock price times a probability element less the present value of the strike times a second probability element. It can be shown that the second probability element is the risk neutral probability that $S(t)$ exceeds K. The first probability element is also the probability that $S(t)$ exceeds K, using now the density obtained on normalizing the product of the stock price with the risk neutral density of the stock price.

We expect the additional parameters of the VG model to be important for option pricing. Risk aversion implies from equation (16) that the risk neutral density of returns is negatively skewed ($\theta < 0$ or $\alpha > 0$), a feature that is missed by the Black Scholes model where symmetry is essentially a consequence of continuity coupled with continuous rebalancing. Regarding kurtosis, a relatively high kurtosis for returns over short periods of time is a well known feature of the statistical return distribution. Furthermore, we suspect that the risk neutral density has an even

greater kurtosis that is reflected in the Black Scholes implied volatility premiums associated with out-of-the money, relative to at-the-money, options.

There are three option pricing formulas nested in the option pricing formula (25). These are a) the VG model, b) the symmetric VG (obtained by restricting θ or α to zero) and c) the Black Scholes model (that results on setting v equal to zero). We report estimates for all three models.

4. The Option Data

The prices used in this study are for the S&P 500 futures options traded at the Chicago Mercantile Exchange (CME). The S&P 500 futures options are cash settled and are listed on a monthly expiration cycle. Option prices are expressed in terms of index units. Each index point represents $500. Strike prices are set at five-point intervals.

The data for the study were obtained from the Financial Futures Institute in Washington D.C. and include all time stamped transaction option prices from January 1992 to September 1994. Closing prices on index futures were also available as was the level of the spot index. To ensure sufficient liquidity and to be immune from problems of nonsynchronous trading, up to 16 options with prices that were time stamped at or near the daily close were selected for each day. Up to four strikes for each of four maturities were selected. Daily data on the three month Treasury Bill rate was obtained from the Federal Reserve in Washington D.C., dividend yields were inferred from the theoretical no-arbitrage relationship between the spot and futures index. All options contracts were viewed as written on the underlying spot index. There were 2824 options selected from the 1992 data file, 3010 from the 1993 file and 2411 from 1994, a total of 8245 option prices.

5. Empirical Performance of the VG model

We begin the analysis by estimating the parameter values of the statistical densities underlying the three nested models of the lognormal, the symmetric VG, and the VG. The data employed was the 691 daily observations of log spot price relatives covering the period from January 1992 to September 1994. For the stock price dynamics under the VG model, we employ the density $h(z)$ of Equation (23) and estimate all four parameters. The parameter θ is set to zero when we estimate the symmetric VG model. The stock price dynamics underlying the Black Scholes model is simply the lognormal density. We employ maximum likelihood estimation for all the estimations. We present the estimated parameter values in Table I.

As shown in the first row of Table I, the annualized mean returns estimated by all three models are very similar. It ranges from a low of 5.69% from the Black Scholes model to a high of 5.91% for the VG model. The estimated asset volatilities are also very similar, ranging from a low of 11.71% for the symmetric VG model to a high of 11.91% for the Black Scholes model. The estimated kurtosis parameter

Table I. Maximum likelihood estimates of the statistical density parameters for daily returns on the standard and poor's stock index based on geometric Brownian motion and the symmetric variance gamma and variance gamma stochastic processes: January 1992–September 1994

Parameter estimated	Geometric Brownian motion	Symmetric variance gamma	Variance gamma
m	0.0596	0.0570	0.0591
	(0.0857)	(0.0685)	(0.0263)
s	0.1191	0.1171	0.1172
	(0.0032)**	(0.0045)**	(0.0044)**
ν		0.0020	0.002
		(0.0003)**	(0.0004)**
α			0.0048
			(0.12)
$\ln \mathcal{L}$	2528.39	2570.36	2569.78
NOBS	691	691	691

ν under the symmetric VG model is 0.002, and it is significantly different from zero. It was noted in Section 2 that for the case $\theta = 0$, this parameter measures an annualized excess percentage kurtosis over 3, the kurtosis of the normal density. From Equations (20) and (18) we observe that the corresponding daily kurtosis is $3[1 + (.002)(365)]$, or 5.19. The VG estimate for ν is also .002 and the estimate for θ, though positive, is insignificant.[16]

On a chi-squared test (taking twice the difference in log likelihoods given in the fifth row of the first two columns of Table I), the lognormal model is strongly rejected in favor of the symmetric VG with a χ_1^2 statistic of 83.94. The VG makes no improvement in the log likelihood over the symmetric VG and we conclude that the log-price relative of the S&P 500 index, statistically follows a symmetric VG process.

