First-order calculus and option pricing

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Abstract

The modern theory of option pricing rests on Itô calculus, which is a second order calculus based on the quadratic variation of a stochastic process. One can instead develop a first order stochastic calculus, which is based on the running minimum of a stochastic process, rather than its quadratic variation. We focus here on the analog of geometric Brownian motion (GBM) in this alternative stochastic calculus. The resulting stochastic process is a positive continuous martingale whose laws are easy to calculate. We show that this analog behaves locally like a GBM whenever its running minimum decreases, but behaves locally like an arithmetic Brownian motion otherwise. We provide closed form valuation formulas for vanilla and barrier options written on this process. We also develop a reflection principle for the process and use it to show how a barrier option on this process can be hedged by a static postion in vanilla options.

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1 Introduction

The modern theory of option pricing rests on Itô calculus, which is a second order calculus based on the quadratic variation of a stochastic process. In the standard Black Scholes model, the focus is on the quadratic variation of the log of the spot price S. Letting W denote the standard Brownian motion (SBM) driving S, the model requires that:

$$\langle \ln{(S/S_0)} \rangle_t = \sigma^2 \langle W \rangle_t, \qquad t \geq 0. \tag{1}$$

In words, the quadratic variation of the log price is positively proportional to the quadratic variation of the driving SBM. Any solution S to (1) is called Geometric Brownian Motion (GBM).

In this paper, we price options using an alternative first order calculus, which replaces the quadratic variation of a process with its running minimum. In particular, we model the underlying price process as a continuous martingale F started at F_0 and evolving so that:

$$\underline{\ln{(F/F_0)}}_t = \sigma \underline{W}_t, \qquad t \geq 0, \tag{2}$$

where $\underline{\ln{(F/F_0)}}_t \equiv_{s\in[0,t]}^{\inf} \ln(F_s/F_0)$ denotes the running minimum of $\ln{(F_t/F_0)}$, σ is a positive constant, and $\underline{W}_t \equiv_{s\in[0,t]}^{\inf} W_s$ denotes the running minimum of W. Comparing (1) with (2), we see that F is defined to be the analog of the GBM S that arises when the quadratic variation of a process is replaced by its running minimum. Since the calculus is first order, σ replaces σ^2 as the proportionality constant.

It is straightforward to solve (2) for the underlying price process F:

$$F_t = F_0 e^{\sigma \underline{W}_t}(1 + \sigma W_t^v), \qquad t \geq 0, \tag{3}$$

where $W_t^v \equiv W_t - \underline{W}_t$ is the running drawup of the SBM W. It is natural to compare this solution with the general solution S of (1):

$$S_t = S_0 e^{-\frac{\sigma^2}{2}\langle W \rangle_t} e^{\sigma W_t}, \qquad t \geq 0, \tag{4}$$

with the parameters S_0 and σ both positive.

Each process arises as the product of three factors. First, each process starts at a positive level, denoted F_0 and S_0 respectively. Second, each starting level is multiplied by a positive decreasing process, with $\sigma \underline{W}_t$ replacing $-\frac{\sigma^2}{2}\langle W \rangle_t = -\frac{\sigma^2}{2} t$ in the argument of an exponential function. Third, each product is further multiplied by a positive submartingale, with $1 + \sigma W_t^v$ in (3) replacing $e^{\sigma W_t}$ in (4) as the multiplier of that product. It follows that both processes start positive and remain positive forever. Whenever the driving SBM W is above its minimum $\underline{W}_t$, the random process $\underline{W}_t$ is locally constant, so the Brownian drawup W_t^v is locally affine in W. It follows from (3) that our new martingale F is also locally affine in the driving SBM W at such times, in stark contrast to S, which always behaves as an exponential function of W. This locally affine behavior of F leads to a downward sloping skew, characteristic of many options markets.

It is clear from (3) that the law of F at a fixed time T depends on the bivariate law of $(\underline{W}_T, W_T^v)$. Fortunately, the reflection principle implies that the bivariate law of $(\underline{W}_T, W_T)$ is known in closed form. As a result, the desired bivariate law is also known, allowing us to develop not only the law of F_T, but also the bivariate law $(\underline{F}_T, F_T)$. The paper shows how one can use this bivariate law to price vanilla and lower barrier options on F in closed form. We also develop a reflection principle for the F process, relating the law of $\underline{F}_T$ to the law of F_T. This reflection principle gives an investor the ability to replicate the payoff of a lower-barrier one-touch on F, by holding a static position in co-terminal puts. We show that this static replicating portfolio of puts is robust to the introduction of independent stochastic volatility, even if the stochastic process generating this volatility is unknown. We argue that these properties of F render our new martingale intuitive, tractable, realistic, and yet simple, making it a suitable springboard for further development.

2 Analysis

We assume no arbitrage and zero interest rates. By the Fundamental Theorem of Asset Pricing, there exists a probability measure $\mathbb{Q}$ under which asset prices are martingales. Financial considerations such as limited liability often restrict attention to positive martingales, with GBM being the standard example. The goal of this paper is to explore the properties of a new positive continuous martingale, which arises as the natural analog of GBM in a first order calculus, which arises when the quadratic variation of a process is replaced by its running minimum. We will show that our new martingale may also be considered as a very special case of the more general class of Azema Yor martingales.

Let W be a standard Brownian motion under a risk-neutral measure $\mathbb{Q}$. Let $\underline{W}_t \equiv_{s\in[0,t]}^{\inf} W_s$ be the running minimum of the W process over the time period $[0,t]$. Let $W_t^v \equiv W_t - \underline{W}_t$ be the running drawup of the W process over the time period $[0,t]$. Trivially, we can decompose W_t additively by:

$$W_t = \underline{W}_t + W_t^v, \qquad t \geq 0.$$

By Skorohod's lemma, there is a unique way to assign the Gaussian probability density of W_t to the ordered pair $(\underline{W}_t, W_t^v)$. For $j \leq 0$ and $k \geq 0$, we find that the bivariate PDF $\mathbb{Q}(\underline{W}_t \in dj, W_t^v \in dk)$ depends on j and k only through their distance $k - j \geq 0$. When we derive the marginal of W_t^v from this bivariate law, we observe that this PDF of W_t^v is folded normal. When we furthermore derive the marginal of $\underline{W}_t$ from the bivariate law, we observe that $-\underline{W}_t$ has the same PDF as W_t^v.

