Generating integrable one dimensional driftless diffusions

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GENERATING INTEGRABLE ONE DIMENSIONAL DRIFTLESS DIFFUSIONS

PETER CARR, PETER LAURENCE AND TAI-HO WANG

Abstract. A criterion on the diffusion coefficient is formulated that allows the classification of driftless time and state dependent diffusions that are integrable in closed form via point transformations. In the time dependent and state dependent case a remarkable intertwining with the inhomogeneous Burger's equation is exploited. The criterion is constructive. It allows us to construct families of driftless diffusions parametrized by a rich class containing several arbitrary funtions for which the solution of any initial value problem can be expressed in closed form. We also derive an elegant form for the masters equation for infinitesimal symmetries, previously considered only in the time homogeneous case.

Résumé Nous présentons une condition nécessaire et suffisante sur le coefficient de diffusion $g(x,t)$ d'une diffusion sans drift, afin que celle-ci puisse se réduire, par des transformations ponctuelles des variables dépendentes et indépendantes, à la forme canonique de Lie $u_t - \frac{1}{2}u_{xx} + \frac{A}{x^2}u = 0$ où $A \in \mathbb{R}$. Lie a démontré que celle-ci est la forme canonique d'une diffusion dont le groupe de symétrie est de dimension quatre ou six. Notre résultat complète donc celui de Lie, en donnant une condition locale intrinsèque sur g rendant possible une telle réduction, ainsi qu'une condition constructive, dans la mesure où elle nous permet de construire de façon explicite la solution fondamentale de l'equation correspondante.

Version Française abrégée

Considérons le problème consistant à trouver la probabilité de transition d'une diffusion

$$dx_t = g(x_t, t)dW_t, \quad t \in [0, T]$$

sur un espace de probabilité filtré $(\Omega, \mathcal{B}, P)$, où W_t est un mouvement Brownien unidimensionel. Il est bien connu que résoudre ce problème équivaut à déterminer la solution fondamentale de l'équation rétrograde

$$u_t + \frac{1}{2}g^2(x,t)u_{xx} = 0 \tag{1}$$

avec condition finale

$$u(\xi, T) = \delta_\xi(x). \tag{2}$$

Un problème de grande importance en physique et en mathématiques financières est de pouvoir exhiber cette solution fondamentale sous une forme explicite. Lie, voulant

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classifier toutes les équations aux dérivées partielles du second ordre qui puissent se résoudre par un processus "d'intégration", a démontré le théorème suivant :

Proposition 1. (**Lie** [11]) *Soit*

$$\mathcal{L}^{a,b,c}u \equiv u_t + a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u = 0 \tag{3}$$

avec $a(x,t) \neq 0$. *L'algèbre de Lie principale* $L_{\mathcal{P}}$ *(c.a.d. l'algèbre de Lie admise par l' équation (3)) ayant pour coefficients a, b, c, admet les opérateurs de symétrie triviaux* $u\frac{\partial}{\partial u}$ *et* $\phi(x,t)\frac{\partial}{\partial u}$*, où* ϕ *est une solution de (3) et peut se mettre sous la forme*

$$v_\tau = v_{yy} + Z(\tau, y)v \tag{4}$$

par le biais d'une transformation, appelée transformation d'équivalence de Lie, soit :

$$y = \alpha(x,t),\ \tau = \beta(t),\ v = \gamma(y,\tau)u(y,\tau),\ \alpha_x \neq 0,\ \beta_t \neq 0. \tag{5}$$

Si l'équation (3) admet une extension de l'algèbre de Lie principale par un opérateur de symétrie supplémentaire, elle se réduit à la forme

$$v_\tau = v_{yy} + Z(y)v. \tag{6}$$

Si l'algèbre s'étend par trois operateurs supplémentaires (la partie finie de l'algèbre est de dimension 4*), elle se réduit à la forme*

$$v_\tau - v_{yy} + \frac{A}{y^2}v = 0 \quad \text{où} \quad A \quad \text{est une constante.} \tag{7}$$

Si $\mathcal{L}_p$ *s'étend par cinq opérateurs, l' équation (3) se réduit à l' équation de la chaleur*

$$v_\tau - v_{yy} = 0. \tag{8}$$

Notre principal résultat est un critère sur le coefficient de diffusion, qui permet de décider quand une diffusion peut se mettre sous une des formes (7) ou (8). Etant donné que les diffusions considérées peuvent, comme dans le cas des diffusions CEV où $g(x,t) = x^{1+\beta}, \beta \in \mathbb{R}$, être dégénérées et que la transformation de Lie-Bluman $y = \int_a^x \frac{1}{g(x',t)}dx' + \zeta(t)$, qui suppose l'intégrabilité de $1/g$, n'est pas dans ces cas-là bien définie, nous introduisons une classe de diffusions dégénérées qui n'est pas la plus générale possible mais qui permet, sans peine, d'appliquer la transformation de Lie-Bluman dans la plupart des cas rencontrés en physique et en mathématiques financières.