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[16] When we start the optimization at the estimates obtained for the symmetric VG, there is no change and the skew is estimated at zero. The reported estimates were obtained by using a starting point that solved the moment equations developed in section 2. As a further check we report the moment estimates for the skewness and kurtosis on our sample. The standard deviation is 0.006233. The third central moment divided by the cube of the standard deviation is −0.2105. The fourth central moment divided by the fourth power of the standard deviation is 7.8463. The moment estimates suggest a small negative skew and kurtosis larger than that provided by the maximum likelihood estimates.

We estimate the parameter values of the risk neutral densities on a weekly basis.[17] Using all of the option prices available for each week, we invert the three option pricing models to estimate the implied volatilities, σ, (for all three models), the implied excess kurtosis or the parameter, ν, (for the two VG models), and the implied skewness or the parameter, θ, (for the VG model). Since the number of option prices ranges from 40 to 80, it is generally not possible to find a single set of parameters that exactly fit all of the option prices. Consequently, these parameters are estimated by the maximum likelihood method.

Specifically, the precise likelihood employed addresses expected heteroskedasticity in option prices for various strikes by using a multiplicative error formulation. Letting w_i be the observed market price on the ith option and letting $\widehat{w}_i$ be the model price we adopted the error model

$$w_i = \widehat{w}_i \exp(\eta\varepsilon_i - \eta^2/2) \tag{29}$$

where it is supposed that the ε_i's are normally distributed with zero mean and unit variance.[18] It is shown in the Appendix, that maximum likelihood estimation is asymptotically equivalent to non-linear least squares on the logarithms, or the minimization of

$$k = \sqrt{\frac{1}{M}\sum_{i=1}^{M}(\ln(w_i) - \ln(\widehat{w}_i))^2}. \tag{30}$$

The average estimated parameter values of the risk neutral densities, along with their standard deviation across the weeks, and the minimum and maximum estimated values over all the weeks are presented in Table II. We also present average values for the estimated log likelihood statistic. As noted earlier the parameters were estimated weekly for 143 weeks.[19]

The average risk neutral volatility estimated for the log normal density is 12.13%. The corresponding values for the symmetric VG and the VG are 13.01% and 11.48%. These are consistent estimates across the models and comparable to the consistently estimated statistical volatility of 12%.

---

[17] We could impose constancy of the risk neutral parameters across time and then estimate jointly the statistical and risk neutral processes from the pooled cross section and time series data on option prices and the underlying index. We, however, allow for time variation in the risk neutral parameters as there is enough option price information at each time point to permit the estimation of this time variation. In this regard we follow the precedents set by Rubinstein (1994), Jackwerth and Rubinstein (1996), Bates (1991), Bates (1996), Stutzer (1996) and Bakshi, Cao and Chen (1997) for the evaluation of the option pricing models.

[18] A multiplicative formulation is employed to preserve the positivity of w_i while allowing ε_i to be unbounded. Similar formulations have recently been employed by Jacquier and Jarrow (1995) and Elliott, Lahaie and Madan (1995).

[19] Exponential transformations were invoked to ensure positivity of the estimates for σ and ν, and the objective function for the estimation was the asymptotic log likelihood. The statistical significance of parameter estimates is directly addressed by likelihood ratio tests.

Table II. Summary statistics of weekly estimates of parameters for S&P 500 stock index risk neutral density function based on he Black/Scholes option pricing model, the symmetric variance gamma and the variance gamma price processes: January 1992–September 1994 (143 Weeks)

Parameter	Mean	Standard deviation	Minimum	Maximum
A. Black Scholes Model				
σ	0.1236	0.0165	0.087	0.171
Mean Log-likelihood	0.0057	0.0026	0.002	0.019
B. Symmetric Variance Gamma Model				
σ	0.1301	0.0199	0.091	0.183
ν	0.1861	0.1391	0.051	0.876
Mean Log-Likelihood	0.0052	0.0024	0.001	0.019
C. Variance Gamma Model				
σ	0.1213	0.0192	0.08	0.1737
ν	0.1686	0.0812	0.0541	0.6790
θ	−0.1436	0.0552	−0.2744	0.0492
Mean Log-Likelihood	0.003	0.0022	0.001	0.015
D. Rejection Percentages At the 5% (1%) Levels				

		Null Hypothesis	
Alternate Hypothesis	Black Scholes	Symm. VG
Symmetric VG Model	37.8% (30.8%)	
VG Model	93.7% (91.6%)	93.7% (91.6%)

The average estimates for ν under the symmetric VG and the VG are, (see Table II rows 4 and 7 of column 1) respectively 18.61% and 16.86%. These are much higher than the statistical estimates of a fifth of a percent. The average estimate for $\theta = -0.1436$. Applying the transformation (16) the average value of risk aversion is 10.59 and the standard error across the 143 weeks is 5.76. Hence, the risk neutral volatility of the VG process is comparable to its statistical volatility and there is a negative skewness to the risk neutral process associated with risk aversion.