We are interested in defining a "geometric" version of W, which emphasizes a first order variation such as $\underline{W}$ rather than the second order variation $\langle W \rangle_t = t$. This task is accomplished by replacing differences in the Brownian minimum by ratios and also by replacing differences in the drawup definition by ratios. The geometric process will be the *product* of its minimum and its drawup, rather than the sum of the two. The starting value of the geometric process will be the multiplicative identity element one, rather than the additive identity element zero. Furthermore, the geometric process stays positive, with zero and infinity as natural boundaries.

Azema and Yor show how one can develop a family of local martingales from a given martingale and its extremum. Consider the Azema Yor local martingale that arises when the driving martingale is an SBM W and its extremum is $\underline{W}$:

$$\hat{N}_t \equiv \Phi(\underline{W}_t) + \phi(\underline{W}_t) W_t^v, \qquad t \geq 0, \tag{5}$$

where $\Phi(x) \equiv \int^x \phi(x')dx'$ is an anti-derivative of $\phi(x)$. Suppose we choose $\phi(x) = e^x$. Then an anti-derivative is $\Phi(x) = e^x$. Let F_t denote the proposed local martingale for this choice:

$$\begin{aligned} F_t &\equiv e^{\underline{W}_t} + e^{\underline{W}_t} W_t^v \\ &= e^{\underline{W}_t}(1 + W_t^v), \qquad t \geq 0. \end{aligned} \tag{6}$$

Notice that this process separates multiplicatively into an exponential function of the Brownian infimum $\underline{W}_t$ and an affine function of the Brownian drawup W_t^v. This is analogous to a standardized GBM S which has the form

$$S_t = e^{-\frac{1}{2}\; W \;\, t} e^{W_t}, \qquad t \geq 0. \tag{7}$$

Since $\underline{W}(0) = W_0^v = 0$, setting $t = 0$ in (6) implies that the process F starts at 1:

$$F_0 = 1. \tag{8}$$

Taking the total derivative in (6) implies that F solves:

$$dF_t = e^{\underline{W}_t} dW_t, \qquad t \geq 0, \tag{9}$$

subject to the initial condition (8). Hence, the process F is a local martingale, and it can be shown that it is also a martingale.

Now let $\underline{F}_t \equiv \inf_{s\in[0,t]} F_s$ denote the running minimum of the F process over $[0, t]$. Since $\underline{W}$ can only decline at the times when $W^v = 0$, we have:

$$\underline{F}_t = e^{\underline{W}_t}, \qquad t \geq 0. \tag{10}$$

Solving for $\underline{W}_t$:

$$\underline{W}_t = \ln \underline{F}_t = \underline{\ln F}_t, \qquad t \geq 0, \tag{11}$$

since $\ln x$ is an increasing function. It follows that we have constructed a continuous martingale whose log has the same minimum as standard Brownian motion. In contrast, the GBM S in (7) is a continuous martingale whose log has the same quadratic variation as standard Brownian motion. Since quadratic variation is a second order variation while the running minimum is first order, the continuous process F defined in (6) will be referred to as the first order martingale. When W is above its minimum, then $\underline{W}$ is locally constant so $F_t = e^{\underline{W}_t}(1 + W_t - \underline{W}_t)$ is locally affine in W. Hence, increments of the continuous martingale F are locally normally distributed. When W is at its minimum and then declines, the continuous martingale F declines exponentially. Hence, these increments of the continuous martingale F are locally lognormally distributed. The Lebesgue measure of the set of times when W is at its minimum is zero. Thus, the continuous martingale F spends 100% of its time behaving locally like an arithmetic Brownian motion (ABM). Nonetheless, (10) indicates that $\underline{F}_t > 0$ and hence the F process is positive. Since the dynamics mimic an ABM almost everywhere, using the F process to describe the dynamics of an underlying results in a downward sloping skew, characteristic of many options markets.

Substituting (10) in (9) implies that F solves the following SDE:

$$dF_t = \underline{F}_t dW_t, \qquad t \geq 0, \tag{12}$$

subject to the initial condition (8). We also have the following differential version of (10):

$$d\underline{F}_t = \underline{F}_t d\underline{W}_t, \qquad t \geq 0. \tag{13}$$

subject to the initial condition:

$$\underline{F}_0 = 1. \tag{14}$$

Recall that the Brownian drawup W_t^v of the SBM was defined as the *difference* between the Brownian level W_t and the Brownian minimum $\underline{W}_t$. Suppose we analogously define the drawup of the first order martingale F as the *ratio* of its level to its minimum, i.e.:

$$F_t^v \equiv \frac{F_t}{\underline{F}_t}, \quad t \geq 0. \tag{15}$$

Then we trivially have that the first order martingale F decomposes multiplicatively as:

$$F_t = \underline{F}_t F_t^v, \quad t \geq 0. \tag{16}$$

Comparing (16) with (6) and (10), we conclude that our first order process' drawup is affine in the Brownian drawup:

$$F_t^v = 1 + W_t^v, \quad t \geq 0. \tag{17}$$

There are no parameters in our definition of the first order process F, but it would be straightforward to introduce two parameters. The first one, naturally labelled F_0, would be the starting value of F and hence is required to be positive. The second parameter, naturally labelled σ, is also required to be positive. The square of this parameter σ^2 has units of one over years and controls the width of the PDF of F_t through $\sigma^2 t$. The definition of the two parameter first order martingale is:

$$F_t = F_0 e^{\sigma \underline{W}_t}(1 + \sigma W_t^v), \qquad t \geq 0. \tag{18}$$

where recall $W_t^v \equiv W_t - \underline{W}_t$. It is natural to compare this two parameter process with the GBM:

$$S_t = S_0 e^{-\frac{1}{2}\sigma^2 t} e^{\sigma W_t}, \qquad t \geq 0. \tag{19}$$

For the first order F process, scaling W by σ is accompanied by scaling $\underline{W}$ by σ. In contrast, for the second order S process scaling W by σ is accompanied by scaling $\langle W \rangle_t = t$ by σ^2.

