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# Optimal rates from eigenvalues

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ABSTRACT

A financial portfolio typically pays dividend based on its value. We show that there is a unique portfolio that pays the maximum dividend rate while remaining solvent, under appropriate assumptions. We also give a characterization of both the portfolio and the optimal dividend rate.



## 1. Introduction

Suppose that one has a model for certain market variables and wants to construct a positive valued financial portfolio that is a function of those market variables and also pays a dividend at a constant yield of $\delta$. It is natural to expect that if $\delta$ is too large than such a portfolio will go bankrupt (for us this simply means that its value would not remain positive). On the other hand, at the time of inception i.e. time zero, if one wants to create a market for such a portfolio[1] then one has to maximize that constant dividend yield $\delta$. Assuming that the portfolio at hand depends on some finite tuple of "recurrent" market variables, like interest rates, inflation rates, ratio of commodity prices etc, we characterize the maximum dividend yield and the corresponding unique portfolio that provides this constant maximum dividend yield, without going bankrupt.

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[1] The example of ETFs come to mind, except that one has to model several other costs etc. to be realistic, we do not do that here.

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More formally, consider a scenario where the value of a portfolio is simply a function of some economic uncertainty factor $X_t$. Suppose that the portfolio pays a continuous dividend at the constant rate of $\delta$, which is to be determined at the formation of a portfolio. We show that if $X_t$ is a strongly recurrent process (Donsker and Varadhan, 1976), for example a $d$-dimensional OU process, then the maximum dividend yield and the portfolio itself can be characterized by the eigenfunction of a certain second order PDE (see Theorem 2.1) .

As far as techniques in the paper are concerned, recently, Hansen and Scheinkman (2009) and Ross (2013) and others (for example Carr and Yu, 2012; Qin and Linetsky, 2014) have shown applications of principal eigenvalues and eigenfunctions and its appropriate generalizations to recovering market beliefs from option prices. Inspired by their ideas, in this note, our methods demonstrate an application of *generalized* principal eigenvalues (Pinsky, 1995) to our problem of calculating maximal dividend yields for financial portfolios.

We remark that Radner and Shepp (1996) and Jeanblanc-Piqué and Shiryaev (1995) did initiate study of the problem of determining optimal dividends for a given financial firm. However, their underlying model is different – theirs is an optimal control problem where the answer is again a dividend payment but it is a diffusion process that has to be optimized, while staving of insolvency indefinitely. Unlike our scenario, it is not a prespecified constant maximum dividend yield that has to be found while avoiding insolvency in the underlying model.

We also remark that, as a first guess, the reader may be tempted to think that the maximum constant dividend yield in question would always equal the long run average of the risk-free rate in the underlying model. However, this is only true if the risk-free rate is modeled as a constant stochastic process. Otherwise, large deviations in that stochastic process will lead to a different value for the maximum dividend yield.[2]

## 2. Our results

Our main result is summarized by the following statement:

**Theorem 2.1.** *Let $X_t$ be an Itô process in $\mathbb{R}^d$ and assume that another Itô process $P(X_t)$ denotes the price, where $P(\cdot)$ is assumed positive and twice differentiable, of a financial portfolio P driven by $X_t$ alone. Suppose P pays a continuously compounded dividend at the constant rate of $\delta$. If $X_t$ is a strongly recurrent process (see Definition 3.4) in the risk neutral measure then the maximum value of $\delta$, for which such a P exists, is given by the critical eigenvalue of a certain elliptic PDE (see (Eq. 4.3)). Moreover, such an optimal portfolio is unique, as far as its dependence on $X_t$, and it corresponds to that positive eigenfunction of the same PDE.*

The crux of the proof of Theorem 2.1 relies on investigating when the PDE in Eq. (4.3) has a unique positive solution. The proof proceeds in two parts. First, we need to show the existence of a portfolio for the optimal dividend rate. The proof of this is based on elementary arguments and is given in Chapter 4 of the textbook by Pinsky (1995). The fundamental existence theorem here states that positive solutions exist at or above the critical eigenvalue but not below this critical eigenvalue (Theorem 4.1). This critical eigenvalue corresponds to the optimal dividend rate.[3] Second, we need to show that the portfolio for the optimal dividend rate is unique. The proof of this relies on the papers by Donsker and Varadhan (1976) on large deviation theory. The idea is to assume that there is more than one positive solution at the critical eigenvalue, these solutions are all candidates for the optimal portfolio, even when the underlying process is strongly recurrent. Next, we prove that the critical eigenvalue increases with the zeroth order term, when the underlying process is strongly recurrent (Lemma 4.5). Furthermore, when there is more than one positive solution at the critical eigenvalue, we construct a positive solution by perturbing the zeroth order term by a positive function, but for the same eigenvalue, which is now less than the critical eigenvalue of the perturbed PDE (Lemma 4.6).

