Static Hedging of Standard Options*

PETER CARR†

Bloomberg L.P. and Courant Institute

LIUREN WU‡

Zicklin School of Business, Baruch College

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†499 Park Avenue, New York, NY 10022; tel: (212) 893-5056; fax: (917) 369-5629; pcarr4@bloomberg.com; http://www.math.nyu.edu/research/carrp/.

‡One Bernard Baruch Way, Box B10-225, New York, NY 10010; tel: (646) 312-3509; fax: (646) 312-3451; Liuren_Wu@baruch.cuny.edu; http://faculty.baruch.cuny.edu/lwu/.

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ABSTRACT

We consider the hedging of options when the price of the underlying asset is always exposed to the possibility of jumps of random size. Working in a single factor Markovian setting, we derive a new spanning relation between a given option and a continuum of shorter-term options written on the same asset. In this portfolio of shorter-term options, the portfolio weights do not vary with the underlying asset price or calendar time. We then implement this static relation using a finite set of shorter-term options and use Monte Carlo simulation to determine the hedging error thereby introduced. We compare this hedging error to that of a delta hedging strategy based on daily rebalancing in the underlying futures. The simulation results indicate that the two types of hedging strategies exhibit comparable performance in the classic Black-Scholes environment, but that our static hedge strongly outperforms delta hedging when the underlying asset price is governed by Merton (1976)'s jump-diffusion model. The conclusions are unchanged when we switch to ad hoc static and dynamic hedging practices necessitated by a lack of knowledge of the driving process. Further simulations indicate that the inferior performance of the delta hedge in the presence of jumps cannot be improved upon by increasing the rebalancing frequency. In contrast, the superior performance of the static hedging strategy can be further enhanced by using more strikes or by optimizing on the common maturity in the hedge portfolio.

We also compare the hedging effectiveness of the two types of strategies using more than six years of data on S&P 500 index options. We find that in all cases considered, a static hedge using just five call options outperforms daily delta hedging with the underlying futures. The consistency of this result with our jump model simulations lends empirical support for the existence of jumps of random size in the movement of the S&P 500 index. We also find that the performance of our static hedge deteriorates moderately as we increase the gap between the maturity of the target call option and the common maturity of the call options in the hedge portfolio. We interpret this result as evidence of additional random factors such as stochastic volatility.

JEL CLASSIFICATION CODES: G12, G13, C52.

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Static Hedging of Standard Options

Over the past two decades, the derivatives market has expanded dramatically. Accompanying this expansion is an increased urgency in understanding and managing the risks associated with derivative securities. In an ideal setting under which the price of the underlying security moves continuously (such as in a diffusion with known instantaneous volatility) or with fixed and known size steps (such as in a binomial tree), derivatives pricing theory provides a framework in which the risks inherent in a derivatives position can be eliminated via frequent trading in only a small number of securities.

In reality, however, large and random price movements happen much more often than typically assumed in the above ideal settings. The last two decades have repeatedly witnessed turmoil in the financial markets such as the 1987 stock market crash, the 1997 Asian crisis, the 1998 Russian default and the ensuing hedge fund crisis, and the tragic event of September 11, 2001. Juxtaposed between these large crises are many more mini-crises, during which prices move sufficiently fast so as to trigger circuit breakers and trading halts. When these crises occur, a dynamic hedging strategy based on small or fixed size movements often breaks down. Worse yet, strategies that involve dynamic hedging in the underlying asset tend to fail precisely when liquidity dries up or when the market makes large moves. Unfortunately, it is during these financial crises such as liquidity gaps or market crashes that investors need effective hedging the most dearly. Indeed, several prominent critics have gone further and blamed the emergence of some financial crises on the pursuit of dynamic hedging strategies.

Perhaps in response to the known deficiencies of dynamic hedging, Breeden and Litzenberger (1978) (henceforth BL) pioneered an alternative approach, which is foreshadowed in the work of Ross (1976) and elaborated on by Green and Jarrow (1987) and Nachman (1988). These authors show that a path-independent payoff can be hedged using a portfolio of standard options maturing with the claim. This strategy is completely robust to model mis-specification and is effective even in the presence of jumps of random size. Its only real drawback is that the class of claims that this strategy can hedge is fairly narrow. First, the BL hedge of a standard option reduces to a tautology. Second, the hedge can neither deal with standard options of different maturities, nor can it deal with path-dependent options. Therefore, the BL strategy is completely robust but has limited range. In contrast, dynamic hedging works for a wide range of claims, but is not robust.

In this paper, we propose a new approach for hedging derivative securities. This approach lies between dynamic hedging and the BL static hedge in terms of both range and robustness. Relative to BL, we place mild structure on the class of allowed stochastic processes of the underlying asset in order to expand the class of claims that can be robustly hedged. In particular, we work in a fairly general one-factor Markovian setting, where the market price of a security is allowed not only to move diffusively, but also to jump randomly to any non-negative value. In this setting, we derive a simple spanning relation between the value of a given European option and the value of a continuum of shorter-term European options. The required position in each of the shorter-term options is proportional to the gamma (second price derivative) that the target option will have when the options in the hedge portfolio expire. As the target option's future gamma does not vary with the passage of time or the change in the underlying price, the weights in the portfolio of shorter-term options are static over the life of these options. Given this static spanning result, no arbitrage implies that the target option and the replicating portfolio have the same value for all times until the shorter term options expire. As a result, we can effectively hedge a long-term option, at least in theory, even in the presence of large random jumps in the security price movement. Furthermore, given the static nature of the strategy, we do not need to rebalance the hedge portfolio until the shorter-term options mature. Therefore, we do not need to worry about market shutdowns and liquidity gaps in the intervening period. The strategy remains viable and can become even more useful when the market is in stress. As an added advantage, the static hedge only requires a correct specification of the underlying price dynamics between the two option maturities. In contrast, delta hedging only succeeds if the model is correctly specified *throughout* the life of the target option.

As transactions costs and illiquidity render the formation of a portfolio with a continuum of options physically impossible, we develop an approximation for the static hedging strategy using only a finite number of options. This discretization of the ideal trading strategy is analogous to the discretization of a continuous-time dynamic trading strategy (e.g., delta hedging). To discretize our static hedge, we choose the strike levels and the associated portfolio weights based on a Gauss-Hermite quadrature method. We use Monte Carlo simulation to gauge the magnitude and distributional characteristics of the hedging error introduced by the quadrature approximation. We compare this hedging error to the hedging error from a delta-hedging strategy based on daily rebalancing with the underlying futures. The simulation results indicate that the two strategies have comparable hedging effectiveness in the classic Black and Scholes (1973) environment. The mean absolute hedging errors are comparable when the two strategies involve the same number of transactions. Nevertheless, since the bid-ask spread is typically lower for the underlying asset than it is for the options, these results favor the delta-hedging strategy.

The conclusion changes when we perform the simulation under the Merton (1976) jump-diffusion environment, in which the underlying asset price can exhibit jumps of random size. In the presence of random jumps, the performance of daily delta hedging deteriorates dramatically, but the performance of the static strategy hardly varies. As a result, under the Merton model, a static strategy with merely three options outperforms delta hedging with daily updating. Further simulations indicate that these results are robust to model misspecification, so long as we perform ad hoc adjustments based on the observed implied volatility. Finally, we also find that increasing the rebalancing frequency in the delta-hedging strategy does not rescue its performance as long as the underlying asset price can jump by a random amount. In contrast, we can further improve the static-hedging performance by increasing the number of strikes used and by choosing the appropriate maturities for the hedge portfolio. We conclude that the superior performance of static hedging over daily delta hedging in the jump model simulations is not due to model misspecification, nor is it due to the approximation error introduced via discrete rebalancing. Rather, this outperformance is due to the fact that delta hedging is inherently incapable of dealing with jumps of random size in the underlying asset price movement. In contrast, our static spanning relation can handle random jumps and our approximation of this spanning relation performs equally well with and without jumps in the underlying asset price process.

To compare the effectiveness of the two types of hedging strategies in practice, we also investigate the historical performance of the two strategies in hedging S&P 500 index options. Based on over six years of data on S&P 500 index options, we find that in all the cases considered, a static hedge using no more than five options outperforms daily delta hedging with the underlying futures. The consistency of this result with our jump model simulations lends empirical support for the existence of jumps of random size in the movement of the S&P 500 index (Aït-Sahalia (2002)).

We also find that our static strategy performs better when the maturity of the options in the hedge portfolio is closer to the maturity of the target option being hedged. As the maturity gap increases, the hedging performance deteriorates moderately, indicating the likely existence of additional random factors such as stochastic volatility.

For clarity of exposition, this paper focuses on hedging a standard European option with a portfolio of shorter-term options under a one-factor Markovian setting. However, the underlying theoretical framework extends readily to the hedging of more exotic, potentially path-dependent options, such as discretely monitored Asian and barrier options, Bermudan options, passport options, cliquets, ratchets, and many other structured notes. We use a globally floored, locally capped, compounding cliquet as an example to illustrate how this option contract with intricate path-dependence can be hedged with a portfolio of European options. The hedging strategy is semi-static in the sense that trades only need to occur at the discrete monitoring dates.

In related literature, the effective hedging of derivative securities has been applied not only for risk management, but also for option valuation and model verification (Bates (2003)). For example, Bakshi, Cao, and Chen (1997), Bakshi and Kapadia (2003), and Dumas, Fleming, and Whaley (1998) use hedging performance to test different option pricing models. Bakshi and Madan (2000) propose a general option-valuation strategy based on effective spanning using basis characteristic securities.

This paper is organized as follows. Section 1 develops the theoretical results underlying our static hedging strategy. Section 2 uses Monte Carlo simulation to enact a wide variety of scenarios under which the market not only moves diffusively, but also jumps randomly. Under each scenario, we analyze the hedging performance of our static strategy and compare it with dynamic hedging. Section 3 applies both strategies to the S&P 500 index options data. Section 4 shows how to extend our theory into the hedging of path-dependent options. Section 5 concludes.

1. Spanning Options with Options

We develop our main theoretical results in this section. Working in a continuous-time one-factor Markovian setting, we show how we can span the risk of a European option by holding a continuum of shorter-term European options. The weights in the portfolio are static as they are invariant to changes in the underlying asset price or the calendar time. We then illustrate how we can use a quadrature rule to approximate the static hedge using a finite number of shorter-term options.

1.1. Assumptions and notation

We assume frictionless markets and no arbitrage. To fix notation, we let S_t denote the spot price of an asset (say, a stock or stock index) at time $t \in [0, \mathcal{T}]$, where $\mathcal{T}$ is some arbitrarily distant horizon. For realism, we assume that the owners of this asset enjoy limited liability, and hence $S_t \geq 0$ at all times. For notational simplicity, we further assume that the continuously compounded riskfree rate r and dividend yield δ are constant. No arbitrage implies that there exists a risk-neutral probability measure $\mathbb{Q}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ such that this instantaneous expected rate of return on every asset equals the instantaneous riskfree rate r. We also restrict our analysis to the class of models in which the risk-neutral evolution of the stock price is Markov in the stock price S and the calendar time t. Our class of models includes local volatility models, e.g., Dupire (1994), and models based on Lévy processes, e.g., Barndorff-Nielsen (1998), Bates (1991), Carr, Geman, Madan, and Yor (2002), Carr and Wu (2003a), Eberlein, Keller, and Prause (1998), Madan and Seneta (1990), and Merton (1976), but does not include stochastic volatility models such as Bates (1996, 2000), Bakshi, Cao, and Chen (1997), Heston (1993), and Hull and White (1987).

We use $C_t(K,T)$ to denote the time-t price of a European call with strike K and maturity T. Our assumption that the state is fully described by the stock price and time implies that there exists a call pricing function $C(S,t;K,T;\Theta)$ such that

$$C_t(K,T) = C(S_t,t;K,T;\Theta), \quad t \in [0,T], K \geq 0, T \in [t, \mathcal{T}]. \tag{1}$$

The call pricing function relates the call price at t to the state variables (S_t,t), the contractual parameters (K,T), and a vector of fixed model parameters Θ.

We use $q(S,t;K,T;\Theta)$ to denote the probability density function of the asset price under the risk-neutral measure $\mathbb{Q}$, evaluated at the future price level K and the future time T and conditional on the stock price starting at level S at some earlier time t. Breeden and Litzenberger (1978) show that this risk-neutral density relates to the second strike derivative of the call pricing function by

$$q(S,t;K,T;\Theta) = e^{r(T-t)} \frac{\partial^2 C}{\partial K^2}(S,t;K,T;\Theta). \tag{2}$$

1.2. Spanning standard European options with shorter-term European options

The main theoretical result of the paper comes from the following theorem, which introduces a new spanning relation between the value of a European option at one maturity and the value of a continuum of European options at some nearer maturity. The practical implication of this theorem is that we can span the risk of a given option by taking a static position in a continuum of shorter-term, usually more liquid, options.