For each fixed time $t \geq 0$ and starting level $F_0 > 0$, we may regard F_t as a function of the parameter σ. Differentiating w.r.t. σ and setting $\sigma = 0$ implies that:

$$\left. \frac{\partial}{\partial \sigma} F_t(\sigma) \right|_{\sigma=0} = \underline{W}_t + W_t^v = W_t, \qquad t \geq 0. \tag{20}$$

Thus this stochastic flow is just the driving SBM W. The well known Brownian scaling property of W_t extends to the pair $(\underline{W}_t, W_t^v)$ so that $(\sigma \underline{W}_t, \sigma W_t^v)$ has the same bivariate law as $(\underline{W}_{\sigma^2 t}, W_{\sigma^2 t}^v)$.

3 Derivation of Bivariate Laws

Let $T > 0$ be a fixed time called the terminal time. In this section, we derive in closed form the bivariate law of the terminal minimum $\underline{F}_T$ and the terminal drawup F_T^v, conditional on the current levels $\underline{F}_t$ and F_t^v at any prior time $t \in [0, T]$. We first develop the corresponding results for standard Brownian motion and then use these to develop the bivariate law of $(\underline{F}_T, F_T^v)$ given $(\underline{F}_t, F_t^v)$.

Consider a down-and-in binary call (DIBC) written on the terminal Brownian drawup W_T^v. For in-barrier $j \leq 0$ and strike $k \geq 0$, the payoff at the terminal time T is $1(\underline{W}_T \leq j, W_T^v \geq k)$. Let τ be the first passage time of the SBM W to the lower barrier $j \leq 0$. As usual, if W never hits j, we set $\tau = \infty$. For $t \in (0, \tau)$, let $DIBC_{[t,T]}(j, k) \equiv \mathbf{Q}_t\{\underline{W}_T \leq j, W_T^v \geq k\}$ denote the value at time t of the DIBC expiring at T. Using a no arbitrage argument, the appendix shows that for $t \in (0, \tau)$, $j \leq 0$, $k \geq 0$:

$$DIBC_{[t,T]}(j, k) = 2\mathbb{Q}_t\{W_T \leq j - k\}. \tag{21}$$

In words, prior to knocking in, the DIBC with barrier $j \leq 0$ and strike $k \geq 0$ has the same value as two co-terminal binary puts on W_T struck at $j - k \leq 0$. The conditional value of the two binary puts on W_T can of course be determined in closed form:

$$2\mathbb{Q}_t\{W_T \leq j - k\} = 2N\left(\frac{j - k - W_t}{\sqrt{T - t}}\right), \qquad t \in [0, T), \tag{22}$$

where $N(z) \equiv \int_{-\infty}^{z} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx$ is the univariate normal distribution function. Equations (21) and (22) together imply that for $t \in (0, \tau), j \leq 0, k \geq 0$:

$$DIBC_{[t,T]}(j, k) \equiv \mathbb{Q}_t\{\underline{W}_T \leq j, W_T^v \geq k\} = 2N\left(\frac{j - k - W_t}{\sqrt{T - t}}\right). \tag{23}$$

So the bivariate law in the middle just depends on the univariate normal distribution function on the RHS. Notice that this bivariate law depends on the time t values of $\underline{W}_t$ and W_t^v only through their sum $W_t = \underline{W}_t + W_t^v$. In contrast, the dependence on $j \leq 0$ and $k \geq 0$ occurs only through the distance $k - j \geq 0$. As a result, subtracting j from both arguments of the DIBC does not affect the result implying that for $t \in (0, \tau)$:

$$\mathbb{Q}_t\{\underline{W}_T \leq 0, W_T^v \geq k - j\} = \mathbb{Q}_t\{W_T^v \geq k - j\} = 2N\left(\frac{j - k - W_t}{\sqrt{T - t}}\right). \tag{24}$$

Hence, the marginal law of W_T^v at $t = 0$ with $W_t = 0$ is folded normal. If we subtract k from both of the arguments of the DIBC in (23), we get that for $t \in (0, \tau)$:

$$\mathbb{Q}_t\{\underline{W}_T \leq j - k, W_T^v \geq 0\} = \mathbb{Q}_t\{\underline{W}_T \leq j - k\} = 2N\left(\frac{j - k - W_t}{\sqrt{T - t}}\right). \tag{25}$$

Comparing (24) and (25), we see that at $t = 0$ with $W_t = 0$, $-\underline{W}_T$ has the same law as W_T^v.