The enhancement of skewness in the risk neutral process relative to the statistical process is an expected consequence of risk aversion in equilibrium. This is easily seen in the context of a simple one period model. For example if $p(x)$ is normal with mean zero and unit variance, and if utility is HARA with marginal utility given by $(1 + e^x)^{-1}$, then q is no longer symmetric and has in fact a fatter left tail.

The risk neutral process also displays increased kurtosis. We view this as related to the well documented evidence on higher implied volatilities in option pricing in general. Unlike the geometric Brownian motion model, that lacks a kurtosis parameter, in the VG model this property of high prices for tail events, is reflected in increased risk neutral kurtosis with volatilities being typically unaffected.

Panel D of Table II presents the results of log likelihood ratio tests of the null hypothesis of the Black Scholes model against the alternate of the symmetric VG and the VG model, and the null hypothesis of the symmetric VG model against the alternate of the VG model. At the one percent level, the Black Scholes model is rejected in 30.8% of the cases given by the 143 weeks in favor of the symmetric VG model, whereas the comparable rejection rate for the Black Scholes model vs the VG model is 91.6%. The symmetric VG is also rejected in favor of the VG in 91.6% of the cases.

The evidence presented so far is statistical and provides some support for the significance of the risk neutral VG process and the additional parameters introduced in this model. In the next section, we analyse further the improvement obtained in the quality of pricing delivered by the VG models by studying the behavior of biases in the pricing errors of the three models.

6. Pricing Performance of the Variance Gamma Option Model

In this section we evaluate the pricing biases of the three models, VG, the symmetric VG and the Black Scholes model. Our results show that the VG model is successful in addressing the pricing biases present in the other two models.

The three models were investigated to determine the quality of prices the models deliver. For each of the 143 weeks we used the parameter estimates of that week to calculate the option price as per each of the three models for all the options of that week and obtained the pricing error. This gave us 8245 pricing errors for each of the three models. The quality of pricing was judged by performing orthogonality tests. For a good model the pricing errors should not exhibit any consistent pattern and they should not be predictable.

With a view to assessing the estimated models in this way we performed a regression analysis on the pricing errors obtained from each model. The explanatory variables for the regression summarized the characteristics of the option. The presence of implied volatility smiles (Bates (1995)), suggests that pricing errors are systematically related to the degree of moneyness, measured by the ratio of the spot index level to the option strike. To allow for the possibility that both out-of-the-money puts and calls may have higher implied volatilities, we employ both the degree of moneyness and its square as an explanatory variable. Implied volatilities are also known to rise with the option maturity and to accomodate this bias we also employ the option maturity as an explanatory variable. In addition we use the level of interest rates as an additional regressor. Consistency with general observations on the Black Scholes implied volatility surface suggests that the coefficient of the

degree of moneyness should be negative, while the coefficients for the square of moneyness and the option maturity should be positive.

The results of these orthogonality tests are presented in Table III. For the Black Scholes model, we observe a high degree of predictability in the pricing errors with an R^2 of 16% given in the fifth row of the first column of Table III. Moreover, there is a moneyness smile with both the linear and the quadratic moneyness coefficients being highly significant: and of the expected signs for a smile effect. This is indicated by rows two and three of column 1 of Table III. There is also a maturity bias indicative of rising implied volatilities with maturity. The pricing errors are also positively related to interest rates as seen from rows 4 and 5. All the independent variables employed are significant. The F statistic reported in row 7 of Table III shows that we must reject the hypothesis of orthogonality of Black Scholes pricing errors to these explanatory variables. The results are consistent with extant evidence on the empirical biases of the Black Scholes model.

We find that the symmetric VG model does not fare much better. The R^2, row 6 of column 2 of Table III, is 17% and the pricing errors are predictable to a similar extent as that of Black Scholes, with an F statistic of 425.068 given in row 7 of column 2. The model appears to over correct for the smile and has significant moneyness coefficients consistent with an inverted smile, (see rows 2 and 3 of column 2 of Table III). There is a positive maturity bias, though the interest rate is now not significant as evidenced by rows 4 and 5 of Table III.