Theorem 1 *Under no arbitrage and the Markovian assumption in (1), the time-*t *value of a European call option maturing at a fixed time* $T \geq t$ *relates to the time-*t *value of a continuum of European call options at a shorter maturity* $u \in [t,T]$ *by*

$$C(S,t;K,T;\Theta) = \int_0^\infty w(\mathcal{K}) C(S,t;\mathcal{K},u;\Theta) d\mathcal{K}, \quad u \in [t,T], \tag{3}$$

for all possible nonnegative values of S *and at all times* $t \leq u$*. The weighting function* $w(\mathcal{K})$ *is given by*

$$w(\mathcal{K}) = \frac{\partial^2}{\partial \mathcal{K}^2} C(\mathcal{K},u;K,T;\Theta). \tag{4}$$

The spanning relation holds for all possible values of the spot price S and at all times up to the expiry of the options in the spanning portfolio. The option weights $w(\mathcal{K})$ are independent of S and t. This property characterizes the static nature of the spanning relation. Under no arbitrage, once we form the spanning portfolio, no rebalancing is necessary until the maturity date of the options in the spanning portfolio.

The weight $w(\mathcal{K})$ on a call option at maturity u and strike $\mathcal{K}$ is proportional to the gamma that the target call option will have at time u, should the underlying price level be at $\mathcal{K}$ then. Since the gamma of a call option typically shows a bell-shaped curve centered near the call option's strike price, the greatest weight go to the options with strikes that are close to that of the target option. Furthermore, as we let the common maturity u of the spanning portfolio approach the target call option's maturity T, the gamma becomes more concentrated around K. In the limit when $u = T$, all of the weight is on the call option of strike K. Equation (3) reduces to a tautology.

Proof. Under the Markovian assumption in (1), we can compute the initial value of the target call option by discounting the expected value it will have at some future date u,

$$\begin{aligned} C(S,t;K,T;\Theta) &= e^{-r(u-t)} \int_0^\infty q(S,t;\mathcal{K},u;\Theta) C(\mathcal{K},u;K,T;\Theta) d\mathcal{K} \\ &= \int_0^\infty \frac{\partial^2}{\partial \mathcal{K}^2} C(S,t;\mathcal{K},u;\Theta) C(\mathcal{K},u;K,T;\Theta) d\mathcal{K}. \end{aligned} \tag{5}$$

The first line follows from the Markovian property. The call option value at any time u depends only on the underlying security's price at that time. The second line results from a substitution of equation (2) for the risk-neutral density function.

We integrate equation (5) by parts twice and observe the following boundary conditions,

$$\begin{aligned}
&\tfrac{\partial}{\partial K}C(S,t;K,u;\Theta)\Big|_{K\to\infty}=0, \quad C(S,t;K,u;\Theta)|_{K\to\infty}=0,\\
&\tfrac{\partial}{\partial S}C(0,u;K,T;\Theta)=0, \quad C(0,u;K,T;\Theta)=0.
\end{aligned} \tag{6}$$

The final result of these operations is as in equation (3). ■

Equation (3) represents a constraint imposed by no-arbitrage and the Markovian assumption on the relation between prices of options at two different maturities. Given that the Markovian assumption is correct, a violation of equation (3) implies an arbitrage opportunity. For example, if we suppose that at time t, the market price of a call option with strike K and maturity T (left hand side) exceeds the price of a gamma weighted portfolio of call options for some earlier maturity u (right hand side), then, conditional on the validity of the Markovian assumption (1), the arbitrage is to sell the call option of strike K and maturity T, and to buy the gamma weighted portfolio of all call options maturing at the earlier date u. The cash received from selling the T maturity call exceeds the cash spent buying the portfolio of nearer dated calls. At time u, the portfolio of expiring calls pays off:

$$\int_0^\infty \frac{\partial^2}{\partial \mathcal{K}^2}C(\mathcal{K},u;K,T;\Theta)(S_u-\mathcal{K})^+ d\mathcal{K}.$$

Integrating by parts twice implies that this payoff reduces to $C(S_u,u;K,T;\Theta)$, which we can use to close the short call position.

To understand the implications of our theorem for risk management, suppose that at time t there are no call options of maturity T available in the listed market. However, it is known that such a call will be available in the listed market by the future date $u \in (t,T)$. An options trading desk could consider writing such a call option of strike K and maturity T to a customer in return for a (hopefully sizeable) premium. Given the validity of the Markov assumption, the options trading desk can hedge away the risk exposure arising from writing the call option over the time period $[t,u]$ using a static position in available shorter-term options. The maturity of the shorter-term options should be equal to or longer than u and the portfolio weight is determined by equation (3). Then at date u, the assumed validity of the Markov condition (1) implies that the desk can use the proceeds from the sale of the shorter-term call options to purchase the T maturity call in the listed market. Thus, this hedging strategy is semi-static in that it involves rolling over call options once. In contrast to a purely static strategy, there is a risk that the Markov condition will not hold at the rebalancing date u. We will continue to use the terser term "static" to describe this semi-static strategy when it is contrasted to a classical dynamic strategy. However, we warn the practically minded reader that our use of this term does not imply that there is no model risk.

Theorem 1 states the spanning relation in terms of call options. The spanning relation also holds if we replace the call options on both sides of the equation by their corresponding put options of the same strike and maturity. The relation on put options can either be proved analogously or via the application of the put-call parity to the call option spanning relation in equation (3).

These static spanning relations stand in sharp contrast to traditional dynamic hedging strategies, which are based on continuous rebalancing of positions in the underlying asset. In what follows, we investigate the effectiveness of the two types of strategies using both Monte Carlo simulation and an empirical study.

1.3. Finite approximation with Gaussian quadrature rules

In practice, investors can neither rebalance a portfolio continuously, nor can they form a static portfolio involving a continuum of securities. Both strategies involve an infinite number of transactions. In the presence of discrete transaction costs, both would lead to financial ruin. As a result, dynamic strategies are only rebalanced discretely in practice. The trading times are chosen to balance the costs arising from the hedging error with the cost arising from transacting in the underlying. Similarly, to implement our static hedging strategy in practice, we need to approximate it using a finite number of call options. The number of call options used in the portfolio is chosen to balance the cost from the hedging error with the cost from transacting in these options.

We approximate the spanning integral in equation (3) by a weighted sum of a finite number (N) of call options at strikes $\mathcal{K}_j$, $j = 1, 2, \cdots, N$,

$$\int_0^\infty w(\mathcal{K})C(S,t;\mathcal{K},u;\Theta)d\mathcal{K} \approx \sum_{j=1}^N \mathcal{W}_j C(S,t;\mathcal{K}_j,u;\Theta), \tag{7}$$

where we choose the strike points $\mathcal{K}_j$ and their corresponding weights based on the Gauss-Hermite quadrature rule.

The Gauss-Hermite quadrature rule is designed to approximate an integral of the form $\int_{-\infty}^{\infty} f(x) e^{-x^2} dx$, where $f(x)$ is an arbitrary smooth function. After some rescaling, the integral can be regarded as an expectation of $f(x)$ where x is a normally distributed random variable with zero mean and variance of two. For a given target function $f(x)$, the Gauss-Hermite quadrature rule generates a set of weights w_i and nodes x_i, $i = 1, 2, \cdots, N$, that are defined by

$$\int_{-\infty}^{\infty} f(x) e^{-x^2} dx = \sum_{j=1}^N w_j f(x_j) + \frac{N!\sqrt{\pi}}{2^N} \frac{f^{(2N)}(\xi)}{(2N)!} \tag{8}$$

for some $\xi \in (-\infty, \infty)$. The approximation error vanishes if the integrand $f(x)$ is a polynomial of degree equal or less than $2N-1$. See Davis and Rabinowitz (1984) for details.

To apply the quadrature rules, we need to map the quadrature nodes and weights $\{x_i, w_j\}_{j=1}^N$ to our choice of option strikes $\mathcal{K}_j$ and the portfolio weights $\mathcal{W}_j$. One reasonable choice of a mapping function between the strikes and the quadrature nodes is given by

$$\mathcal{K}(x) = Ke^{x\sigma\sqrt{2(T-u)}+(\delta-r-\sigma^2/2)(T-u)}, \tag{9}$$

where σ^2 denotes the annualized variance of the log asset return. This choice is motivated by the gamma weighting function under the Black-Scholes model, which is given by

$$\mathcal{W}(\mathcal{K}) = \frac{\partial^2 C(\mathcal{K}, u; K, T; \Theta)}{\partial \mathcal{K}^2} = e^{-\delta(T-u)} \frac{n(d_1)}{\mathcal{K}\sigma\sqrt{T-u}}, \tag{10}$$

where $n(\cdot)$ denotes the probability density of a standard normal and the standardized variable d_1 is defined as

$$d_1 \equiv \frac{\ln(\mathcal{K}/K) + (r-\delta+\sigma^2/2)(T-u)}{\sigma\sqrt{T-u}}.$$

We can then obtain the mapping in (9) by replacing d_1 with $\sqrt{2}x$, which can also be regarded as a standard normal variable.

Thus, given the Gauss-Hermite quadrature $\{w_j, x_j\}_{j=1}^N$, we choose the strike points as

$$\mathcal{K}_j = Ke^{x_j\sigma\sqrt{2(T-u)}+(\delta-r-\sigma^2/2)(T-u)}. \tag{11}$$

The portfolio weights are then given by

$$\mathcal{W}_j = \frac{w(\mathcal{K}_j)\mathcal{K}_j'(x_j)}{e^{-x_j^2}} w_j = \frac{w(\mathcal{K}_j)\mathcal{K}_j\sigma\sqrt{2(T-t)}}{e^{-x_j^2}} w_j\,. \tag{12}$$

Conceivably, we can use different methods for the finite approximation. The Gauss-Hermite quadrature method chooses both the optimal strike levels and the optimal weight under each strike. This method is applicable to a market where options at many different strikes are available, such as the S&P 500 index options market at the Chicago Board of Options Exchange (CBOE). On the other hand, for some over-the-counter options markets where only a few fixed strikes are available, it would be more appropriate to use an approximation method that takes the strike points as given and only solves for the weight for each strike.

2. Simulation Analysis Based on Popular Models

Consider the problem faced by the writer of a call option on a certain stock. For concreteness, suppose that the call option matures in one year and is written at-the-money. The writer intends to hold this short position for a month, after which the option position will be closed. During this month, the writer can hedge his market risk using various exchange traded liquid assets such as the underlying stock, futures, and/or options on the same stock.

We compare the performance of the following two strategies: (i) a static hedging strategy using one-month standard options, and (ii) a dynamic delta hedging strategy using the underlying stock futures. The static strategy is based on the spanning relation in equation (3) and is approximated by a finite number of options, with the portfolio strikes and weights determined by the quadrature method. The dynamic strategy is discretized by rebalancing the futures position daily. The choice of using futures instead of the stock itself for the delta hedge is intended to be consistent with our empirical study in the next section on S&P 500 index options. For these options, direct trading in the 500 stocks comprising the index is impractical. Practically all delta hedging is done in the very liquid index futures market.

Given our assumption of constant interest rates and dividend yields, the simulated performances of the delta hedges based on the stock or its futures are almost identical. Hence, this choice does not affect our results.

We compare the performance of the above two strategies based on Monte Carlo simulation. For the simulation, we consider two data generating processes: the benchmark Black-Scholes model (BS) and the Merton (1976) jump-diffusion model (MJ). Under the objective measure, $\mathbb{P}$, the stock price dynamics in the two models follows the stochastic differential equations,

$$\begin{aligned} \text{BS:} \quad dS_t/S_t &= \mu dt + \sigma dW_t, \\ \text{MJ:} \quad dS_t/S_t &= (\mu - \lambda g)\, dt + \sigma dW_t + dJ(\lambda), \end{aligned} \tag{13}$$

where W denotes a standard Brownian motion in both models, and $J(\lambda)$ in the MJ model denotes a compound Poisson jump process with constant intensity $\lambda > 0$. Conditional on a jump occurring, the MJ model assumes that the log price relative is normally distributed with mean μ_j and variance σ_j, with the mean percentage price change induced by a jump given by $g = e^{\mu_j + \frac{1}{2}\sigma_j^2} - 1$.

We specify the data generating processes in equation (13) under the objective measure $\mathbb{P}$. To price the relevant options and to compute the weights in the hedge portfolios, we also need to specify their respective risk-neutral $\mathbb{Q}$-dynamics,

$$\begin{aligned} \text{BS:} \quad dS_t/S_t &= (r-q)dt + \sigma dW_t^*, \\ \text{MJ:} \quad dS_t/S_t &= (r - q - \lambda^* g^*)dt + \sigma dW_t^* + dJ^*(\lambda^*), \end{aligned} \tag{14}$$

where W^* denotes a standard Brownian motion under measure $\mathbb{Q}$. The compound Poisson process under measure $\mathbb{Q}$, J^*, is assumed to have constant intensity $\lambda^* > 0$. Conditional on a jump occurring, the jump size is normally distributed with mean μ_j^* and variance σ_j^2. See Bates (1991) for an equilibrium economy that supports such a measure change. For the simulation, we benchmark the parameter values of the two models to the S&P 500 index. We set $\mu = 0.10$, $r = 0.06$, and $\delta = 0.02$ for both models. We further set $\sigma = 0.27$ for the Black-Scholes model and $\lambda = \lambda^* = 2.00$, $\mu_j = \mu_j^* = -0.10$, $\sigma_j = 0.13$, and $\sigma = 0.14$ for the Merton jump-diffusion model.

