Explicit constructions of martingales calibrated to given implied volatility smiles*

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Abstract

The construction of martingales with given marginal distributions at given times is a recurrent problem in financial mathematics. From a theoretical point of view, this problem is well-known as necessary and sufficient conditions for the existence of such martingales have been described. Moreover several explicit constructions can even be derived from solutions to the Skorokhod embedding problem. However these solutions have not been adopted by practitioners, who still prefer to construct the whole implied volatility surface and use the explicit constructions of calibrated (jump-) diffusions, available in the literature, when a continuum of marginal distributions is known.

In this paper, we describe several new constructions of calibrated martingales, which do not rely on a potentially risky interpolation of the marginal distributions but only on the input marginal distributions. These calibrated martingales are intuitive since the continuous-time versions of our constructions can be interpreted as time-changed (jump-) diffusions. Moreover, we show that the valuation of claims, depending only on the values of the underlying process at maturities where the marginal distributions are known, can be extremely efficient in this setting. For example, path-independent claims of this type can be valued by solving a finite number of ordinary (integro-) differential equations. Finally, an example of calibration to the S&P 500 market is provided.

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1 Introduction

The construction of martingales with constrained marginal distributions is an omnipresent problem in mathematical finance. Indeed, the assumption of absence of arbitrage often translates into the fact that the underlying asset must be modeled by a martingale in some measure (see [26] and [27]). Moreover, when pricing exotic contingent claims, whose hedging requires the use of European options, one needs to have a model, which is consistent with the prices of these options.

In this paper, we assume that European option prices are known at a finite number of maturities, or, equivalently, that the marginal distributions of the underlying asset are known at these maturities (see [9]) and we address the problem of constructing a martingale with these marginal distributions.

This problem was first tackled by [42], who described necessary and sufficient conditions on the marginal distributions for the existence of a calibrated martingale. In the case of two marginal distributions - which is considered in this paper without loss of generality - Strassen's theorem can be stated as follows:

Theorem 1 (Strassen's theorem (1965)) *If two probability distributions μ_1 and μ_2 on $\mathbb{R}_+^* \equiv \{x \in \mathbb{R} \,|\, x > 0\}$ have the same finite first moment, X_0, and satisfy:*

$$\int_0^{+\infty} f(x)\,\mu_1(dx) \leqslant \int_0^{+\infty} f(x)\,\mu_2(dx) \tag{1}$$

for every convex function f on $\mathbb{R}_+^*$ (where the integrals are possibly infinite), then there exist two random variables, X_1 and X_2, defined on the same probability space, such that:

- *the probability distribution of X_i is μ_i for $i \in \{1, 2\}$,*
- *the process (X_i) is a martingale, i.e.*

$$\mathbb{E}[X_2 | X_1] = X_1. \tag{2}$$

Constructive proofs of this result have been known for a while, as all solutions to the Skorokhod embedding problem[1], which accept non trivial initial laws, correspond to constructions of calibrated martingales. Examples of such constructions include those of [15] (which generalizes [21] in particular), [5] (see [29]), [39] (see [31]), [32] (see [17]), [47] (see [17]) and [41] (see [19]). However most of these solutions are extremal in some sense and are not natural candidates for a pricing model. For instance, the construction of [5] (resp. [39]) maximizes (resp. minimizes) the law of the maximum of the stopped Brownian motion amongst all embeddings, while the construction of [41] minimizes the price of options on the stopping time. (These extremal properties can actually be used to find model-free bounds for the price of derivative securities as well as sub/super-replicating strategies. We refer to [30] and the references therein for more details.)

However, to the best of our knowledge, the majority of practitioners prefer to interpolate the given implied volatility smiles to obtain a whole implied volatility surface[2] and then use the explicit constructions of calibrated continuous-time martingales available in the literature. We refer to [20] and [22] when the underlying process is a diffusion, to [4] and [12] when it is a jump-diffusion and, finally, to [13] when it is a deterministically time-changed Lévy process.

The purpose of this paper is to describe several new constructions of calibrated martingales, (i) which do not require the construction of the whole implied volatility surface and, (ii) which

[1]We refer to [38] for a survey on this problem and a discussion of over twenty solutions.
[2]See [36], [1], [33] and the references therein for more details.

are intuitive and realistic enough for practitioners. Our constructions can actually be interpreted as the discrete-time counterparts of *local volatility models* (Section 4), of *local volatility models with jumps* (Section 5.2), of finite activity *local Lévy models* (Section 5.3) and of *localizable jump-diffusion models* (Section 5.4). Indeed, they use transition distributions, which are those of (jump-) diffusions, sampled at independent and exponentially distributed random times (see Section 2) and all the techniques, which were developed in the above papers can be transposed to our discrete-time setting.

Moreover, our constructions are not confined to discrete-time martingales. Indeed, if the distributions μ_1 and μ_2 correspond to given maturities T_1 and T_2, such that $T_0 \equiv 0 < T_1 < T_2$, we also explain how to construct a continuous-time martingale (X_t^c), which starts at X_0 and such that the distribution of $X_{T_i}^c$ is μ_i, for $i \in \{1, 2\}$. And, as one could expect, (X_t^c) can be understood as a time-changed (jump-) diffusion[3].

Finally, as far as pricing is concerned, the only technique, which is missing is the valuation of claims on (X_t^c) using backward partial integro-differential equations, since this process is not a Markov process a priori. However, $(X_i) \equiv \left(X_{T_i}^c\right)$ is a Markov process and the valuation of path-independent claims on (X_i) can be performed by solving a finite number of ordinary (integro-) differential equations. Moreover Monte Carlo schemes, very similar to those used for jump-diffusions, are available for valuating complex claims on (X_i), (X_t^c) or in a multinomial setting.

This paper is organized as follows. Section 2 defines Sampled Jump-Diffusion (SJD) distributions, which will be used later as transition distributions of the calibrated martingale. Section 3 describes a necessary condition linking the parameters of a SJD distribution and the marginal distributions μ_1 and μ_2. Section 4 exhibits sufficient conditions for sampled diffusion transition distributions to define a calibrated martingale. Section 5 does the same, when the transition distributions are those of a sampled jump-diffusion, and transpose to the discrete-time setting the techniques used by [4], [13] and [12]. Section 6 addresses the problem of constructing a continuous-time martingale, calibrated to the μ_i. Section 7 explains how the valuation of European style path-independent claims can be reduced to the valuation of ordinary (integro-) differential equations. Section 8 gives an example of calibration to S&P 500 option quotes. Section 9 concludes.

2 Definition and properties of Sampled Jump-Diffusion distributions

To construct the distributions of $X_2 | X_1$, we first consider a positive time-homogeneous Markov process (F_t), whose generator $\mathcal{G}$, is, for $f \in C_0^\infty\left(\mathbb{R}_+^*\right)$ – the space of infinitely differentiable functions with compact support on $\mathbb{R}_+^*$:

$$\mathcal{G}f(x) \equiv \frac{1}{2}x^2\sigma^2(x)\, f''(x) + \lambda(x)\int_0^{+\infty}\left(f(y) - f(x) - f'(x)(y-x)\right)\gamma(x,y)\, dy, \tag{3}$$

where $x\sigma(x)$ corresponds to the diffusion coefficient, λ is the intensity of the counting process and $\gamma(x,.)$ is the density of $(F_t | (F_{t^-} = x) \,\&\, (\Delta F_t \neq 0))$.

We assume in the following that:

- σ is a positive, bounded and continuous function,
- λ is a non-negative, bounded and continuous function,

[3]Note that our constructions encompass continuous processes, discontinuous processes with finite (resp. infinite) activity.

- If $\lambda = 0$, γ is a non-negative function such that:
    - $\gamma(x, .)$ is a probability density on $\mathbb{R}_+^*$ for every $x \in \mathbb{R}_+^*$,
    - $\int_0^{+\infty} \frac{y}{x} \gamma(x, y)\, dy$ is bounded by above by a constant A and is continuous,
    - $\int_\Gamma \frac{\ln\left(\frac{y}{x}\right)}{1+\left(\ln\left(\frac{y}{x}\right)\right)^2} \gamma(x, y)\, dy$ is continuous in x for every Γ, Borel set of $\mathbb{R}_+^*$.

The above assumptions ensure that the process (F_t) exists and is a strong Markov process. Indeed the martingale problem associated to its logarithm is well-posed thanks to [43], who generalized to processes with Lévy generators the seminal results of [44] and [45]. (See Proposition 2.1 in [16] for the details.)

Moreover, if we further assume, in the case $\lambda = 0$, that:

$$
{}_0^{+\infty} \frac{y}{x} \left| \ln\left(\frac{y}{x}\right) \right| \gamma(x, y)\, dy \qquad B < +\infty, \tag{4}
$$

then (F_t) is a martingale since it satisfies the sufficient conditions for a positive local martingale to be a martingale described by [35]. (See Proposition 2.3 in [16] for the details.)

We now have all the elements to construct $X_2 | X_1$. Indeed, if τ denotes a random time, which has an exponential distribution of parameter 1 and is independent of (F_t), we define, as suggested in the introduction:

$$
(X_2 | X_1 = x_1) \overset{law}{\equiv} (F_\tau | F_0 = x_1). \tag{5}
$$

The distributions of $X_2 | X_1$ will be called *Sampled Jump-Diffusion (SJD) distributions* in the following. Notice that this concept generalizes the one of *generalized Laplace distributions*, introduced in [12], which corresponds to the particular case where (F_t) is a diffusion.

In the above mentioned paper, the authors show in particular that generalized Laplace distributions are centered and the Green's functions of linear second-order differential operators (see Appendix A for more details). As expected, SJD distributions have similar properties.

Indeed, notice that

$$
\mathbb{E}[X_2 | X_1 = x_1] = x_1, \tag{6}
$$

since τ is independent of (F_t), which is a positive martingale. Furthermore, if $P \in C_0^\infty(\mathbb{R}_+^*)$, the forward and backward Kolmogorov equations are respectively:

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\mathbb{E}[P(F_t) | F_0]\right) &= \mathbb{E}[\mathcal{G}P(F_t) | F_0], & (7)\\
\frac{\partial}{\partial t}\left(\mathbb{E}[P(F_t) | F_0]\right) &= \mathcal{G}\mathbb{E}[P(F_t) | F_0]. & (8)
\end{aligned}
$$

Multiplying them by e^{-t} and integrating over $\mathbb{R}_+^*$ gives, using integration by parts:

$$
\begin{aligned}
\mathbb{E}[([\mathcal{I} - \mathcal{G}] P)(F_\tau) | F_0] &= P(F_0), & (9)\\
[\mathcal{I} - \mathcal{G}] \mathbb{E}[P(F_\tau) | F_0] &= P(F_0), & (10)
\end{aligned}
$$

i.e.

$$
\begin{aligned}
\mathbb{E}[([\mathcal{I} - \mathcal{G}] P)(X_2) | X_1] &= P(X_1), & (11)\\
[\mathcal{I} - \mathcal{G}] \mathbb{E}[P(X_2) | X_1] &= P(X_1). & (12)
\end{aligned}
$$

The above equations are based on the fact that the resolvent of the generator $\mathcal{G}$ is linked to the Laplace transform of the associated transition semigroup (see Proposition 2.1 in [23], p. 10) and will prove to be decisive in the following. Indeed, Equation (11) will be used in the next section to derive a necessary condition for (X_i) to be calibrated and Equation (12) will be used in Section 7 for the pricing of path-independent claims.

3 A necessary condition for calibration

In this section, we derive heuristically a necessary condition that must be satisfied by σ, λ and γ when X_1 and X_2 are calibrated respectively to the call price functions C_1 and C_2.