Definition 1. *Soit* $I = (l,r) \subset \mathbb{R}$ *un intervalle, pouvant être non borné. Soit* $\mathcal{L}u = u_t - \frac{1}{2}g^2(x,t)u_{xx} = 0$ *une diffusion sur l' intervalle* I*. Supposons que* $g(x,t) \geq 0$ *soit continu sur* I*. Définissons de façon itérative un recouvrement fini* $\cup_i[l_i,r_i] = \cup_i I_i = I$ *de* I *avec pour centres associés* m_i*, selon le procédé suivant:*

- *Choisissons* m_1 *avec* $g(m_1,t) \neq 0$ *et définissons* l_1 *et* r_1 *par*

$$l_1 = \inf\left\{x \in I : \int_{m_1}^x \frac{dx'}{g(x',t)} > -\infty\right\}, \quad r_1 = \sup\left\{x \in I : \int_{m_1}^x \frac{dx'}{g(x',t)} < +\infty\right\}.$$

  et posons $R_1^- = \int_{m_1}^{l_1} \frac{dx'}{g(x',t)}, R_1^+ = \int_{m_1}^{r_1} \frac{dx}{g(x',t)}$.
- *Ayant défini* m_i, $R_i^\pm$ *et* I_i *pour* $i \leq i_0$*, définissons* m_{i_0+1} *en choisissant* $m_{i_0+1} \in I \setminus \cup_{i=1}^{i_0} I_i$ *avec* $g(m_{i_0+1},t) > 0$ *et par la suite en procédant comme ci-dessus.*

Definition 2. *Nous disons qu'une diffusion* $\mathcal{L}u = 0$ *est modérement degénérée si* l_i, r_i *et* m_i *peuvent être choisis indépendement du temps et si ce recouvrement est fini.*

Proposition 2. *Soit* $\mathcal{L}u = 0$ *une diffusion modérement degénérée avec recouvrement associé* $\mathcal{C} = \{I_i = [l_i, r_i] : i = 1, \cdots, n\}$. *Soit* $H^{(i)}(y,t) = \int_{D^{(i)}}^{y} \tilde{g}^{(i)}(y',t)dy' + m_i$, *avec* $\tilde{g}^{(i)}$ *satisfaisant* $\tilde{g}^{(i)}(Y^{(i)}(x,t),t) = g(x,t)$, $y = Y^{(i)}(x,t) = \int_{m_i}^{x} \frac{1}{g(x',t)}dx' + D^{(i)}(t)$. *Pour chaque* $I_i \in \mathcal{C}$,

- *Une condition nécessaire et suffisante afin qu'une diffusion modérement degénérée puisse se réduire à une forme canonique à quatre dimension est qu'il existe* $\lambda^{(i)} = 0$ *et des coefficients* $A^{(i)}, B^{(i)}, C^{(i)}, D^{(i)}$, *tels que* $\ddot{D}^{(i)} - 2A^{(i)}D^{(i)} = B^{(i)}$ *et tels que pour chaque* i, $H^{(i)}$ *satisfait l'EDP*

$$H_t - \frac{1}{2}H_{yy} + \beta^{(i)}H_y = 0 \quad \text{pour } y \in (R_i^-, R_i^+) \setminus \{D^{(i)}(t)\}, \tag{9}$$

et les conditions

$$H(D^{(i)} + R_i^-, t) = l_i, H(D^{(i)}, t) = m_i, H(D^{(i)} + R_i^+, t) = r_i$$

où $\beta^{(i)} = -(\log \alpha^{(i)})_y$, *avec* $\alpha^{(i)}$ *satisfaisant l' équation*

$$\alpha_t - \frac{1}{2}\alpha_{yy} + \left(\frac{\lambda^{(i)}}{(y - D^{(i)}(t))^2} + A^{(i)}(t)y^2 + B^{(i)}(t)y + C^{(i)}(t)\right)\alpha = 0 \tag{10}$$

pour $y \in (R_i^-, R_i^+) \setminus \{D^{(i)}(t)\}$. *Notons que le produit d'une solution de (9) et d'une solution de (10), c.à.d.* $H^{(i)}\alpha^{(i)}$, *satisfait aussi à (10) pour* $y = D^{(i)}(t)$.

Ce résultat peut être utilisé pour exhiber de nouvelles classes de diffusions dont la solution fondamentale peut s'exprimer sous forme explicite. Une simple extension de la méthode au cas avec drift met en lumière la structure de groupe sous-jacente du résultat de Feller [8] et une extension de certains résultats bien connus pour les procéssus CEV, dans le cas où leur coefficients peuvent dépendre du temps. Voir Exemples 1 et 2 de la version anglaise.

English Version

Consider the problem of determining the transition probability of a one-dimensional diffusion

$$dx_t = g(x_t, t)dW_t, \quad t \in [0, T], \quad x_0 = \xi,$$

where W_t is a standard Brownian motion with respect to an underlying probability space $(\Omega, \mathcal{B}, P)$. As is well known, under general conditions, this problem is equivalent to finding the solution of the backward Kolmogorov equation for $u(\xi, t, \eta, T)$

$$u_t + \frac{1}{2}g^2(\xi, t)u_{\xi\xi} = 0 \tag{11}$$

with terminal condition

$$u(\xi, 0, \eta, T) = \delta_\eta(\xi). \tag{12}$$

In the setting of more general one dimensional diffusions, Sophus Lie [11] discovered a classification of second order differential equations in two variables. A detailed statement of this classification in the parabolic case can be found in the French version of this note. Lie's main result is that the symmetry algebra $\mathcal{L}_P$ of the equation