In each simulation, we generate a time series of daily underlying asset prices according to an Euler approximation of the respective data generating process. The starting value for the stock price is set to \$100. We consider a hedging horizon of one month and simulate paths over this period. We assume that there are 21 business days in a month. To be consistent with the empirical study on S&P 500 index options in the next section, we think of the simulation as starting on a Wednesday and ending on a Thursday four weeks later, spanning a total of 21 week days and 29 actual days. The hedging performance is recorded and, when needed, updated only on week days, but the interest earned on the money market account is computed based on actual over 360.

At each week day, we compute the relevant option prices based on the realization of the security price and the specification of the risk-neutral dynamics. For the dynamic delta hedge, we also compute the delta each day based on the risk-neutral dynamics and rebalance the portfolio accordingly. For both strategies, we monitor the hedging error (profit and loss) at each week day based on the simulated security price and the option prices. The hedging error at each date t, e_t, is defined as the difference between the value of the hedge portfolio and the value of the target call option being hedged,

$$\begin{aligned} e_t^D &= B_{t-\Delta t} e^{r\Delta t} + \Delta_{t-\Delta t} \left(F_t - F_{t-\Delta t} \right) - C(S_t, t; K, T); \\ e_t^S &= \mathcal{W}_j C(S_t, t; \mathcal{K}_j, u) + B_0 e^{rt} - C(S_t, t; K, T), \end{aligned} \tag{15}$$

where the superscripts D and S denote the dynamic and static strategies, respectively, Δ_t denotes the delta of the target call option with respect to the futures price at time t, Δt denotes the time interval between stock trades, and B_t denotes the time-t balance in the money market account. The balance includes the receipts from selling the one-year call option, less the cost of initiating and possibly changing the hedge portfolio. In the case of the static hedging strategy, under no arbitrage, the value of the portfolio of the shorter-term options should be equal to the value of the long term target option, and hence B_0 should be zero. However, since we use a finite number of call options in the static hedge to approximate the spanning relation, the money market account captures the value difference due to the approximation error, which is normally very small. No rebalancing is needed in the static strategy.

Under each model, the delta is given by the partial derivative $\partial C(S,t;K,T;\Theta)/\partial F$, with $F = Se^{(r-\delta)(T-t)}$ denoting the forward/futures price. If an investor could trade continuously, this delta hedge removes all of the risk in the BS model. The hedge does not remove all risk in the MJ model, but has nonetheless emerged as the market standard for implementing delta-hedges in jump models. The hedge portfolio for the static strategy is formed based on the weighting function $w(\mathcal{K})$ in equation (4) implied by each model, the Gauss-Hermite quadrature nodes and weights $\{x_i, w_i\}$, and the mapping from the quadrature nodes and weights to the option strikes and weights, as given in equations (11) and (12). In computing the strike points, the annualized variance is $v = \sigma^2$ for the Black-Scholes model and $v = \sqrt{\sigma^2 + \lambda\left(\mu_j^2 + \sigma_j^2\right)}$ for the Merton jump-diffusion model. Given the chosen model parameters, $v \doteq 0.27^2$ for both models. Appendix A details the option pricing formulae, the formulae for delta $\partial C(S,t;K,T;\Theta)/\partial F$, and the weighting functions $w(\mathcal{K})$ for both models.

2.1. Hedging comparison under the diffusive Black-Scholes world

Table 1 reports the summary statistics of the simulated hedging errors, based upon 1,000 simulations. Panel A in Table 1 summarizes the results based on the Black-Scholes model. Entries are the summary statistics of the hedging errors at the last step (at the end of the 21 business days) based on both strategies. For the dynamic strategy (the last column), we perform daily updating. For the static strategy, we consider hedge portfolios with $N = 3, 5, 10, 15, 21$ one-month options. If the transaction cost for a single option is comparable to the transaction cost for revising a position in the underlying security, we would expect that the transaction cost induced by buying 21 options at one time is close to the cost of rebalancing a position in the underlying stock 21 times. Hence, it would be of interest to compare the performance of daily delta hedging with the performance of the static hedge with 21 options.

Our simulations of the underlying geometric Brownian motion indicate that the daily updating strategy beats the static strategy with 21 options in terms of the standard error, the root mean squared error (RMSE), the mean absolute error (MAE), and the mean short fall (MSF). The static strategy with 21 options does slightly better in terms of maximum profit or loss (Min and Max). Overall, the two strategies are comparable with a slight edge to the dynamic strategy. Since the stock market is much more liquid than the stock options market, the simulation results favor the dynamic delta strategy over the static strategy, if indeed stock prices move as in the Black-Scholes world.

The hedging errors from the two strategies show different distributional properties. The kurtosis of the hedging errors from the dynamic strategy is larger than that from all the static strategies. The kurtosis is 4.68 for the dynamic hedging errors, but is below two for errors from all the static hedges. Therefore, when an investor is particularly concerned about avoiding large losses, the investor may prefer the static strategy.

The last row shows the accuracy of the Gauss-Hermite quadrature approximation of the integral in pricing the target options. Under the Black-Scholes model, the theoretical value of the target call option is \$12.35, which we put under the dynamic hedging column. The approximation error is about one cent when applying a 21-node quadrature. The approximation error increases as the number of quadrature nodes declines in the approximation.

2.2. Hedging comparison in the presence of random jumps as in the Merton world

In Table 1, Panel B shows the hedging performance under the Merton jump-diffusion model. For ease of comparison, we present the results in the same format as in Panel A for the Black-Scholes model. The performance of all the static strategies are comparable to their corresponding cases under the Black-Scholes world. If anything, most of the performance measures for the static strategies become slightly better under the Merton jump-diffusion case. In contrast, the performance of the dynamic strategy deteriorates dramatically as we move from the diffusion-based Black-Scholes model to the jump-diffusion process of Merton (1976). The standard error and the root mean squared error increase by a factor of ten for the dynamic strategy. The mean absolute error increases by a factor of four. As a result, the performance of the dynamic strategy is worse than the static strategy with only three options.

The distributional differences between the hedging errors of the two strategies become even more obvious under the Merton model. The kurtosis of the static hedge errors remains small (below six), but the kurtosis of the dynamic hedge errors explodes from 4.68 in the BS model to 59.79 in the MJ model. The maximum loss from the dynamically hedged portfolio is \$12.12, even larger than the initial revenue from writing the call option (\$11.99). In contrast, the maximum loss is less than two dollars for the static hedge with only three call options.

Figure 1 plots the simulated sample paths and the corresponding hedging errors under the two models, the BS model in the left panels and the Merton model in the right panels. In the top two panels of Figure 1, we compare the simulated sample paths of the underlying security price under the two models. The daily movements under the Black-Scholes model are usually small, but the Merton-jump diffusion model generates both small and large movements.

The middle two panels in Figure 1 compare the sample paths of the hedging errors from the static hedging strategy using ten options. We apply the same scale for ease of comparison. Although the sample paths of the static hedging errors look different under the two models, the relative magnitudes of the errors are similar. The performance of the static hedging strategy is relatively insensitive to the specification of the underlying process.

The bottom two panels illustrate the sample paths of the dynamic hedging error under the two models. Under the Black-Scholes model, the dynamic hedging errors are smaller than the static hedging errors (the scale of the graphs remains the same); but under the Merton jump-diffusion model, the hedging errors from the dynamic strategy become so much larger that we have to adopt a much larger scale in plotting the error paths (right, bottom panel). The large hedging errors from the dynamic strategy correspond to the large moves in the underlying security price.

Another interesting feature is that, under the Merton model, most of the large errors from the dynamic strategy are negative, irrespective of the direction of the large move in the underlying security price. The reason is that the option price function exhibits positive convexity with the underlying futures price. Under a large movement, the value of the delta portfolio is always below the value of the option contract. Therefore, most of the large hedging errors for selling an option contract are losses (negative values).

The daily delta hedging strategy performs reasonably well under the diffusion-based Black-Scholes model, but fails miserably when the underlying price jumps randomly. In contrast, the performance of the static hedging strategy with a few shorter-term options is much less sensitive to the nature of the underlying price process. These simulation results parallel what theory predicts for continuously rebalanced delta hedges and for static hedges with a continuum of short term options. The continuously revised delta hedge is not designed to handle jumps of random size, but the static hedge with a continuum of short-term options takes these jumps in stride. The discretizations needed to implement both strategies do not change the result that introducing jumps destroys the effectiveness of the delta strategy, but has little impact on our static hedging strategy.

2.3. Effects of model uncertainty and misspecification

We perform the above simulation under the assumption that the hedger knows exactly under which model the options are priced. In practice, however, we can only use different models to approximately fit market option prices. Hence, model uncertainty is an inherent part of both pricing and hedging. To investigate the sensitivity of the hedging performance to model misspecification, we further compare the two types of hedging strategies when the hedger does not know the data generating process and must develop a hedging approach in the absence of this information. We assume that the actual underlying asset prices and the option prices are generated from the Merton jump diffusion model, but the hedger forms the hedge portfolios using the Black-Scholes model, with an ad hoc adjustment using the observed option implied volatility.

For the static strategy, we compute the weighting function $w(\mathcal{K})$ based on the Black-Scholes model, but use the eleven-month at-the-money option implied volatility as the input for annualized volatility. For the dynamic strategy, we compute the daily delta based on the Black-Scholes formula using the implied volatility of the target call option as the volatility input. In practice, updating Black-Scholes deltas based on the market observed implied volatilities is in wide use as an ad hoc defense against model risk. Also, empirical studies in, for example, Engle and Rosenberg (2002), Jackwerth and Rubinstein (1996), and Bollen and Raisel (2003) have generally found that this approach works as well or better than the alternative approach of estimating a sophisticated model and delta-hedging with it.

We summarize the hedging performance in Panel C of Table 1. For the dynamic strategy, as long as we compute the delta based on the market implied volatility, the impact of model misspecification is minimal. For the static strategy, we observe some slight deterioration in performance when there are more than ten option contracts in the hedge portfolio, but the performance actually improves slightly when fewer option contracts are used in the hedge portfolio. Overall, model misspecification is not an over-riding concern in hedging.

These remarkable results illustrate that, in hedging, being able to span the right space is much more important than specifying the right parametric model. Even if an investor has perfect knowledge of the stochastic process governing the underlying asset price, and hence can compute the perfectly correct delta, a dynamic strategy in the underlying asset still fails miserably when the underlying asset price can jump by a random amount. In contrast, as long as an investor uses a few short-term call options of different strikes in the hedge portfolio, the hedging error is about the same regardless of whether jumps can occur or not. This result holds even if the investor does not know which model to use to pick the appropriate strikes and portfolio weights.

2.4. Effects of rebalancing frequency in delta hedging

In the above simulations, we approximate the sample paths of the underlying stock price process using an Euler approximation with daily time steps and consider dynamic delta strategies with daily updating.

We are interested in knowing how much of the failure of the delta-hedging strategy under the Merton jump-diffusion model is due to this somewhat arbitrary choice of discretization step.

Under the Black-Scholes environment, the dependence of the delta hedging error on the discretization step has been studied extensively in, for example, Black and Scholes (1972),Boyle and Emanuel (1980), Bhattacharya (1980), Figlewski (1989), Galai (1983), Leland (1985), and Toft (1996). Several of these authors show that, under the Black-Scholes environment, the standard deviation of the hedging error arising from discrete rebalancing over a time step of length Δt declines to zero slowly like $O(\sqrt{\Delta t})$. Thus, doubling the trading frequency reduces the standard deviation by about thirty percent. In contrast, the discretization error in the Gaussian quadrature method is $\frac{N!\sqrt{\pi}}{2^N}\frac{f^{(2N)}(\xi)}{(2N)!}$. This error drops by much more when the number of strikes N is doubled. Indeed, our simulations indicate that the standard deviation of the hedging error drops rapidly as the number of strikes increases.

This subsection focuses on relating the delta-hedging error to the rebalancing frequency under the Merton-jump diffusion model. We also simulate the Black-Scholes model as a benchmark reference. Table 2 shows the impacts of the rebalancing frequency on the hedging performance under three different cases: (A) the Black-Scholes model, (B) the Merton jump-diffusion model, assuming that the hedger knows the underlying data generating process, and (C) an ad hoc Black-Scholes delta hedging under the Merton world, assuming that the hedger does not have knowledge of the data generating process. We consider rebalancing frequencies from once per day, to twice, five times, and ten times per day. To ease comparisons, we perform all the hedging exercises on the same simulated sample paths. To accommodate the more frequent rebalancing, we now simulate the sample paths based on the Euler approximation with a time interval of one-tenth of a business day. The slight differences between the dynamic hedging with daily updating in this table and in Table 1 reflects this difference in the simulation of the sample paths.