The VG model performs much better. The R^2 (row 6, column 3 of Table III) is 0.001 and there appears to be no predictability, at least by the simple explanators considered. The F statistic is not significant. There is also no moneyness bias to note. The $t - statistics$ of rows 2 and 3 of column 3 are near zero. The positive maturity bias is also reduced and, if any, there is a slight negative maturity bias (see row 4 of column 3). On these orthogonality tests, the VG model appears to deliver acceptable option prices.

7. Conclusion

This paper presents a new option valuation formula, that nests the Black Scholes formula as a parametric special case, based on asset price dynamics characterized by the variance gamma process, obtained by evaluating Brownian motion with drift at a random time given by a gamma process. The pricing performance of the new model is compared on S&P 500 option data with the Black Scholes model and with a symmetric special case of the new model. In contrast to traditional Brownian motion, the VG process is a pure jump process with an infinite arrival rate of jumps, but unlike Brownian motion (that also has infinite motion), the process has finite variation and can be written as the difference of two increasing processes, each giving separately the market up and down moves. The resulting option pricing model has two parameters in addition to the asset volatility that allow for skewness and excess kurtosis in the risk-neutralized density. Theoretically skewness is a

Table III. Regression Results on the Predictability of S&P 500 Stock Index Futures Option Pricing Errors Based on the Black Scholes Model, the Symmetric Variance Gamma and the Variance Gamma Model: January 1992–September 1994

The regression specification is

$$PE_i = \alpha_0 + \alpha_1 MNY_i + \alpha_2 (MNY_i)^2 + \alpha_3 MAT_i + \alpha_4 INT_i + \varepsilon_i$$

where PE_i is the model pricing error, MNY_i is the ratio of the index level to the strike price of option i, $(MNY_i)^2$ is square of MNY_i, MAT_i is the option maturity and INT_i is the risk free interest rate. t-values are in parentheses

Explanatory variable	Black scholes	Symmetric VG	Variance gamma
α_0	59.005	−228.3113	0.0476
(Constant)	(4.254)**	(−22.367)**	(0.007)
α_1	−138.0872	435.7121	−0.4693
(Moneyness)	(−5.00)**	(21.711)**	(−0.020)
α_2	78.4561	−207.2874	0.5378
(Moneyness^2)	(5.703)**	(020.904)**	(0.077)
α_3	2.5739	1.4843	−0.6241
(Maturity)	(8.242)**	(5.012)**	−(2.310)*
α_4	10.4648	3.3723	2.9540
(Interest rate)	(3.787)**	(1.272)	(1.201)
ADJ-R^2	0.0.161	0.171	0.001
NOBS	8245	8245	8245
F-Stat$_{(4,8240)}$	394.460	425.068	2.649

*, **, Indicates significance at the 1% (5%) level.

consequence of risk aversion in facing the risks of price jumps that is not addressed in the Black Scholes model by assumption of continuity. Excess kurtosis is also a result of jumps and is reflected in risk premia on deep in- and out-of-the money options. Closed form formulas are developed for European option prices when the risk neutral dynamics is given by the VG process.

Estimates of the statistical and risk neutral VG process, and the Black Scholes special case are obtained for S&P 500 futures index option data. All parameters were estimated by the maximum likelihood method. We provide evidence that, while the statiscal density of the underlying index return is symmetric, the risk neutral density implied option data is negatively skewed (confirming risk aversion), with significant excess kurtosis. For the null of Black Scholes against the VG alternate, the null of Black Scholes is strongly rejected by likelihood ratio tests.

We also show that the superior performance of the VG model is reflected in orthogonality tests conducted on the pricing errors. Option pricing errors from the Black Scholes model and the symmetric special case of the VG model are observed to be correlated with the degree of moneyness and the maturity of the options. Orthogonality tests show that the VG model is relatively free of these biases.

Appendix

DERIVATION OF EQUATION (8)

The characteristic function (7) may be written as the product of the following two characteristic functions,

$$\phi_{\gamma_p(t)}(u) = \left[\frac{1}{1 - i(\nu_p/\mu_p)u} \right]^{\frac{\mu_p^2}{\nu_p} t}$$

and

$$\phi_{-\gamma_n(t)}(u) = \left[\frac{1}{1 + i(\nu_n/\mu_n)u} \right]^{\frac{\mu_n^2}{\nu_n} t}$$

with μ_p, μ_n, ν_p and ν_n satisfying

$$\frac{\mu_p^2}{\nu_p} = \frac{\mu_n^2}{\nu_n} = \frac{1}{\nu}, \tag{A1}$$

$$\frac{\nu_p \nu_n}{\mu_p \mu_n} = \frac{\sigma^2 \nu}{2}, \tag{A2}$$

$$\frac{\nu_p}{\mu_p} - \frac{\nu_n}{\mu_n} = \theta \nu. \tag{A3}$$

It follows that the VG process is the difference of two gamma processes with mean rates μ_p, μ_n and variance rates ν_p, ν_n respectively.

