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LMI criteria for robust chaos synchronization of a class of chaotic systems

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Abstract

Based on the Lyapunov stability theory and LMI technique, a new sufficient criterion, formulated in the LMI form, is established in this paper for chaos robust synchronization by linear-state-feedback approach for a class of uncertain chaotic systems with different parameters perturbation and different external disturbances on both master system and slave system. The new sufficient criterion can guarantee that the slave system will robustly synchronize to the master system at an exponential convergence rate. Meanwhile, we also provide a criterion to find out proper feedback gain matrix K that is still a pending problem in literature [H. Zhang, X.K. Ma, Synchronization of uncertain chaotic systems with parameters perturbation via active control, Chaos, Solitons and Fractals 21 (2004) 39–47]. Finally, the effectiveness of the two criteria proposed herein is verified and illustrated by the chaotic Murali–Lakshmanan–Chua system and Lorenz systems, respectively.

Keywords: Parameters perturbation; Robust synchronization; LMI (linear matrix inequality); Chaotic system

1. Introduction

Since the seminal work by Pecora and Carroll [1], chaos synchronization has been extensively investigated in the past decades [2–15]. Numerous methods have been proposed to cope with the chaos synchronization, such as sampled-data feedback synchronization method [3], impulsive control method [4], adaptive design method [5,6], invariant manifold method [7], backstepping design method [8], sliding mode control method [9,10], and other control methods [11–15].

Recently, Jiang and Zheng [16] established a LMI criterion for linear-state-feedback based chaos synchronization of a class of chaotic systems. However, the criterion proposed in [16] did not consider the parameters perturbation and external disturbances. Although the simulation shows that the proposed method in [16] is robust to the external noise, no rigorous mathematical proof was provided. In fact, system parameters are inevitably disturbed by the external force or other factors. Based on active technique, Zhang and Ma [17] proposed a controller to cope with it. However, the proposed method in [17] also did not consider the external disturbances on both master and slave systems and

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the parameters perturbation on two systems are identical, not different. Although [17] give out a theorem to judge whether two systems can be synchronized, no effective method was proposed to find out the gain matrix K. In some real synchronization systems, the parameters perturbations on master and slave systems are inevitably existing and not identical. In this case, the application of the above methods are limited in Refs. [16,17], and it is very important for real chaotic systems to implement chaos synchronization instead.

In this paper, we will take the time-varying perturbation of parameters and external disturbances into full account for the synchronization problem of two uncertain chaotic systems. Based on linear-state-feedback technique, an easily verified sufficient condition, formulated in the LMI form, is developed for chaos robust synchronization, which is applicable to a large class of general uncertain chaotic systems. The proposed criterion can ensure chaos synchronization at an exponential convergence rate. On the other hand, a corollary is developed to solve the gain matrix K that is a pending problem in literature [17]. Finally, the effectiveness and feasibility of the proposed LMI criteria are numerically verified.

The paper is organized as follows. In Section 2, the chaos synchronization problem under investigation is presented. In Section 3, two LMI criteria are presented for robust chaos synchronization for a class of uncertain chaotic systems. In Section 4, the effectiveness of the proposed criteria are illustrated by the chaotic Murali–Lakshmanan–Chua system [16] and Chen chaotic system [17], respectively. Finally, some concluding remarks are included in Section 5.

Notation: The notation $\|x\|$ is the Euclidean norm of the vector x, while $\|A\| = \sqrt{\rho(A^{\mathrm{T}}A)}$ denotes the spectral norm of the matrix A, where $\rho(M)$ denotes the spectral radius of the matrix M.