[2] The reader may eventually check that the quantum harmonic oscillator can serve as a simple toy example to illustrate this remark.

[3] It will be the negation of the optimal dividend rate.

However, no solutions are supposed to exist below the critical eigenvalue and so we obtain a contradiction. We remark that our main contribution is not in developing some fundamental mathematics, in fact most of the technical ideas existed in some form or the other in the earlier papers on large deviation theory (Donsker and Varadhan, 1975; 1976; Berestycki et al., 1994), but we did need to develop clever ways for putting them together and applying the results to financial theory.

## 3. Preliminaries

Most of the definitions below are adapted from Donsker and Varadhan (1976) and Pinsky (1995). But see also Varadhan (1978) and the references therein. The technical assumptions in this section are also from the above sources and will be needed for our statements, since they rely crucially on their results.

Let $X_t$ be an Itô diffusion in $\mathbb{R}^d$ of the form

$$dX_t = b(X_t)dt + a(X_t)dW_t, \tag{3.1}$$

where $W_t$ is $d$-dimensional standard Brownian motion in the risk neutral measure.[4] Let $p(t, x, dy)$ be the transition probability of $X_t$ and $D$ its state space. We assume that $p_t$ is Feller and that $D$ is $\mathbb{R}^d$. Throughout, we shall assume that $T_t$ is strongly continuous, it maps the bounded continuous functions to itself, and that $T_t f \geq 0$ when $f \geq 0$.

**Definition 3.1.** Define the semigroup of operators $\{T_t\}$ as:

$$(T_t f)(x) := \int_D f(y)p(t, x, dy). \tag{3.2}$$

Let $C(D)$ be the set of continuous functions on $D$. Throughout, let $L$ be the infinitesimal generator of the semigroup associated with $X_t$ acting on the domain $\mathcal{D} \subseteq C(D)$. In other words,

$$L = \sum_{i,j=1}^{d} \frac{a_{ij}(x)}{2} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial}{\partial x_i}. \tag{3.3}$$

**Assumption 3.2.** L satisfies the following assumption: For every closed $S \subset \mathbb{R}^d$, $a_{ij}$, $b_i \in C^{1,\alpha}(S)$ and $\Sigma a_{ij}(x)v_i v_j > 0$ for all $v \in \mathbb{R}^d \setminus \{0\}$ and $x \in S$.[5]

**Definition 3.3.** Let $C_L$ be the cone of all positive harmonic functions i.e., $C_L := \{u(x) \in C^{2,\alpha}(D) : Lu(x) = 0, u(x) > 0 \ in \ D\}$.[6] $L$ on $D$ is said to subcritical if it possesses a Green's function and therefore $C_L$ is non-empty. $L$ on $D$ is said to be critical if it does not possess a Green's function but $C_L$ is non-empty. Otherwise, $L$ is said to be supercritical. Moreover, when $D$ is not a compact manifold without boundaries, which is the case here, $L$ is supercritical is equivalent to $C_L$ being empty.

For Markov chains, it is known that positive recurrence is enough to ensure the existence of an invariant measure and conversely transience implies non-existence of any invariant measure. Analogously, Donsker and Varadhan (1976) define a stricter criterion than positive recurrence, which ensures the existence of invariant measure for Markov chains. Their corresponding recurrence criterion for stochastic processes is defined below.

**Definition 3.4.** Suppose there exists a $v(x)$ on $D$ such that $\{x \in D: v(x) \geq l\}$ is a compact set for each $l > -\infty$ and that there exists a sequence $\{u_n\}$ in $\mathcal{D}$ satisfying:

1. $u_n \geq 1$ for all $n$ and all $x \in D$
2. For each compact set $W \subset D$, $\sup_{x \in W} \sup_n u_n(x) < \infty$

[4] We avoid vector notation i.e., bold faces, for the coefficients but the meaning is clear from the context. Note that we assume that $a$ and $b$ are functions of $X_t$ alone.