DERIVATION OF EQUATIONS (9), (10), (11) AND (12)

These relations are obtained by solving the equations (A1), (A2) and (A3). Solving (A2) and (A3) we obtain that

$$\frac{\nu_p}{\mu_p} = \frac{1}{2} \left[\sqrt{\theta^2 \nu^2 + 2\sigma^2 \nu} + \frac{\theta \nu}{2} \right]$$

$$\frac{\nu_n}{\mu_n} = \frac{1}{2} \left[\sqrt{\theta^2 \nu^2 + 2\sigma^2 \nu} - \frac{\theta \nu}{2} \right]$$

The result then follows from (A1).

DERIVATION OF EQUATIONS (18), (19) AND (20)

Conditional on the gamma time change, g, the VG variate, $X(t)$, over an interval of length t, is normally distributed with mean θg and variance $\sigma^2 g$. Hence, we may write

$$X(t) = \theta g + \sigma \sqrt{g}z \qquad (A4)$$

where z is a standard normal variate, independent of the gamma random variable g that has mean t and variance vt.

Computing expectations we obtain that

$$E[X(t)] = \theta t.$$

Let $x = X(t) - E[X(t)]$, then we may write

$$x = \theta(g - t) + \sigma\sqrt{g}z.$$

It follows on squaring and computing expectations that

$$E[x^2] = \theta^2 vt + \sigma^2 t,$$

and hence (18) holds.

Computing the cube of x, we have

$$x^3 = \theta^3(g-t)^3 + 3\theta^2(g-t)^2\sigma\sqrt{g}z + 3\theta(g-t)\sigma^2 gz^2 + \sigma^3 g^{\frac{3}{2}}z^3.$$

Taking expectations we obtain that

$$E[x^3] = \theta^3 E[(g-t)^3] + 0 + 3\sigma^2\theta vt + 0.$$

The expectation of $(g - t)^3$ may be obtained by explicit integration of the gamma density. On explicit integration we have that

$$\begin{aligned}
E[g^3] &= v^3(2 + \frac{t}{v})(1 + \frac{t}{v})\frac{t}{v} \\
&= t^3 + 3vt^2 + 2v^2 t.
\end{aligned}$$

It follows that

$$\begin{aligned}
E[(g-t)^3] &= t^3 + 3vt^2 + 2v^2 t - 3(vt + t^2)t + 3t^3 - t^3 \\
&= 2v^2 t.
\end{aligned}$$

The result of Equation (19) follows on substitution in the expression for $E[x^3]$.

For the fourth moment we note on expanding x^4 and taking expectations that

$$E[x^4] = \theta^4 E[(g-t)^4] + 6\sigma^2\theta^2 E[(g-t)^2 g] + 3\sigma^4 E[g^2].$$

The expectation of the g^4 may be explicitly computed by integration and is

$$
\begin{aligned}
E[g^4] &= (3\nu + t)(2\nu + t)(\nu + t)t \\
&= 6\nu^3 t + 11\nu^2 t^2 + 6\nu t^3 + t^4.
\end{aligned}
$$

The result for (20) follows by substitution and collecting terms.

Proof of Theorem 1. We may write the density of z, as the conditional density given g times the marginal density of g, with the random variable g integrated out. Hence,

$$
h(z) = \int_0^\infty \frac{\exp\left(-\dfrac{1}{2\sigma^2 g}\left(z - mt - \dfrac{t}{\nu}\ln(1 - \theta\nu - \sigma^2\nu/2) - \theta g\right)^2\right)}{\sigma\sqrt{2\pi g}} X
$$
$$
\frac{g^{\frac{t}{\nu}-1}\exp\left(-\dfrac{g}{\nu}\right)}{\nu^{\frac{t}{\nu}}\Gamma\left(\dfrac{t}{\nu}\right)} dg,
$$

because z is normally distributed with mean $mt + t/\nu\ln(1 - \theta\nu - \sigma^2\nu/2) + \theta g$ and variance $\sigma^2 g$ conditional on the gamma variate g. By Gradshetyn and Ryzhik (1970) 3.471.9 this form is integrable with the result given by (23).