$$\mathcal{L}^{a,b,c}u \equiv u_t - a(x,t)u_{xx} - b(x,t)u_x - c(x,t)u = 0, \qquad a > 0 \tag{13}$$

is the direct sum of two components $\mathcal{L}_P = \mathcal{L}_P^f \oplus \mathcal{L}_P^\infty$, where $\mathcal{L}_P^\infty$ is generated by symmetry operators of the form $\phi(x,t)\frac{\partial}{\partial u}$ for ϕ an arbitrary solution of (13). Lie showed that $\mathcal{L}_P^f$ is one, two, four or six dimensional. All such equations can be reduced, by a transformation called Lie's equivalence transformation, hereafter LET (also known as a linear *point transformation*),

$$y = \alpha(x,t),\ \tau = \beta(t),\ v = \gamma(y,\tau)u(y,\tau),\ \alpha_x = 0,\ \beta_t = 0, \tag{14}$$

to the form

$$v_t - \frac{1}{2}v_{yy} + Z(t,y)v = 0, \tag{15}$$

called *Lie's canonical form.* Z is called the *potential term.* Actually (13) can be reduced to the form (15) without making a time change using a particular equivalence transformation, the Lie-Bluman transformation defined in the sequel (see (17-18)). If the finite component of the symmetry group is at least two dimensional Lie showed that (13) can be put in canonical form with a potential term $Z(y)$ independent of t. At the other extreme, if $\mathcal{L}_P^f$ is six dimensional, the canonical form is the heat equation, ie. $Z = 0$. All diffusion equations possessing a four dimensional symmetry group can be transformed, by an equivalence transformation, to the form

$$v_t - \frac{1}{2}v_{yy} + \frac{\lambda}{y^2}v = 0 \tag{16}$$

where λ is a nonzero constant. The fundamental solution of the above equation can be shown to be (see [10])

$$F(t,y,\xi) = K\sqrt{\frac{y\xi}{t}}\frac{e^{-\frac{\xi y}{t}}}{\sqrt{t}}\mathcal{I}\left(\kappa, \frac{\xi y}{t}\right)\exp\left(-\frac{(y-\xi)^2}{2t}\right),$$

where $\kappa = \pm\frac{\sqrt{1+8\lambda}}{2}$, K is a normalization constant that ensures $\lim_{t\to 0^+}\int_{\Re} F(t,y,\xi)d\xi = 1$ and $\mathcal{I}(\kappa,\cdot)$ is a modified Bessel function of order κ. It is this class that will be the main focus of the present paper which will be devoted to fully characterizing those diffusion coefficients which are associated with the four dimensional symmetry class and will also provide a constructive method for determining such coefficients. On the other hand, no *canonical form*, i.e. no special choice(s) of potential Z, has been identified to date characterizing *all* equations possessing a symmetry group of dimension 2, corresponding to the smallest non-trivial symmetry algebra. In this note we concentrate on the important case $b = c = 0$ of a driftless diffusion. Our results readily extend beyond this case taking on however a more complicated and less intuitive form. Although we do not classify driftless diffusions with two dimensional symmetry group, we also provide in this note, for the first time in the time-dependent case, an elegant form for the "master equation" (see (34)) and the latter is a significant step towards finding new classes of driftless diffusions with two dimensional symmetry group and towards unifying previous special cases.

1. Main Results

It is natural to ask the following questions:

- *Question* : Characterize the diffusion coefficients $g(x,t)$ for which the corresponding equation can be reduced to canonical form (16) for $\lambda = 0$ or for $\lambda = 0$ by a point transformation.

Important examples, such as the case $g(x,t) = x^{1+\beta}$, $\beta \in \mathbb{R}$, $x \geq 0$ that plays a central role in mathematical finance, are mappable to the four dimensional class and are at the same time *degenerate diffusions*. Introducing the Lie-Bluman transformation (LBT), a *special* Lie equivalence transformation that can be used to transform (13) to canonical form (15)

$$y = Y(x,t) = \int_0^x \frac{1}{g(x',t)} dx' + \zeta(t) \quad \text{change of independent variable,} \tag{17}$$

$$v = w \exp \int_\zeta^y \left[Y_t + \frac{1}{2} \frac{\tilde{g}_y}{\tilde{g}} \right] \quad \text{change of dependent variable,} \tag{18}$$

$$\text{where} \quad \tilde{g}(y,t) = g(Y^{-1}(x,t),t) \text{ and } \zeta(t) \text{ is an arbitrary function of } t, \tag{19}$$

we observe that when applying LBT to a diffusion degenerate at zero (for instance CEV processes mentioned above) there are issues of non-integrability for values of $\beta \geq 0$. That is why below we will introduce a slight modification of the Lie-Bluman methodology that is better suited to deal with such degeneracies. Note also that although LBT can be used to transform a diffusion equation with a diffusion coefficient in a certain class into canonical form (15) with a particular potential Z associated to a given symmetry class, this does not exclude that there may be a larger class of diffusion coefficients that may be put into the same canonical form by a *different* equivalence transformation. We are now in a position to state our main results. We begin with a definition whose purpose is to split up the domain on which the diffusion is being considered into several maximal subdomains in which LBT can be applied. The following definition is flexible enough to account for most applications found in practice. If the diffusion coefficient is pathological enough, the definition needs to be generalized to include time dependent centers and countably infinite coverings.