Our simulation of the Black-Scholes model is consistent with the results in previous studies. As the updating frequency increases from once to two, five, and ten times per day, the standard error of the hedging error reduces from 0.10 to 0.07, 0.04 and to 0.03, adhering fairly closely to the $\sqrt{\Delta t}$ rule.

However, this speed of improvement in hedging performance is no longer valid when the underlying data generating process follows the Merton jump-diffusion model, irrespective of whether the hedger knows the model or not. In the case when the process is known (Panel B), the standard error of the hedging errors remains around 1.02 – 1.03 as we increase the rebalancing frequency. In the ad hoc rebalancing case (Panel C), the standard error hovers around 0.88 – 0.93 and exhibits no obvious dependence on the rebalancing frequency. Therefore, we conclude that the failure of the delta hedging strategy under the Merton model is neither due to model misspecification, nor due to infrequent updating, but due to its inherent incapability in spanning risks associated with jumps of random size.

We note that the Achilles heel of delta hedging in jump models is not the large size of the movement per se, but rather the randomness of the jump size. For example, Cox and Ross (1976) and Dritschel and Protter (1999) show that dynamic delta hedging can span all risks arising in their pure jump models. Under these jump models, the jump size is known just prior to any jump. This is analogous to the binomial model where only two subsequent asset prices are possible. Under both cases, delta hedging can remove all risks. Therefore, it is the a priori randomness in the jump size that creates the difficulty in dynamic delta hedging.

2.5. Effects of target and hedging instrument choice

For concreteness, the above simulations focus on the hedging of a one-year call option with one-month options in the static portfolio. In this subsection, we compare the hedging performance when we choose different target options being hedged and different maturities for the options in the static hedge portfolio. In theory, if we use a continuum of options at a certain maturity, the spanning is perfect regardless of the exact maturity choice for the hedge portfolio. In practice, however, the Gaussian quadrature approximation error may vary with the target strike and with the maturities used. The simulation analyzes how the hedging error introduced by the quadrature approximation varies over different choices of target and hedging options. Along the same lines, we also analyze how the dynamic hedging error varies with the choice of the target option.

Table 3 summarizes the results of this exercise. To save space, we only report static hedges with three and five options and compare their performance with that of delta hedging with daily updating. First, we investigate the impact of varying the target option maturity given the same hedging instruments. We choose target option maturities of two months, four months, and 12 months. For the static hedging strategy, as we lower the target option's maturity, the hedging errors become smaller in the Black-Scholes model, but slightly larger in the Merton jump-diffusion model. We conjecture that these variations in performance are related to the different accuracies of the quadrature approximation for different integrands.

For the dynamic strategy, the hedging errors are larger for hedging shorter term options than for hedging longer term options under all simulated scenarios. This deteriorating performance with declining maturity is probably linked to the gamma of the target option. The shorter the maturity, the larger is the gamma for an at-the-money option. Since we can regard the delta strategy as a linear approximation, the hedging error normally increases with gamma, especially in the presence of large moves.

Our static spanning relation allows the use of different maturities in forming the static hedge portfolios. Thus, holding the same one-year option as the target option, we also compare how different maturity options fare in spanning the risk of this target option. Under all three scenarios, we find that the hedging performance improves quite significantly when the maturity of the hedging options increases. Under the Black-Scholes environment, the standard error of the hedging error is 0.66 when we use five one-month options to hedge the one-year option. This performance is much worse than daily delta hedging, which generates a standard error of 0.10. However, as we replace the one-month options in the portfolio by two-month options and then by four-month options, the performance of the static hedge improves quite dramatically. The standard error of the hedging error declines to 0.25 when using two-month options and to 0.04 when using four-month options. Thus, when longer-term options are liquid and available in the market, we can further improve the performance of the static hedging strategy such that it outperforms daily delta hedging even under the Black-Scholes environment. Comparing this to Table 2, we see that to achieve the static hedging error of 0.04, a dynamic delta strategy must be updated five to ten times per day.

The same trend follows under the Merton jump-diffusion world. Under the Merton world, the standard error of the hedging error is 0.47 when hedging one-year options with five one-month options. The standard error reduces to 0.29 when using five two-month options and to 0.16 when using five four-month options, which is much smaller than the standard error of the dynamic hedging error (1.05) under daily updating.

The simulation exercises illustrate that when the underlying asset price can jump by a random amount, our static strategy with a few appropriately chosen options delivers much smaller hedging errors than the dynamic delta strategy. But probably the biggest advantage of the static strategy lies in its flexibility. For the same target option, we have the freedom to choose options at different maturities to form the hedge portfolio. Furthermore, while the Gauss-Hermite quadrature rule provides a convenient way in performing finite approximations, there is ample room left for future research in developing better approximating schemes that can potentially further improve the performance of the static strategy.

The fact that a static hedging strategy with merely three to five options can outperform a dynamic strategy with daily updating is remarkable. In addition to the above mentioned flexibility and potentially reduced transaction costs due to fewer transactions, there are other advantages in implementing the static strategy. First, since the static hedge employs neither short stock positions nor substantial borrowing,[1] it is not subject to either short sales restrictions or leverage constraints. In contrast, delta hedges of options always involve a short position in either the risky asset or a riskfree bond, and hence always face one of these restrictions. Furthermore, the use of a static hedge also allows one to economize on the monitoring costs (e.g., paying for traders and real time data feeds) associated with dynamic rebalancing. These costs are much larger in practice than typically assumed in theory and potentially explain the current situation where dynamic hedging is usually only performed by specialized institutions.

3. Hedging S&P 500 Index Options: An Applied Example

The simulation study in the previous section compares the performance of the two different types of hedging strategies under controlled conditions. In this section, we investigate the historical performance of the two strategies in hedging the sale of S&P 500 index options. Although the simulation allows us to benchmark the magnitude of the approximation error in various Markov models, only an empirical study can gauge the likely effectiveness of the two types of hedging strategies in practice. Furthermore, since the simulations indicate that the two strategies exhibit comparable performance

[1] The money market account induced by the approximation error for the static strategy is normally very small, and can be reduced to zero via a rescaling of portfolio weights without much effect on the hedging performance.

when the stock price follows geometric Brownian motion, but the static strategy has superior performance when the stock price follows Merton's jump diffusion process, the relative performance of the two strategies in the past can also serve as an indirect test on whether the S&P 500 index has moved purely diffusively or has also experienced jumps of random size.

3.1. Data and estimation

The data on S&P 500 index options are obtained from OptionMetrics, a financial research and consulting firm specializing in econometric analysis of the options markets. The "Ivy DB" data set from OptionMetrics is the first widely-available, up-to-date, and comprehensive source of high-quality historical price and implied volatility data for the US equity and index options markets. Encompassing six years of data, Ivy DB contains accurate historical prices of options and their associated underlying instruments, correctly calculated implied volatilities, and option sensitivities. The index options data we have obtained from OptionMetrics are from January 1996 to August 2002. They are standard European options on the spot index and are listed at the Chicago Board of Options Exchange (CBOE). The data set includes, among other information, the closing quotes on each options contract (bid and ask) and implied volatilities based on the mid quote. Also included in the data set is a unique option contract identifier to facilitate the tracking of an option contract over time. The underlying index level at close, the interest rate curve, and the projected dividend yield for the calculation of implied volatility are also supplied by OptionMetrics. Our hedging exercises are based on the mid option price quotes.

In parallel with the hedging exercises in the simulation studies, we perform month-long hedging exercises on the index options. The S&P 500 index options expire on the Saturday following the third Friday. Since the terminal payoff is computed based on the opening price on that Friday morning, trades and quotes on the expiring options effectively stop on the preceding Thursday. Hence, we start the hedging exercise each month 30 days prior to the expiring Friday, which is a Wednesday. The available number of one-month option contracts at each of the starting dates ranges from 48 to 142, half of them call options and half of them put options. From these starting dates, we can perform month long hedging exercises for 79 non-overlapping months, from January 1996 to July 2002. Sampling properties of the hedging errors can then be computed from the 79 hedging experiments. To be comparable with the simulations, we normalize the option prices and hedging errors as percentages of the underlying index level at the starting date of each month.

At each starting date, we classify options into four maturity groups, matching those used in the simulations: (i) one-month options, (ii) two-month options, (iii) options with maturities four to six months, and (iv) options with maturities 12 to 17 months. The variations in maturities in the last two maturity groups necessary to obtain a monthly series because we do not have four- and twelve-month options in all months. As in the simulations, we use the first three groups (one, two, and four month options) in forming static hedge portfolios and the last three maturity groups (two, four, and twelve month options) for the target option being hedged. We choose the target option as the one with the strike price nearest to the spot index level at the starting date.

Since we do not know the true data generating process nor the option pricing model underlying the market prices, we adopt the ad hoc strategy using the Black-Scholes model. For the dynamic strategy, we delta hedge with the underlying futures based on the Black model, using the observed implied volatility to compute the delta. For the static strategy, we form the portfolio according to the Black-Scholes formula, using the at-the-money implied volatility of the appropriate maturity as the needed volatility input. Our simulations indicate that these ad hoc hedging strategies perform about as well as the hedges conducted when the true process is known.

When quotes at the appropriate strikes are not available, we use the nearest available strike contract as a replacement.[2] For the static strategy, we can pick any number of shorter term options based on the quadrature rule. However, a large order quadrature rule often requires some deep out-of-the-money or deep in-the-money option contracts that are not available on the market. Thus, we focus on analyzing the performance of the static hedge with only three to five option contracts.

We follow both strategies for 29 actual days, running from the starting date to the Thursday of the fourth following week, the last day of trading for the one-month options used in the static hedge. For the static strategy, we only need to track the price of the short term options at each date and record the difference between the price of the hedge portfolio and the price of the target call option. When there is a discrepancy between the price of the target call option and the cost of the quadrature-determined hedge portfolio at the starting date, we also monitor the typically small money market account balance. For the dynamic strategy, we need to compute a new delta at each date based on the newly observed underlying price level and implied volatility and perform the appropriate rebalancing. For obvious reasons, we do not rebalance during weekends, holidays, or other market closures. For ease of comparison, we align the hedging errors based on the week days of each week and then compute the sample properties of the hedging errors at each week day.

3.2. Static versus dynamic hedging in practice

Table 4 presents the summary statistics for the hedging errors of the various hedging exercises on S&P 500 index options. To ease comparisons, we present the results in a similar format to those from the simulations summarized in Table 3. As in the simulation exercise, we represent the option prices and hedging errors as percentages of the underlying index level at the starting date of each month.

[2] The weight, however, is not adjusted.

We consider the hedging of three maturity groups of target options: (i) two-month calls, (ii) four-to-six month calls, and (iii) 12-month and longer calls. We find that the performance varies with the maturity group of the option being hedged. First, we consider the hedging of the sale of a two-month call option. Daily delta hedging with the underlying futures generates hedging errors with a standard error of 0.57, a root mean squared error of 0.59, and a mean absolute error of 0.47. The corresponding statistics for the static strategy with three one-month options are 0.27, 0.27, and 0.21, respectively, less than half of the corresponding values for the dynamic strategy. Using five one-month options makes the hedging errors even smaller. Therefore, a static hedge with just three one-month options significantly outperforms daily delta hedging in reducing the risks associated with writing two-month call options.

In hedging the sale of a four-to-six month call option, the dynamic hedging strategy generates a standard error of 0.65 and a mean absolute error of 0.49. In contrast, the standard error from the static strategy with three one-month call options is 0.62, slightly smaller than the dynamic strategy; but the mean absolute error from this static strategy is 0.50, slightly larger than that from the dynamic strategy. Overall, the performance from the two strategies are on par in hedging the sale of a four-to-six month call option. When using five one-month options rather than three, the static strategy generates smaller errors than the dynamic strategy.

When hedging the sale of a call option with a time-to-maturity of 12 months or longer, the dynamic strategy generates a standard error of 0.88 for the hedging error. The mean absolute error is smaller at 0.64. Both numbers are larger than those for hedging a shorter-term call option. This dynamic hedging performance is better than the static strategy with three one-month call options, but on par with the static strategy with five one-month call options. The standard error from the static strategy with three one-month call options is 1.06, larger than 0.88 from the dynamic strategy. The standard error from the static strategy with five one-month call options is 0.87, about the same as that from the dynamic strategy.

Consistent with the results observed in the simulations, the performance of the static strategy improves if we increase the time-to-maturity of the options in the hedge. In hedging the sale of a 12-month or longer call option, the standard error of the hedging errors from the static strategy with three call options declines from 1.06 to 0.74 and then to 0.44, as the time-to-maturity of the three call options in the hedge portfolio increases from one month to two months, and then to four-to-six months. We also observe a similar reduction when using five call options in the static hedge portfolio.