Proof of Theorem 2. First conditioning on the random time g, the conditional option value is obtained from the conditional normality as in Madan and Milne (1991, equation (6.5)), noting that the coefficient α in that paper is ζs) as a Black Scholes type formula, with the option value, $c(g)$, being

$$
\begin{aligned}
c(g) =\ & S(0)\left(1 - \frac{\nu(\alpha+s)^2}{2}\right)^{\frac{t}{\nu}}\exp\left(\frac{(\alpha+s)^2 g}{2}\right) \\
& \times N\left(\frac{d}{\sqrt{g}} + (\alpha+s)\sqrt{g}\right) \\
& - K\exp(-rt)\left(1 - \frac{\nu\alpha^2}{2}\right)^{\frac{t}{\nu}}\exp\left(\frac{\alpha^2 g}{2}\right) \\
& \times N\left(\frac{d}{\sqrt{g}} + \alpha\sqrt{g}\right),
\end{aligned}
\tag{A5}
$$

where N is the cumulative distribution function of the standard normal variate and d is given by equation (26). The call option price, $c(S(0); K, t)$ is obtained on integrating with respect to the gamma density, specifically,

$$c(S(0); K, t) = \int_0^\infty c(g) \frac{g^{\frac{t}{\nu}-1} \exp(-\frac{g}{\nu})}{\nu^{\frac{t}{\nu}} \Gamma(\frac{t}{\nu})} dg. \tag{A6}$$

Making the change of variable $y = g/\nu$ and defining $\gamma = \frac{t}{\nu}$, $c_1 = \frac{\nu(\alpha+s)^2}{2}$, $c_2 = \frac{\nu\alpha^2}{2}$ we may write that

$$
\begin{aligned}
c(S(0); K, t) = \int_0^\infty \Bigg\{ & S(0) (1-c_1)^\gamma \exp(c_1 y) N\left(\frac{d/\sqrt{\nu}}{\sqrt{y}} + (\alpha+s)\sqrt{\nu}\sqrt{y}\right) \\
& -K \exp(-rt) (1-c_2)^\gamma \exp(c_2 y) N\left(\frac{d/\sqrt{\nu}}{\sqrt{y}} + \alpha\sqrt{\nu}\sqrt{y}\right) \Bigg\} \\
& \frac{y^{\gamma-1} \exp(-y)}{\Gamma(\gamma)} dy
\end{aligned}
$$

where we may now write

$$d = \frac{\ln(S(0)\exp(rt)/K)}{s} + \frac{\gamma}{s} \ln\left(\frac{1-c_1}{1-c_2}\right).$$

Consider the general form

$$\int_0^\infty \exp(cy) N\left(\frac{a}{\sqrt{y}} + b\sqrt{y}\right) \frac{y^{\gamma-1}\exp(-y)}{\Gamma(\gamma)} dy$$

and note on making the change of variable $u = (1-c)y$ that this integral is equal to

$$\int_0^\infty N\left(\frac{a\sqrt{1-c}}{\sqrt{u}} + \frac{b}{\sqrt{1-c}}\sqrt{u}\right) \frac{u^{\gamma-1}\exp(-u)}{(1-c)^\gamma \Gamma(\gamma)} du.$$

Hence if we define the function

$$\Psi(a, b, \gamma) = \int_0^\infty N\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right) \frac{u^{\gamma-1}\exp(-u)}{\Gamma(\gamma)} du.$$

we may write the call option directly in terms of Ψ as

$$
\begin{aligned}
c(S(0); K, t) = {} & S(0)\Psi\left(d\sqrt{\frac{1-c_1}{\nu}}, (\alpha+s)\sqrt{\frac{\nu}{1-c_1}}, \gamma\right) - \\
& K\exp(-rt)\Psi\left(d\sqrt{\frac{1-c_2}{\nu}}, \alpha\sqrt{\frac{\nu}{1-c_2}}, \gamma\right).
\end{aligned}
$$

A closed form for the option price is obtained on developing a closed form for the function $\Psi(a, b, \gamma)$. For this we first develop expressions for the derivatives of Ψ with respect to a and b. Taking derivatives and integrating using Gradshetyn and Ryzhik (1970) 3.471.9 we obtain that

$$\Psi_a = \frac{2\exp(-ab)}{2\pi\Gamma(\gamma)} \left(\frac{a^2/2}{1+b^2/2}\right)^{\frac{\gamma}{2}-\frac{1}{4}} K_{\gamma-1/2}\left(|a|\sqrt{2+b^2}\right)$$

where K_α is the modified bessel function of the second kind of order α. By a similar calculation we obtain

$$\Psi_b = \frac{2\exp(-ab)}{2\pi\Gamma(\gamma)} \left(\frac{a^2/2}{1+b^2/2}\right)^{\frac{\gamma}{2}+\frac{1}{4}} K_{\gamma+1/2}\left(|a|\sqrt{2+b^2}\right)$$