Let us apply Equation (11) to the payoff $P(x) = (x-K)^+$ (with $K > 0$), even though a call payoff does not belong to the domain of $\mathcal{G}$:

$$\mathbb{E}\left[\left([\mathcal{I}-\mathcal{G}](x-K)^+\right)(X_2)\ \ X_1\right] = (X_1-K)^+. \tag{13}$$

Moreover,

$$\begin{aligned} & [\mathcal{I}-\mathcal{G}](x-K)^+ \\ = \ & (x-K)^+ - \frac{1}{2}x^2\sigma^2(x)\,\delta(x-K) \\ & -\lambda(x)\int_0^{+\infty}\left((y-K)^+ - (x-K)^+ - H(x-K)(y-x)\right)\gamma(x,y)\,dy, \end{aligned} \tag{14}$$

where H is the Heaviside function. If we denote by $\Psi_\gamma(x,K)$ the double tail associated to the density $\gamma(x,.)$:

$$\Psi_\gamma(x,K) \equiv \begin{cases} \int_K^{+\infty}\left(\int_u^{+\infty}\gamma(x,v)\,dv\right)du & \text{if } K > x \\ \frac{1}{2}\left(\int_K^{+\infty}\left(\int_u^{+\infty}\gamma(K,v)\,dv\right)du + \int_0^K\left(\int_0^u\gamma(K,v)\,dv\right)du\right) & \text{if } K = x \\ \int_0^K\left(\int_0^u\gamma(x,v)\,dv\right)du & \text{if } K < x \end{cases} \tag{15}$$

then we obtain, using put-call parity:

$$[\mathcal{I}-\mathcal{G}](x-K)^+ = (x-K)^+ - \frac{1}{2}x^2\sigma^2(x)\,\delta(x-K) - \lambda(x)\,\Psi_\gamma(x,K). \tag{16}$$

Therefore, Equation (13) can be rewritten as:

$$\mathbb{E}\left[(X_2-K)^+ - \frac{1}{2}X_2^2\sigma^2(X_2)\,\delta(X_2-K) - \lambda(X_2)\,\Psi_\gamma(X_2,K)\ \ X_1\right] = (X_1-K)^+. \tag{17}$$

and taking expectations, we get:

$$C_2(K) - C_1(K) = \frac{1}{2}K^2\sigma^2(K)\,\mathbb{P}[X_2 = K] + \mathbb{E}\left[\lambda(X_2)\,\Psi_\gamma(X_2,K)\right]. \tag{18}$$

If we further assume that X_2 admits a density, we have, in the spirit of [9], $\mathbb{P}[X_2 = K] = \frac{\partial^2 C_2}{\partial K^2}(K)$ and the above equation becomes:

$$C_2(K) - C_1(K) = \frac{1}{2}K^2\sigma^2(K)\frac{\partial^2 C_2}{\partial K^2}(K) + \int_0^{+\infty}\frac{\partial^2 C_2}{\partial x^2}(x)\,\lambda(x)\,\Psi_\gamma(x,K)\,dx. \tag{19}$$

Notice that the problem of finding σ, λ and γ, which satisfy this equation, is very similar to the one of calibrating a (jump-) diffusion to an implied volatility surface. The only difference is that the time derivative is replaced by a time difference. For instance, when $\lambda = 0$, the discrete-time counterpart of the well-known *Dupire's formula* (see [22]) is:

$$\sigma^2(K) = \frac{C_2(K) - C_1(K)}{\frac{1}{2}K^2\frac{\partial^2 C_2}{\partial K^2}(K)}. \tag{20}$$

4 Sufficient conditions in the case of generalized Laplace distributions

In this section, we derive sufficient conditions on the marginal distributions μ_1 and μ_2, for the existence of centered transition densities of the generalized Laplace type (see Appendix A). As explained in the previous section, this construction can be interpreted as a discrete-time equivalent of *local volatility models* developed by [20] and [22].

Proposition 2 *Let us denote by C_1 (resp. C_2) the call price function associated to the probability distribution μ_1 (resp. μ_2) on $\mathbb{R}_+^*$:*

$$C_i(K) \equiv \int_0^{+\infty} (x-K)^+ \mu_i(dx) \text{ for } i \in \{1,2\}. \tag{21}$$

We assume that:

- *μ_1 and μ_2 have the same finite first moment X_0,*
- *$C_1(K) < C_2(K)$ for $K \in \mathbb{R}_+^*$,*
- *μ_1 admits a continuous density f_1 or is a Dirac function in X_0,*
- *μ_2 admits a continuous and positive density f_2,*
- *the following equation:*
$$\sigma^2(K) = \frac{C_2(K) - C_1(K)}{\frac{1}{2}K^2 f_2(K)} \tag{22}$$
defines a bounded function σ^2 on $\mathbb{R}_+^*$.

Then there exists a unique family of generalized Laplace densities $(p(.\,|x_1))$ associated to the function $x^2\sigma^2(x)$, i.e. of functions, which solves:

$$\left(\mathcal{I} - \frac{\partial^2}{\partial x_2^2}\frac{x_2^2\sigma^2(x_2)}{2}\right) p(x_2|x_1) = \delta(x_2 - x_1), \tag{23}$$

$$\frac{\partial}{\partial x_2}\left(\frac{x_2^2\sigma^2(x_2)}{2} p(x_2|x_1)\right) \to_{0 \text{ or } +\infty} 0, \tag{24}$$

for $(x_1, x_2) \in (\mathbb{R}_+^*)^2$. Moreover,

$$\int_0^{+\infty} x_2 p(x_2|x_1)\, dx_2 = x_1, \tag{25}$$

$$\int_0^{+\infty} p(x_2|x_1)\, \mu_1(dx_1) = f_2(x_2). \tag{26}$$

Proof. See Appendix B. ∎

Remark 3 *A related construction is described in [36], where the authors use an optimal control framework to show that, under given smoothness conditions, there exists a (unique) time-homogeneous diffusion, which is a consistent martingale. However, one benefit of stopping the time-homogeneous diffusion at an independent exponentially distributed random time rather than at a fixed time is that we get an explicit formula for the diffusion coefficient (see Equation (22)) and do not need to use (involved) optimal control techniques to recover it.*

Remark 4 *Equation (23) is only valid in distribution. However, the first derivative in Equation (24) exists everywhere except in x_1. Therefore, considering its limit when x_2 goes to 0 or $+\infty$ is valid. We refer to Appendix A for more details on generalized Laplace densities.*

In the above proposition, the assumption that σ^2 is a bounded function is not straightforward to test on the call price functions C_i. The next lemma describes sufficient conditions on f_2 alone, which are based on Karamata's theory. We refer to [7] for a clear and exhaustive description of this theory. (See Appendix C for a reminder of the main definitions.)

Lemma 5 *With the notations of Proposition 2, if f_2 has upper Matuszewska index $\alpha\left(f_2\right) < -2$, then:*

$$\sigma^2\left(K\right) =_{+\infty} O\left(1\right).$$

If $\widetilde{f}_2\left(x\right) \equiv f_2\left(1/x\right)$ has upper Matuszewska index $\alpha\left(\widetilde{f}_2\right) < 1$, then:

$$\sigma^2\left(K\right) =_{0} O\left(1\right).$$

Proof. See Appendix D. ■

Remark 6 *Note that the above assumptions are not very restrictive since f_2 is a probability density with finite first moment. For instance, simple computations show that if f_2 is a log-normal density, then $\alpha\left(f_2\right) = \alpha\left(\widetilde{f}_2\right) = -\infty$ and the assumptions of the above lemma are therefore satisfied.*

5 Sufficient conditions in the case of SJD distributions

In this section, we describe general sufficient conditions for SJD distributions to be suitable transition distributions for (X_i). Furthermore, we adapt to the discrete-time setting the techniques, developed in [4], [13] and [12], to calibrate a jump-diffusion to a continuum of marginal densities.

Remark 7 *The motivation behind having SJD transition distributions and not only generalized Laplace transition distributions is the same as the one of [4] when they overlayed a local volatility dynamics with a jump process. Indeed, when the underlying asset exhibits jumps, the only way to calibrate short maturity option prices with a continuous path model is to input unrealistic parameters. We also refer to Chapter 5 of [25] for more details about why jumps are sometimes needed.*

5.1 General sufficient conditions

To describe sufficient conditions in the case of SJD transition distributions is more complicated than in the case of general Laplace distributions. Indeed, in the latter case, we knew sufficient conditions for the associated boundary value problems to have a unique solution and that generalized Laplace distributions admitted smooth enough densities. The situation is completely different here. First, to the best of our knowledge, Green's functions associated to the operator $[\mathcal{I} - \mathcal{G}]$ or its adjoint on the *unbounded* set $\mathbb{R}_+^*$ did not receive as much attention in the general case as in the case where $\mathcal{G}$ is a linear second-order differential operator. Moreover, sufficient conditions for SJD distributions to admit densities does not seem to be readily available in the literature. To tackle this problem, a possibility could have been to use the sufficient conditions for jump-diffusions to admit smooth densities. (We refer to [6], [46], [34] and [14].) However, all these results are valid for processes, which are strong solutions of stochastic differential equations with jumps, and as remarked in [6], to translate results in this setting to the setting of the martingale problem of Section 2 is not painless.

Proposition 8 *Let us denote by C_1 (resp. C_2) the call price function associated to the probability distribution μ_1 (resp. μ_2) on $\mathbb{R}_+^*$:*

$$C_i(K) \equiv \int_0^{+\infty} (x-K)^+ \mu_i(dx) \ \text{ for } i \in \{1,2\}\,. \tag{27}$$

We assume (A_1), that is:

- *μ_1 and μ_2 have the same finite first moment X_0,*
- *$C_1(K) < C_2(K)$ for $K \in \mathbb{R}_+^*$,*
- *μ_1 admits a continuous density f_1 or is a Dirac function in X_0,*
- *μ_2 admits a continuous and positive density f_2.*

We further assume (A_2), i.e.

- *σ is a positive, bounded and continuous function on $\mathbb{R}_+^*$,*
- *λ is a non-negative, bounded and continuous functions on $\mathbb{R}_+^*$,*
- *γ is a function from $\left(\mathbb{R}_+^*\right)^2$ to $\mathbb{R}_+$ such that:*
  - *$\gamma(x,.)$ is a probability density for $x \in \mathbb{R}_+^*$,*
  - *$\int_{\mathbb{R}_+^*} \frac{y}{x}\gamma(x,y)\, dy$ is continuous on $\mathbb{R}_+^*$,*
  - *$\int_\Gamma \frac{\ln\left(\frac{y}{x}\right)}{1+\left(\ln\left(\frac{y}{x}\right)\right)^2}\gamma(x,y)\, dy$ is continuous in x for every Γ, Borel set of $\mathbb{R}_+^*$,*
  - *there exists $A < +\infty$ such that, for $x \in \mathbb{R}_+^*$,*

$$\int_0^{+\infty} \frac{y}{x} \max\left(1,\ \ln\left(\frac{y}{x}\right)\right) \gamma(x,y)\, dy \leq A, \tag{28}$$

- *λ, γ and σ satisfy Equation (19), where Ψ_γ is defined by Equation (15).*

Then there exists a positive strong Markov martingale (F_t), which is right-continuous with left limits, whose generator $\mathcal{G}$ is, for $f \in C_0^\infty\left(\mathbb{R}_+^*\right)$:

$$\mathcal{G}f(x) \equiv \frac{1}{2}x^2\sigma^2(x) f''(x) + \lambda(x) \int_0^{+\infty} \left(f(y) - f(x) - f'(x)(y-x)\right)\gamma(x,y)\, dy. \tag{29}$$

If we further assume (A_3), i.e. the distribution of $F_\tau | F_0 = x_1$ – where τ is an exponentially distributed random time with mean 1, independent of (F_t) – admits a smooth enough density $p(.|x_1)$, such that the following limit conditions hold:

$$p(x_2|x_1)\frac{\partial}{\partial x_1}\left(\frac{1}{2}x_1^2\sigma^2(x_1) f_2(x_1)\right) \underset{x_1 \to 0 \text{ or } +\infty}{\to} 0, \tag{30}$$

$$\frac{\partial p}{\partial x_1}(x_2|x_1)\left(\frac{1}{2}x_1^2\sigma^2(x_1) f_2(x_1)\right) \underset{x_1 \to 0 \text{ or } +\infty}{\to} 0, \tag{31}$$

$$p(x_2|x_1)\left(x_1\mu(x_1) f_2(x_1)\right) \underset{x_1 \to 0 \text{ or } +\infty}{\to} 0, \tag{32}$$

then p is a suitable transition density between T_1 and T_2:

$$\int_0^{+\infty} x_2 p\left(x_2 | x_1\right) dx_2 = x_1, \tag{33}$$

$$\int_0^{+\infty} p\left(x_2 | x_1\right) \mu_1\left(dx_1\right) = f_2\left(x_2\right). \tag{34}$$

Remark 9 *An intuitive necessary condition implied by Equation (19) is:*

$$\sigma^2\left(K\right) \quad \frac{C_2\left(K\right) - C_1\left(K\right)}{\frac{1}{2}K^2 \frac{\partial^2 C_2}{\partial K^2}\left(K\right)}, \tag{35}$$

where we recognize on the RHS the expression for σ^2, which allows us to fit the marginal distributions when the transition densities are of the generalized Laplace type (see Section 4). Informally, the presence of jumps forces the diffusion component of the sampled process to be less volatile than the one of a sampled diffusion, which recovers the option prices.