2. Problem formulation

Assume that there are two identical uncertain chaotic systems with parameters perturbation and external disturbances on both master system and slave system. The master system is given by

$$\dot{x} = (A + \Delta A_1(t))x + f(x, t) + d_1(t) \tag{1}$$

where $x \in R^n$ denotes the state vector, A is an $n \times n$ system matrix, $\Delta A_1(t)$ is an $n \times n$ perturbation matrix bounded by $\|\Delta A_1(t)\| \le \delta_1$, $d_1(t)$ is external disturbance bounded by $\|d_1(t)\| \le d_1$ and $f(x, t)$ is a nonlinear continuous function, satisfying Lipschitz condition, namely

$$\|f(x, t) - f(y, t)\| \le L\,\|x - y\| \tag{2}$$

and the slave system is described as

$$\begin{aligned} \dot{\widehat{x}} &= (A + \Delta A_2(t))\,\widehat{x} + f(\widehat{x}, t) + d_2(t) + B(\widehat{y} - y) + \alpha \\ y &= Kx, \qquad \widehat{y} = K\widehat{x} \end{aligned} \tag{3}$$

where $\widehat{x} \in R^n$ denotes the state vector, $K \in R^{1\times n}$ is a feedback gain, $B \in R^{n\times 1}$ is chosen such that (A, B) is controllable, $\Delta A_2(t)$ is an $n \times n$ perturbation matrix bounded by $\|\Delta A_2(t)\| \le \delta_2$, $d_2(t)$ is external disturbance bounded by $\|d_2(t)\| \le d_2$ and $\alpha \in R^{n\times 1}$ is a nonlinear input.

From the above, there must exist two positive constants δ_3, d, such that $\|\Delta A_1(t) - \Delta A_2(t)\| \le \delta_3$ and $\|d_1(t) - d_2(t)\| \le d$.

Define error $e = \widehat{x} - x$, from (3), (2) and (1), we can get the following error dynamical system.

$$\dot{e} = (A + BK)e + f(\widehat{x}, t) - f(x, t) + d_2(t) - d_1(t) + \Delta A_2(t)\,\widehat{x} - \Delta A_1(t)x + \alpha. \tag{4}$$

Our main objective is to find an appropriate feedback gain K, B and nonlinear input α such that the trajectory of the slave system (3) exponentially asymptotically approaches the master system (1) and finally implements synchronization, in the sense that

$$\lim_{t\to\infty} \|e\| = 0. \tag{5}$$

Remark 1. The introduction of α is just to obviate the impact of external disturbance and parameters perturbation.

Lemma 1. *Let x, y be real vectors of appropriate dimensions, A, $B(t)$ are real matrices of appropriate dimensions, $\|B(t)\| \le r$ and with any scalar $\varepsilon > 0$, we have*

$$2x^{\mathrm{T}}A^{\mathrm{T}}B(t)y \le \varepsilon x^{\mathrm{T}}A^{\mathrm{T}}Ax + \frac{r^2}{\varepsilon}y^{\mathrm{T}}y. \tag{6}$$

Proof. Since

$$\left(\sqrt{\varepsilon}Ax - \frac{1}{\sqrt{\varepsilon}}B(t)y\right)^{\mathrm{T}}\left(\sqrt{\varepsilon}Ax - \frac{1}{\sqrt{\varepsilon}}B(t)y\right) \ge 0 \tag{7}$$

and expanding the left of the (7), we obtain

$$\begin{aligned} x^{\mathrm{T}}A^{\mathrm{T}}B(t)y + y^{\mathrm{T}}B^{\mathrm{T}}(t)Ax &\le \varepsilon x^{\mathrm{T}}A^{\mathrm{T}}Ax + \frac{1}{\varepsilon}y^{\mathrm{T}}B^{\mathrm{T}}(t)B(t)y \\ &\le \varepsilon x^{\mathrm{T}}A^{\mathrm{T}}Ax + \frac{r^2}{\varepsilon}y^{\mathrm{T}}y. \end{aligned}$$

This completes the proof. □

3. Main results

Theorem 1. *If a suitable matrix B is chosen such that (A, B) is controllable, a suitable feedback gain K and nonlinear input α, such that*

$$P(A+BK) + (A+BK)^{\mathrm{T}}P + (\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)P^2 + (\varepsilon_1^{-1}\delta_2^2 + \varepsilon_3^{-1}L^2)I + 2\eta P < 0 \tag{8}$$

$$\alpha = \frac{-(\varepsilon_2^{-1}\delta_3^2 x^{\mathrm{T}}x + \varepsilon_4^{-1}d^2)}{2\|e\|^2}P^{-1}e \tag{9}$$

where P is a positive definite symmetric matrix, I is the identity matrix, ε_1, ε_2, ε_3, ε_4 and η are positive constants, then the error dynamical system (4) is globally exponentially stable, implying that the coupled system (1) and (3) are globally exponentially synchronized.