Overall, the performance of static hedging with three to five call options is on par with or better than the performance of daily delta hedging. In addition, the performance of our static strategy can be further improved by choosing slightly longer maturities for the options in the hedge portfolio. Therefore, the static strategy not only works in theory and in simulations, but it also works on historical data, at least for S&P 500 index options, one of the most actively traded derivative contracts.

3.3. Implications for the index movement

By comparing the hedging results from the Monte Carlo simulations with those from the historical data, we can draw inferences on the type of stochastic process underlying the S&P 500 index movement. The issues that we can draw inferences on include (i) whether the stock index movement displays jumps of random size and (ii) whether the risk-neutral stock index process is Markovian in the spot index level and the calendar time.

Our Monte Carlo simulation indicates that the dynamic hedging strategy works very well in the Black-Scholes environment, but this dynamic strategy deteriorates dramatically under Merton's jump-diffusion model so that a static hedge with only three options has a hedging performance on par with or even better than daily delta hedging. For example, The last column of Table 3 shows that the standard error for hedging 12-month options using daily delta hedging is 0.10 under the Black-Scholes model, but 1.05 under the Merton model, more than ten times larger. In contrast, the standard error from the static strategy with three one-month options is 1.00 under the Black-Scholes model, but 0.72 under the Merton model. Comparing these numbers to those for S&P 500 index options in Table 4, we find that the standard error from the static strategy with three one-month options is 1.06, and that from the dynamic strategy is 0.88, only slightly smaller. Overall, the performance difference is much closer to that under the Merton jump-diffusion case than under the purely diffusive Black-Scholes world. Therefore, we infer that the S&P 500 index movement may have jumps of random magnitudes. This result is consistent with the findings from many parametric studies, e.g., Bates (2000) and Bakshi, Cao, and Chen (1997) and also with the results from the more generic tests such as in Aït-Sahalia (2002) and Carr and Wu (2003b).

Figure 2 depicts the normalized sample paths of the S&P 500 index level over the 79 month-long hedging experiments. The four major breaks in the sample paths reflect the four weekends of the month. There may also be other breaks due to holidays. When we compare this to the simulated sample paths under the Black-Scholes model and the Merton model (See Figure 1), we see that the sample paths of the index show both small and large movements. However, the jumps are not as dramatic as those shown in the simulated paths of the Merton jump-diffusion model.

The hedging performance on S&P 500 index options does not always match the simulated results on the Merton jump-diffusion model. Under the Merton model, the simulated hedging error is smaller for hedging longer-term options than for hedging shorter-term options. This holds for both static and dynamic strategies. For example, Panel B of Table 3 shows that the standard error of the hedging errors from the static strategy with three one-month options is 0.84 for hedging two-month options, 0.77 for hedging four-month options, and 0.72 for hedging 12-month options. We also observe a similar downward trend under the dynamic strategy. However, the trend is quite the opposite when hedging S&P 500 index options. As shown in Table 4, the magnitude of the hedging errors increases when the maturity of the target option increases. This observation holds for both the static strategy and, to a lesser extent, the dynamic strategy. When hedging with three one-month options, the standard error of the hedging errors is 0.27 for hedging two-month index options, 0.62 for hedging four-to-six month index options, but 1.06 for hedging 12-month or longer index options, a large increase. A similar trend also exist for the hedging errors under the dynamic strategy.

Figure 3 shows the impact of the maturity of the target index option on the hedging performance. In the figure, we plot the 79 sample paths of the hedging errors for the hedging of near-the-money index options at maturities of (i) two months (top row), (ii) four-to-six months (middle row), and (iii) one year or longer (bottom row). The horizontal axis is the actual number of days forward. Again, the four breaks in the sample paths represent the four weekends during the month-long monitoring of the hedging performance. We also observe occasional path breaks during the week days, which can be either due to holidays or missing data. We record the performance of the static hedge only when the market quotes for all the relevant options (the options in the hedge portfolio and the target call option) are available.

The three panels on the left depict the hedging errors based on the static hedge portfolio using three one-month options. The panels in the middle are from static hedging with five one-month options, and those on the right are errors from on daily delta hedging with futures. For ease of comparison, we apply the same scale on all panels in the figure. For all three hedging strategies, the magnitudes of the hedging errors increase with the maturity of the target option, more so for the two static strategies than for the dynamic strategy. These results stand in sharp contrast to what we have observed from simulating either the Merton model or the Black-Scholes model.

The different results between the Monte Carlo simulation of single-factor Markovian processes and the historical record of S&P500 index options prompt us to conjecture that the underlying index movement may not be Markovian in itself. Additional sources of risk could exist other than the underlying index level. These risks could also affect the index option prices. One such risk could be stochastic volatility. When there are additional sources of risk, the hedging performance of both strategies should deteriorate as the maturity of the target call option increases away from the maturity of the hedge portfolio. For the pure delta hedging strategy, the effects of neglecting to vega (partial derivative against volatility) hedge would become more pronounced as we increase the target call option's maturity and hence its vega. For the static hedging strategy, the difference between the vega of the hedge portfolio and the vega of the target call option also increases with the maturity gap between the target call option and the call options in the hedge portfolio. As the call options in the static hedge approach the expiry date, the resulting payoffs from the maturing options are purely determined by the realized index level and do not depend on any other state variables such as volatility. However, the value at that time of the unexpired target call option will be sensitive to factors other than price, which will result in replication error. The magnitude of this replication error increases as this sensitivity of the target call option increases.

Given the same target index option, we also observe that the performance of the static hedge improves as we increase the maturity of the options in the portfolio. Figure 4 shows this phenomenon. In this figure, we plot the sample paths of the hedging errors from the static hedge of a target call option with maturity one-year or longer. From left to right, the time to maturity of the options in the hedge portfolio increases from one month (left panel) to two months (middle panel) and then to four-to-six months (right panel). The top panels are based on portfolios with three options, and the bottom panels are based on portfolios with five options. From left to right, as the time to maturity of the options in the hedge portfolio increases, the magnitude of the hedging error declines. This result is consistent with the simulation results, confirming that longer term options are more effective in spanning the target option when a quadrature approximation is applied in setting up the portfolio. Furthermore, when there exists additional sources of risk such as stochastic volatility, the reduced maturity gap between the target option and the options in the static hedge also reduces the exposure of the hedge portfolio to these additional risks.

4. Semi-Static Hedging of Path-Dependent Options

For ease of exposition, the focus of this paper thus far has been on static hedging of standard European options. It is clear that our results extend to path-independent European claims. When comparing this extension to the classical result of Breeden and Litzenberger (1978), we make the extra assumption that the underlying security price follows a one-factor Markovian process. The gain from this extra assumption is that we no longer require the target claim and its hedge portfolio of options to expire at the same time, a necessary requirement in the BL result. Therefore, we can effectively hedge a path-independent claim with a portfolio of shorter-term European options, as we have elaborated on in the case of a European option in the previous sections. In this section, we show that we can also form semi-static hedges of path-dependent options with European options, provided that the path is discretely monitored.

Hedging path-dependent options is not possible under the BL framework. *Dynamically* hedging path-dependent options is plausible in theory, but for many path-dependent claims, the reality of jumps often destroys the effectiveness of these hedges in practice. Our semi-static hedging theory takes jumps in stride, as we now show.

We consider the wide class of contingent claims whose single payoff at the fixed time T depends on a finite number ($n < \infty$) of points of the price path of a single underlying asset

$$V_T = f(S_{t_0}, S_{t_1}, \ldots, S_{t_n}), \tag{16}$$

where $t_0 = 0$ and $t_n = T$. We label the times $t_0, t_1, \ldots, t_n$ as monitoring times. The payoff structure in equation (16) excludes various continuously monitored Asian and barrier options, or American claims. Although we can always discretize a continuous problem, the analysis of this section assumes that we can trade at each fixed monitoring time t_i in options maturing at t_{i+1}.

To simplify the discretely monitored payoff function in equation (16), we note that for many claims, we can capture the path-dependence by one or more summary statistics. In what follows, we will work with a single summary statistic, but it should be clear how to extend the analysis to multiple such statistics. A single summary statistic captures the path-dependence of a claim if we can write the final payoff of the claim recursively as follows,

$$V_T = \phi(H_T), \tag{17}$$

where

$$H_{t_i} = g_i(H_{t_{i-1}}, S_{t_{i-1}}, S_{t_i}), \quad i = 1, \ldots, n, \tag{18}$$

where $\phi(\cdot)$ and $g_i(\cdot)$ are known functions, H is the single summary statistic, and H_0 and S_0 are known constants. Examples in this class include discretely monitored Asian and barrier options, Bermudan, passport, and cliquet options, and many structured notes. A concrete example which we will focus on is a globally-floored, locally-capped, compounding cliquet with discrete monitoring,

$$V_T = S_0 \max[L, H_T], \tag{19}$$

with

$$H_{t_i} = H_{t_{i-1}} \left[\left(\frac{S_{t_i}}{S_{t_{i-1}}} \right) \wedge U \right], \quad i = 1, \ldots, n, \tag{20}$$

where L is the global floor, $U > 1$ is the local cap, and n denotes the number of monitoring periods. Here, $H_0 = 1$, and S_0 is known. In practice, L is typically chosen to be one so that the annualized return is always positive. A typical value of the local cap U is 1.35 so that the maximum return for any year cannot exceed 35 percent.

We assume the same one-factor Markovian setting as in equation (1). To hedge the discretely monitored options as described by the payoff function in (17) and (18), we assume that at each discrete time t_i, we can take static positions in European options of all strikes and maturing at t_{i+1}, for $i = 0, 1, \ldots, n-1$. Given this access to markets, the algorithm for valuing a path-dependent option of the specified type is as follows.

At time t_{n-1}, conditioning on the history to that time $H_{t_{n-1}}$ and the contemporaneous stock price $S_{t_{n-1}}$, and from (17) and (18) with $i = n$, the final payoff becomes a known function of only the final stock price,

$$V_T = \phi(H_T) = \phi(g_n(H_{t_{n-1}}, S_{t_{n-1}}, S_T)) \equiv f_n(S_T; H_{t_{n-1}}, S_{t_{n-1}}), \tag{21}$$

where the last two arguments of f_n are known due to the conditioning.

Following Breeden and Litzenberger (1978), we can span the final payoff using options maturing at time T,

$$
\begin{aligned}
f_n(S_T;H_{t_{n-1}},S_{t_{n-1}}) \quad = \quad & f_n(\kappa_n;H_{t_{n-1}},S_{t_{n-1}}) + f_n'(\kappa_n;H_{t_{n-1}},S_{t_{n-1}})[(S_T-\kappa_n)^+ - (\kappa_n - S_T)^+] + \qquad (22) \\
& \int_0^{\kappa_n} f_n''(\mathcal{K};H_{t_{n-1}},S_{t_{n-1}})(\mathcal{K}-S_T)^+ d\mathcal{K} + \int_{\kappa_n}^{\infty} f_n''(\mathcal{K};H_{t_{n-1}},S_{t_{n-1}})(S_T-\mathcal{K})^+ d\mathcal{K},
\end{aligned}
$$

where the expansion point $\kappa_n \geq 0$ can be any convenient constant separating the put options from the call options. A common choice is the forward price $\kappa_n = F_0(T)$.

We can value this contingent-claim at time t_{n-1} by taking conditional expectations on both sides of equation (22) under the risk-neutral measure $\mathbb{Q}$ and then discounting the expectation by the constant riskfree rate. We can then represent the value of this claim in terms of the riskfree rate and the contemporaneous option prices,

$$
\begin{aligned}
V_{t_{n-1}}^{f_n} \quad = \quad & e^{-r(T-t_{n-1})} f_n(\kappa_n;H_{t_{n-1}},S_{t_{n-1}}) + f_n'(\kappa_n;H_{t_{n-1}},S_{t_{n-1}})[C_{t_{n-1}}(\kappa_n,T) - P_{t_{n-1}}(\kappa_n,T)] \\
& + \int_0^{\kappa_n} f_n''(\mathcal{K};H_{t_{n-1}},S_{t_{n-1}}) P_{t_{n-1}}(\mathcal{K},T) d\mathcal{K} + \int_{\kappa_n}^{\infty} f_n''(\mathcal{K};H_{t_{n-1}},S_{t_{n-1}}) C_{t_{n-1}}(\mathcal{K},T) d\mathcal{K}. \qquad (23)
\end{aligned}
$$

Therefore, at the last time step t_{n-1}, we can replicate the contingent claim using a portfolio of standard European options maturing at the same time. This result is the same as in Breeden and Litzenberger (1978) and does not need the Markovian assumption.