[5] $C^{1,\alpha}$ denotes differentiable functions that are Hölder continuous with exponent $\alpha$.

[6] In this particular definition, $L$ is meant to include zeroth order terms.

3. For each $x \in D$,

$$\lim_{n\to\infty}\left(\frac{Lu_n}{u_n}\right)(x) = v(x) \tag{3.4}$$

4. For some $C < \infty$,

$$\sup_{n,x}\left(\frac{Lu_n}{u_n}\right)(x) \leq C \tag{3.5}$$

then $X_t$ (corresponding to $L$) is said to be strongly recurrent.

**Assumption 3.5.** Throughout, we will assume that $X_t$ is strongly recurrent.

In particular, Standard Brownian Motion is not strongly recurrent but the Ornstein-Uhlenbeck process is strongly recurrent (see Section 9 of Donsker and Varadhan, 1976).

**Definition 3.6.** Let $\mathcal{D}^+$ be functions with a positive lower bound in $\mathcal{D}$. Define the rate functional $I(\mu)$ as:

$$I(\mu) := -\inf_{u\in\mathcal{D}^+}\int_D\left(\frac{Lu}{u}\right)(x)\mu(dx). \tag{3.6}$$

## 4. Maximal dividend rates

Our goal is to study the existence and uniqueness of a portfolio $P$ or rather its price process, which evolves as a positive valued Itô diffusion $P(X_t)$, with the following characteristics:[7]

1. $P$ can be traded.
2. $P$ pays continuous dividends proportional to its price and at the constant rate of $\delta$ to its holder. We want to maximize $\delta$, while keeping $P(\cdot)$ positive.[8]

Suppose that we construct another portfolio $\bar{P}$, which was based on the same underlying $X_t$ as $P$, but which pays no dividends. In a time interval $[t, t+dt]$, the percentage change in price $P$ is related to $\bar{P}$ and $\delta$ as follows:

$$\frac{d\bar{P}_t}{\bar{P}_t} = \delta dt + \frac{dP_t}{P_t}. \tag{4.1}$$

However, $\bar{P}$ is what is termed as a self-financing portfolio, and mathematically, the expected percentage value of $\bar{P}$ in the risk-neutral measure will equal the risk free rate. Therefore,

$$\text{drift}\left(\delta dt + \frac{dP_t}{P_t}\right) = r(X_t). \tag{4.2}$$

By Itô's formula we get that $P(X_t)$, viewed as a function of $x$, obeys the following elliptic PDE:

$$Lu(x) - r(x)u(x) + \delta u(x) = 0, \tag{4.3}$$

where $L$ is the infinitesimal generator defined in Section 3.

Given processes $X_t$ and $r(X_t)$ in the risk-neutral measure, and a fixed choice of $\delta$, the existence and uniqueness of $P(X_t)$ corresponds to the existence and uniqueness of a positive solution to the PDE (4.3). In finance terms, if the positive valued solution $u(\cdot)$ does not exist then all solutions must reach zero, hence there is no solvent portfolio which pays dividend $\delta$, given $X_t$ and $r(X_t)$. Our first problem now is to investigate the PDE for the maximum value of $\delta$ at which a positive solution exists. The following statement is immediate from Theorem 3.2 in Chapter 4 of Pinsky (1995).

[7] Below $P_t$ is short hand for $P(X_t)$.

[8] We do not need the maximal $\delta$ to be positive for our goal to be non-trivial. The portfolio may not exist even if $\delta$ is negative – one may charge fees, but if the risk-free rate is negative then it is possible the fees are not high enough to compensate and $P(\cdot)$ may not exist.

**Theorem 4.1** (Pinsky, 1995)**.** *Let $L$ be an elliptic operator with domain $D \subseteq \mathbb{R}^d$ as mentioned before and $V(x)$ be continuous and bounded from above, then there exists a $\lambda_c$ such that $L + V - \lambda$ is supercritical for $\lambda < \lambda_c$, subcritical for $\lambda > \lambda_c$ and either subcritical or critical for $\lambda = \lambda_c$. Equivalently,*

$$Lu(x) + V(x)u(x) - \lambda u(x) = 0 \tag{4.4}$$

*has at least one positive solution for $\lambda \geq \lambda_c$ and no positive solutions otherwise.*[9]

Now, if we replace $V(x) = -r(x)$ and $\lambda = -\delta$ in Eq. (4.4), then we obtain Eq. (4.3). Moreover, as long as $r(x)$ is bounded from below,[10] Theorem 4.1 implies that Eq. (4.3) has a positive solution as long as $\delta \leq \delta_c$ (for some critical $\delta_c$). We note that $\lambda_c$ is referred to as the *generalized principal eigenvalue* of $L$. Throughout this paper we will merely assume that $r(x)$ and $V(x)$ are bounded functions, they need not be sign definite.