Definition 3. *Let $I = (l,r) \subset \mathbb{R}$ be an open (possibly infinite or semi-infinite) interval. Let $\mathcal{L}u = u_t - \frac{1}{2}g^2(x,t)u_{xx} = 0$ be a diffusion on the interval I. Assume that $g(x,t) \geq 0$ is continuous on I. Define iteratively a finite covering $\cup_i [l_i, r_i] = \cup_i I_i = I$ of I with collection of centers m_i as follows (with the convention that $[l_i, r_i] = (l, r_i]$ if $l_i = l$ and $[l_i, r_i] = [l_i, r)$ if $r_i = r$ hereafter):*

- *Choose m_1 where $g(m_1, t) = 0$ and define l_1, r_1 and then R_1^-, R_1^+ by*

$$l_1 = \inf \left\{ x \in I : \int_{m_1}^x \frac{dx'}{g(x',t)} > -\infty \right\}, \quad r_1 = \sup \left\{ x \in I : \int_{m_1}^x \frac{dx'}{g(x',t)} < +\infty \right\}$$

$$R_1^- = \int_{m_1}^{l_1} \frac{dx'}{g(x',t)}, \quad R_1^+ = \int_{m_1}^{r_1} \frac{dx'}{g(x',t)}.$$

- *Having defined m_i, $R_i^{\pm}$ and I_i for $i \leq i_0$ define m_{i_0+1} by picking $m_{i_0+1} \in I \setminus \cup_{i=1}^{i_0} I_i$ with $g(m_{i_0+1}, t) > 0$ and proceed as before.*

Definition 4. *We say that $\mathcal{L}u = 0$ is a moderately degenerate diffusion if the l_i, r_i and m_i's can be chosen independently of time and if the covering is finite.*

Proposition 3. *Let $\mathcal{L}u = 0$ be a moderately degenerate diffusion as in Definition 4 with associated natural covering $\mathcal{C} = \{I_i = [l_i, r_i] : i = 1, \cdots, n\}$. Let $H^{(i)}(y,t) = \int_{D^{(i)}}^{y} \tilde{g}^{(i)}(y',t)dy' + m_i$, where $\tilde{g}^{(i)}$ satisfies $\tilde{g}^{(i)}(Y^{(i)}(x,t),t) = g(x,t)$ and $y = Y^{(i)}(x,t) = \int_{m_i}^{x} \frac{1}{g(x',t)}dx' + D^{(i)}(t)$. For each $I_i \in \mathcal{C}$,*

(1) *a necessary and sufficient criterion for a moderately degenerate driftless diffusion to be reducible to a four dimensional canonical form is that there exist a $\lambda^{(i)} = 0$, coefficients $A^{(i)}, B^{(i)}, C^{(i)}, D^{(i)}$ with $\ddot{D}^{(i)} - 2A^{(i)}D^{(i)} = B^{(i)}$ such that $H^{(i)}$ satisfies the partial differential equation*

$$H_t - \frac{1}{2}H_{yy} + \beta^{(i)}H_y = 0, \quad \text{for} \quad y \in \left(R_i^-, R_i^+\right) \setminus \{D^{(i)}(t)\}, \tag{20}$$

subject to the conditions (with $l_i \leq m_i \leq r_i$)

$$H(D^{(i)} + R_i^-, t) = l_i, H(D^{(i)}, t) = m_i, H(D^{(i)} + R_i^+, t) = r_i$$

where $\beta^{(i)} = -(\log \alpha_2^{(i)})$ and where $\alpha_2^{(i)}$ satisfies the equation

$$\alpha_t - \frac{1}{2}\alpha_{yy} + \left(\frac{\lambda^{(i)}}{(y - D^{(i)}(t))^2} + A^{(i)}(t)y^2 + B^{(i)}(t)y + C^{(i)}(t)\right)\alpha = 0 \tag{21}$$

for $y \in (R_i^-, R_i^+) \setminus \{D^{(i)}(t)\}$. Note that the product of a solution of (20) and a solution of (21), i.e. $\alpha_1^{(i)} := H^{(i)}\alpha_2^{(i)}$, also satisfies (21) when $y = D^{(i)}(t)$, hence $H^{(i)}$ is the ratio of two solutions of (21).

(2) *a necessary and sufficient criterion for a moderately degenerate driftless diffusion to be reducible to a six dimensional canonical form is that $H^{(i)}$ satisfies the partial differential equation*

$$H_t - \frac{1}{2}H_{yy} + \beta^{(i)}H_y = 0, \quad \text{for } y \in (R_i^-, R_i^+) \setminus \{D^{(i)}(t)\} \tag{22}$$

subject to the conditions

$$H(D^{(i)} + R_i^-, t) = l_i, H(D^{(i)}, t) = m_i, H(D^{(i)} + R_i^+, t) = r_i$$

where $\beta^{(i)} = -(\log \alpha^{(i)})_y$ and $\alpha^{(i)}$ satisfies the equation

$$\alpha_t - \frac{1}{2}\alpha_{yy} + \left(A^{(i)}(t)y^2 + B^{(i)}(t)y + C^{(i)}(t)\right)\alpha = 0. \tag{23}$$

Here $A^{(i)}, B^{(i)}$ and $C^{(i)}$ are arbitrary time dependent functions. The general solution of equation (22) can be expressed as the ratio $\frac{\alpha_1^{(i)}}{\alpha_2^{(i)}}$ of two arbitrary solutions $\alpha_1^{(i)}$ and $\alpha_2^{(i)}$ of equation (23) where $\beta^{(i)}$ in (22) is given by $\beta^{(i)} = -(\log \alpha_2^{(i)})_y$.