However, to be able to replicate the claim at any other time steps, we need the one-factor Markovian assumption. Substitution of the Markovian property (1) into equation (23) implies that the time-t_{n-1} value of this contingent claim is a known function of $H_{t_{n-1}}$ and $S_{t_{n-1}}$,

$$
V_{t_{n-1}}^{f_n} \quad = \quad e^{-r(T-t_{n-1})} f_n(\kappa_n;H_{t_{n-1}},S_{t_{n-1}}) + f_n'(\kappa_n;H_{t_{n-1}},S_{t_{n-1}})[C(S_{t_{n-1}},t_{n-1};\kappa_n,T;\Theta) - P(S_{t_{n-1}},t_{n-1};\kappa_n,T;\Theta)]
$$

$$
\begin{aligned}
&+ \int_0^{\kappa_n} f_n''(\mathcal{K};H_{t_{n-1}},S_{t_{n-1}})P(S_{t_{n-1}},t_{n-1};\mathcal{K},T;\Theta)d\mathcal{K} + \int_{\kappa_n}^{\infty} f_n''(\mathcal{K};H_{t_{n-1}},S_{t_{n-1}})C(S_{t_{n-1}},t_{n-1};\mathcal{K},T;\Theta)dK \\
\equiv\ & V(H_{t_{n-1}},S_{t_{n-1}},t_{n-1}). \qquad (24)
\end{aligned}
$$

Now, we step back to time t_{n-2} and condition on the history to that time $H_{t_{n-2}}$ and the contemporaneous stock price $S_{t_{n-2}}$. From the Markovian representation in (24) and the definition of the history summary statistic in (18) with $i = n-1$, we can write the time-t_{n-1} value of this claim as a known function of only the contemporaneous stock price at t_{n-1},

$$
V_{t_{n-1}}^{f_n} = V(H_{t_{n-1}},S_{t_{n-1}},t_{n-1}) = V(g_{n-1}(H_{t_{n-2}},S_{t_{n-2}},S_{t_{n-1}}),S_{t_{n-1}},t_{n-1}) \equiv f_{n-1}(S_{t_{n-1}};H_{t_{n-2}},S_{t_{n-2}}), \qquad (25)
$$

where $H_{t_{n-2}}$ and $S_{t_{n-2}}$ are known through the conditioning. Therefore, at time t_{n-2}, we can simply regard $f_{n-1}(S_{t_{n-1}};H_{t_{n-2}},S_{t_{n-2}})$ as the terminal payoff of a one-step claim, expressed as a function of the terminal stock price $S_{t_{n-1}}$. We can again follow Breeden and Litzenberger (1978) and replicate this payoff using options maturing at t_{n-1}, analogous to the steps in equations (22) and (23). Furthermore, we can again exploit the Markovian assumption in (1) and derive the new value function $V(H_{t_{n-2}},S_{t_{n-2}},t_{n-2})$ and the new target payoff function $f_{n-2}(S_{t_{n-2}};H_{t_{n-3}},S_{t_{n-3}})$ by performing operations analogous to (24) and (25).

We repeat the procedure until we obtain the value function at time 0. For this final iteration, we only need to condition on the known values of H_0 and S_0.

Therefore, the semi-static hedging of this path-dependent claim goes as follows. At time 0, we use a portfolio of European options maturing at time t_1 to span the value function of the claim. At time t_1, we collect the receipts from the expiring options in the hedge portfolio and form another hedge portfolio maturing at time t_2. This procedure continues until time $T = t_n$, when the payoff from the hedge portfolio formed at time t_{n-1} matches the payoff from the path-dependent claim. The hedging is static and no portfolio rebalancing is needed in between monitoring times. But at each monitoring step, the options in the hedge portfolio expire and a new hedge portfolio needs to be formed. Thus, the rebalancing frequency matches the monitoring frequency, reflecting the semi-static nature of the strategy.

5. Conclusion

Dynamic hedging has been widely used due to its flexibility in hedging a wide class of contingent claims. However, the performance of this strategy deteriorates dramatically in the presence of jumps of random size. The static hedging strategy introduced by Breeden and Litzenberger (1978) addresses this model risk, but can only be applied to a narrow range of payoffs. In this paper, we propose a new approach that is more robust than dynamic hedging and covers a much wider class of claims than BL. For simplicity, we illustrate our theory when the target claim is a European option. Since a perfect static hedge requires a continuum of options in the hedge portfolio, we develop a discrete approximation of the static hedge and test its effectiveness using historical data and Monte Carlo simulations.

The simulation results indicate that the static hedge approximation has about the same effectiveness as delta hedging with daily rebalancing in the Black-Scholes environment. However, when the simulated underlying price process can also experience jumps of random size, the performance of the delta hedge deteriorates dramatically. In contrast, the performance of our static option hedge is relatively insensitive to the change from a purely diffusive process to a jump diffusion. The conclusions are unchanged when we switch to ad hoc static and dynamic hedging practices necessitated by a lack of knowledge of the driving process. Further simulation indicates that increasing the rebalancing frequency cannot improve the inferior performance of the delta hedge in the presence of random jumps, but the superior performance of the static hedging strategy can be further enhanced by using more strikes or by optimizing on the common maturity in the hedge portfolio. As a result, the static hedge can achieve superior risk reduction with as few as three options in the hedge portfolio.

Accompanying the superior performance of the static hedge are the potentially lower transaction and monitoring costs. Furthermore, since delta hedging also requires short positions in either the risky asset or the money market account, complications can arise from short sales restrictions and leverage constraints. Neither complication arises in our static hedging strategy.

To investigate how our static strategy performs in a realistic setting, we investigate its effectiveness in hedging S&P 500 index options and compare its performance with daily delta hedging with the index futures. We find that the superior performance of our static hedge found in the simulations of the Merton model also extends to the index options data. This finding lends indirect support to the existence of jumps of random size as part of the S&P 500 index dynamics. We also find that the hedging errors from both the static and the dynamic strategies become larger when the maturity of the target call increases, indicating the potential existence of additional risk factors affecting option prices. One such risk can come from stochastic volatility. Hence, based on the availability and liquidity of the relevant option contracts, future research should be directed towards developing and testing static hedging strategies which account for a second risk factor. Such strategies would in general involve simultaneous positions in multiple strikes and maturities.

Although we focus on the hedging of a standard European option for ease of exposition, our theoretical results extend readily to the semi-static hedging of exotic options, including discretely monitored path-dependent options. We provide a summary of the theory underlying this semi-static hedging strategy. Once data for path-dependent option prices become available, a line for future research is to investigate the practical effectiveness of the strategy in real situations.

Appendix A. Option Pricing and Hedging under BS and MJ models

Appendix A.1. The Black-Scholes model

Under the Black-Scholes model, the spot price follows geometric Brownian motion under measure $\mathbb{Q}$,

$$dS_t/S_t = (r-\delta)dt + \sigma dW_t^*.$$

The time-t value of a European call with strike K and maturity T is given by

$$C(S,t;K,T;\Theta) = Se^{-\delta(T-t)}N(d_1(S,t;K,T;r,\delta,\sigma)) - Ke^{-r(T-t)}N(d_1(S,t;K,T;r,\delta,\sigma) - \sigma\sqrt{T-t}),$$

where $N(\cdot)$ denotes the standard normal distribution function and

$$d_1(S,t;K,T;r,\delta,\sigma) = \frac{\ln(S/K) + (r-\delta+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}. \tag{A1}$$

The delta and strike weighting functions are given by

$$\begin{aligned}
\Delta &\equiv \frac{\partial C(S,t;K,T;\Theta)}{\partial F} = e^{-r(T-t)}N(d_1(S,t;K,T;r,\delta,\sigma)),\\
w(\mathcal{K}) &= \frac{\partial^2 C(\mathcal{K},u;K,T;\Theta)}{\partial \mathcal{K}^2} = e^{-\delta(T-u)}\frac{n(d_1(\mathcal{K},u;K,T;r,\delta,\sigma))}{\mathcal{K}\sigma\sqrt{T-u}},
\end{aligned}$$

where $n(\cdot)$ denotes the probability density function of a standard normal,

$$n(d_1) = \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}.$$

Given the Gauss-Hermite quadrature $\{x_j, w_j\}_{j=1}^N$, the static portfolio strikes and weights are given by,

$$\begin{aligned}
\mathcal{K}_j &= Ke^{x_j\sigma\sqrt{2(T-u)}+(\delta-r-\sigma^2/2)(T-u)},\\
\mathcal{W}_j &= \frac{w(\mathcal{K}_j)\mathcal{K}_j\sigma\sqrt{2(T-u)}}{e^{-x_j^2}}w_j = \frac{e^{-\delta(T-u)}}{\sqrt{\pi}}w_j.
\end{aligned}$$

Note that given the definition of $\mathcal{K}_j$, we have $d_1(\mathcal{K}_j, u; K, T) = \sqrt{2}x_j$ and hence

$$n(d_1(\mathcal{K}_j, u; K, T; r, \delta, \sigma)) = \frac{1}{\sqrt{2\pi}} e^{-x_j^2}.$$

The cancellation then follows for the portfolio weight $\mathcal{W}_j$.

Appendix A.2. The Merton jump-diffusion model

The Merton (1976) jump-diffusion model assumes the following risk-neutral dynamics for the underlying security price movement,

$$dS_t/S_t = (r - \delta - \lambda^* g^*)\, dt + \sigma dW_t^* + dJ^*(\lambda^*),$$

where dJ^* denotes a compound Poisson jump with intensity λ^*. Conditional on a jump occurring, the log price relative is normally distributed with mean μ_j^* and variance σ_j^2. Conditional on a jump occurring, the mean percentage price change is given by $g^* = \left(e^{\mu_j^* + \sigma_j^2/2} - 1\right)$.

The price of a European call can be written as a weighted average of the Black-Scholes call pricing functions, with the weights given by the Poisson distribution,

$$C(S,t;K,T;\Theta) = e^{-r(T-t)} \sum_{n=0}^{\infty} \Pr(n) \left[S e^{(r_n - \delta)(T-t)} N(d_{1n}(S,t;K,T)) - KN(d_{1n}(S,t;K,T) - \sigma_n \sqrt{T-t}) \right],$$

where $\Pr(n)$ is the Poisson probability mass function, which gives the probability of having n jumps between time t and T,

$$\Pr(n) \quad = \quad e^{-\lambda^*(T-t)} \frac{(\lambda^*(T-t))^n}{n!},$$

$d_{1n}(S,t;K,T)$ is defined as

$$d_{1n}(S,t;K,T) = \frac{\ln(S/K) + (r_n - \delta - \sigma_n^2/2)(T-t)}{\sigma_n \sqrt{T-t}},$$

with

$$
\begin{aligned}
r_n &= r - \lambda^* g^* + n(\mu_j^* + \sigma_j^2/2)/(T-t), \\
\sigma_n^2 &= \sigma^2 + n\sigma_j^2/(T-t).
\end{aligned}
$$

In Merton's jump diffusion model, the delta and strike weighting function are given by

$$
\begin{aligned}
\Delta &= e^{-2r(T-t)} \sum_{n=0}^{\infty} \Pr(n) e^{r_n(T-t)} N(d_{1n}(S,t;K,T)), \\
w(\mathcal{K}) &= e^{-r(T-u)} \sum_{n=0}^{\infty} \Pr(n) e^{(r_n-\delta)(T-u)} \frac{n(d_{1n}(\mathcal{K},u;K,T))}{\mathcal{K}\sigma_n\sqrt{T-u}},
\end{aligned}
$$

We define the strike price points based on the Gauss-Hermite quadrature $\{x_j, w_j\}_{j=1}^N$ as follows,

$$
\mathcal{K}_j = K e^{x_j\sqrt{2v(T-u)}+(\delta-r-v/2)(T-u)},
$$

where

$$
v = \sigma^2 + \lambda^* \left((\mu_j^*)^2 + \sigma_j^2\right),
$$

is the annualized variance of the asset return under measure $\mathbb{Q}$. The portfolio weights are then given by

$$
\mathcal{W}_j = \frac{w(\mathcal{K}_j)\mathcal{K}_j \sqrt{2v(T-u)}}{e^{-x_j^2}} w_j.
$$

Note that in this case we no longer have the equality $d_{1n} = \sqrt{2}x_j$ as $v \neq \sigma_n^2$. One can regard v as a weighted average of σ_n^2 for all n's.