The previous discussion already implies the existence result in Theorem 2.1. It now remains to study uniqueness of the portfolio $P$ and characterize the maximum dividend rate $\delta$. This will take much of the space in the rest of the paper.

The following lemma relies on the results in Donsker and Varadhan (1976) (see also Varadhan, 1978).

**Lemma 4.2.** *Let $L$ be strongly recurrent. For the operator $L + V$, the critical value $\lambda_c$ is given by:*

$$\lambda_c(V) = \sup_{\mu \in \mathcal{M}_D} \left( \int_D V(x)\mu(dx) - I(\mu) \right), \tag{4.5}$$

*where $\mathcal{M}_D$ is the set of probability measures supported on $D$, the state space of $X_t$.*

**Proof.** As mentioned before, the theory behind generalized principal eigenvalues is very well developed and so before going into the argument it seems worthwhile to discuss what is being proven here.[11]

Theorem 4.4 in Chapter 4 of Pinsky (1995) characterizes $\lambda_c$ as follows. □

**Theorem 4.3** (Pinsky, 1995)**.** *Let $V$ be continuous and bounded, then:*

$$\lambda_c(D, V) = \sup_{\substack{A \subset\subset D, \\ \partial A \text{ is } C^{2,\alpha}}} \lim_{t \to \infty} \frac{1}{t} \log \sup_{y \in A} \mathbb{E}_y \left[ \exp\left( \int_0^t V(X_s) ds \right); \tau_A > t \right], \tag{4.6}$$

*where $A \subset\subset D$ stands for $A$ is bounded and the closure of $A$ is a proper subset of $D$, and $\tau_A$ is the first exit time of $X_t$ from $A$.*

On the other hand, for strongly recurrent processes Donsker and Varadhan (1976) provide the following characterization.

**Theorem 4.4** (Donsker and Varadhan, 1976)**.** *Let $L$ be strongly recurrent then*

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}_y \left[ \exp\left( \int_0^t V(X_s) ds \right) \right] = \sup_{\mu \in \mathcal{M}_D} \left( \int_D V(x)\mu(dx) - I(\mu) \right), \quad y \in D, \tag{4.7}$$

*where $V(x)$ is continuous.*

Hypothetically, if one could interchange the order of the outer sup and lim in Eq. (4.6) then one obtains Eq. (4.7) and the statement of our lemma follows. However, it is not true that one can simply switch the order and a counterexample is provided by Brownian Motion with constant drift (see also Donsker and Varadhan, 1976). Suppose that $V \equiv 0$ so that the RHS of Eq. (4.6) evaluates to:

$$\lambda_c(D, V) = \sup_{\substack{A \subset\subset D, \\ \partial A \text{ is } C^{2,\alpha}}} \lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}_x[\tau_A > t], \quad x \in D, \tag{4.8}$$

[9] Throughout, we assume no boundary or initial value condition. However, Linearity of Eq. (4.4) and positive scaling ensures that we can equally well assume the time 0 value of $P$ as specifying the boundary condition.

[10] Typically, this is true since the short rate is usually non-negative.

[11] More detailed sources for such a discussion are Chapter 4 of Pinsky (1995) and the papers by Donsker and Varadhan (1976); 1975).

which clearly depends on the drift of the Brownian Motion. On the other hand, the LHS of Eq. (4.7) is a constant independent of the drift. Clearly, the two cannot be equal. Therefore, there is more to it and the assumption of strong recurrence indeed suffices to ensure the lemma. The rest of the proof uses a limiting argument on the compact subsets of $D$ (cf. Berestycki et al., 1994) and relies on one of the results in Donsker and Varadhan (1976).