Proof. See Appendix E. ■

Of course, there are many ways of choosing λ, γ and σ so that the assumptions (A_2) – and Equation (19) in particular – are satisfied. In the following sections, we mimic, in a discrete-time setting, the strategies described respectively by [4], [13] and [12].

5.2 A solution à la Andersen and Andreasen (2000)

In [4], the authors calibrate a jump-diffusion process to an implied volatility surface through the diffusion coefficient once the jump intensity and the jump distributions have been specified. We adopt a similar approach in this subsection by first fixing λ, γ, then defining σ via Equation (19), i.e.

$$\sigma^2\left(x\right) \equiv \frac{C_2\left(x\right) - C_1\left(x\right) - \int_0^{+\infty} \frac{\partial^2 C_2(y)}{\partial y^2} \lambda\left(y\right) \Psi_\gamma\left(y, x\right) dy}{\frac{1}{2}x^2 \frac{\partial^2 C_2}{\partial x^2}\left(x\right)}. \tag{36}$$

However, for this recipe to be valid, the above equation needs, at least, to define a positive quantity and it is clear that not all pairs (λ, γ) will do. The purpose of the next lemma is to give sufficient conditions on γ for the existence of a non-trivial λ and therefore σ satisfying the assumptions of Proposition 8.

Lemma 10 *With the notations of Proposition 8, we assume that:*

- *μ_1, μ_2 and γ satisfy the assumptions (A_1) and (A_2),*
- $$\begin{aligned} P_{2,\gamma}\left(x\right) &\equiv \int_0^{+\infty} \left(\int_0^{+\infty} \left(x - v\right)^+ \gamma\left(u, v\right) dv \right) f_2\left(u\right) du \\ &=_0 O\left(C_2\left(x\right) - C_1\left(x\right)\right), \end{aligned} \tag{37}$$
- $$\begin{aligned} C_{2,\gamma}\left(x\right) &\equiv \int_0^{+\infty} \left(\int_0^{+\infty} \left(v - x\right)^+ \gamma\left(u, v\right) dv \right) f_2\left(u\right) du \\ &=_{+\infty} O\left(C_2\left(x\right) - C_1\left(x\right)\right), \end{aligned} \tag{38}$$

then there exists a positive, bounded and continuous function λ such that:

$$\left(C_2\left(x\right) - C_1\left(x\right)\right) > \int_0^{+\infty} \frac{\partial^2 C_2\left(y\right)}{\partial y^2} \lambda\left(y\right) \Psi_\gamma\left(y, x\right) dy. \tag{39}$$

If we further assume that f_2 (resp. $\widetilde{f}_2(x) \equiv f_2(1/x)$) has upper Matuszewska index $\alpha(f_2) < -2$ (resp. $\alpha\left(\widetilde{f}_2\right) < 1$), then the function σ, defined by Equation (36), λ and γ satisfy the assumptions (A_2).

Remark 11 *Note that $P_{2,\gamma}(x)$ (resp. $C_{2,\gamma}(x)$) is the price of a put (resp. call) option struck at x, when the risk-neutral distribution is the one of a random variable Γ, such that the density of $\Gamma|\, X_2 = x_2$ is $\gamma(x_2, .)$ and the density of X_2 is f_2.*

Proof. See Appendix F. ■

5.3 A solution à la Carr et al. (2004)

Another approach to calibrate a jump-diffusion to an implied volatility surface is the one of [13], where the authors consider a stochastic differential equation, which is not driven by a Brownian motion but by a general Lévy process and explain how to recover the local speed function from option prices.

In our setting, their method would translate into fixing the diffusion coefficient σ, the jump-densities γ and finding a function λ, which would satisfy Equation (19). However, for their method to apply, we need to specify a special structure for γ, i.e.

$$\gamma(x, y) \equiv \frac{1}{x} g\left(\frac{y}{x}\right), \tag{40}$$

where g is a probability density on $\mathbb{R}_+^*$. Note that g is the density of $\frac{F_t}{F_{t^-}}$ $\Delta F_t = 0$ and that, in this case:

$$\Psi_\gamma(x, y) = x\Psi_g\left(\frac{y}{x}\right), \tag{41}$$

where Ψ_g is the double tail associated to g, i.e.

$$\Psi_g(x) \equiv \begin{cases} \int_x^{+\infty}\left(\int_u^{+\infty} g(v)\, dv\right) du & \text{if } x > 1 \\ \frac{1}{2}\left(\int_1^{+\infty}\left(\int_u^{+\infty} g(v)\, dv\right) du + \int_0^x \left(\int_0^u g(v)\, dv\right) du\right) & \text{if } x = 1 \\ \int_0^x \left(\int_0^u g(v)\, dv\right) du & \text{if } x < 1 \end{cases}. \tag{42}$$

Consequently, Equation (19) becomes:

$$C_2(y) - C_1(y) - \frac{1}{2} y^2 \sigma^2(y) \frac{\partial^2 C_2}{\partial y^2}(y) = \int_0^{+\infty} \frac{\partial^2 C_2}{\partial x^2}(x) \lambda(x) x \Psi_g\left(\frac{y}{x}\right) dx, \tag{43}$$

and changing variables to consider log-prices, we get:

$$\begin{aligned} & C_2(\exp(y')) - C_1(\exp(y')) - \frac{1}{2}(\exp(y'))^2 \sigma^2(\exp(y')) \frac{\partial^2 C_2}{\partial y^2}(\exp(y')) \\ = & \int_{-\infty}^{+\infty} \frac{\partial^2 C_2}{\partial x^2}(\exp(x')) \lambda(\exp(x')) (\exp(x'))^2 \Psi_g(\exp(y' - x'))\, dx', \end{aligned} \tag{44}$$

with $y = \exp(y')$ and $x = \exp(x')$. Therefore, the above equation becomes:

$$A(y') = \int_{-\infty}^{+\infty} B(x') C(y' - x')\, dx', \tag{45}$$

if A, B and C are defined by:

$$\begin{aligned} A\left(y'\right) &\equiv C_2\left(\exp\left(y'\right)\right) - C_1\left(\exp\left(y'\right)\right) - \frac{1}{2}\exp\left(2y'\right)\sigma^2\left(\exp\left(y'\right)\right)\frac{\partial^2 C_2}{\partial y^2}\left(\exp\left(y'\right)\right), && (46)\\ B\left(x'\right) &\equiv \frac{\partial^2 C_2}{\partial x^2}\left(\exp\left(x'\right)\right)\lambda\left(\exp\left(x'\right)\right)\left(\exp\left(x'\right)\right)^2, && (47)\\ C\left(z'\right) &\equiv \Psi_g\left(\exp\left(z'\right)\right). && (48) \end{aligned}$$

Since the RHS of Equation (45) is a convolution, we obtain, by taking the Fourier transform on both sides:

$$B\left(x'\right) = \mathcal{F}^{-1}\left(\frac{\mathcal{F}A}{\mathcal{F}C}\right)\left(x'\right), \tag{49}$$

where $\mathcal{F}f$ (resp. $\mathcal{F}^{-1}f$) denotes the (resp. inverse) Fourier transform of f. Finally, we remind you that:

$$\lambda\left(x\right) = \frac{B\left(\ln\left(x\right)\right)}{x^2\frac{\partial^2 C_2}{\partial x^2}\left(x\right)}. \tag{50}$$

using Equation (47). The above considerations lead to the following lemma.

Lemma 12 *With the notations of Proposition 8, we assume* (A_1) *and that:*

- f_2 *satisfies the following growth condition:*

$$\int_1^{+\infty} x\left(\ln\left(x\right)\right)^{\alpha_2} f_2\left(x\right) dx < +\infty \text{ with } \alpha_2 > 1, \tag{51}$$

- g *is a continuous probability density on* $\mathbb{R}_+^*$ *such that:*

$$\begin{aligned} \int_1^{+\infty} x\left(\ln\left(x\right)\right)^{\alpha_g} g\left(x\right) dx &< +\infty \text{ with } \alpha_g > 1, && (52)\\ \int_0^{1} x\left|\ln\left(x\right)\right| g\left(x\right) dx &< +\infty, && (53) \end{aligned}$$

- σ *is a positive, continuous and bounded function on* $\mathbb{R}_+^*$*, such that:*

$$\sigma^2\left(x\right) \quad \frac{C_2\left(x\right) - C_1\left(x\right)}{\frac{1}{2}x^2 f_2\left(x\right)}. \tag{54}$$

In that case, we define A *(resp.* C*) via Equation (46) (resp. (48)) and denote by* $\mathcal{F}A$ *(resp.* $\mathcal{F}C$*) its Fourier transform, which is well-defined.*

We further assume that:

- $\mathcal{F}A/\mathcal{F}C \in L^1\left(\mathbb{R}\right)$,
- $B \equiv \mathcal{F}^{-1}\left(\mathcal{F}A/\mathcal{F}C\right) \in L^1\left(\mathbb{R}\right)$,
- *the following equation:*

$$\lambda\left(x\right) \equiv \frac{B\left(\ln\left(x\right)\right)}{x^2\frac{\partial^2 C_2}{\partial x^2}\left(x\right)} \tag{55}$$

defines a non-negative, bounded and continuous function.

Then, σ, λ and $\gamma(x,.) \equiv \frac{1}{x} g\left(\frac{\cdot}{x}\right)$ satisfies the assumptions (A_2) of Proposition 8.

Proof. See Appendix G. ■

Remark 13 *It will probably be difficult to check in practice if the last three sufficient conditions of Lemma 12 are satisfied or not.*

5.4 A solution à la Carr and Cousot (2007a)

Another recipe to calibrate a jump-diffusion to an implied volatility surface is the one described in [12], where the authors explain how using jump-densities of the generalized Laplace type greatly simplifies both pricing and calibration. Indeed, in this case, the forward and backward Kolmogorov equations can be transformed into partial differential equations, what makes the calibration and the pricing of the same order of complexity as in a diffusion setting.

As expected, it is possible to translate this technique to the discrete-time framework. Indeed, if γ are generalized Laplace densities associated to a given function a^2, then, according to Lemma 21 in Appendix A:

$$\Psi_\gamma(x,y) = \frac{a^2(x)}{2}\gamma(y,x) = \frac{a^2(y)}{2}\gamma(x,y), \tag{56}$$

and Equation (19) becomes:

$$\int_0^{+\infty} \frac{\partial^2 C_2}{\partial x^2}(x)\,\lambda(x)\,\frac{a^2(x)}{2}\gamma(y,x)\,dx = C_2(y) - C_1(y) - \frac{1}{2}y^2\sigma^2(y)\,\frac{\partial^2 C_2}{\partial y^2}(y). \tag{57}$$

Consequently, applying the operator $\left[\mathcal{I} - \frac{a^2(y)}{2}\frac{\partial^2}{\partial y^2}\right]$ yields:

$$\frac{\partial^2 C_2}{\partial y^2}(y)\,\lambda(y)\,\frac{a^2(y)}{2} = \mathcal{I} - \frac{a^2(y)}{2}\frac{\partial^2}{\partial y^2}\left(C_2(y) - C_1(y) - \frac{y^2\sigma^2(y)}{2}\frac{\partial^2 C_2}{\partial y^2}(y)\right), \tag{58}$$

since $\left[\mathcal{I} - \frac{a^2(y)}{2}\frac{\partial^2}{\partial y^2}\right]\gamma(y,x) = \delta(y-x)$. This equation can be rewritten as:

$$\frac{a^2(y)}{2} = \frac{C_2(y) - C_1(y) - \frac{1}{2}y^2\sigma^2(y)\frac{\partial^2 C_2}{\partial y^2}(y)}{\frac{\partial^2}{\partial y^2}\left(C_2(y) - C_1(y) - \frac{1}{2}y^2\sigma^2(y)\frac{\partial^2 C_2}{\partial y^2}(y)\right) + \frac{\partial^2 C_2}{\partial y^2}(y)\,\lambda(y)}, \tag{59}$$

or

$$\lambda(y) = \frac{\left[\mathcal{I} - \frac{a^2(y)}{2}\frac{\partial^2}{\partial y^2}\right]\left(C_2(y) - C_1(y) - \frac{1}{2}y^2\sigma^2(y)\frac{\partial^2 C_2}{\partial y^2}(y)\right)}{\frac{a^2(y)}{2}\frac{\partial^2 C_2}{\partial y^2}(y)}. \tag{60}$$

The above equations suggest two strategies:

- the first one (henceforth Strategy A) would be to choose first σ and λ, and take a^2 according to Equation (59);
- the second one (henceforth Strategy Λ) would be to take λ via Equation (60) once a^2 and σ have been chosen.