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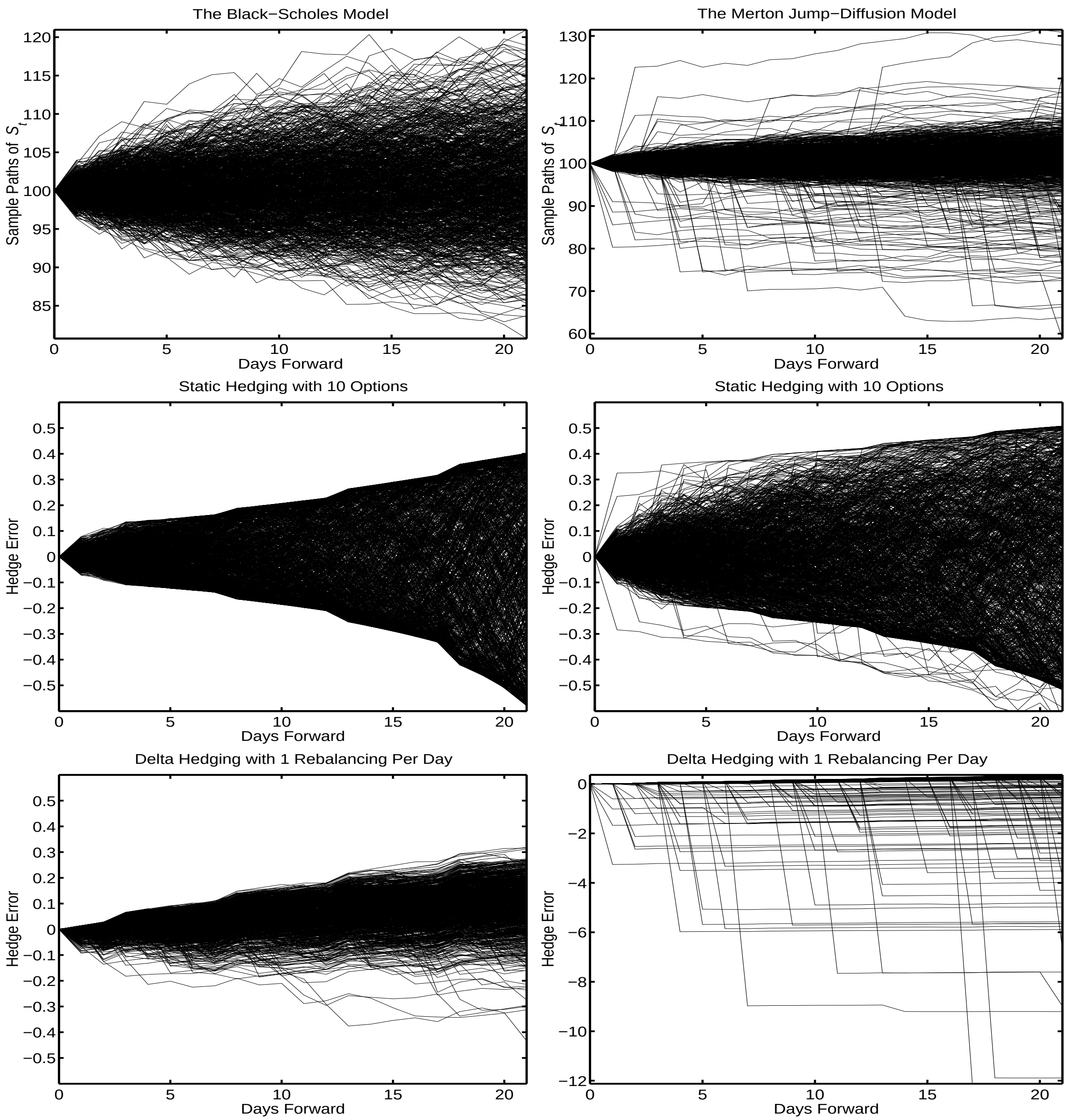

Figure 1
Hedging performance under different sample paths

The two panels in the first row depict the simulated sample paths for the underlying security price movement based on the Black-Scholes model (left) and the Merton jump-diffusion model (right). The two panels in the second row depict the sample paths of the hedging errors from the static hedging strategy with ten option contracts under the two models. The last row depicts the corresponding sample paths of the hedging errors from the dynamic delta strategy with the underlying futures and daily updating.

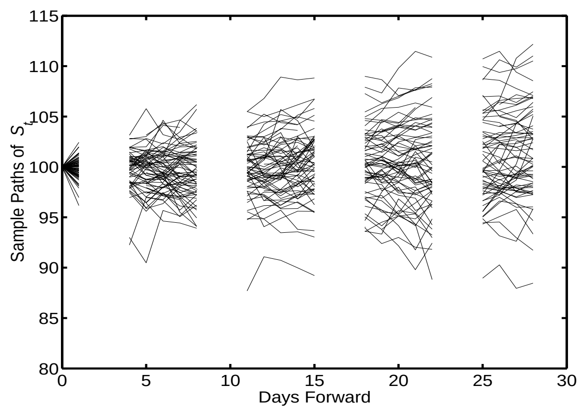

Figure 2
Normalized sample paths of the S&P 500 index
Plots are the sample paths of the S&P 500 index level over the month-long horizon of the hedging exercises. We normalize the index level to 100 at the start of each hedging exercise, and align the paths based on week days, starting on a Wednesday and ending on a Thursday four weeks later, spanning 29 days.

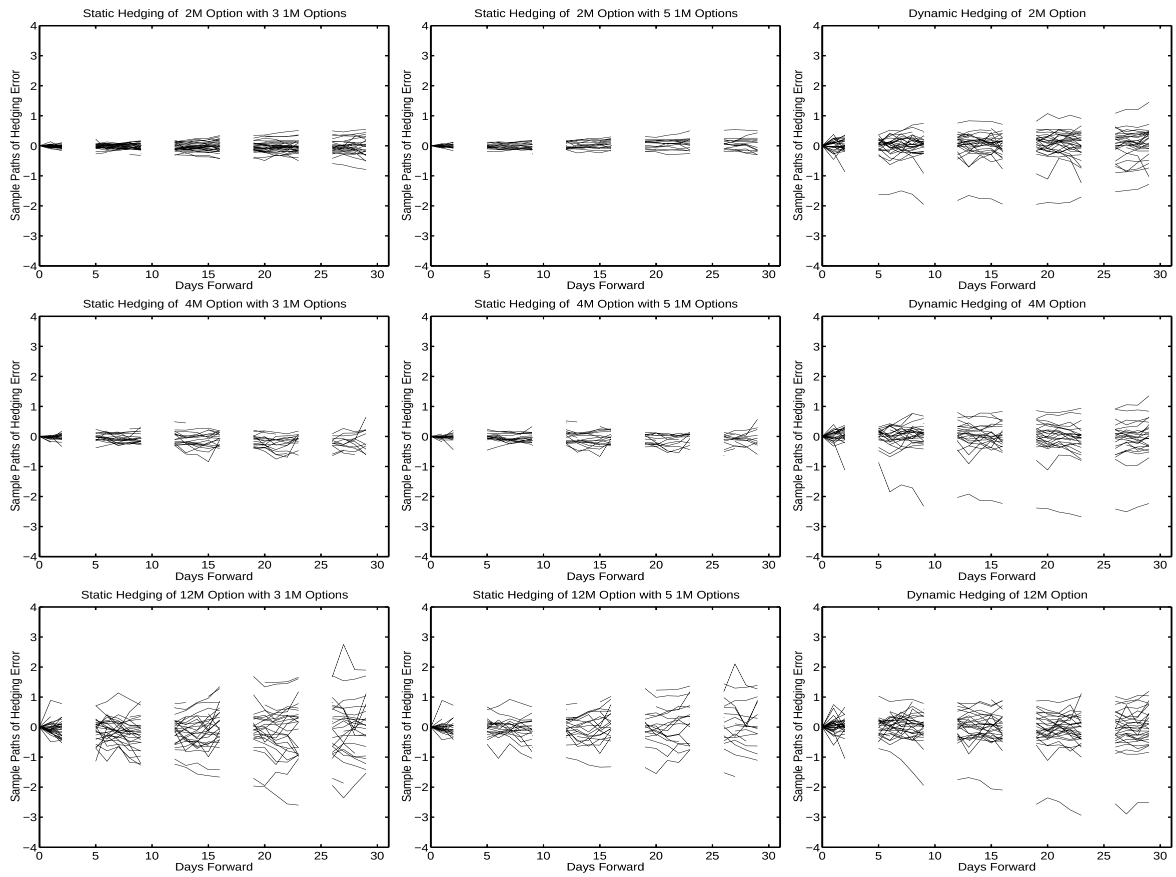

Figure 3

Hedging errors from static and dynamic strategies

Plots are the sample paths of the hedging errors based on (i) static strategies with a portfolio of three one-month options (left column), (ii) static strategies with a portfolio of five one-month options (middle column), and (iii) dynamic delta hedging strategies with the underlying futures and daily updating (right column). The options being hedged are at the money and have maturities of (i) two months (top row), (ii) four-to-six months (middle row), and (iii) one year and longer (bottom row).

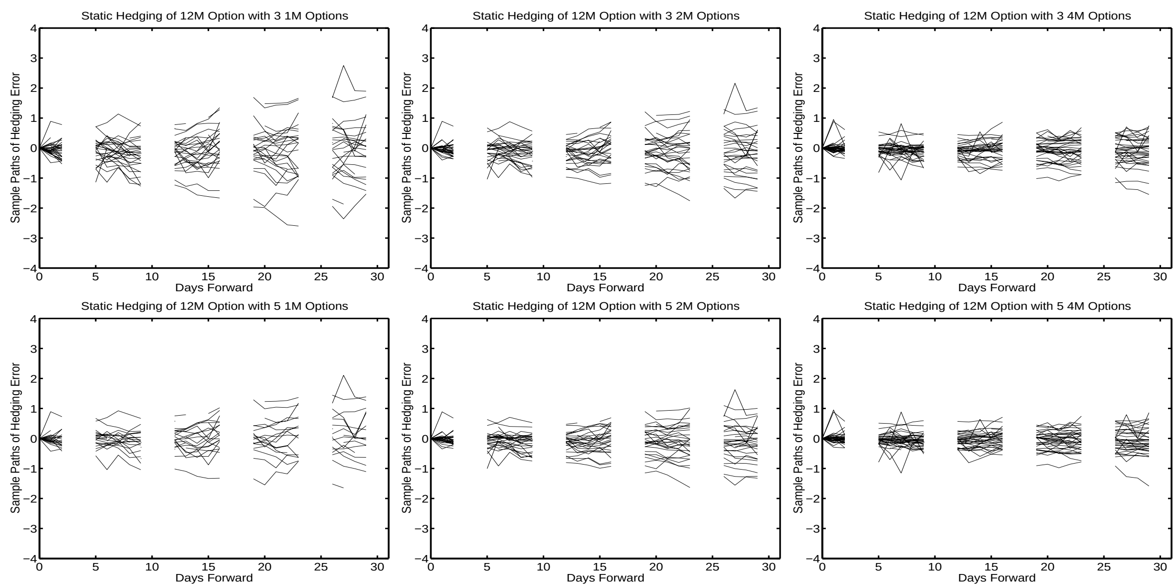

Figure 4

Errors of static hedging with options of different maturities

Plots are the sample paths of the hedging errors for the static hedging strategy with a portfolio of three options. The time to maturity of the options being hedged is one year or longer. The time to maturity of options in the hedge are (i) one month (left), (ii) two months (middle), and (iii) four-to-six months (right). The hedge portfolio contains three options in the first row, and five options in the second row.

Table 1

Simulated hedge performance comparisons of static and dynamic strategies

Hedge Error	Static with Options					Dynamic with
No. of Assets	3	5	10	15	21	Underlying
A. The Black-Scholes Model						
Mean	-0.00	0.01	-0.02	0.02	0.01	0.10
Std Err	1.00	0.66	0.33	0.20	0.14	0.10
RMSE	1.00	0.66	0.33	0.20	0.14	0.14
MAE	0.86	0.57	0.29	0.18	0.13	0.12
MSF	-0.43	-0.28	-0.15	-0.08	-0.06	-0.01
Min	-1.62	-1.13	-0.58	-0.38	-0.25	-0.43
Max	1.86	0.93	0.40	0.24	0.17	0.32
Skewness	0.01	-0.26	-0.32	-0.61	-0.48	-0.84
Kurtosis	1.87	1.79	1.70	1.96	1.78	4.68
Call Value	11.72	12.20	12.34	12.37	12.36	12.35
B. The Merton Jump-Diffusion Model						
Mean	-0.01	0.00	-0.01	0.02	0.02	0.07
Std Err	0.72	0.47	0.31	0.16	0.12	1.05
RMSE	0.72	0.47	0.31	0.16	0.12	1.05
MAE	0.53	0.35	0.27	0.13	0.10	0.48
MSF	-0.27	-0.18	-0.14	-0.05	-0.04	-0.20
Min	-1.73	-1.28	-0.67	-0.58	-0.41	-12.12
Max	2.84	1.48	0.51	0.20	0.14	0.37
Skewness	0.56	-0.16	-0.10	-1.77	-1.65	-6.82
Kurtosis	5.23	4.07	1.83	5.74	5.15	59.79
Call Value	9.52	11.14	11.93	12.09	12.06	11.99
C. Ad Hoc Black-Scholes Hedge under the Merton World						
Mean	-0.00	0.01	-0.02	0.01	0.01	0.03
Std Err	0.61	0.38	0.46	0.19	0.21	1.04
RMSE	0.61	0.38	0.46	0.19	0.21	1.04
MAE	0.48	0.29	0.39	0.12	0.15	0.45
MSF	-0.24	-0.14	-0.21	-0.05	-0.07	-0.21
Min	-1.57	-1.30	-0.72	-0.92	-0.82	-12.48
Max	1.63	1.10	0.85	0.68	0.65	2.08
Skewness	-0.26	-0.43	0.12	-0.43	-0.61	-6.45
Kurtosis	3.13	4.07	1.87	9.23	5.61	56.05
Call Value	11.53	11.96	11.84	12.07	12.03	11.99

Table 1 (Continued)

Entries report the summary statistics (mean, standard error, root mean squared error (RMSE), mean absolute error (MAE), mean short fall (MSF), minimum, maximum, skewness, and kurtosis) of the hedging error of a one-year call option based on both static and dynamic strategies. The hedging error is defined as the difference between the value of the hedge portfolio and the value of the target call option being hedged at the closing of the month-long hedging exercise. The static hedge portfolios consist of several one-month call options with strikes and weights determined by the static relation and a Gauss-Hermite quadrature approximation. The portfolios are then hold for one month without rebalancing. The dynamic hedge portfolio is a simple delta hedging with the underlying futures, but with daily rebalancing. The statistics are computed based on 1,000 simulated paths of the Black-Scholes model (Panel A) and the Merton jump-diffusion model (Panel B), assuming that the hedger knows the exact model in forming the portfolios. In Panel C, the sample paths and option prices are simulated based on the Merton model, but we assume that the hedger does not know this information and is formed to form the hedge portfolios based on the Black-Scholes forumla, with ad hoc adjustments to accommodate the observed implied volatility. The last row of each panel reports the value of the target call option approximated by the quadrature method, with the theoretical value given under the dynamic hedging column.