To evaluate $\Psi(a, b, \gamma)$ we choose a path of integration in (a, b) space along which the arguments of the bessel functions are constant. Hence, consider the path of integration

$$b(t) = t \qquad\qquad -\infty < t < b$$

$$a(t) = \frac{sign(a)|a|\sqrt{2+b^2}}{\sqrt{2+t^2}} \qquad -\infty < t < b.$$

noting by construction that along the path of integration $|a(t)|\sqrt{2+b(t)^2} = |a|\sqrt{2+b^2} = c$. We may then write

$$\Psi(a, b, \gamma) = \int_{-\infty}^{b} \left[\Psi_a a'(t) + \Psi_b b'(t)\right] dt, \tag{A7}$$

for when $b = -\infty$, $a = 0$ and we know that $\Psi(0, -\infty, \gamma) = 0$. Substituting required expressions and evaluating (A5) we get that

$$c(S(0); K, t) = \int_{-\infty}^{b} 2\frac{\exp\left(-\frac{sign(a)ct}{\sqrt{2+t^2}}\right)}{2\pi\Gamma(\gamma)} \left\{ \left(\frac{c^2}{(2+t^2)^2}\right)^{\frac{\gamma}{2}+\frac{1}{4}} K_{\gamma+\frac{1}{2}}(c) \right.$$

$$\left. - \left(\frac{c^2}{(2+t^2)^2}\right)^{\frac{\gamma}{2}-\frac{1}{4}} K_{\gamma-\frac{1}{2}}(c)\frac{sign(a)ct}{(2+t^2)^{3/2}} \right\} dt$$

On simplifying and making the change of variable $y = \frac{t}{\sqrt{2+t^2}}$ we obtain that

$$c(S(0); K, t) = \int_{-1}^{\frac{b}{\sqrt{2+b^2}}} \frac{c^{\gamma+\frac{1}{2}}}{2\pi\Gamma(\gamma)2^{\gamma-1}}$$

$$\left\{ K_{\gamma+\frac{1}{2}}(c)\exp(-sign(a)cy)\left(1-y^2\right)^{\gamma-1} \right.$$

$$\left. -sign(a)K_{\gamma-\frac{1}{2}}(c)\exp(-sign(a)cy)\left(1-y^2\right)^{\gamma-1}y \right\} dy$$

The answer follows on obtaining the integrals,

$$H_1(u, c, \gamma) = \int_{-1}^{u} \exp(-cy)(1 - y^2)^{\gamma-1} dy$$

and

$$H_2(u, c, \gamma) = \int_{-1}^{u} \exp(-cy)(1 - y^2)^{\gamma-1} y \, dy.$$

We then have that

$$\Psi(a, b, \gamma) = \frac{c^{\gamma+\frac{1}{2}}}{2\pi\Gamma(\gamma)2^{\gamma-1}} \left\{ K_{\gamma+\frac{1}{2}}(c) H_1 \left(\frac{b}{2+b^2}, \text{sign}(a)c, \gamma \right) \right.$$

$$\left. - \text{sign}(a) K_{\gamma-\frac{1}{2}}(c) H_2 \left(\frac{b}{2+b^2}, \text{sign}(a)c, \gamma \right) \right\}. \qquad \text{(A8)}$$

To reduce the integral H_1 in terms of the special functions of mathematics we make the change of variable $x = \frac{1+y}{1+u}$ to obtain that

$$H_1(u, c, \gamma) = \exp(c)(1+u)^{\gamma} 2^{\gamma-1} \int_0^1 \exp(-c(1+u)x)$$

$$\times \left(1 - \frac{1+u}{2} x \right)^{\gamma-1} x^{\gamma-1} dx.$$

The degenerate hypergeometric function of two variables, Φ has the integral representation (Humbert (1920))

$$\Phi(\alpha, \beta, \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}$$

$$\times \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta} \exp(uy) du.$$

We may therefore obtain H_1 in terms of Φ, specifically,

$$H_1(u, c, \gamma) = \exp(c)(1+u)^{\gamma} 2^{\gamma-1} \frac{1}{\gamma} \Phi$$

$$\times \left(\gamma, 1-\gamma, 1+\gamma; \frac{1+u}{2}, -c(1+u) \right). \qquad \text{(A9)}$$