But, before attempting to find sufficient conditions for these strategies in the following sections, we need to ensure that the generalized Laplace distributions γ satisfy the assumptions of Proposition 8. This is the aim of the following lemma.

Lemma 14 *If a^2 is a continuous function on $\mathbb{R}_+^*$, which satisfies:*

$$m^2 \quad \frac{a^2(x)}{x^2} \quad M^2, \tag{61}$$

for given $m, M \in \mathbb{R}_+^*$, then there exists a unique family of generalized Laplace distributions γ associated to the function a^2. Moreover, the assumptions (A_2) of Proposition 8, which only concern γ, are satisfied.

Proof. This lemma corresponds to the particular case of Lemma 2.5 in [16], where a^2 is time-homogeneous. ■

5.4.1 Strategy A

In this section, we focus on Strategy A, which is the discrete-time equivalent of the strategy by the same name developed in [12].

Lemma 15 *With the notation of Proposition 8, we assume (A_1) and that:*

- *μ_1 admits a continuous density f_1,*
- *σ is a positive, bounded and continuous function on $\mathbb{R}_+^*$, such that:*

$$\sigma^2(x) < \frac{C_2(x) - C_1(x)}{\frac{1}{2}x^2 \frac{\partial^2 C_2}{\partial x^2}(x)} \tag{62}$$

- *λ is a non-negative, bounded and continuous function on $\mathbb{R}_+^*$,*
- *the following equation defines a continuous and positive function[4] a^2:*

$$\frac{a^2(x)}{2} = \frac{C_2(x) - C_1(x) - \frac{1}{2}x^2\sigma^2(x)\frac{\partial^2 C_2}{\partial x^2}(x)}{\frac{\partial^2}{\partial x^2}\left(C_2(x) - C_1(x) - \frac{1}{2}x^2\sigma^2(x)\frac{\partial^2 C_2}{\partial x^2}(x)\right) + \lambda(x)\frac{\partial^2 C_2}{\partial x^2}(x)}, \tag{63}$$

  which satisfies the growth conditions of Equation (61).

Then, γ, the unique family of generalized Laplace distributions associated to a^2, σ and λ satisfy the assumptions (A_2) of Proposition 8.

Proof. See Appendix H. ■

Note that a possible specification of σ, which ensures that Equation (62) is satisfied, is:

$$\sigma^2(x) = \alpha^2 \frac{C_2(x) - C_1(x)}{\frac{1}{2}x^2 f_2(x)} \tag{64}$$

with $0 < \alpha^2 < 1$. Since we recognize on the RHS, the expression for σ^2, which allows one to fit option prices using a sampled diffusion (see Section 4), the parameter α^2 allows one to control how much of the smile should be explained by the diffusion component of (F_t).

[4] In particular we assume that $x \to \frac{1}{2}x^2\sigma^2(x)\frac{\partial^2 C_2}{\partial x^2}(x)$ is twice continuously differentiable and that the denominator of the fraction on the RHS of Equation (63) is positive.

Moreover, in this case, Equation (63) greatly simplifies:

$$\frac{a^2(x)}{2} = \frac{(C_2(x) - C_1(x))}{(f_2(x) - f_1(x)) + \beta^2 \lambda(x) f_2(x)} \tag{65}$$

with $\beta^2 = 1/\left(1-\alpha^2\right)$. The next lemma, which uses concepts belonging to Karamata's theory (see Appendix C), describes sufficient conditions for these specifications to be valid.

Lemma 16 *With the notations of Proposition 8, we assume* (A_1) *and that:*

- μ_1 *admits a continuous density* f_1,
- f_2 *has bounded decrease and upper Matuszewska index* $\alpha(f_2) < -2$,
- $\widetilde{f}_2(x) \equiv f_2(1/x)$ *has bounded decrease and upper Matuszewska index* $\alpha\left(\widetilde{f}_2\right) < 1$,
- *There exists* $M > 1$, *such that* $f_2(x) > M f_1(x)$ *in a neighborhood of* 0 *and* $+\infty$.

Then there exists a positive function λ, *such that* σ, *defined by Equation (64),* a^2, *defined by Equation (65) and* λ *satisfy the assumptions of Lemma 15.*

Proof. See Appendix I. ■

5.4.2 Strategy Λ

In this section, we describe sufficient conditions for the feasibility of Strategy Λ, which is the discrete-time counterpart of the strategy by the same name introduced in [12].

Lemma 17 *With the notations of Proposition 8, we assume* (A_1) *and that:*

- μ_1 *admits a continuous density* f_1,
- σ *is a positive, bounded and continuous function such that, for* $x \in \mathbb{R}_+^*$:

$$\sigma^2(x) \quad \frac{C_2(x) - C_1(x)}{\frac{1}{2}x^2 \frac{\partial^2 C_2}{\partial x^2}(x)}, \tag{66}$$

- a^2 *is a continuous function, which satisfies:*

$$m^2 \quad \frac{a^2(x)}{x^2} \quad M^2 \tag{67}$$

  for $x \in \mathbb{R}_+^*$, *with* $m, M > 0$,

- *the following equation*

$$\lambda(x) = \frac{\left[\mathcal{I} - \frac{a^2(x)}{2}\frac{\partial^2}{\partial x^2}\right]\left(C_2(x) - C_1(x) - \frac{1}{2}x^2\sigma^2(x)\frac{\partial^2 C_2}{\partial x^2}(x)\right)}{\frac{a^2(x)}{2}\frac{\partial^2 C_2}{\partial x^2}(x)} \tag{68}$$

  defines a continuous, non-negative and bounded function[5].

[5]In particular we assume that $x \to \frac{1}{2}x^2\sigma^2(x)\frac{\partial^2 C_2}{\partial x^2}(x)$ is twice continuously differentiable.

Then, γ, the unique generalized Laplace distributions associated to a^2, σ and λ satisfy the assumptions (A_2) of Proposition 8.

Proof. This proof is in every point similar to the one of Lemma 15 (see Appendix H) and is left to the reader. ■

Note that a specification of σ and a^2, which greatly simplifies Equation (68), is:

$$\sigma^2(x) = \alpha^2 \frac{C_2(x) - C_1(x)}{\frac{1}{2}x^2 f_2(x)}, \tag{69}$$

$$\frac{a^2(x)}{2} = \beta^2 \frac{C_2(x) - C_1(x)}{f_2(x)}, \tag{70}$$

with $\alpha^2, \beta^2 \in (0,1)$. Indeed, in this case, Equation (68) reduces to:

$$\lambda(x) = \left(1-\alpha^2\right)\left(\frac{1}{\beta^2} - 1\right) + \frac{f_1(x)}{f_2(x)} \quad . \tag{71}$$

Moreover the parameters α^2 and β^2 have a clear interpretation: α^2 allows one to explain how much of the smile should be explained by the diffusion component of (F_t) and β^2 allows one to control the size of its jumps.

The next lemma describes sufficient conditions for this simplified strategy to work.

Lemma 18 *With the notations of Proposition 8, we assume (A_1) and that:*

- *μ_1 admits a continuous density f_1,*
- *f_2 has bounded decrease and upper Matuszewska index $\alpha(f_2) < -2$,*
- *$\widetilde{f_2}(x) \equiv f_2(1/x)$ has bounded decrease and upper Matuszewska index $\alpha\left(\widetilde{f_2}\right) < 1$,*
- *There exists $M > 1$, such that $f_2(x) > M f_1(x)$ in a neighborhood of 0 and $+\infty$.*

Then Equations (69), (70) and (71) define respectively functions σ, a^2 and λ, which satisfy the assumptions of Lemma 17.

Proof. See Appendix J. ■

6 Construction of continuous-time consistent martingales

Since the transition distributions of the process (X_i) are those of continuous-time processes sampled at random times, one could naturally imagine that it is possible to construct a non-trivial continuous-time process, with the same transition distributions, using a proper time change. The purpose of this section is to make this intuition more precise.

We assume that:

- $\mu_0 \equiv \delta(x - x_0)$ with $x_0 \in \mathbb{R}_+^*$,
- μ_1 (resp. μ_2) admits a positive and continuous density f_1 (resp. f_2) and has a first moment equal to x_0,

- For $x > 0$,

$$(x_0 - x)^+ < \int_0^{+\infty} (y - x)^+ f_1(y)\, dy < \int_0^{+\infty} (y - x)^+ f_2(y)\, dy. \tag{72}$$

Using Sections 4 or 5, we know sufficient conditions on f_1 and f_2, for the existence of positive strong Markov martingales $\left(\left(F_t^i\right)\right)_{i\in\{1,2\}}$, such that the process (X_i) defined by:

$$X_0 \equiv x_0, \tag{73}$$

$$X_1 | X_0 \equiv \left(F^1_{\tau_1} \mid F^1_0 = X_0\right), \tag{74}$$

$$X_2 | X_1 \equiv \left(F^2_{\tau_2} \mid F^2_0 = X_1\right), \tag{75}$$

where (τ_1) and (τ_2) are two exponentially distributed random variables of mean 1, independent respectively of $\left(F_t^1\right)$ and $\left(F_t^2\right)$, is a martingale, which is calibrated to the distributions $(\mu_i)_{0\leq i\leq 2}$.

Consequently, if we want to construct a continuous-time martingale (X_t^c), which starts at x_0 in $T_0 \equiv 0$ and has marginal density f_1 (resp. f_2) at time $T_1 > 0$ (resp. $T_2 > T_1$), then a possibility is:

$$\begin{aligned} &X^c_{T_0} \equiv x_0, \\ &X^c_t \equiv \left(F^1_{\mathcal{T}^1_{t-T_0}} \mid F^1_0 = X^c_0\right) \quad \text{for } t \in (T_0, T_1], \\ &X^c_t \equiv \left(F^2_{\mathcal{T}^2_{t-T_1}} \mid F^2_0 = X^c_{T_1}\right) \quad \text{for } t > T_1, \end{aligned} \tag{76}$$

where $\left(\mathcal{T}_t^i\right)$ is an adapted and increasing process, independent of $\left(F_t^i\right)$, starting at 0, and, whose marginal distribution at time $T_i - T_{i-1}$ is an exponential distribution of parameter 1, for $i \in \{1, 2\}$.

Note that the process (X_t^c) is indeed a martingale, because the processes $\left(\left(F_t^i\right)\right)_{i\in\{1,2\}}$ are martingales and the time changes $\left(\left(\mathcal{T}_t^i\right)\right)_{i\in\{1,2\}}$ are independent. Furthermore, by construction,

$$X^c_{T_i} \mid X^c_{T_{i-1}} \overset{law}{=} X_i | X_{i-1}, \tag{77}$$

for $i \in \{1, 2\}$. Consequently, the fact that (X_i) is calibrated to the marginal distributions (μ_i) implies that (X_t^c) is too.

A simple candidate for $\left(\mathcal{T}_t^i\right)$ is a Gamma process $(\Gamma_t^{\mu_i,\nu_i})$, independent of all the other processes. We remind you that $(\Gamma_t^{\mu_i,\nu_i})$ is an increasing pure-jump Lévy process, whose Lévy measure is defined by:

$$k_{\mu_i,\nu_i}(x)\, dx = \frac{\mu_i^2 \exp\left(-\frac{\mu_i}{\nu_i} x\right)}{\nu_i x} 1_{\{x>0\}} dx, \tag{78}$$

where μ_i (resp. ν_i) is the mean (resp. variance) rate. Its marginal density at time t is a Gamma distribution of mean $\mu_i t$ and variance $\gamma_i t$. Since an exponential distribution is a special case of a Gamma distribution, we just need to chose the parameters according to our needs, that is:

$$\mu_i = \nu_i = \frac{1}{T_i - T_{i-1}}. \tag{79}$$

A second simple candidate for $\left(\mathcal{T}_t^i\right)$ is a linear interpolation of the random time τ_i:

$$\mathcal{T}_t^i \equiv \frac{t}{T_i - T_{i-1}} \tau_i. \tag{80}$$

For more elaborated constructions of increasing processes - or time changes - with given marginal distributions, we refer to [16].