Theorem 6.1 in Chapter 3 of Pinsky (1995) and Donsker and Varadhan (1975) together provide the following characterization of the principal eigenvalue, when the domain $D_i$ is compact.

$$\lambda_c(D_i, V) = \sup_{\mu_i \in \mathcal{M}_{D_i}} \left( \int_{D_i} V(x)\mu_i(dx) - I(\mu_i) \right). \tag{4.9}$$

Consider any increasing sequence of compact domains, each a subset of $D$, so that $D_0 \subset D_1 \ldots \subset D_i \subset D_{i+1} \ldots$ and the sequence goes to $D$. Now as $i \to \infty$, Theorem 4.1 in Chapter 4 of Pinsky (1995) shows that $\lambda_c(D_i, V)$ tends to the generalized principal eigenvalue $\lambda_c(D, V)$ of $L$ on $D$.

Lemma 7.1 in Donsker and Varadhan (1976) shows that if $L$ is strongly recurrent then $\{\mu \in \mathcal{M}_D : I(\mu) \le l\}$ is a compact set. Furthermore, since $V(x)$ is assumed to be continuous and bounded, $\Phi$, where $\Phi(\mu) := \int_D V(x)\mu(dx)$, attains its extremum values (Weierstrass' extreme value theorem). Therefore, under the assumption of strong recurrence, the supremum in the RHS of Eq. (4.5) is attained for some $\mu'$. Similarly, for each $D_i$ there exists a $\mu_i$ such that the RHS of Eq. (4.9) attains its supremum at $\mu_i$.

Let $\Psi(D, \mu')$ denote the RHS of Eq. (4.7) at the suprema i.e.,

$$\Psi(D, \mu') := \left( \int_D V(x)\mu'(dx) + \inf_{u \in \mathcal{D}^+} \int_D \frac{Lu}{u}\mu'(dx) \right), \tag{4.10}$$

and similarly we define $\Psi(D_i, \mu_i)$. Furthermore, as $i \to \infty$, $\Psi(D_i, \mu_i)$ is non-decreasing and by its definition $\Psi(D, \mu') \ge \Psi(D_i, \mu_i)$, for all $i$. In the domain $D_i$ define $\Psi_{D_i}(D, \mu')$ as

$$\Psi_{D_i}(D, \mu') := \left( \int_{D_i} V(x)\mu'(dx) + \inf_{u \in \mathcal{D}^+} \int_{D_i} \frac{Lu}{u}\mu'(dx) \right) \tag{4.11}$$

i.e., it is the same as $\Psi(D, \mu')$ but with the integrals restricted to $D_i$. Then we have $\Psi(D_i, \mu_i) \ge \Psi_{D_i}(D, \mu')$. Furthermore, we have assumed that $V(x)$ is bounded, and if we select a sequence of functions $\{u_n\}$, as in the definition of strong recurrence, we observe that $\frac{Lu_n}{u_n}$ is bounded for $u_n \in \mathcal{D}^+$ and $n \in \mathbb{N}$. Therefore, as $i \to \infty$, $\Psi_{D \backslash D_i}(D, \mu')$ i.e., $\Psi(D, \mu')$ with the integrals restricted to $D \backslash D_i$ goes to zero. Hence, the difference:

$$\Psi(D, \mu') - \Psi(D_i, \mu_i) \tag{4.12}$$

goes to zero.

We have shown that RHS of Eq. (4.5) equals the limit of the RHS of Eq. (4.9) as $i \to \infty$, and the latter equals the generalized principal eigenvalue. Hence our lemma is proved.

**Lemma 4.5.** *Given operator $L + V$ and a positive function $w(x)$, such that the diffusion corresponding to $L$ is strongly recurrent, we have:*

$$\lambda_c(V + \varepsilon w) > \lambda_c(V), \qquad \varepsilon > 0. \tag{4.13}$$

**Proof.** Recall that, under the assumption of strong recurrence, the supremum in the RHS of Eq. (4.5) is attained. Let $\mu_0$ be such that:

$$\lambda_c(V) = \int_D V(x)\mu_0(dx) - I(\mu_0). \tag{4.14}$$

Similarly,

$$\lambda_c(V + \varepsilon w) = \int_D (V(x) + \varepsilon w)\mu_0'(dx) - I(\mu_0') \ge \int_D V(x)\mu_0(dx) - I(\mu_0) + \int_D \varepsilon w \mu_0'(dx) > \lambda_c(V).$$

The second to last inequality follows from the properties of the supremum in the definition of $\lambda_c$ and the last inequality follows from the positivity of $\varepsilon w$. $\square$

**Lemma 4.6.** *If $Lu(x) + V(x)u(x) - \lambda_c(V)u(x) = 0$ possesses at least two positive solutions then $Lu(x) + (V(x) + w(x))u(x) - \lambda_c(V)u(x) = 0$ possesses at least one positive solution, for some judicious choice of function $w(x)$, where $w(x)$ is positive.*

**Proof.** Given

$$Lu(x) + V(x)u(x) - \lambda_c(V)u(x) = 0, \tag{4.15}$$

assume that, without loss of generality, $\lambda_c(V) = 0$. Let $u_1(x)$ and $u_2(x)$ be the two distinct solutions of Eq. (4.15) with $\lambda_c(V) = 0$.