We shall restrict ourselves to any of the subintervals $I \in \mathcal{C}$ in the following therefore the dependence of the index i will be suppressed for notational simplicity.

Remark 1. *Having determined the solution $H(y,t)$ one obtains the diffusion coefficient $\tilde{g} = H_y$ in the (implicit) y variable, notes that $H(y,t) = \int_{D(t)}^{y} \tilde{g}(y',t)dy' + m$ so $y = Y(x,t) \Leftrightarrow x = H(y,t)$, i.e. one inverts the now known function $H(y,t)$ and then sets $g(x,t) = \tilde{g}(Y(x,t),t)$. The additional conditions on the endpoints and center of the I_i interval ensure that such a process is a self-consistent one. In some cases the resulting H will be a weak solution (appropriately defined) of the equation throughout the entire interval but this will not generally be the case. We also remark that, if the conditions on $g(x,t)$ are strengthened to $g \in C^1$ then by construction , $R_i^- = -\infty$ except possibly for the leftmost subinterval and $R_i^+ = +\infty$ except possibly for the rightmost subinterval.*

Remark 2. *In the theory of the Schroedinger equations and reduction to Liouville form a key role is played by the equation $\{z,r\} = 2J(r)$ where $\{z,r\}$ is the Schwarzian derivative. A classical theorem due to Schwarz* [9] *says that the solution of this equation can be expressed as the ratio of two independent solutions of $\psi''(t) + J(r)\psi = 0$. As far as we can tell this observation (constructive procedure), in the context of parabolic equations with time dependent coefficients, appears to be new even in the context of reducibility to the heat equation. A related remark, generalizing results in Albanese et al* [1] *to the time dependent case, was made by Albanese* [3]*. Lastly note that the relation between (22) and (23) holds independently of the particular potential chosen and so may prove to be valuable in the case of $2D$ symmetry group as well.*

Proposition 4. *(Time independent case) In the special case in which $g(x,t)$ is time independent, Proposition 3 can be put in the following form. Let $v = \log(\tilde{g})_y$.*

(1) *A necessary and sufficient condition for a moderately degenerate driftless diffusion with time independent diffusion coefficient to have a four dimensional symmetry group is that*
$$v_y - \frac{1}{2}v^2 = \frac{\lambda}{(y-D)^2} + A(y-D)^2 + C,$$
where $\lambda = 0$, A, C and D are constants.
(2) *A necessary and sufficient condition for a moderately degenerate driftless diffusion with time independent diffusion coefficient to have a six dimensional symmetry group is that*
$$v_y - \frac{1}{2}v^2 = Ay^2 + By + C,$$
where A and B and C are constants.

The diffusion coefficient as a function of x, t can then be reconstructed in the same way as described in Remark 1 above.

<u>**Discussion**</u> Note that Proposition 3 allows the construction of a rich class of solvable driftless diffusions. This richness is characterized by the freedom to choose the time dependent parameters A, B, C, D in (21) and (23) as well as by the freedom

in the choice of initial condition α_1^0 and α_2^0 of the solutions in (21) and (23). (An illustration is given in Proposition 5.) In the case of the reduction to the heat equation, closely related results to the above have been obtained by Bluman [4]. In the case of time independent g, there are related results by Albanese and Campolieti [2].

Proposition 5. *Let $J = (0, \infty)$, $\lambda \geq -\frac{1}{8}$, $\kappa = \frac{\sqrt{1+8\lambda}}{2}$ and α_i^0, $i = 1, 2$, be positive integrable functions defined on J and*

$$\alpha_1(y,t) = \frac{\sqrt{y}e^{-\frac{y^2}{2t}}}{t} \int_0^\infty \mathcal{I}_\kappa \left(\frac{\xi y}{t}\right) \sqrt{\xi} \exp\left(-\frac{\xi^2}{2t}\right) \alpha_1^0(\xi) d\xi,$$

$$\alpha_2(y,t) = \frac{\sqrt{y}e^{-\frac{y^2}{2t}}}{t} \int_0^\infty \mathcal{K}_\kappa \left(\frac{\xi y}{t}\right) \sqrt{\xi} \exp\left(-\frac{\xi^2}{2t}\right) \alpha_2^0(\xi) d\xi,$$

where $\mathcal{I}_\kappa$ and $\mathcal{K}_\kappa$ are the modified Bessel functions of first and second kind of order κ respectively. Define

$$H(y,t) := \frac{\alpha_1(y,t)}{\alpha_2(y,t)} = \frac{\int_0^\infty \mathcal{I}_\kappa\left(\frac{y\xi}{t}\right)\sqrt{\xi}e^{-\frac{\xi^2}{2t}}\alpha_1^0(\xi)d\xi}{\int_0^\infty \mathcal{K}_\kappa\left(\frac{y\xi}{t}\right)\sqrt{\xi}e^{-\frac{\xi^2}{2t}}\alpha_2^0(\xi)d\xi}.$$