Finally, notice that the process (X_t^c) is a priori not Markov but the process (X_i) is. As a consequence, the valuation of contingent claims depending only on $\left(X_{T_i}^c\right)$ is much simpler, as explained in the following section.

Remark 19 *In the case where there is only one maturity, this section, along with Section 4, explains how to time-change a driftless diffusion to match a given marginal distribution at a given time. Consequently, using the classical Dambis-Dunbins-Schwarz theorem, we know how to sample a Brownian motion to match a given marginal distribution - what is exactly the purpose of the Skorokhod embedding problem (see [38]). We refer to [11] for a more detailed description of this solution. See also the recent working paper [18] on the same subject.*

7 Valuation of path-independent claims

In this section, we focus on the valuation of path-independent claims on (X_i), using ordinary (integro-) differential equations (O(I)DEs). Actually, for the sake of simplicity, we only describe the valuation of European style claims, but a similar treatment is possible for American style ones.

With the notations of the previous section, we denote by $\mathcal{G}_i$ the generator of the process $\left(F_t^i\right)$, for $i \in \{1,2\}$:

$$\mathcal{G}_i f(x) \equiv \frac{1}{2} x^2 \sigma_i^2(x) f''(x) + \lambda_i(x) \int_0^{+\infty} \left(f(y) - f(x) - f'(x)(y-x)\right) \gamma_i(x,y)\, dy. \tag{81}$$

Then, if V_i^f denotes the prices, at time T_i, of a claim expiring at time T_2 and, whose payoff is a given function f, we have:

$$V_i^f(x) \equiv \mathbb{E}\left[f(X_2) | X_i = x\right], \tag{82}$$

and the purpose of this section is to show that V_0^f can be computed by solving a finite number of O(I)DEs.

First, V_2^f clearly satisfies: $V_2^f(x) = f(x)$. Second, by conditioning, we have:

$$V_{i-1}^f(x) = \mathbb{E}\left[\mathbb{E}\left[f(X_2)| X_i\right] | X_{i-1} = x\right] = \mathbb{E}\left[V_i^f(X_i) \middle| X_{i-1} = x\right]. \tag{83}$$

Therefore, applying the backward operator $[\mathcal{I} - \mathcal{G}_i]$ on the above equation, we obtain, heuristically, that:

$$[\mathcal{I} - \mathcal{G}_i] V_{i-1}^f(x) = V_i^f(x), \tag{84}$$

using Equation (12).

Note that Equation (84) is a linear ODE in V_{i-1}^f, if the transition distributions are of the general Laplace type:

$$V_{i-1}^f(x) - \frac{1}{2} x^2 \sigma_i^2(x) \frac{\partial^2 V_{i-1}^f}{\partial x^2}(x) = V_i^f(x), \tag{85}$$

but an OIDE if the transition distributions are general SJD distributions:

$$\begin{aligned} V_i^f(x) &= V_{i-1}^f(x) - \frac{1}{2} x^2 \sigma_i^2(x) \frac{\partial^2 V_{i-1}^f}{\partial x^2}(x) \\ &\quad - \lambda_i(x) \int_0^{+\infty} \left(V_{i-1}^f(y) - V_{i-1}^f(x) - \frac{\partial V_{i-1}^f}{\partial x}(x)(y-x)\right) \gamma_i(x,y)\, dy. \end{aligned} \tag{86}$$

However, if the jump-densities γ_i are of the generalized Laplace type and are associated to a function a_i^2, as in Section 5.4, then this OIDE can be transformed into an ODE. Indeed, since the density $\gamma_i(x,.)$ integrates to 1 and has a mean equal to x, Equation (86) can be transformed into:

$$\int_0^{+\infty} V_{i-1}^f(y)\gamma_i(x,y)\,dy = \frac{V_{i-1}^f(x) - V_i^f(x) - \frac{1}{2}x^2\sigma_i^2(x)\frac{\partial^2 V_{i-1}^f}{\partial x^2}(x) + \lambda_i(x)V_{i-1}^f(x)}{\lambda_i(x)}, \tag{87}$$

if $\lambda_i > 0$. And applying the operator $\left[\mathcal{I} - \frac{a_i^2(x)}{2}\frac{\partial^2}{\partial x^2}\right]$ on the above equation yields:

$$\frac{a_i^2(x)}{2}\frac{\partial^2 V_{i-1}^f}{\partial x^2}(x) = \left[\mathcal{I} - \frac{a_i^2(x)}{2}\frac{\partial^2}{\partial x^2}\right]\left(\frac{V_{i-1}^f(x) - V_i^f(x) - \frac{1}{2}x^2\sigma_i^2(x)\frac{\partial^2 V_{i-1}^f}{\partial x^2}(x)}{\lambda_i(x)}\right), \tag{88}$$

after a simple simplification, since $\left[\mathcal{I} - \frac{a_i^2(x)}{2}\frac{\partial^2}{\partial x^2}\right]\gamma_i(x,y) = \delta(x-y)$.

Consequently, the valuation of European path-independent claims can be performed, at least heuristically, by solving, backward in time, a finite number of inhomogeneous ODEs (resp. OIDEs) if the transition distributions are of the generalized Laplace (resp. SJD) type. Moreover, to use generalized Laplace distributions as jump densities of general SJD distributions not only simplifies the calibration procedure, as we saw in Section 5.4, but also the valuation of claims, as the general valuation OIDE of Equation (86) was transformed, in this special case, into the simpler ODE of Equation (88).

8 A numerical example: SPX quotes

In this section, we give an example of calibration to S&P 500 option quotes, using the model developed in Section 4, where the transition distributions are of the generalized Laplace type.

The market data are SPX quotes as of the close on September 15, 2005. To construct marginal distributions, which are (almost perfectly) consistent with these quotes, we fit the different volatility smiles using the Stochastic Volatility Inspired (SVI) parametrization described in [25], p. 37. The SVI fits to the SPX quotes[6] are displayed in Figure 1. The corresponding marginal distributions (in the forward measure) are displayed in Figure 2.

Remark 20 *In this paper, we do not tackle the (difficult) problem of constructing, at each maturity, a smooth volatility smile, which is consistent with given market quotes. However we ensure that the constructed model will be calibrated, exactly, to the input marginal distributions.*

A quick check shows that Proposition 2 applies in this example. Consequently, valid transition distributions between the maturities T_{i-1} and T_i are generalized Laplace distributions associated to the function σ_i^2 defined by:

$$\sigma_i^2(x) = \frac{C_i(x) - C_{i-1}(x)}{\frac{1}{2}x^2\frac{\partial^2 C_i}{\partial x^2}(x)}. \tag{89}$$

Notice that we did not really pay attention to the time dimension so far. Indeed, to ease notations, we decided to sample (jump-) diffusions at exponential random times of mean 1, but this mean is arbitrary and can be set to $T_i - T_{i-1}$. In this case, the only change is that the associated

[6]We would like to thank Jim Gatheral for communicating the SPX quotes and the corresponding SVI fits he used in his book (see [25]).

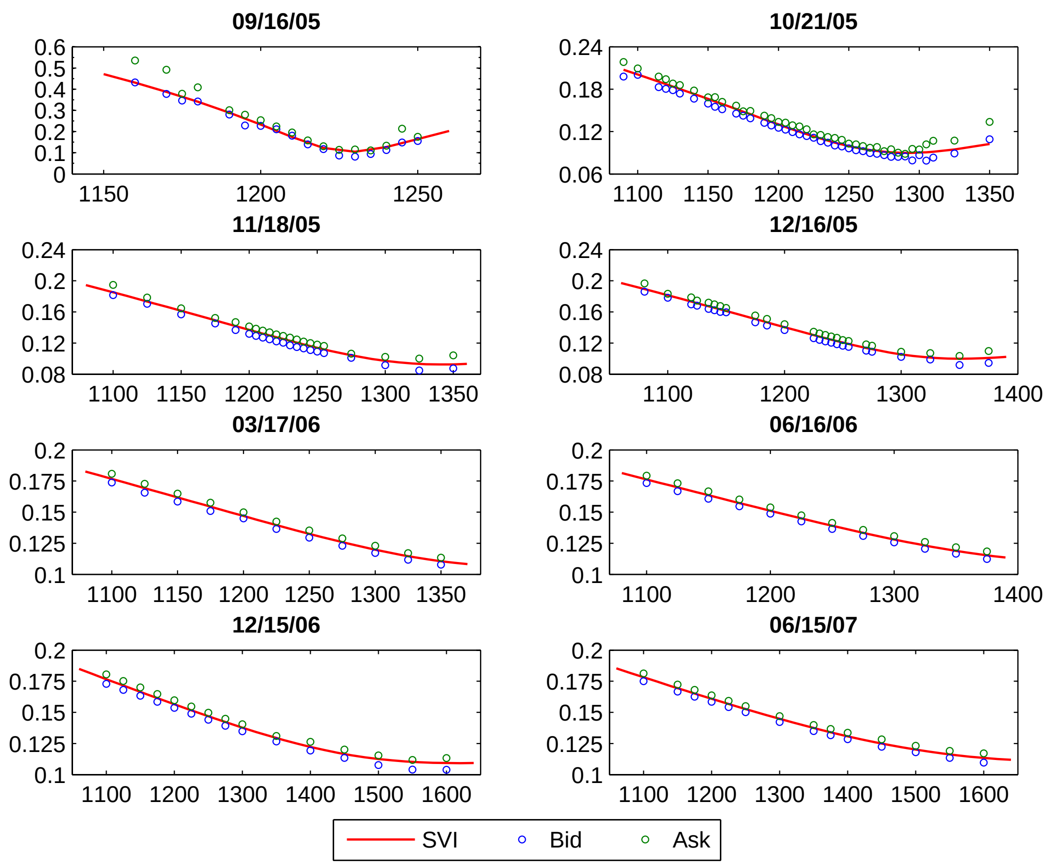

Figure 1: SVI fits to SPX quotes as of the close on September 15, 2005.

generator $\mathcal{G}_i$ (see Equation (3)) should be replaced by $\overline{\mathcal{G}_i} \equiv \mathcal{G}_i / (T_i - T_{i-1})$. These considerations lead us to consider the following rescaled quantities:

$$\overline{\sigma_i}(x) \equiv \frac{\sigma_i(x)}{\sqrt{T_i - T_{i-1}}}. \tag{90}$$

Since we have analytical expressions for the call price and for the density at every maturities, computing the above functions $\overline{\sigma}_i$ is straightforward. These functions are displayed in Figure 3 between the 10^{-5} quantiles of the T_i density. Note that this choice is only esthetic and that these functions can be computed until much smaller quantiles without facing any numerical instability.

A first reassuring remark is that the functions $\overline{\sigma}_i$ are of the same order of magnitude as the SPX implied volatilities and look like admissible local volatility functions. To have an idea of the behavior of the model, we draw the at-the-money forward transition distributions[7] in Figure 4. These transitions densities have been obtained by using the definition of generalized Laplace distributions as the Green's functions of a differential operator and the algorithm described in [40] to numerically solve linear second-order boundary value problem on infinite intervals.

[7] Note that these transition densities do not need to have a particular shape since generalized Laplace densities are structurally able to match a very large class of densities – see Section 4. In particular, the shape of the transition distribution from T_1 to T_2 displayed in Figure 4 is not surprising since the function $\overline{\sigma_2}$ decreases a lot at the right of the forward.

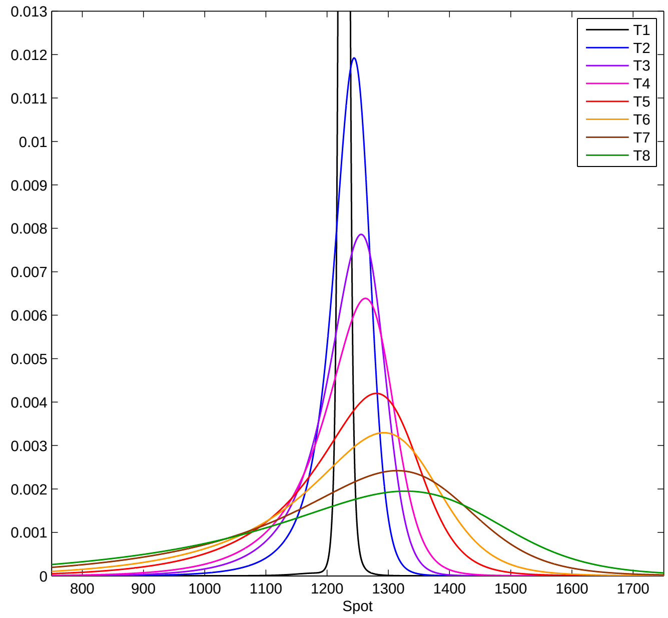

Figure 2: Marginal densities.