Table 2
Effect of different rebalancing frequencies on the dynamic delta hedge

Statistics	Reblancing Frequency Per Day			
	1	2	5	10
	A. The Black-Scholes Model			
Mean	0.11	0.11	0.11	0.11
Std Err	0.10	0.07	0.04	0.03
RMSE	0.15	0.13	0.12	0.12
MAE	0.13	0.12	0.11	0.11
MSF	-0.01	-0.00	-0.00	0.00
Min	-0.36	-0.15	-0.03	0.02
Max	0.32	0.28	0.22	0.19
Skewness	-0.77	-0.41	-0.34	-0.16
Kurtosis	4.21	3.23	3.12	2.86
	B. The Merton Jump-Diffusion Model			
Mean	0.09	0.09	0.09	0.09
Std Err	1.02	1.03	1.03	1.02
RMSE	1.02	1.03	1.03	1.03
MAE	0.49	0.49	0.49	0.49
MSF	-0.20	-0.20	-0.20	-0.20
Min	-11.78	-11.84	-11.72	-11.72
Max	0.38	0.37	0.35	0.35
Skewness	-6.27	-6.30	-6.24	-6.20
Kurtosis	50.97	51.40	50.34	49.79
	C. Ad Hoc Black-Scholes Hedge under the Merton World			
Mean	0.08	0.09	0.09	0.07
Std Err	0.88	0.92	0.93	0.88
RMSE	0.88	0.93	0.93	0.88
MAE	0.41	0.41	0.42	0.41
MSF	-0.17	-0.16	-0.16	-0.17
Min	-10.13	-10.19	-10.08	-10.07
Max	1.21	8.97	9.43	0.76
Skewness	-6.31	-4.74	-4.43	-6.25
Kurtosis	52.66	54.10	53.41	51.59

Table 2 (Continued)

Entries report the summary statistics (mean, standard error, root mean squared error (RMSE), mean absolute error (MAE), mean short fall (MSF), minimum, maximum, skewness, and kurtosis) of the hedging error of a one-year call option based on a dynamic delta hedge with different rebalancing frequencies. The hedging error is defined as the difference between the value of the hedge portfolio and the value of the target call option at the closing time of the month-long exercise. The statistics are computed based on 1,000 simulated paths of the Black-Scholes model (Panel A) and the Merton jump-diffusion model (Panel B)assuming that the hedger knows the exact model in forming the portflios. In Panel C, the sample paths and option prices are simulated based on the Merton model, but we assume that the hedger does not know this information and is formed to form the hedge portfolios based on the Black-Scholes forumla, with ad hoc adjustments to accommodate the observed implied volatility.

Table 3
Effect of different target and hedging instrument choice

Target Mat	2	4	12	12	12	2	4	12	12	12	2	4	12
Strategy	Static with Three Options					Static with Five Options					Daily Delta		
Instruments	1	1	1	2	4	1	1	1	2	4	Underlying Futures		
A. The Black-Scholes Model													
Mean	-0.02	-0.07	-0.00	-0.02	-0.01	-0.01	-0.03	0.01	-0.00	-0.00	0.30	0.21	0.11
Std Err	0.28	0.56	1.00	0.50	0.15	0.14	0.33	0.66	0.25	0.04	0.26	0.18	0.10
RMSE	0.28	0.57	1.00	0.50	0.15	0.14	0.33	0.66	0.25	0.04	0.40	0.27	0.15
MAE	0.24	0.50	0.86	0.43	0.12	0.12	0.29	0.57	0.22	0.03	0.34	0.23	0.13
MSF	-0.13	-0.28	-0.43	-0.22	-0.07	-0.06	-0.16	-0.28	-0.11	-0.02	-0.02	-0.01	-0.01
Min	-0.50	-0.90	-1.62	-0.59	-0.16	-0.26	-0.56	-1.13	-0.31	-0.04	-1.11	-0.77	-0.42
Max	0.32	0.79	1.86	1.22	0.43	0.18	0.41	0.93	0.44	0.07	0.97	0.63	0.33
Skewness	-0.42	-0.01	0.01	0.70	0.93	-0.31	-0.24	-0.26	0.36	0.54	-0.62	-0.76	-0.83
Kurtosis	1.73	1.60	1.87	2.37	2.90	1.81	1.61	1.79	1.75	1.99	4.14	4.45	4.64
Target Call	4.68	6.60	11.72	11.94	12.20	4.70	6.77	12.20	12.28	12.34	4.71	6.81	12.35
B. The Merton Jump-Diffusion Model													
Mean	-0.10	-0.09	-0.01	-0.02	-0.02	-0.06	-0.05	0.00	-0.00	-0.01	0.23	0.16	0.09
Std Err	0.84	0.77	0.72	0.50	0.27	0.40	0.46	0.47	0.29	0.16	2.55	1.92	1.05
RMSE	0.85	0.77	0.72	0.50	0.27	0.40	0.46	0.47	0.29	0.16	2.56	1.93	1.05
MAE	0.68	0.64	0.53	0.35	0.14	0.34	0.37	0.35	0.21	0.06	1.33	0.95	0.49
MSF	-0.39	-0.36	-0.27	-0.19	-0.08	-0.20	-0.21	-0.18	-0.11	-0.03	-0.55	-0.40	-0.20
Min	-1.20	-1.20	-1.73	-0.81	-0.33	-0.69	-0.74	-1.28	-0.72	-0.64	-25.22	-19.83	-12.11
Max	6.08	2.76	2.84	2.49	1.95	1.87	1.26	1.48	1.25	0.98	1.14	0.76	0.38
Skewness	1.29	0.51	0.56	1.76	4.76	0.11	0.66	-0.16	1.13	4.44	-5.51	-5.85	-6.81
Kurtosis	7.89	2.65	5.23	9.43	29.43	2.67	3.38	4.07	7.10	26.23	39.24	43.87	59.66
Target Call	3.76	4.80	9.52	9.76	10.15	3.72	5.33	11.14	11.19	11.18	4.11	6.34	11.99
C. Ad Hoc Black-Scholes Hedge under the Merton World													
Mean	-0.10	-0.07	-0.01	-0.02	-0.02	-0.06	-0.04	0.00	-0.00	-0.00	0.18	0.11	0.05
Std Err	0.84	0.55	0.63	0.46	0.27	0.40	0.31	0.41	0.26	0.14	2.26	1.73	1.04
RMSE	0.85	0.55	0.63	0.46	0.27	0.40	0.31	0.41	0.26	0.14	2.26	1.73	1.05
MAE	0.68	0.43	0.45	0.31	0.11	0.34	0.25	0.32	0.19	0.06	1.18	0.85	0.47
MSF	-0.39	-0.25	-0.23	-0.17	-0.06	-0.20	-0.14	-0.16	-0.10	-0.03	-0.50	-0.37	-0.21
Min	-1.20	-0.81	-1.42	-0.70	-0.65	-0.69	-0.53	-1.10	-0.67	-0.59	-22.01	-17.43	-12.46
Max	6.08	1.59	2.53	2.38	1.87	1.87	0.89	1.12	1.07	0.83	2.36	2.16	2.10
Skewness	1.29	1.06	0.81	2.16	5.12	0.11	0.47	-0.42	0.92	4.13	-5.38	-5.68	-6.45
Kurtosis	7.89	4.26	5.93	11.28	31.09	2.67	3.25	3.53	6.27	24.24	38.74	42.88	55.98
Target Call	3.76	4.94	10.03	9.94	10.24	3.72	5.80	11.55	11.38	11.34	4.11	6.34	11.99

Table 3 (Continued)

Entries report the summary statistics (mean, standard error, root mean squared error (RMSE), mean absolute error (MAE), mean short fall (MSF), minimum, maximum, skewness, and kurtosis) of the hedging errors when hedging different target options and using different hedging hedging instruments. The first row denotes the maturity of the target at-the-money option being hedged. The second row denotes the strategy, and the third row denotes the maturity of the options in the case of the static hedging strategy. The statistics are computed based on 1,000 simulated paths of the Black-Scholes model (Panel A) and the Merton jump-diffusion model (Panel B)assuming that the hedger knows the exact model in forming the portflios. In Panel C, the sample paths and option prices are simulated based on the Merton model, but we assume that the hedger does not know this information and is formed to form the hedge portfolios based on the Black-Scholes forumla, with ad hoc adjustments to accommodate the observed implied volatility. The last row of each panel reports the value of the target call option approximated by the quadrature method, with the theoretical value given under the dynamic hedging column.

Table 4
Static and dynamic hedging of S&P 500 index options

Target Mat	2	4	12	12	12	2	4	12	12	12	2	4	12
Strategy	Static with Three Options					Static with Five Options					Daily Delta		
Instruments	1	1	1	2	4	1	1	1	2	4	Underlying Futures		
Mean	-0.03	-0.13	-0.03	-0.09	0.00	-0.00	-0.10	-0.01	-0.07	0.01	0.16	0.03	-0.01
Std Err	0.27	0.62	1.06	0.74	0.44	0.23	0.50	0.87	0.64	0.42	0.57	0.65	0.88
RMSE	0.27	0.63	1.05	0.74	0.44	0.23	0.51	0.86	0.64	0.42	0.59	0.65	0.87
MAE	0.21	0.50	0.85	0.61	0.35	0.18	0.38	0.69	0.52	0.33	0.47	0.49	0.64
MSF	-0.12	-0.31	-0.44	-0.35	-0.17	-0.09	-0.24	-0.35	-0.29	-0.16	-0.15	-0.23	-0.32
Min	-0.79	-1.78	-2.90	-1.95	-1.55	-0.62	-1.55	-2.32	-1.97	-1.58	-1.28	-2.23	-2.51
Max	0.55	1.34	1.90	1.34	0.76	0.50	1.09	1.56	1.10	0.86	1.45	1.42	1.76
Skewness	-0.19	-0.21	-0.35	-0.22	-0.87	-0.28	-0.51	-0.46	-0.45	-0.74	-0.49	-0.78	-0.76
Kurtosis	3.00	3.20	2.81	2.51	4.16	3.12	3.70	2.91	2.89	4.51	3.34	4.47	3.83
Target Call	3.63	5.92	10.49	10.41	11.47	3.68	6.16	10.94	10.76	11.52	3.65	6.14	11.58

Entries report the summary statistics (mean, standard error, root mean squared error (RMSE), mean absolute error (MAE), mean short fall (MSF), minimum, maximum, skewness, and kurtosis) of the hedging errors for the hedging exercises on S&P 500 index options. The maturities (in months) of target options being hedged are given in the first row. They are near-the-money options. The hedging strategy is either static with a portfolio of three options, five options, or dynamic with the underlying futures and daily updating. The maturity of the options in the static hedge portfolio (in months) are given in the third row. The statistics are computed based on the 79 non-overlapping month long hedging exercises over a sample period of six years (from January 1996 to August 2002). The hedging errors are computed in percentages of the spot index level at the starting date of each exercise. The hedging strategies are designed based on the Black model with ad hoc adjustments to the observed implied volatilities. The last row reports the sample average of the value of the target call option approximated by the quadrature-based hedge portfolio. Numbers under the dynamic hedging columns are the sample average of the observed target call option price, all in percentages of the underlying spot index level at the starting date of each month.