Differentiating (23) w.r.t. j and k yields the following simple closed form formula for the bivariate PDF of the Brownian Minimum and Brownian Drawup:

$$\mathbb{Q}_t\{\underline{W}_T \in dj, W_T^v \in dk\} = b(j, k)djdk$$

where:

$$b(j,k) \equiv \sqrt{\frac{2}{\pi(T-t)^3}}(k-j-W_t)e^{-\frac{(k-j-W_t)^2}{2(T-t)}}, \qquad j \leq 0, k \geq 0.$$

Armed with these results, we now derive the conditional bivariate PDF of the pair $(\underline{F}_t, F_t^v)$. Evaluating (11) and (17) at $t = T$ implies that:

$$\underline{F}_T = e^{\underline{W}_T}, \qquad F_T^v = 1 + W_T^v, \qquad t \geq 0.$$

Using the standard change of variables formula, it follows that for $J \in (0, 1], K \geq 1$ and $t \in (0, \tau)$, the conditional bivariate PDF of the pair $(\underline{F}_t, F_t^v)$ is given by:

$$\mathbb{Q}_t\{\underline{F}_T \in dJ, F_T^v \in dK\} = f(J,K)dJdK, \text{ where:}$$

$$f(J,K) \equiv \sqrt{\frac{2}{\pi(T-t)^3}}\,(K-1-\ln(J)-W_t)\,\frac{e^{-\frac{1}{2(T-t)}(K-1-\ln J - W_t)^2}}{J}, \text{ with:}$$

$$W_t = \ln(\underline{F}_t) + F_t^v - 1. \tag{26}$$

3.1 Marginal of the First Order Martingale

At the terminal time T, the first order martingale F_T is just the product of its terminal minimum $\underline{F}_T$ and its terminal drawup F_T^v:

$$F_T = \underline{F}_T F_T^v.$$

Since the conditional bivariate law of $(\underline{F}_T, F^v)$ is known, the law of the product is $\mathbb{Q}_t\{F_T \in dF\}$:

$$\begin{aligned} &= \int_0^1 f\left(J, \frac{F}{J}\right) dJ \\ &= \int_0^1 \sqrt{\frac{2}{\pi(T-t)^3}}\left(\frac{F}{J} - 1 - \ln(J) - W_t\right)\frac{e^{-\frac{1}{2(T-t)}\left(\frac{F}{J}-1-\ln(J)-W_t\right)^2}}{J} dJ. \end{aligned} \tag{27}$$

Hence, the transition PDF of F_T is given by the bounded integral in (27). This integral cannot be calculated in closed form, so evaluating the PDF requires quadrature. However, we will see in the next section that iterating the integration does not introduce additional quadratures.

4 Pricing Lower Barrier Claims and Calls

Since the first order martingale is driven by a single SBM, one can replicate any contingent claim written on its path via dynamic trading in the underlying risky asset and in a riskless asset. No arbitrage implies that the value of such a path-dependent claim is given by the cost of this replicating portfolio. In this section, we develop closed form pricing formulas for this replication cost for a few contingent claims. In particular, we focus on some simple lower barrier options and on standard calls.

4.1 One-Touch

We start with a One-Touch with a lower barrier. We will present a semi-static hedge for this claim in the next section. In this subsection, we just focus on providing a simple closed form pricing formula assuming zero interest rates.

Let $OT_t(L,T)$ denote the arbitrage-free value of a One-Touch at time $t \in [0,T]$:

$$OT_t(L,T) \equiv E_t 1(\underline{F}_T \leq L), \tag{28}$$

for any barrier $L \leq F_0$. We trivially have:

$$OT_t(L,T) = 1(\underline{F}_t \leq L) + 1(\underline{F}_t > L) E_t 1(\underline{F}_T \leq L), \tag{29}$$

for all $t \in [0,T]$, and for all $L \leq F_0$. Since the underlying F is our first order martingale:

$$1(\underline{F}_T \leq L) = 1(\underline{W}_T \leq \ln(L)), \tag{30}$$

for all $L \in (0,1]$. Substituting (30) in (29) implies that:

$$OT_t(L,T) - 1(\underline{F}_t \leq L) = 1(\underline{F}_t > L) E_t 1(\underline{W}_T < \ln(L)) = 1(\underline{F}_t > L) 2N\left(\frac{\ln(L) - W_t}{\sqrt{T-t}}\right), \tag{31}$$

where $N(z)$ is the standard normal distribution function and W_t is given in (26).

4.2 Down-and-In Call

We next turn to the pricing of a Down-and-In Call assuming zero interest rates. We first establish some preliminary model-free results concerning the value of a down-and-in call. We then assume that its underlying is our first order martingale F, allowing us to strengthen our conclusions substantially.

A down-and-in call (DIC) with strike price K and maturity date T pays its owner $(F_T - K)^+$ at T, so long as the underlying F process has touched or crossed some lower barrier $L \leq F_0$ prior to T. If F fails to touch or cross L prior to T, then the DIC expires worthless. Let $DIC_t(L,K,T)$ denote the arbitrage-free value of a Down-and-In Call at time $t \in [0,T]$:

$$DIC_t(L,K,T) \equiv E_t 1(\underline{F}_T \leq L)(F_T - K)^+, \tag{32}$$

for any barrier $L \leq F_0$ and strike price $K \geq L$. Setting $L = F_0$ results in the arbitrage-free value of a vanilla call:

$$C_t(K,T) \equiv E_t (F_T - K)^+, \tag{33}$$

for any time $t \in [0,T]$ and strike price $K \geq F_0$. The two option prices are trivially related by:

$$DIC_t(L,K,T) = 1(\underline{F}_t \leq L) C_t(K,T) + 1(\underline{F}_t > L) E_t 1(\underline{F}_T \leq L)(F_T - K)^+, \tag{34}$$

for all $t \in [0,T]$, $L \leq F_0$, and $K \geq L$. This result just says that before the barrier is hit, the arbitrage-free value of the DIC is given by its expected payoff, while after the barrier is hit, the DIC value coincides with the vanilla call value.