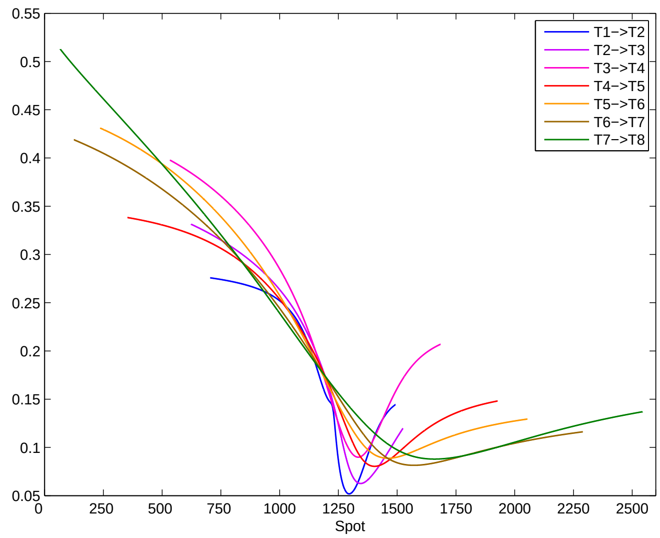

Figure 3: The functions $\overline{\sigma_i}$.

The corresponding forward smiles are displayed in Figure 5. Note that the model is quite stable in time since the at-the-money forward smiles are quite similar to today's smiles.

Finally, we draw some paths of the continuous-time process, with the same transition distributions, obtained by linear interpolation of independent sampling times (see Section 6). The paths are displayed in Figure 6 and the corresponding time changes in Figure 7.

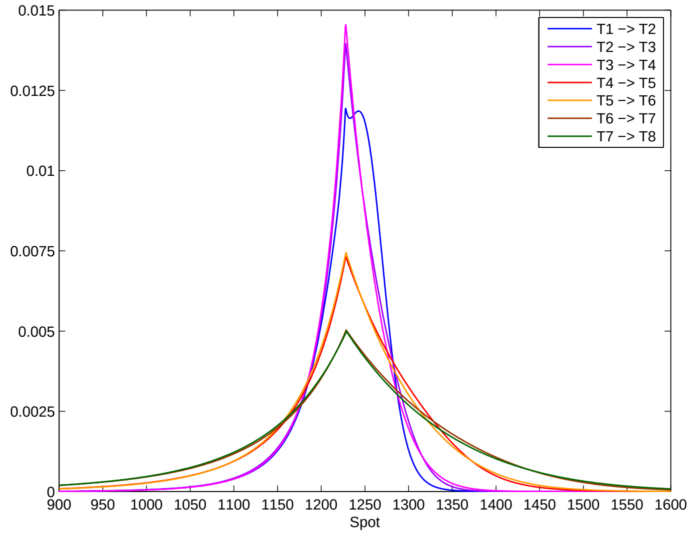

Figure 4: At-the-money forward transition densities.

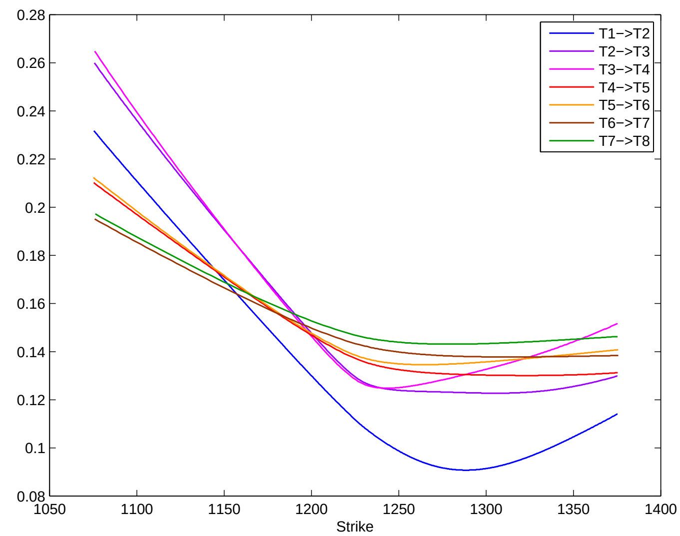

Figure 5: At-the-money forward smiles.

9 Conclusion

In this paper, we described several constructions of discrete-time martingales consistent with a finite set of marginal distributions. Moreover, we explained how to extend these constructions to continuous-time martingales (with a solution to the Skorokhod embedding problem as a by-product – see Remark 19).

The advantages of these constructions over those of [20], [22], [4] and [12] are three-fold. First, once the marginal distributions at traded maturities are known, there is no need to interpolate them in time to get the whole implied volatility surface. One could actually see our constructions as a way of interpolating in an arbitrage-free and robust way given implied volatility smiles. Second,

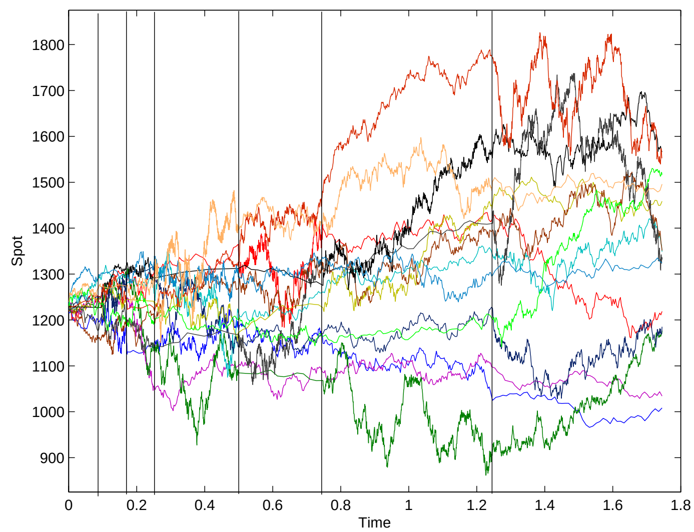

Figure 6: Different paths of the continuous-time process.

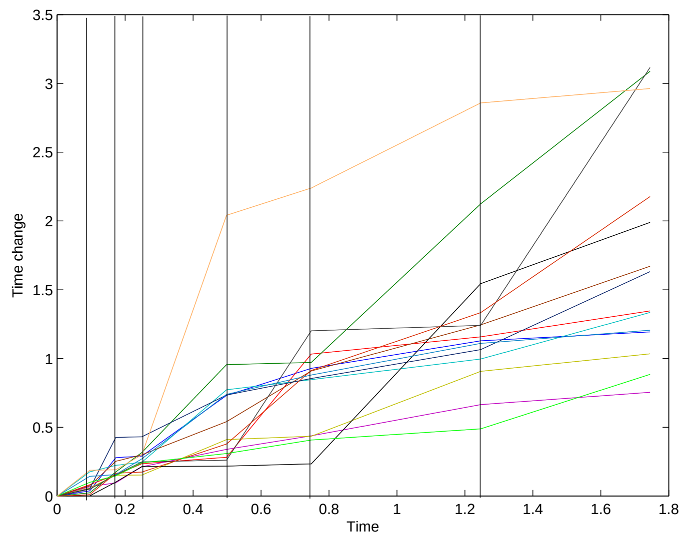

Figure 7: Time changes corresponding to the paths of Figure 6.

the constructed models are local stochastic volatility models with jumps and, to the best of our knowledge, are the first models of this kind to be structurally able to fit given implied volatility smiles. Third, as explained in Section 6, there is some room in the specification of the time changes when constructing the continuous-time martingales. This freedom should be studied further but could potentially be exploited to calibrate volatility products, when available. For instance, a lead that we did not pursue here is to introduce some correlation between the time change and the underlying process by making the mean of the exponentially distributed time change depend on the initial value of the corresponding (jump-) diffusion process.

Finally, we focused in this paper on sampling (jump-) diffusions but the same recipe can be applied to other processes. For instance, we could have considered as a starting point a driftless diffusion with a Brownian time change[8]. Indeed, this kind of Markov process has a fourth order differential generator and a constant mean. Consequently sampling it at an independent exponentially distributed random time would have allowed us to construct centered transition densities, which would have been the Green's function of fourth order differential operators. Another application is the construction of increasing processes calibrated to (a finite number of) marginal distributions described in [16] – where the techniques described in this paper and in [12] are also applied to the construction of increasing processes.

[8]See [10], [3], [2] and the references therein for more details about applying a Brownian time change to a given Markov process.

Appendix A Reminders about generalized Laplace distributions

The Laplace distribution, whose density is defined, for $\lambda > 0$, by:

$$p_\lambda(x) \equiv \frac{1}{2\lambda} \exp\left(-\frac{|x|}{\lambda}\right), \tag{91}$$

can be interpreted as the distribution of $\left(\sqrt{2}\lambda W_\tau\right)$, where (W_t) is a Brownian motion and τ is an independent random time with an exponential distribution of mean 1. To define *generalized Laplace distributions*, we sample a general driftless diffusion (S_t), satisfying:

$$dS_t = a(S_t)\, dW_t, \tag{92}$$

instead of the multiple of a Brownian motion.

The lemma below, along with its proof, can be found in [12] and in [16] and are recalled here for the sake of completeness.

Lemma 21 *If a^2 is a continuous and positive function on $\mathbb{R}_+^*$, which satisfies:*

$$J_0 \equiv \int_0^1 \left(\int_x^1 \frac{2}{a^2(y)} dy\right) dx = +\infty, \tag{93}$$

then, for every $y \in \mathbb{R}_+^*$, there exists a unique function $x \to G(x, y)$, which satisfies:

$$\mathcal{I} - \frac{a^2(x)}{2}\frac{\partial^2}{\partial x^2} \quad G(x, y) = \frac{a^2(y)}{2}\delta(x - y), \tag{94}$$

as well as the following boundary conditions:

$$\frac{\partial G}{\partial x}(x, y) \underset{x \to 0 \text{ or } +\infty}{\to} 0. \tag{95}$$

Moreover, $x \to G(x, y)$ is positive and continuous on $\mathbb{R}_+^*$ as well as convex, twice differentiable and increasing (resp. decreasing) on $(0, y]$ (resp. $[y, +\infty)$). It also possesses the following symmetry:

$$G(x, y) = G(y, x). \tag{96}$$

Furthermore, if we define $g(x, y) \equiv \frac{2G(x,y)}{a^2(y)}$, then $y \to g(x, y)$ is a probability density and g satisfies:

$$\mathcal{I} - \frac{\partial^2}{\partial y^2}\frac{a^2(y)}{2} \quad g(x, y) = \delta(y - x), \tag{97}$$

as well as

$$\mathcal{I} - \frac{a^2(x)}{2}\frac{\partial^2}{\partial x^2} \quad g(x, y) = \delta(x - y). \tag{98}$$

Finally, if we further assume that:

$$K_0 \equiv \int_0^1 \frac{x\, dx}{a^2(x)} = +\infty, \tag{99}$$

$$K_{+\infty} \equiv \int_1^{+\infty} \frac{x\, dx}{a^2(x)} = +\infty, \tag{100}$$

then:

$$G(x,y) \underset{x\to 0 \ or \ +\infty}{\to} 0, \tag{101}$$

$$x\frac{\partial G}{\partial x}(x,y) \underset{x\to 0 \ or \ +\infty}{\to} 0, \tag{102}$$

and $y \to g(x,y)$ is a probability density of mean x.

Proof.