Then we have, for all $t \geq 0$, $H(0+,t) = 0$, $H(\infty,t) = \infty$, $m_1 = 0$ and for the parameters in the potential $A = B = C = D = 0, \lambda \geq -\frac{1}{8}$ and $H_y(y,t) > 0$ for $y \in J$. Note that the change of variable $x = H(y,t)$ transforms the positive half line J onto itself. Define $\tilde{g}$ by H_y and note that

$$\tilde{g}(y,t) = \frac{H(y,t)}{t}\left[\frac{\int_0^\infty \mathcal{I}_{\kappa+1}\left(\frac{y\xi}{t}\right)\xi^{3/2}e^{-\frac{\xi^2}{2t}}\alpha_1^0(\xi)d\xi}{\int_0^\infty \mathcal{I}_\kappa\left(\frac{y\xi}{t}\right)\sqrt{\xi}e^{-\frac{\xi^2}{2t}}\alpha_1^0(\xi)d\xi} + \frac{\int_0^\infty \mathcal{K}_{\kappa+1}\left(\frac{y\xi}{t}\right)\xi^{3/2}e^{-\frac{\xi^2}{2t}}\alpha_2^0(\xi)d\xi}{\int_0^\infty \mathcal{K}_\kappa\left(\frac{y\xi}{t}\right)\sqrt{\xi}e^{-\frac{\xi^2}{2t}}\alpha_2^0(\xi)d\xi}\right].$$

Hence, according to Proposition 3, we obtain a class of diffusion coefficients $g(x,t)$ (parametrized by the initial conditions α_1^0 and α_2^0) on the positive half line, which can be transformed to the canonical form (16) for the equations of four dimensional symmetry group, by expressing $\tilde{g}$ in the original variables, i.e. by letting $g(x,t) = \tilde{g}(H^{-1}(x,t),t)$.

Recently Spichak and Stognii [15] have given a complete answer in a complementary setting concerning forward Kolmogorov equations with *trivial principal part* (diffusion coefficient equal $\frac{1}{2}$) and arbitrary time dependent drift: Find all drift coefficients b for diffusions of the form $\mathcal{L}^{-1/2,-b,-b_x}u = \partial_\tau u - \frac{1}{2}u_{xx} - (b(x,t)u)_x = 0$, which can be reduced by point transformations to the form (16) for $\lambda = 0$ and for $\lambda = 0$. They have established the following proposition

Proposition 6. *(**Spichak and Stognii**) The class of operators of the form $\mathcal{L}^{-1/2,-b,-b_x}$ admitting a four-dimensional algebra of invariance is described by the condition*

$$b_t + \frac{1}{2}b_{xx} + bb_x = \frac{\lambda}{(x - H(t))^3} + F(t)x + G(t), \tag{24}$$

where $\lambda = 0$ and where $H(t), F(t), G(t)$ are arbitrary functions restricted only by the condition $G = H'' - FH$.

In their paper, Spichak and Stognii start with the general forward equation $u_t - (au)_{xx} - (bu)_x = 0$ and "reduce" this more general equation to that with $a = \frac{1}{2}$ and arbitrary b by appealing to a *random time change*, due to Dynkin. Thus the role of the original diffusion coefficient in the reducibility is not detectable in their approach. Moreover, since stochastic time changes cannot, to our knowledge, be recast as point or equivalence transformations, Spichak and Stognii's results cannot easily be adapted to obtain a criterion of the kind presented in the present paper and regarding reducibility criteria on the diffusion coefficient. In the case of reducibility to the heat equation general criteria were given by Bluman [4]. However, in his approach, the decoupling into H equation in y variable and generalization of the Schwarz procedure does not explicitly appear .

2. Idea of proofs of main results

In the proof of the main results there is an interaction between the following ingredients

- LBT (17)-(18) is a key ingredient in establishing the *sufficiency* part of Proposition 3.
- The Lie-Ovsiannikov equivalence transformation (LOT) defined below (see (26)). This is needed to establish the *necessity* of our criterion in Proposition 3. The key property thereof that is exploited is that the Lie-Ovsiannikov transformations are the *most general* transformations that map an equation in Lie canonical form back into the same form (with a, in general, different potential). Combining this with LBT leads to a simple proof of necessity.