Let $f(x) := \sqrt{u_1(x)}$, $g(x) := \sqrt{u_2(x)}$ and let $v(x) := f(x)g(x)$.

$$\begin{aligned} Lv &= \left(\frac{a(x)}{2}\nabla^2 + b(x)\nabla\right) fg \\ &= \frac{a(x)}{2}\left(2\nabla f \nabla g + f\nabla^2 g + g\nabla^2 f\right) + b(x)(f\nabla g + g\nabla f) \end{aligned} \tag{4.16}$$

Since $Lu_1(x) + Vu_1(x) = 0$ and $Lu_2(x) + Vu_2(x) = 0$, we have:

$$V \cdot f(x)g(x) = -\frac{g}{f}Lf^2, \quad V \cdot f(x)g(x) = -\frac{f}{g}Lg^2. \tag{4.17}$$

Therefore,

$$\begin{aligned} -2V \cdot v(x) &= \left(\frac{g}{f}Lf^2 + \frac{f}{g}Lg^2\right) \\ &= a(x)\left(g\nabla^2 f + \frac{g}{f}(\nabla f)^2 + f\nabla^2 g + \frac{f}{g}(\nabla g)^2\right) \\ &\quad + b(x)(2g\nabla f + 2f\nabla g). \end{aligned} \tag{4.18}$$

Adding Eqs. (4.16) and (4.18) and canceling out the common terms, we get:

$$\begin{aligned} Lv(x) + V \cdot v(x) &= \frac{a(x)}{2}\left(\frac{g}{f}(\nabla f)^2 + \frac{f}{g}(\nabla g)^2 + 2\nabla f \nabla g\right) \\ &= -\left\langle \frac{a(x)}{2} fg\left(\frac{\nabla g}{g} - \frac{\nabla f}{f}\right), \left(\frac{\nabla g}{g} - \frac{\nabla f}{f}\right)\right\rangle. \end{aligned}$$

We have shown that $v(x)$ will satisfy:

$$Lv + (V(x) + w(x))v(x) = 0, \tag{4.19}$$

where $w(x)$ is positive unless $u_1$ is proportional to $u_2$. □

Theorem 4.1 shows that below a critical value of $\lambda$ there are no positive solutions to Eq. (4.4). Lemma 4.5 shows that $\lambda_c$ is increasing as a functional of the zeroth order term. Finally, assuming the existence of two different positive solutions to Eq. (4.4) at $\lambda = \lambda_c$, Lemma 4.6 constructs a positive solution for a PDE with a positive perturbation to the zeroth order term, one where $\lambda < \lambda_c$. However, the last three statements provide a contradiction and so the assumption in Lemma 4.6 is false i.e., for strongly recurrent processes Eq. (4.3) has a unique solution at the critical eigenvalue.

### *4.1. Is strong recurrence necessary?*

So far, we have shown that strong recurrence of the underlying driver suffices to uniquely characterize the optimal portfolio and dividend rate. The following lemma shows that mere positive recurrence of the underlying diffusion process is not enough for a unique characterization.

**Lemma 4.7.** *There exists $X_t$, which is positively recurrent but not strongly recurrent, and if $L$ is the generator of $X_t$ then at $\lambda = \lambda_c$, the corresponding equation as 4.4, for appropriate choice of $V(x)$, has more than one positive solution.*

**Proof.** The example is in one dimension over $\mathbb{R}$. Let

$$L = \frac{\partial^2}{\partial x^2} - \frac{2x}{1+x^2}\frac{\partial}{\partial x}, \tag{4.20}$$

and let $V(x) := \frac{-4}{1+x^2}$. First, observe that $L$ is positive recurrent, and this can be checked, for example, using the criterion in Corollary 1.11 in Chapter 5 of Pinsky (1995). Moreover, this choice of $L$ and $V$ ensures that:

$$h(x) = (1+x^2)^2 \tag{4.21}$$

is a solution of $L_V u(x) = 0$, where the equation we have in mind is:

$$L_V u(x) := \frac{\partial^2 u(x)}{\partial x^2} - \frac{2x}{1+x^2}\frac{\partial u(x)}{\partial x} - \frac{4u(x)}{1+x^2} = 0. \tag{4.22}$$

Now, we need to show that: (1) There exists another positive solution to Eq. (4.22), and (2) There are no positive solutions to Eq. 4.22 if the RHS is replaced by $-\lambda u$ (for $\lambda > 0$), or equivalently that $\lambda_c = 0$.