First, since a^2 is positive and continuous, we clearly have:

$$J_{+\infty} \equiv \int_1^{+\infty} \left(\int_1^x \frac{dy}{a^2(y)} \right) dx = +\infty. \tag{103}$$

Therefore, the results concerning G are mainly consequences of results, which can be found in [24] and [37]. Indeed, in this case, two solutions u_1 and u_2 of:

$$\mathcal{I} - \frac{a^2(x)}{2}\frac{\partial^2}{\partial x^2} \quad f(x) = 0, \tag{104}$$

exist and satisfy:

$$u_1'(x) \underset{x\to 0}{\to} 0, \tag{105}$$

$$u_2'(x) \underset{x\to +\infty}{\to} 0. \tag{106}$$

Moreover, u_1 (resp. u_2) is positive, convex and increasing (resp. decreasing). Besides, u_1 and u_2 are independent solutions and their Wronskian – denoted by $W[u_i]$ – is a positive constant. Consequently, it allows us to define:

$$G(x,y) \equiv \frac{u_1(\min(x,y))\, u_2(\max(x,y))}{W[u_i]}, \tag{107}$$

and all the properties of G follow easily from this definition.

Now let us focus on the properties of g. Equation (97) is a direct consequence of the definition of g and of Equation (94). Equation (98) is proved by using the definition of g – Equation (94) – and the symmetry property of G – Equation (96). To prove that g integrates to 1, we just need to integrate Equation (97) over $\mathbb{R}_+^*$ and use the boundary conditions of Equation (95).

If we further assume Conditions (99) and (100), then:

$$u_1(x) \underset{x\to 0}{\to} 0, \tag{108}$$

$$u_2(x) \underset{x\to +\infty}{\to} 0, \tag{109}$$

and Equation (101) follows.

To prove Equation (102), we obtain an estimate of u_2' in terms of u_2, using Theorem 1 in [28]. Indeed, u_2 is positive, decreasing and satisfies Equation (104), therefore:

$$xu_2'(x) \quad 8u_2\left(\frac{x}{2}\right), \tag{110}$$

for x big enough and the conclusion follows, since the equivalent result for u_1 is a consequence of Equation (105).

Finally, to prove that the mean of $y \to g(x,y)$ is x, we multiply Equation (97) by y, integrate over $\mathbb{R}_+^*$ to obtain:

$$\int_{\mathbb{R}_+^*} y g(x,y)\, dy - \int_{\mathbb{R}_+^*} y \frac{\partial^2 G}{\partial y^2}(x,y)\, dy = x,$$

and we use the boundary conditions of Equations (101) and (102) to show that the second integral is zero by using two integrations by parts. ■

Appendix B Proof of Proposition 2

The existence and the properties of the generalized Laplace densities $(p(.\,|x_1))$ are a simple consequence of the fact that $a^2(x) \equiv x^2\sigma^2(x)$ satisfies the assumptions of Lemma 21 in Appendix A. The only exception is Equation (26). Note that Equation (22) can be transformed into:

$$C_1(K) = \left(\mathcal{I} - \frac{a^2(K)}{2}\frac{\partial^2}{\partial K^2}\right) C_2(K)$$

because $f_2(K) = \frac{\partial^2 C_2}{\partial K^2}(K)$. Differentiating this equation twice, we obtain:

$$\frac{\partial^2 C_1}{\partial K^2}(K) = \left(\mathcal{I} - \frac{\partial^2}{\partial K^2}\frac{a^2(K)}{2}\right) f_2(K).$$

Finally, we have:

$$f_2(K) = \int_0^{+\infty} p(K\,|x_1)\, \frac{\partial^2 C_1}{\partial x_1^2}(x_1)\, dx_1$$

since $\frac{\partial}{\partial K}\left(\frac{a^2(K)}{2} f_2(K)\right) \to_{0 \text{ or } +\infty} 0$ and using the uniqueness of the associated boundary value problem.

Remark 22 *In Proposition 2, the assumption 'σ^2 is bounded' could have been replaced by the less constraining conditions:*

$$\int_0^1 \left(\int_x^1 \frac{f_2(y)}{C_2(y) - C_1(y)}\, dy\right) dx = +\infty, \quad (111)$$

$$\int_0^1 \frac{x f_2(x)}{C_2(x) - C_1(x)}\, dx = +\infty, \quad (112)$$

$$\int_1^{+\infty} \frac{x f_2(x)}{C_2(x) - C_1(x)}\, dx = +\infty, \quad (113)$$

and the above proof would have remained valid. However, we ensure with the assumptions of Proposition 2 that the generalized Laplace densities are those of a sampled diffusion, which is a martingale. And this will have its importance in Section 6 when we will explain how to construct continuous-time martingales with marginal distributions μ_1 and μ_2.

Appendix C Reminders about Karamata's theory

For the sake of completeness, we state here some definitions and properties taken from [7] about Karamata's theory.

Definition 23 *Let f be a positive function defined in a neighborhood of $+\infty$. Its upper Matuszewska index $\alpha(f)$ is the infimum of those α for which there exists a constant $C = C(\alpha)$ such that for each $\Lambda > 1$,*

$$\frac{f(\lambda x)}{f(x)} \quad C(1 + o(1))\lambda^{\alpha} \text{ when } x \to +\infty, \text{ uniformly in } \lambda \in [1, \Lambda];$$

Its lower Matuszewska index $\beta(f)$ is the supremum of those β for which, for some $D = D(\beta) > 0$ and all $\Lambda > 1$,

$$\frac{f(\lambda x)}{f(x)} \geqslant D(1 + o(1))\lambda^{\beta} \text{ when } x \to +\infty, \text{ uniformly in } \lambda \in [1, \Lambda].$$

If $\alpha(f) < +\infty$ (resp. $\beta(f) > -\infty$), we say that f has bounded increase (resp. decrease).

A way to compute $\alpha(f)$, $\beta(f)$ or to check if f has bounded increase (resp. decrease) is given by the following lemma, which is a simple rewriting of Theorem 2.1.5 and Corollary 2.1.6 in [7]. We will need the following notations:

$$\Psi(f, \lambda) \equiv \limsup_{x \to +\infty} \sup_{\mu \in [1, \lambda]} \frac{f(\mu x)}{f(x)}, \qquad f^{*}(\lambda) \equiv \limsup_{x \to +\infty} \frac{f(\lambda x)}{f(x)},$$

$$\Psi_{-}(f, \lambda) \equiv \Psi(1/f, \lambda), \qquad f_{*}(\lambda) \equiv \liminf_{x \to +\infty} \frac{f(\lambda x)}{f(x)}.$$

Lemma 24 *Let f be a positive function defined in a neighborhood of $+\infty$. We have the following equivalence:*

$$\begin{aligned} f \text{ has bounded increase} \quad &\Leftrightarrow \quad \Psi(f, .) < +\infty \\ f \text{ has bounded decrease} \quad &\Leftrightarrow \quad \Psi_{-}(f, .) < +\infty \end{aligned}$$

Moreover, if $\Psi(f, .) < +\infty$ or $\Psi_{-}(f, .) < +\infty$, then

$$\alpha(f) = \lim_{\lambda \to +\infty} \left(\frac{\log(f^{*}(\lambda))}{\log(\lambda)} \right) = \inf_{\lambda > 1} \left(\frac{\log(f^{*}(\lambda))}{\log(\lambda)} \right),$$

and

$$\beta(f) = \lim_{\lambda \to +\infty} \left(\frac{\log(f_{*}(\lambda))}{\log(\lambda)} \right) = \inf_{\lambda > 1} \left(\frac{\log(f_{*}(\lambda))}{\log(\lambda)} \right).$$

Appendix D Proof of Lemma 5

Using the Potter-type bounds of Proposition 2.2.1. in [7], p. 72, $\exists \varepsilon_0, A_0, M_0 > 0$, s.t. $\forall y \geqslant x \geq M_0$,

$$\frac{f_2(y)}{f_2(x)} \quad A_0 \left(\frac{x}{y} \right)^{2 + \varepsilon_0}.$$

Consequently,

$$\sigma^2(x) \quad \frac{C_2(x)}{\frac{1}{2} x^2 f_2(x)} = 2 \frac{\int_x^{+\infty} \left(\int_y^{+\infty} \frac{f_2(z)}{f_2(x)} dz \right) dy}{x^2} \quad 2A_0 \frac{\int_x^{+\infty} \left(\int_y^{+\infty} \left(\frac{x}{z} \right)^{2 + \varepsilon_0} dz \right) dy}{x^2} = \frac{2A_0}{\varepsilon_0 (1 + \varepsilon_0)}.$$

Likewise, using the Potter-type bounds, we have: $\exists \varepsilon_\infty, A_\infty, M_\infty > 0$, s.t. $\forall y \geqslant x \geqslant M_\infty$,

$$\frac{\widetilde{f}_2(y)}{\widetilde{f}_2(x)} \quad A_\infty \left(\frac{y}{x}\right)^{1-\varepsilon_\infty}.$$

Therefore, $\forall y \quad x \quad 1/M_\infty$,

$$\frac{f_2(y)}{f_2(x)} \quad A_\infty \left(\frac{x}{y}\right)^{1-\varepsilon_\infty}.$$

Consequently, if P_i denotes the put price function associated to the distribution μ_i, we have, using put-call parity:

$$\sigma^2(x) = \frac{P_2(x) - P_1(x)}{\frac{1}{2}x^2 f_2(x)} \quad \frac{P_2(x)}{\frac{1}{2}x^2 f_2(x)} = 2\frac{\int_0^x \left(\int_0^y \frac{f_2(z)}{f_2(x)}\, dz\right) dy}{x^2} \quad \frac{A_\infty}{\varepsilon_\infty(\varepsilon_\infty + 1)}.$$

Appendix E Proof of Proposition 8

The existence and the properties of (F_t) are a consequence of Propositions 2.1 and 2.3 in [16] as discussed in Section 2. Since (F_t) is a martingale, Equation (33) is satisfied.

As far as Equation (34) is concerned, we have:

$$\frac{\partial^2 C_1}{\partial x_1^2}(x_1) = [\mathcal{I} - \mathcal{G}^*]\, f_2(x_1),$$

where $\mathcal{G}^*$ is the adjoint of $\mathcal{G}$, by differentiating Equation (19) twice. Consequently, integrating by parts and using the limit conditions of Equations (30), (31) and (32), we obtain:

$$\begin{aligned}
\int_0^{+\infty} p(x_2|\, x_1) \frac{\partial^2 C_1}{\partial x_1^2}(x_1)\, dx_1 &= \int_0^{+\infty} p(x_2|\, x_1)(\mathcal{I} - \mathcal{G}^*)\, f_2(x_1)\, dx_1 \\
&= \int_0^{+\infty} (\mathcal{I} - \mathcal{G})\, p(x_2|\, x_1)\, f_2(x_1)\, dx_1.
\end{aligned}$$

But remember that the probability distribution of (F_t) also satisfies the backward Kolmogorov equation, which yields $(\mathcal{I} - \mathcal{G})\, p(x_2|\, x_1) = \delta(x_2 - x_1)$ once the Carson-Laplace transform has been taken on both sides (see Equation (12)). Equation (34) follows.

Appendix F Proof of Lemma 10

Let us first consider a positive, bounded and continuous function l. Then

$$\begin{aligned}
&\int_0^{+\infty} \frac{\partial^2 C_2(y)}{\partial y^2} l(y)\, \Psi_\gamma(y, x)\, dy \\
= &\int_0^x \frac{\partial^2 C_2(y)}{\partial y^2} l(y)\, \Psi_\gamma(y, x)\, dy + \int_x^{+\infty} \frac{\partial^2 C_2(y)}{\partial y^2} l(y)\, \Psi_\gamma(y, x)\, dy \\
= &\int_0^x \frac{\partial^2 C_2(y)}{\partial y^2} l(y) \left(\int_0^{+\infty} (u - x)^+ \gamma(y, u)\, du\right) dy \\
&+ \int_x^{+\infty} \frac{\partial^2 C_2(y)}{\partial y^2} l(y) \left(\int_0^{+\infty} (x - u)^+ \gamma(y, u)\, du\right) dy,
\end{aligned}$$

since $\Psi_\gamma(y,x)$ represents the price of an out-of-the-money option, struck at x, when the risk-neutral density is $\gamma(y,.)$. Moreover,

$$\begin{aligned}
\int_0^{+\infty} \frac{\partial^2 C_2(y)}{\partial y^2} l(y)\, \Psi_\gamma(y,x)\, dy & \quad \int_0^{x} \frac{\partial^2 C_2(y)}{\partial y^2} l(y) \left(\int_0^{+\infty} u\gamma(y,u)\, du \right) dy + (\sup l)\, P_{2,\gamma}(x) \\
& \quad A \int_0^{x} \frac{\partial^2 C_2(y)}{\partial y^2} l(y)\, y\, dy + (\sup l)\, P_{2,\gamma}(x) \\
& \equiv \mathcal{A}_0 + \mathcal{B}_0,
\end{aligned}$$

since the price of a call is always less than or equal to the forward price and l is bounded. Therefore, by taking l small enough in a neighborhood of 0, it is possible to obtain $\mathcal{A}_0 =_0 o(\mathcal{B}_0)$ and therefore:

$$\int_0^{+\infty} \frac{\partial^2 C_2(y)}{\partial y^2} l(y)\, \Psi_\gamma(y,x)\, dy =_0 O(C_2(x) - C_1(x)), \tag{114}$$

using Equation (37).