Next consider the problem of pricing a Down-and-In Call written on our first order martingale F. We will show that since:

$$F_T = e^{\underline{W}_T}(1 + W_T^v), \tag{35}$$

and:

$$\underline{F}_T = e^{\underline{W}_T}, \tag{36}$$

the payoff of a DIC written on F can be related to the payoff from a claim written on $\underline{W}_T$ and W_T^v. For $L \in (0, 1]$ and $K_c \geq L$:

$$\begin{aligned} 1(\underline{F}_T \leq L)(F_T - K_c)^+ &= \int_0^L \delta(\underline{F}_T - J)J\left(1 + W_T^v - \frac{K_c}{J}\right)^+ dJ \\ &= \int_0^L \delta(e^{\underline{W}_T} - e^{\ln(J)})J(W_T^v - k(K_c/J))^+ dJ, \end{aligned} \tag{37}$$

where

$$k(x) \equiv x - 1. \tag{38}$$

Now:

$$\delta(e^{\underline{W}_T} - e^{\ln(J)}) = \delta(\underline{W}_T - \ln(J))\frac{1}{J} \tag{39}$$

from the properties of Dirac delta functions. Substituting (39) in (37) implies that the payoff of a DIC written on F can be related to the payoff from a claim written on $\underline{W}_T$ and W_T^v:

$$1(\underline{F}_T \leq L)(F_T - K)^+ = \int_0^L \delta(\underline{W}_T - \ln(J))(W_T^v - k(K_c/J))^+ dJ. \tag{40}$$

Multiplying (40) by $1(\underline{F}_t > L)$ and taking conditional expectations at time t:

$$1(\underline{F}_t > L)E_t 1(\underline{F}_T \leq L)(F_T - K)^+ = 1(\underline{F}_t > L)\int_0^L E_t\delta(\underline{W}_T - \ln(J))(W_T^v - k(K_c/J))^+ dJ. \tag{41}$$

Substituting (41) in (34) implies:

$$DIC_t(L, K, T) - 1(\underline{F}_t \leq L)C_t(K, T) = 1(\underline{F}_t > L)\int_0^L E_t\delta(\underline{W}_T - \ln(J))(W_T^v - k(K_c/J))^+ dJ. \tag{42}$$

We now show how to eliminate W_T^v from (42). The appendix proves that for any $j \leq 0$ and $k \geq 0$:

$$1(\underline{W}_t > j)E_t 1(\underline{W}_T \leq j)(W_T^v - k)^+ = 1(\underline{W}_t > j)2E_t(j - k - W_T)^+. \tag{43}$$

In words, a Down-and-In Call written on W_T^v with in-barrier j and strike k has the same pre-touch value as 2 co-terminal puts written on W_T struck at $j - k$. Suppose that $t \in [0, \tau]$, so we have $\underline{W}_t > j$. Since $1(\underline{W}_t > j) = 1$, differentiating both sides of (43) w.r.t. j implies that for $t \in [0, \tau]$:

$$1(\underline{W}_t > j)E_t\delta(\underline{W}_T - j)(W_T^v - k)^+ = 1(\underline{W}_t > j)2E_t 1(W_T \leq j - k). \tag{44}$$

However, when $\underline{W}_t > j$, it is well known that the reflection principle implies that:

$$2E_t 1(W_T \leq j - k) = E_t 1(\underline{W}_T \leq j - k). \tag{45}$$

In words, 2 digital puts have the same pre-touch value as a co-terminal One-Touch. Substituting (45) in (44) implies that for $t \in [0, \tau]$:

$$1(\underline{W}_t > j) E_t \delta(\underline{W}_T - j)(W_T^v - k)^+ = 1(\underline{W}_t > j) E_t 1(\underline{W}_T \leq j - k). \tag{46}$$

Since $J \leq L \in (0, 1]$, we have $\ln(J) \leq 0$, and since $K \geq L$, we have $k(K_c/J) \geq 0$. Evaluating (46) at $j = \ln(J)$ and at $k = k(K, J)$, substitution into (42) implies that we can eliminate W_T^v:

$$\begin{aligned} DIC_t(L, K, T) - 1(\underline{F}_t \leq L) C_t(K_c, T) &= 1(\underline{F}_t > L) \int_0^L E_t 1(\underline{W}_T \leq \ln(J) - k(K_c/J)) dJ \qquad (47) \\ &= 1(\underline{F}_t > L) \int_0^L 2N\left(\frac{\ln(J) - k(K_c/J) - W_t}{\sqrt{T-t}}\right) dJ, \end{aligned}$$

where W_t is given in (26). The price of a standard call struck at $K_c \geq 1$ is just the special case when $L = 1$:

$$C_t(K_c, T) = \int_0^1 2N\left(\frac{\ln(J) - k(K_c/J) - W_t}{\sqrt{T-t}}\right) dJ. \tag{48}$$

We can now state the main result of this subsection. For $L \in (0, 1]$ and $K_c \geq L$, a DIC on the first order martingale F is priced by:

$$DIC_t(L, K_c, T) = 1(\underline{F}_t \leq L) C_t(K_c, T) + 1(\underline{F}_t > L) \int_0^L 2N\left(\frac{\ln(J) - k(K_c/J) - W_t}{\sqrt{T-t}}\right) dJ, \tag{49}$$

where $C_t(K_c, T)$ is given by (48), W_t is given by (26), and where $k(K_c/J)$ is given by (38). Hence, the problem of valuing a DIC has been reduced to a single quadrature. If we differentiate the DIC value twice with respect to its strike K_c, we obtain the joint risk-neutral PDF of F_T and $\underline{F}_T$ as a single quadrature.

5 Semi-Static Replication of a One-Touch

As mentioned, the arbitrage-free approach to pricing is based on establishing the existence of a replicating portfolio and determining its initial cost. The standard approach for replication requires the ability to trade continuously in the underlying risky asset and in a riskless asset. For some path-dependent contingent claims (eg. barrier options and lookbacks) and for some dynamics (eg. Geometric Brownian motion), there may exist an alternative replicating strategy which just involves static positions in co-terminal options. In this section, we show that for a lower barrier one-touch written on our first order martingale F, there is a static hedge involving just co-terminal put options. The hedge is established at the time the one-touch is sold, and the hedge is liquidated if the barrier is first touched before expiry. At this first hitting time, the value of the portfolio of co-terminal puts is guaranteed by the model to be worth one, regardless of the actual value of the hitting time. If the barrier is not touched by expiry, the portfolio of co-terminal puts is guaranteed to expire worthless because all of the put options are struck below the barrier. We refer to this type of replication as "semi-static", since the put options in the hedge may be sold before they mature.