To see that (1) is true, we can use the criteria given in Theorem 1.4 in Chapter 5 of Pinsky (1995) to observe that a Green's function exists for Eq. (4.22). Moreover, in one dimension, the criterion given in Proposition 1.3 in Chapter 5 of Pinsky shows that such an equation i.e., $L_V u(x) = 0$, must have two independent positive solutions.

To see that (2) is true, we will show that $\lambda_c = 0$ for $L_V$. We will work with $L_V^h$, the $h$-transform of $L_V$ (see for eg. Chapter 4 in Pinsky (1995)), and show that $\lambda_c = 0$ for $L_V^h$. Since the eigenvalues of $L_V^h$ and $L_V$ coincide (Theorem 3.3 in Chapter 4 of Pinsky (1995)), we will obtain the desired result.

In one dimension, $L_V^h u(x)$ is defined as follows:

$$L_V^h u(x) = Lu(x) + a(x)\frac{h'(x)}{h(x)} \cdot \frac{\partial u(x)}{\partial x} + \frac{L_V h(x)}{h(x)} \cdot u(x). \tag{4.23}$$

In our case, $a(x) = 2$ and $h(x)$ is a solution of $L_V u(x) = 0$, so there is no zeroth order term and we have:

$$L_V^h u(x) = Lu(x) + a(x)\frac{h'(x)}{h(x)} \cdot \frac{\partial u(x)}{\partial x} = \frac{\partial^2 u(x)}{\partial x^2} + \frac{6x}{1+x^2}\frac{\partial u(x)}{\partial x}. \tag{4.24}$$

Finally, we need the following observation, which is Exercise 5.4 in Pinsky (1995).

**Observation 4.8.** Let $L = \frac{1}{2}\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}$ on $(0, \infty)$. If $\lim_{x\to\infty}(b^2(x) + b'(x)) = 0$, then $\lambda_c = 0$.

Although our domain is $(-\infty, \infty)$, Eq. (4.24) remains unchanged when one replaces $x$ by $-x$. In other words, any positive solution over $(0, \infty)$ is a positive solution over $(-\infty, \infty)$ and vice versa. Hence, Observation 4.8 remains applicable in our case even when the domain is $(-\infty, \infty)$ and a simple calculation shows $\lambda_c$ is indeed 0. □

### *4.2. Future work: portfolios with costs*

Consider a portfolio $P_t$, which pays a continuously compounded dividend at the constant rate $\delta$ (to be determined), but which also charges a constant cash amount of $\exp\{k\}$ (known in advance). Using the standard risk-neutral approach, such a portfolio would then be represented by the inhomogeneous PDE:

$$Lu(x) - r(x)u(x) + \delta u(x) = k. \tag{4.25}$$

We now want to maximize the payment $\delta$ so that the equation still has a unique positive solution. As long as $k$ is positive the existence of a Green's function is enough to ensure the existence of a positive solution to Eq. (4.25). However, observe that any positive scalar multiple of a positive solution to the homogeneous PDE:

$$Lu(x) - r(x)u(x) + \delta u(x) = 0. \tag{4.26}$$

can be added to the positive solution of Eq. (4.25) and it still remains a positive solution of the latter. Therefore, a necessary criterion for recovering a unique positive valued portfolio from Eq. (4.25) is that Eq. (4.26) have no positive solutions - so $\delta > \delta_c$ in Eq. (4.26). We leave the question of characterizing when a portfolio is uniquely recoverable, in the presence of a cost function, for future consideration.

## Disclaimers

The views represented herein are the authors own views and do not necessarily represent the views of the authors employers and are not a product of authors employers or their affiliates. The aim of this paper is to present academic research by the authors and not to suggest any investing ideas. The authors do not bear responsibility for any loss if anyone interprets it otherwise.

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