Proposition 7. *Given two diffusions in canonical form*

$$u_t - \frac{1}{2}u_{yy} + Z(t,y)u = 0\,(*), \quad u_{\bar{t}} - \frac{1}{2}u_{\bar{y}\bar{y}} + \bar{Z}(\bar{t},\bar{y})\bar{u} = 0\,(**) \tag{25}$$

there exists a special class of point transformation which map (*) to (**):

$$\bar{t} = \quad a^2(t)dt, \qquad \bar{y} = a(t)y + b(t), \quad u = \bar{u}\exp\left(\frac{\dot{a}}{2a}y^2 + \frac{\dot{b}}{a}y + c\right), \tag{26}$$

where a, b and c are three arbitrary functions of t, transforms

$$u_t - \frac{1}{2}u_{yy} + Z(t,y)u = 0 \quad \textit{to} \quad \bar{u}_{\bar{t}} - \frac{1}{2}\bar{u}_{\bar{y}\bar{y}} + \bar{Z}\bar{u} = 0 \tag{27}$$

in the new variables. The potentials $\bar{Z}$ and Z are related by

$$a^2\bar{Z} = Z + \frac{a\ddot{a} - 2\dot{a}^2}{2a^2}y^2 + \frac{a\ddot{b} - 2\dot{a}\dot{b}}{a^2}y + \dot{c} - \frac{\dot{a}}{2a} - \frac{\dot{b}^2}{2a^2}.$$

The importance of the latter transformation is illustrated by the right up and down arrows in diagram **1**.

Corollary 1. *(Lie-Ovsiannikov) In the special case where* $\bar{Z}(\bar{t},\bar{y}) = \frac{\lambda}{\bar{y}^2}$*, the most general* $Z(t,y)$ *equivalent to it is of the form*

$$\frac{\lambda}{(y+\frac{b}{a})^2} - \frac{a\ddot{a}-2\dot{a}^2}{2a^2}y^2 - \frac{a\ddot{b}-2\dot{a}\dot{b}}{a^2}y - \dot{c} + \frac{\dot{a}}{2a} + \frac{\dot{b}^2}{2a^2}, \tag{28}$$

where $a(t), b(t), c(t)$ *are arbitrary functions.*

Corollary 2. *In the special case where* $\bar{Z}(\bar{t},\bar{y}) = \frac{A}{\bar{y}^2}$*, the most general* $Z(t,y)$ *equivalent to it is of the form*

$$\frac{\lambda}{(y-D(t)^2)} + A(t)y^2 + B(t)y + C(t), \tag{29}$$

where

$$B(t) = \ddot{D} - 2AD, \tag{30}$$

but A, B, C, D *are otherwise arbitrary.*

Remark 3. *Corollary 1 is proved in* [14]*. Corollary 2 appears to be new. It explains the appearance of restriction (30) in Proposition 3.*

The strategy we will use in order to give a complete characterization of *all diffusion coefficients* that can be mapped by *some* arbitrary Lie equivalence transformation (ie. LET but not necessarily LBT) to the canonical form (16) is to exploit the above ingredients in a way illustrated by the following diagram.

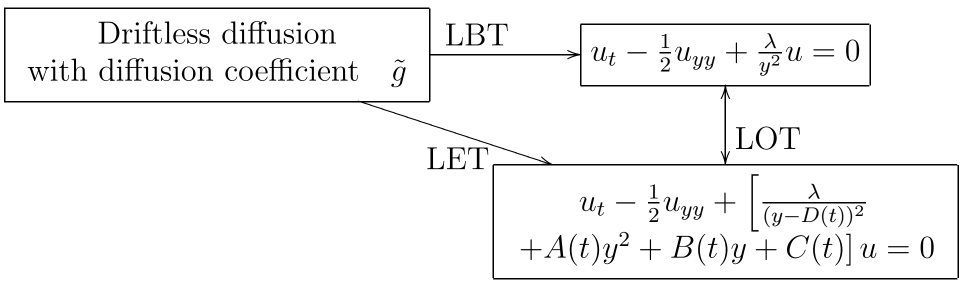

Diagram 1

Outline of the proof of Proposition 3.

Part I: Sufficiency The proof of sufficiency is done by applying the following argument subinterval by subinterval, hence we shall suppress the dependence of the index i in the following for notational simplicity. To establish the sufficiency of conditions (20)-(21) for any subinterval $I \in \mathcal{C}$, we need to show that whenever these conditions are satisfied, it is possible to find a point transformation that maps the equation (11) to an equation of the form (21) for $v = e^{\int \beta}u$. This is shown by using the LBT to map the original equation to the equation

$$v_t - \frac{1}{2}v_{yy} + \quad \frac{1}{2}\tilde{\beta}^2 - \frac{1}{2}\tilde{\beta}_y + \partial_t \quad {}_{D(t)}^{\;y} \tilde{\beta}dy \quad v = 0,$$

where $\beta = y_t + \frac{1}{2}\frac{\tilde{g}_y}{\tilde{g}}$. A key step is to re-express β in the (y,t) variables as $\tilde{\beta} = -\frac{H_t - \frac{1}{2}H_{yy}}{H_y}$ where $H = \int_{D(t)}^{y} \tilde{g} + m$. In order for the potential term to equal the target potential term (29), we need to enforce the condition

$$\frac{1}{2}\tilde{\beta}^2 - \frac{1}{2}\tilde{\beta}_y + \partial_t \int_{D(t)}^{y} \tilde{\beta} dy = \frac{\lambda}{(y-D)^2} + A(t)y^2 + B(t)y + C(t). \tag{31}$$

Now, note that (31) is an inhomogeneous *viscous Burger's equation* for $\tilde{\beta}$. Using the Hopf-Cole transformation $\tilde{\beta} = -\log(\alpha)_y$ is transformed to (21) for α.