The second integral, for H_2 is similarly obtained and is given by

$$H_2(u, c, \gamma) = \frac{\exp(c)(1+u)^{1+\gamma} 2^{\gamma-1}}{1+\gamma} \Phi$$

$$\times (1+\gamma, 1-\gamma, 2+\gamma; \frac{1+u}{2}, -c(1+u)) - H_1(u, c, \gamma).$$

$$\text{(A10)}$$

Substituting (A9) and (A10) into (A8) yields the representation of $\Psi(a, b, \gamma)$ in terms of the modified bessel function of the second kind and the degenerate hypergeometric function of two variables as

$$\Psi(a, b, \gamma) = \frac{c^{\gamma+\frac{1}{2}} \exp(\text{sign}(a)c)(1+u)^{\gamma}}{\overline{2\pi}\,\Gamma(\gamma)\gamma} X \tag{A11}$$

$$K_{\gamma+\frac{1}{2}}(c)\Phi\left(\gamma, 1-\gamma, 1+\gamma; \frac{1+u}{2}, -\text{sign}(a)c(1+u)\right)$$

$$-\text{sign}(a)\frac{c^{\gamma+\frac{1}{2}} \exp(\text{sign}(a)c)(1+u)^{1+\gamma}}{\overline{2\pi}\,\Gamma(\gamma)(1+\gamma)} X$$

$$K_{\gamma-\frac{1}{2}}(c)\Phi\left(1+\gamma, 1-\gamma, 2+\gamma; \frac{1+u}{2}, -\text{sign}(a)c(1+u)\right)$$

$$+\text{sign}(a)\frac{c^{\gamma+\frac{1}{2}} \exp(\text{sign}(a)c)(1+u)^{\gamma}}{\overline{2\pi}\,\Gamma(\gamma)\gamma} X$$

$$K_{\gamma-\frac{1}{2}}(c)\Phi\left(\gamma, 1-\gamma, 1+\gamma; \frac{1+u}{2}, -\text{sign}(a)c(1+u)\right) \quad ,$$

where $c = |a|\,\overline{2+b^2}$ and $u = \frac{b}{\overline{2+b^2}}$.

DERIVATION OF THE LIKELIHOOD FUNCTION FOR RISK NEUTRAL ESTIMATION

The log likelihood function of the data w_i for $i = 1, \cdots M$ under the model given by Equation (29) is

$$\ln \mathcal{L} = -\frac{1}{2}\sum_{i=1}^{M}\left(\frac{\ln(w_i) - \ln(\widehat{w}_i)}{\eta} + \frac{\eta}{2}\right)^2$$
$$-\frac{M\ln(2\pi)}{2} - M\ln\eta - \sum_{i=1}^{M}\ln(w_i).$$

The partial with respect to η gives on simplification the equation

$$k^2 = \frac{1}{M}\sum_{i=1}^{M}(\ln(w_i) - \ln(\widehat{w}_i))^2 = \eta^2 + \frac{\eta^4}{4}, \tag{A12}$$

and this equation may be solved for η to obtain

$$\eta = \sqrt{2(\overline{1+k^2} - 1)}. \tag{A13}$$

For purposes of model comparison or estimation, the terms not involving parameters may be dropped and this leaves, for twice the log likelihood the expression

$$2 \ln \mathcal{L} = -\sum_{i=1}^{M} \left[\frac{\ln(w_i) - \ln(\widehat{w}_i)}{\eta} + \frac{\eta}{2} \right]^2 - M \ln \eta^2$$

Completing the square, dividing by M and simplifying we get that

$$\frac{2 \ln \mathcal{L}}{M} = -\frac{1}{M} \sum_{i=1}^{M} \left[\frac{\ln(w_i) - \ln(\widehat{w}_i)}{\eta} \right]^2 -$$

$$\frac{1}{M} \sum_{i=1}^{M} (\ln(w_i) - \ln(\widehat{w}_i)) - \frac{\eta^2}{4} - \ln \eta^2 \qquad (A14)$$

Substituting for the sum of squares of log differences from (A12) and using the definition of ξ as the average of log price relatives we obtain ignoring constants that

$$\frac{2 \ln \mathcal{L}}{M} = -\ln \eta^2 - (\xi + \frac{\eta^2}{2}). \qquad (A15)$$

Substituting from (A13) into (A15) yields the expression employed for likelihood ratio test comparisons

$$2 \ln \mathcal{L} = -M \left[\ln \left(2(\sqrt{1+k^2} - 1) \right) + (\xi + \sqrt{1+k^2} - 1) \right]. \qquad (A16)$$

For the estimation we observe that for large M the limiting value of ξ is by hypothesis $\eta^2/2$ and hence (A15) shows that asymptotically maximum likelihood estimation is equivalent to minimizing η or by (A13), k.

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