Consider a static position in a portfolio of co-terminal European options of maturity T. Suppose we never use in-the-money options, hence we use only at-the-money (ATM) and out-of-the-money (OTM) puts and calls. By allowing arbitrary static positions in a continuum of positive strikes, one can construct any terminal payoff $h(F_T)$ of the terminal first order martingale F_T. In particular, using initially OTM put options struck at a level $K \in (0, 1]$ and below, one can construct the payoff $(\ln K - \ln F_T)^+ \equiv \ln^+(K/F_T)$, which we refer to as "put on the log". The initial goal of this section is to prove that for $K \in (0, 1]$:

$$E_0^{\mathbb{Q}}[1 + \ln^+(K/F_T)] = E_0^{\mathbb{Q}} W \left(e \frac{K}{\underline{F}_T} \right), \tag{50}$$

where $W(\cdot)$ denotes the Lambert W function. In words, the path-dependent claim paying $W\left(e\frac{K}{\underline{F}_T}\right)$ at its maturity date T with strike price $K < F_0 = 1$ has the same initial price as the co-terminal path-independent claim paying $1+\ln^+(K/F_T)$. We may consider (50) as the analog of the reflection principle for the first order martingale F. The next section shows that the relative pricing result in (50) survives intact when the underlying first order martingale F is generalized into *any* Ocone martingale. Put another way, the introduction of independent stochastic volatility with unknown dynamics does not change the relative pricing result in (50).

Of course, the contingent claims on each side of (50) do not trade outright in financial markets. However, Mellin transforms can be used to invert for the PDF of $\underline{F}_T$ implicit on the RHS of (50). Integration then yields the CDF of $\underline{F}_T$ which is just the value of a one touch. Since the LHS can be interpreted as the price of a portfolio of bonds and OTM put options, the inversion and subsequent integration gives a relative pricing relation between a one touch and a portfolio of bonds and co-terminal OTM puts. We indicate how this latter relative pricing relationship can be used to provide a semi-static replicating portfolio for a one touch.

For $K \in (0, 1]$, the payoff $(\ln K - \ln F_T)^+$ from the put on the log struck at $\ln K$ arises by combining the payoff from $\frac{1}{K}$ puts on F_T struck at K with the payoffs from $\frac{dJ}{J^2}$ puts on F_T struck at all strikes below K:

$$(\ln K - \ln F_T)^+ = \frac{1}{K}(K - F_T)^+ + \int_0^K \frac{1}{J^2}(J - F_T)^+ dJ. \tag{51}$$

Let $P_0(K)$ be the initial value of a European put struck at K and maturing at T. It follows from no arbitrage that for $K \in (0, 1]$:

$$E_0^{\mathbb{Q}}[1 + (\ln K - \ln F_T)^+] = 1 + \frac{1}{K}P_0(K) + \int_0^K \frac{1}{J^2}P_0(J)dJ. \tag{52}$$

In words, the path-independent payoff $[1 + (\ln K - \ln F_T)^+]$ has the same initial value as a static position in:

1. a co-terminal bond paying \$1 at T
2. $\frac{1}{K}$ co-terminal puts on F_T struck at K with each put paying $(K - F_T)^+$ at T, and:
3. an infinitesimal position in $\frac{dJ}{J^2}$ co-terminal puts struck at J, for each $J \in (0, K)$.

For any $K \geq 0$, the payoff from the put on the log struck at $\ln K$ decomposes into the sum of the payoffs from all binary puts on the log struck below $\ln K$:

$$(\ln K - \ln F_T)^+ = \int_{-\infty}^{\ln K} 1(\ln F_T \leq \ell) d\ell. \tag{53}$$

Evaluating (6) at t =T and taking logs implies that for the first order martingale:

$$\ln F_T = \underline{W}_T + \ln(1 + W_T^v). \tag{54}$$

Substituting (54) into (53) implies:

$$\begin{aligned}
(\ln K - \ln F_T)^+ &= \int_{-\infty}^{\ln K} 1(\underline{W}_T + \ln(1 + W_T^v) \leq \ell) d\ell \\
&= \int_{-\infty}^{\ln K} \int_0^{\infty} 1(\underline{W}_T + \ln(1 + k) \leq \ell)\delta(W_T^v - k) dk d\ell,
\end{aligned} \tag{55}$$

by the law of total probability. Thus, the payoff from the put on the log arises by combining the payoffs from a static position in co-terminal down-and-in butterfly spreads written on the terminal Brownian drawup. From no arbitrage, the put on the log also has the same value as an ensemble of co-terminal down-and-in butterfly spreads written on the terminal Brownian drawup:

$$E_0^{\mathbb{Q}} \ln^+(K/F_T) = \int_{-\infty}^{\ln K} \int_0^{\infty} E_0^{\mathbb{Q}} 1(\underline{W}_T \leq \ell - \ln(1 + k))\delta(W_T^v - k) dk d\ell. \tag{56}$$

Recall that the Appendix showed that for any barrier $j \leq 0$ and $k \geq 0$:

$$E_0^{\mathbb{Q}} 1(\underline{W}_T \leq j) 1(W_T^v > k) = 2E_0^{\mathbb{Q}} 1(W_T < j - k) = E_0^{\mathbb{Q}} 1(\underline{W}_T < j - k)$$

by the reflection principle. Differentiating w.r.t. k and negating:

$$E_0^{\mathbb{Q}} 1(\underline{W}_T \leq j)\delta(W_T^v - k) = E_0^{\mathbb{Q}} \delta(\underline{W}_T - (j - k)). \tag{57}$$