Likewise, since $\left(\int_0^{+\infty} (x-u)^+ \gamma(y,u)\, du\right) \quad x$, it is possible to take l small enough in a neighborhood of $+\infty$, so that:

$$\int_0^{+\infty} \frac{\partial^2 C_2(y)}{\partial y^2} l(y)\, \Psi_\gamma(y,x)\, dy =_{+\infty} O(C_2(x) - C_1(x)), \tag{115}$$

using Equation (38).

We are now in a position to conclude. Indeed, $C_2 - C_1$ (resp. $\int_0^{+\infty} \frac{\partial^2 C_2(y)}{\partial y^2} l(y)\, \Psi_\gamma(y,.)\, dy$) is a continuous function on $\mathbb{R}_+^*$ and admits a minimum (resp. maximum) on any compact interval. Therefore, using Equations (114) and (115), $\lambda = \alpha l$ with $\alpha > 0$ small enough satisfies Equation (39).

The second part of the lemma is a simple consequence of the fact that:

$$\frac{C_2(x) - C_1(x) - \int_0^{+\infty} \frac{\partial^2 C_2(y)}{\partial y^2} \lambda(y)\, \Psi_\gamma(y,x)\, dy}{\frac{1}{2} x^2 \frac{\partial^2 C_2}{\partial x^2}(x)} \quad \frac{C_2(x) - C_1(x)}{\frac{1}{2} x^2 \frac{\partial^2 C_2}{\partial x^2}(x)},$$

and of Lemma 5.

Appendix G Proof of Lemma 12

This proof has three main steps.

Step 1: We show that the Fourier transforms of A and C are well-defined.

First $C \in L_1(\mathbb{R})$, since $z \to \frac{\Psi_g(z)}{z} \in L_1(\mathbb{R}_+^*)$. Indeed,

$$\frac{\Psi_g(z)}{z} \quad \begin{cases} 1 & \text{if } z < 1 \\ \frac{\int_0^{+\infty} y g(y)\, dy}{z} & \text{if } z \in (1,2) \\ \frac{\int_1^{+\infty} y (\ln(y))^{\alpha_g} g(y)\, dy}{z (\ln(z))^{\alpha_g}} & \text{if } z > 2 \end{cases}$$

using Equation (52). Likewise,

$$
\begin{aligned}
C_2(z) - C_1(z) & \quad C_2(z) - (X_0 - z)^+ \\
& = \begin{cases} \int_0^{+\infty} f_2(y)(z-y)^+ \, dy & \text{if } z < X_0 \\ \int_0^{+\infty} f_2(y)(y-z)^+ \, dy & \text{if } z > X_0 \end{cases} \\
& \quad \begin{cases} z & \text{if } z < X_0 \\ \int_0^{+\infty} y f_2(y) \, dy & \text{if } z \in (X_0, \max(X_0, 2)) \\ \frac{\int_1^{+\infty} f_2(y) y (\ln(y))^{\alpha_2} \, dy}{(\ln(z))^{\alpha_2}} & \text{if } z > \max(X_0, 2) \end{cases}
\end{aligned}
$$

using Equation (51). Therefore $\frac{C_2(z)-C_1(z)}{z} \in L_1(\mathbb{R}_+^*)$ and, as a result, $A \in L_1(\mathbb{R})$.

Step 2: We show that the assumptions of (A_2), which only concern γ, are satisfied.

First $\gamma(x,.) \equiv \frac{1}{x} g\left(\frac{\cdot}{x}\right)$ is clearly a probability density since g is one. Second,

$$
\int_0^{+\infty} \frac{y}{x} \gamma(x,y) \, dy = \int_0^{+\infty} \frac{y}{x} g\left(\frac{y}{x}\right) \frac{dy}{x} = \int_0^{+\infty} u g(u) \, du
$$

is indeed a bounded and continuous function since g has a finite first moment. Third,

$$
\int_0^{+\infty} \frac{y}{x} \ln\left(\frac{y}{x}\right) \gamma(x,y) \, dy = \int_0^{+\infty} u \left|\ln(u)\right| g(u) \, du < +\infty
$$

using Equations (52) and (53). Finally, if Γ is a Borel set of $\mathbb{R}_+^*$, let us consider a sequence (x_n), which goes to x as n goes to $+\infty$. Then, we have:

$$
\begin{aligned}
& \int_\Gamma \frac{\ln\left(\frac{y}{x}\right)}{1+\left(\ln\left(\frac{y}{x}\right)\right)^2} \gamma(x,y) \, dy - \int_\Gamma \frac{\ln\left(\frac{y}{x_n}\right)}{1+\left(\ln\left(\frac{y}{x_n}\right)\right)^2} \gamma(x_0,y) \, dy \\
= & \int_{\frac{\Gamma}{x}} \frac{\ln(u)}{1+(\ln(u))^2} g(u) \, du - \int_{\frac{\Gamma}{x_n}} \frac{\ln(u)}{1+(\ln(u))^2} g(u) \, du \\
& \int_{\frac{\Gamma}{x}\Delta\frac{\Gamma}{x_n}} \frac{|\ln(u)|}{1+(\ln(u))^2} g(u) \, du \\
& \frac{1}{2} \int_{\frac{\Gamma}{x}\Delta\frac{\Gamma}{x_n}} g(u) \, du,
\end{aligned}
$$

where $\frac{\Gamma}{x} \equiv \left\{\frac{y}{x} \text{ s.t. } y \in \Gamma\right\}$ and $A \Delta B \equiv (A \cup B) \cap \overline{(A \cap B)}$. Finally, the facts that $1_{\frac{\Gamma}{x}\Delta\frac{\Gamma}{x_n}}(u) \to 0$ as $n \to +\infty$ and that $g \in L_1(\mathbb{R}_+^*)$ allows us to conclude that this integral is continuous using the dominated convergence theorem.

Step 3: We show that Equation (19) is satisfied.

Since $B, \mathcal{F}A/\mathcal{F}C \in L_1(\mathbb{R})$, we have (see Theorem 8 in [8], p. 19):

$$
\mathcal{F}A = \mathcal{F}B\mathcal{F}C
$$

Using the theorems about the Fourier transform of a convolution (see Theorem 2 in [8], p. 6.) and the above theorem about inversion of the Fourier transform, we get:

$$
A = B * C.
$$

Finally, reproducing backward the computations of the beginning of this section gives the desired conclusion.

Appendix H Proof of Lemma 15

Thanks to Lemma 14, we only need to prove that Equation (19) is valid. This can be done by rewriting Equation (63), as follows:

$$\mathcal{I} - \frac{a^2(x)}{2}\frac{\partial^2}{\partial x^2}\left(C_2(x) - C_1(x) - \frac{1}{2}x^2\sigma^2(x)\frac{\partial^2 C_2}{\partial x^2}(x)\right) = \frac{a^2(x)}{2}\lambda(x)\frac{\partial^2 C_2}{\partial x^2}(x).$$

Since $\left(C_2(x) - C_1(x) - \frac{1}{2}x^2\sigma^2(x)\frac{\partial^2 C_2}{\partial x^2}(x)\right) \to_{0 \text{ or } +\infty} 0$, we have, by unicity of the associated value problem:

$$C_2(x) - C_1(x) - \frac{1}{2}x^2\sigma^2(x)\frac{\partial^2 C_2}{\partial x^2}(x) = {}_{0}^{+\infty}\ \lambda(y)\frac{\partial^2 C_2}{\partial x^2}(y)\frac{a^2(y)}{2}\gamma(x,y)\,dy$$

and the conclusion follows using Equation (56).

Appendix I Proof of Lemma 16

The proof has five main steps.

Step 1: We show that σ satisfies the assumptions of Lemma 15.

The fact that σ is a positive, bounded and continuous function on $\mathbb{R}_+^*$ is a consequence of Lemma 5.

Step 2: We introduce a candidate for the function λ.

$\frac{f_1}{f_2}$ is bounded since it is a continuous function, bounded in a neighborhood of 0 and $+\infty$. Consequently, if l is a positive, bounded and continuous function, such that:

$$l(x) > \frac{\varepsilon + \sup\left(\frac{f_1(x)}{f_2(x)}\right) - 1}{\beta^2},$$

where ε is a positive constant, then the function:

$$A_l^2(x) \equiv \frac{1}{x^2}\frac{(C_2(x) - C_1(x))}{(f_2(x) - f_1(x)) + \beta^2 l(x) f_2(x)},$$

is a well-defined, positive and continuous function on $\mathbb{R}_+^*$.

Step 3: We show that A_l^2 is bounded by two positive constants in a neighborhood of $+\infty$.

First, by reducing this neighborhood, we can assume that $f_2(x) > M f_1(x)$. Therefore,

$$C_2(x) \geqslant (C_2(x) - C_1(x)) \geqslant \left(1 - \frac{1}{M}\right)C_2(x),$$

and

$$(1 + \beta^2(\sup l))f_2(x) \geqslant (f_2(x) - f_1(x)) + \beta^2 l(x) f_2(x) \geqslant \left(1 - \frac{1}{M}\right)f_2(x).$$

Consequently,

$$\frac{\left(1 - \frac{1}{M}\right)}{(1 + \beta^2(\sup l))}\frac{1}{x^2}\frac{C_2(x)}{f_2(x)} \leq A_l^2(x) \qquad \frac{M}{(M-1)}\frac{1}{x^2}\frac{C_2(x)}{f_2(x)}$$

Second $\frac{1}{x^2}\frac{C_2(x)}{f_2(x)}$ is bounded by above in a neighborhood of $+\infty$, since $\alpha(f_2) < -2$ (see the proof of Lemma 5 in Appendix D). Third, this quantity is also bounded by below by a positive constant because f_2 has bounded decrease. Indeed, using Proposition 2.2.1 in [7], p. 72, there exist $\varepsilon_\infty < -2$, $A_\infty, M_\infty > 0$, s.t. $\forall v \geqslant x \geqslant M_\infty$,

$$\frac{f_2(v)}{f_2(x)} \geqslant A_\infty \left(\frac{v}{x}\right)^{\varepsilon_\infty}.$$

Therefore, integrating twice in v, we get:

$$\frac{C_2(x)}{f_2(x)} \geqslant A_\infty \int_x^{+\infty} \left(\int_u^{+\infty} \left(\frac{v}{x}\right)^{\varepsilon_\infty} dv \right) du = \frac{A_\infty}{(\varepsilon_\infty + 1)(\varepsilon_\infty + 2)} x^2.$$

Step 4: We show that A_l^2 is bounded by two positive constants in a neighborhood of 0.

This step is actually left to the reader as it is entirely similar to Step 3. As one could expect, its proof is based on the facts that $\alpha\left(\widetilde{f_2}\right) < 1$ and that $\widetilde{f_2}$ has bounded decrease.

Step 5: We conclude.

Since A_l^2 is a positive, continuous function, which is bounded, by below and above, by positive constants in neighborhoods of 0 or $+\infty$, there exist two positive constants $m_l, M_l > 0$, such that:

$$m_l \leqslant A_l^2(x) \leqslant M_l.$$

As a consequence, taking $\lambda = l$ is sufficient.

Appendix J Proof of Lemma 18

Since $\frac{f_1}{f_2}$ is continuous, bounded in a neighborhood of 0 and $+\infty$, it is bounded on $\mathbb{R}_+^*$. Consequently, λ is bounded and continuous. Moreover, it is positive since $\alpha^2, \beta^2 \in (0,1)$. Finally, $\frac{C_2(x)-C_1(x)}{x^2 f_2(x)}$ is bounded by two positive constants using the proof of Lemma 16 in Appendix I.

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