Part II: Necessity To show necessity one can argue as follows: If the original driftless diffusion has a four dimensional symmetry group, any equivalence transformation that reduces it to canonical form, will necessarily reduce it to a canonical form with potential (29). This is due to the fact that equivalence transformations preserve the order of the symmetry group and equations in canonical form which have four dimensional symmetry group necessarily have a potential of precisely the form (29).

Example 1. Singular diffusion. *Feller* [8] *considered the (singular) diffusion equation of the form*

$$u_t - (axu)_{xx} + ((bx+c)u)_x = 0 \tag{32}$$

on the half line $\{0 < x < \infty\}$*, where* a, b, c *are constants,* $a > 0$*. Note that (32) is a moderately degenerate diffusion with* $I = (0, \infty)$*. By applying the change of variables (17) and (18), (32) is transformed to the following canonical form in Proposition 4.*

$$v_t - \frac{1}{2}v_{yy} + \left[\frac{1}{2}\left(\frac{c}{a} - \frac{1}{2}\right)\left(\frac{c}{a} - \frac{3}{2}\right)\frac{1}{y^2} + \frac{b^2}{8}y^2 + \frac{bc}{2a}\right] v = 0,$$

where $y = \int_0^x \frac{1}{\sqrt{2ax}} dx' = \sqrt{\frac{2x}{a}}$*,* $y \in (0, \infty)$*. Hence the fundamental solution given in* [8] *(see Lemma 9) can be derived by reversing the transformation (modulo the consideration of the boundary conditions). In the cases that* $c = \frac{a}{2}$ *or* $c = \frac{3a}{2}$*, (32) is six dimensional and hence can be further transformed to the heat equation by LOT.*

Example 2. Time dependent CEV process. *A shifted time dependent CEV process generalizing processes widely used in mathematical finance of the form*

$$u_t - \frac{1}{2}\sigma^2(t)(S + \alpha(t))^\beta u_{SS} - (r(t) - d(t))(S + \alpha(t))u_S + ru = 0 \tag{33}$$

when reduced to canonical form by an LBT transformation is easily seen to have a potential of the form (29) with $D = 0$ *and* $B = 0$*. Thus its fundamental solution can be expressed in closed form providing an extension and easy proof of a result obtained recently by Lo and Hui* [12]*. Though (33) is not driftless unless* $r = d$ *it can be treated by trivial modifications of the results in Proposition 3.*

Master Equation for symmetry group

$$X = \tau(t)\partial_t + \xi(x,t)\partial_x + \phi(x,u,t)\partial_u, \qquad \phi = u\hat{\beta}(x,t)$$

is, as usual, the infinitesimal generator of the symmetry group. It is easily shown that ξ has the form $\frac{\tau_t(t)}{2} g(x,t)G(x,t) - g(x,t)G_t(x,t)\tau(t) + c(t)$ where $G(x,t) = \int_m^x \frac{1}{g(x',t)} dx'$.

One then shows that the master equation for the determination of the symmetry group is of the form:

$$g(p_{1t} - q_{1x})c - c_{tt} + g(p_{5t} - q_{5x})\tau + g(p_{3t} + p_5 - q_{3x})\tau_t + \frac{1}{2}G\tau_{ttt} = 0,$$

where $q_i = \frac{1}{2}g^2 p_{ix}$ and

$$p_1 = \frac{g_{xx}}{2} - \frac{g_t}{g^2} \quad p_2 = -\frac{1}{g} \quad p_4 = -\frac{1}{2g}G,$$
$$p_3 = -\frac{1}{2}\frac{g_t G}{g^2} + \frac{1}{2}\frac{G_t}{g} + \frac{1}{4}g_{xx}G + \frac{1}{4}\frac{g_x}{g},$$
$$p_5 = \frac{G_{tt}}{g} + \frac{g_t G_t}{g^2} + \frac{1}{2}\frac{g_{xt}}{g} - \frac{1}{2}g_{xx}G_t.$$

This generalizes the results of Cicogna and Vitali (see equation 14, p. 454 in [7] and [6]) to the time dependent case. Next one studies the master equation $\hat{\beta}_{xt} = \hat{\beta}_{tx}$ and finds after a considerable amount of computation and manipulation that it can be cast into the following elegant form

$$c(U_y - \partial_t[\tilde{\beta}(D(t),t)]) - (y-D)c_{tt} + (U_t + D'U_y - D'''(y-D) + D''D')\tau$$
$$+ \quad \frac{y-D}{2}U_y + U - \frac{3}{2}D''(y-D) + \frac{(D')^2}{2} \quad \tau_t + \frac{(y-D)^2}{4}\tau_{ttt} = \Sigma(t) \qquad (34)$$

where $U = \frac{1}{2}\tilde{\beta}^2 - \frac{1}{2}\tilde{\beta}_y + \partial_t \int_{D(t)}^{y} \tilde{\beta} dy$ as in (31), $\tilde{\beta} = -\frac{H_t - \frac{1}{2}H_{yy}}{H_y}$ as in Proposition 3 and $\Sigma(t)$ is an arbitrary function of t. This simple form of the masters equation appears to be new even in the time independent case. Since in the case of two dimensional symmetry there is no all inclusive form for the solvable cases of the canonical potential our hope is that this new form of the master equation can be exploited in future work in determining new classes of solvable two dimensional driftless diffusions.

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