We now suppose that $K \in (0, 1)$ so that $\ell - \ln(1 + k) \leq 0$ for all $k \geq 0$. Evaluating (57) at $j = \ell - \ln(1 + k)$ and substituting the result in (56) implies that:

$$E_0^{\mathbb{Q}} \ln^+(K/F_T) = \int_{-\infty}^{\ln K} \int_0^{\infty} E_0^{\mathbb{Q}} \delta(\underline{W}_T - (\ell - \ln(1 + k) - k) dk d\ell. \tag{58}$$

By Fubini and the symmetry of the Dirac delta function:

$$
\begin{aligned}
E_0^{\mathbb{Q}} \ln^+(K/F_T) &= E_0^{\mathbb{Q}} \int_0^{\infty} \int_{-\infty}^{\ln K} \delta(\ell - (\ln((1+k)e^k) + \underline{W}_T)) d\ell dk \\
&= E_0^{\mathbb{Q}} \int_0^{\infty} \int_{-\infty}^{\infty} \delta(\ell - (\ln((1+k)e^k) + \underline{W}_T)) 1(\ell \leq \ln K) d\ell dk \\
&= E_0^{\mathbb{Q}} \int_0^{\infty} 1(\ln((1+k)e^k) + \underline{W}_T \leq \ln K) dk, \qquad (59)
\end{aligned}
$$

using the sifting property of the Dirac delta function. Exponentiating inside the indicator function implies that:

$$
E_0^{\mathbb{Q}} \ln^+(K/F_T) = E_0^{\mathbb{Q}} \int_0^{\infty} 1\left((k+1)e^{k+1} \leq e\frac{K}{\underline{F}_T}\right) dk. \qquad (60)
$$

Recall that the Lambert W function $W(x)$ is the inverse of xe^x and is increasing for $x > 0$. It follows that:

$$
E_0^{\mathbb{Q}} \ln^+(K/F_T) = E_0^{\mathbb{Q}} \int_0^{\infty} 1\left(k+1 \leq W\left(e\frac{K}{\underline{F}_T}\right)\right) dk = E_0^{\mathbb{Q}} W\left(e\frac{K}{\underline{F}_T}\right) - 1. \qquad (61)
$$

Hence, the path-independent payoff $1 + \ln^+\left(\frac{K}{F_T}\right)$ has the same initial price $\pi(K)$ as the path-dependent payoff $W\left(e\frac{K}{\underline{F}_T}\right)$, i.e.:

$$
\pi(K) \equiv E_0^{\mathbb{Q}}\left[1 + \ln^+\left(\frac{K}{F_T}\right)\right] = E_0^{\mathbb{Q}} W\left(e\frac{K}{\underline{F}_T}\right). \qquad (62)
$$

Evaluating (25) at $k - j = \ln J$ and at $t = 0$ with $W_0 = 0$, the distribution function of $\underline{F}_T$ is known in closed form:

$$
\mathbb{Q}_0\{\underline{W}_T \leq \ln J\} = \mathbb{Q}_0\{\underline{F}_T \leq J\} = 2N\left(\frac{\ln J}{\sqrt{T}}\right), \qquad J \in (0, 1]. \qquad (63)
$$

Differentiating w.r.t. J implies that the PDF of $\underline{F}_T$ is also known in closed form:

$$
\mathbb{Q}_0\{\underline{F}_T \in dJ\} = \sqrt{\frac{2}{\pi T}} e^{-\frac{(\ln J)^2}{2T}} \frac{dJ}{J}, \qquad J \in (0, 1]. \qquad (64)
$$

Hence, the expected value on the RHS of (62) can be written as an explicit integral:

$$
\pi(K) = E_0^{\mathbb{Q}} W\left(e\frac{K}{\underline{F}_T}\right) = \int_0^1 W\left(e\frac{K}{J}\right) \sqrt{\frac{2}{\pi T}} e^{-\frac{(\ln J)^2}{2T}} \frac{dJ}{J}. \qquad (65)
$$

The RHS of (65) is recognized as a multiplicative convolution in K of the function $w(\cdot) \equiv W(e\cdot)$ with the function $q(\cdot) \equiv \sqrt{\frac{2}{\pi T}} e^{-\frac{(\ln \cdot)^2}{2T}} 1(\cdot \in (0, 1))$, i.e.:

$$
\pi(K) = (w * q)(K), \qquad K > 0, \qquad (66)
$$

where the $*$ denotes multiplicative convolution. The Mellin transform of a function $f(J), J > 0$ is defined by:

$$\mathcal{M}_f(s) \equiv \int_0^\infty J^{s-1} f(J) dJ, \tag{67}$$

for s in some region in the complex plane where the integral converges. Taking the Mellin transform of both sides of (66):

$$\mathcal{M}_\pi(s) = \mathcal{M}_w(s)\mathcal{M}_q(s). \tag{68}$$

In words, the Mellin transform of π is just the product of the Mellin transform of w and the Mellin transform of q. In principle, the Mellin transform on the LHS of (68) can be directly observed from the smile. From Corless, Jeffrey, and Knuth [8], the Mellin transform of the Lambert W function on the RHS is:

$$\mathcal{M}_W(s) \equiv \int_0^\infty J^{s-1} W(J) dJ = \frac{(-s)^{-s}\Gamma(s)}{s}. \tag{69}$$

By a simple change of variables, the Mellin transform of $w(\cdot) \equiv W(e\cdot)$ is:

$$\mathcal{M}_w(s) \equiv \int_0^\infty J^{s-1} W(eJ) dJ = \frac{\Gamma(s)}{s(-es)^s}. \tag{70}$$

Substituting (70) in (67) implies:

$$\mathcal{M}_\pi(s) = \frac{\Gamma(s)}{(-es)^s s}\mathcal{M}_q(s). \tag{71}$$

One can solve (71) for the the Mellin transform of q:

$$\phi(s) \equiv \mathcal{M}_q(s) = \frac{s(-es)^s}{\Gamma(s)}\mathcal{M}_\pi(s). \tag{72}$$

Let $OT(L) \equiv \int_0^L q(J)dJ$ be the value of a one-touch with lower barrier $L \in (0, 1]$. The Mellin transform of the indefinite integral OT is related to the Mellin transform of the function q being integrated by:

$$\mathcal{M}_{OT}(s) = -\frac{\phi(s+1)}{s} = -\frac{(-e(s+1))^{s+1}}{\Gamma(s+1)}\mathcal{M}_\pi(s+1). \tag{73}$$

The Inverse Mellin transform of a function $f(s)$ is defined as:

$$\{\mathcal{M}^{-1} f\}(J) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} J^{-s} f(s) ds, \tag{74}$$

where c is any real such that the integral converges. Applying the inverse Mellin transformation to (73), we have:

$$OT(L) \equiv \mathbb{Q}_0\{\underline{F}_T \leq L\} = \left\{\mathcal{M}^{-1}\left(-\frac{(-es)^{s+1}}{\Gamma(s+1)}\mathcal{M}_\pi(s+1)\right)\right\}(L), \qquad L \in (0, 1]. \tag{75}$$

While this representation is certainly more complicated than integrating the simple formula for the PDF of $\underline{F}_T$ in (64), the advantage of the representation is that it holds unchanged when the dynamics of the underlying process are generalized to entertain independent stochastic volatility. We explore this further in the next section.

To replicate the payoff of a one-touch written on the first order martingale F, (75) indicates that one initially buys co-terminal puts for all strikes below some critical level. If the underlying first order martingale does not touch L by expiry, then all of these puts expire worthless. On the other hand, if the underlying first order martingale does touch L at some time prior to expiry, then at the first passage time, this portfolio of puts should be sold. The revenue generated is just sufficient to buy a bond paying one dollar at T. Once this bond is purchased, the payoff to the one-touch is replicated.

The next section shows that this semi-static replication result survives intact if the SBM W driving the first order martingale is generalized into an Ocone martingale.

6 Extension to Independent Stochastic Volatility

7 Summary and Extensions

We developed a positive continuous martingale denoted by F and analyzed it. In particular, we gave a closed form formula for the bivariate PDF $(\underline{F}_T, F_T^v)$. We showed that the PDF and CDF of F_T are both given by a bounded integral. We also explored the implication of symmetry for our first order martingale.

One can generalize our first order martingale by replacing the exponential function by the distribution function of a nonpositive random variable. One can treat these PAY martingales as the skeleton of some more complicated process arising by time change. If the time change is continuous and independent, we have the same link between the law of the level and the joint law of the minimum and drawup. If the stochastic clock is a subordinator, Weiner Hopf factorization can be explored. When two correlated SBM's act as drivers, we still have tractability since the joint law of the two minima is known. In the interests of brevity, these extensions are best left for future research.

8 Appendix

Consider a Down-and-In Binary Call written on W_T^v, the terminal drawup of an SBM, with in-barrier $j \le 0$ and strike price $k \ge 0$. The payoff at time T is $1(\underline{W}_T \le j, W_T^v \ge k)$ and prior to knocking in, the price is $1(\underline{W}_t > j)E_t 1(\underline{W}_T \le j, W_T^v \ge k)$. In this appendix, we will show that for $j \le 0$ and $k \ge 0$:

$$1(\underline{W}_t > j)E_t 1(\underline{W}_T \le j, W_T^v \ge k) = 1(\underline{W}_t > j)2E_t 1(W_T \le j - k). \tag{76}$$

In words, we claim that prior to knocking in, the Down-and-In Binary Call described above has the same price as 2 binary puts on W_T struck at $j - k$. The reason is that the two binary puts can be used to replicate the payoff of the DIBC. Suppose that for some $t \in [0, T]$, we have $\underline{W}_t > j$. If W stays above j between t and T, then the 2 puts expire worthless, as does the DIBC. If instead, W hits j between t and T, then at the first passage time, one of the binary puts can be sold. By the symmetry of SBM, the revenue generated from the sale of the binary put struck at $j - k$ is just sufficient to buy one binary call struck at $j + k$. After purchasing the call, the investor has a binary call and a binary put on W which are equally OTM. If the minimum does not sink further by T, then the binary call part of the position provides the desired payoff. If the minimum does sink further, then at each time that the minimum decreases, the whole position is sold and a new one is purchased which is centered at the running minimum. By the symmetry of SBM, the revenue generated by rolling down the binary put strike is just sufficient to cover the cost of rolling down the binary call strike. Again, the binary call part of the position provides the desired payoff.

We now also show that for $j \leq 0$ and $k \geq 0$:

$$1(\underline{W}_t > j) E_t 1(\underline{W}_T \leq j)(V_T^w - k)^+ = 1(\underline{W}_t > j) 2 E_t (j - k - W_T)^+. \tag{77}$$

In words, we claim that prior to knocking in, a Down-and-In Call written on the terminal drawup of an SBM has the same price as 2 puts struck at $j - k$. The reason is that the two puts can be used to replicate the payoff of the DIC. Suppose that for some $t \in [0, T]$, we have $\underline{W}_t > j$. If W stays above j between t and T, then the 2 puts expire worthless, as does the DIC. If instead, W hits j between t and T, then at the first passage time, one of the puts can be sold. By the symmetry of SBM, the revenue generated from the sale of the put struck at $j - k$ is just sufficient to buy one call struck at $j + k$. After purchasing the call, the investor has a strangle on W centered at j. If the minimum does not sink further by T, then the call part of the strangle provides the desired payoff. If the minimum does sink further, then at each time that the minimum decreases, the strangle is sold and a new one is purchased which is centered at the running minimum. By the symmetry of SBM, the revenue generated by rolling down the put strike is just sufficient to cover the cost of rolling down the call strike. Again, the call part of the strangle provides the desired payoff.

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