Proof. We select the Lyapunov function as

$$V = e^{\mathrm{T}}Pe. \tag{10}$$

Differentiating the (10) with respect to t along the trajectory of system (4), we have

$$\begin{aligned} \dot{V} &= ((A+BK)e + f(\widehat{x},t) - f(x,t) + d_2(t) - d_1(t) + \Delta A_2(t)\widehat{x} - \Delta A_1(t)x + \alpha)^{\mathrm{T}}Pe \\ &\quad + e^{\mathrm{T}}P((A+BK)e + f(\widehat{x},t) - f(x,t) + d_2(t) - d_1(t) + \Delta A_2(t)\widehat{x} - \Delta A_1(t)x + \alpha) \\ &= e^{\mathrm{T}}(P(A+BK) + (A+BK)^{\mathrm{T}}P + \Delta A_2^{\mathrm{T}}(t)P + P\Delta A_2(t))e + 2(f(\widehat{x},t) - f(x,t))^{\mathrm{T}}Pe \\ &\quad + 2e^{\mathrm{T}}P\alpha + 2e^{\mathrm{T}}P(\Delta A_2(t) - \Delta A_1(t))x + 2e^{\mathrm{T}}P(d_2(t) - d_1(t)). \end{aligned}$$

By using Lemma 1, we have

$$2e^{\mathrm{T}}P\Delta A_2(t)e \le \varepsilon_1 e^{\mathrm{T}}P^2e + \varepsilon_1^{-1}\delta_2^2 e^{\mathrm{T}}e \tag{11}$$

$$2e^{\mathrm{T}}P(\Delta A_2(t) - \Delta A_1(t))x \le \varepsilon_2 e^{\mathrm{T}}P^2e + \varepsilon_2^{-1}\delta_3^2 x^{\mathrm{T}}x \tag{12}$$

$$2(f(\widehat{x},t) - f(x,t))^{\mathrm{T}}Pe \le \varepsilon_3 e^{\mathrm{T}}P^2e + \varepsilon_3^{-1}L^2e^{\mathrm{T}}e \tag{13}$$

$$2e^{\mathrm{T}}P(d_2(t) - d_1(t)) \le \varepsilon_4 e^{\mathrm{T}}P^2e + \varepsilon_4^{-1}d^2. \tag{14}$$

Since (8), (9) and (11)–(14) hold, we have

$$\begin{aligned} \dot{V} &\le e^{\mathrm{T}}(P(A+BK) + (A+BK)^{\mathrm{T}}P + (\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)P^2 + (\varepsilon_1^{-1}\delta_2^2 + \varepsilon_3^{-1}L^2)I)e \\ &\le -2\eta e^{\mathrm{T}}Pe = -2\eta V < 0. \end{aligned}$$

Based on the Lyapunov stability theory, the error dynamical system (4) is globally exponentially stable. Thus, two systems (1) and (3) are globally exponentially synchronized. This completes the proof. □

Remark 2. It is easy to find that the α will approach infinity as e approaches zero. Hence, the Eq. (9) is just conceptual formulae. In practice, we can set the lower norm bound for e to prevent α from approaching infinity. If α is replaced by α^* such that

$$\alpha^* = \begin{cases} \alpha & \|e\| \geq r \\ \dfrac{-(\varepsilon_2^{-1}\delta_3^2 x^{\mathrm{T}} x + \varepsilon_4^{-1} d^2)}{2r^2} P^{-1} e & \|e\| < r \end{cases}$$

where r is an adjustable parameter, then the error will be bounded by $\|e\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} r$ eventually (proof see Appendix). Therefore, our method can synchronize two uncertain systems within finite accuracy. In a practical sense, it is meaningful.

Remark 3. The aim of introduction of positive constants $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ is to increase the flexibility of the design.

Remark 4. From (9), we find that if $d = 0$, i.e., $d_1(t) = d_2(t)$ and $\delta_3 = 0$, i.e., $\Delta A_1(t) = \Delta A_2(t)$ then the extra nonlinear input $\alpha = 0$, which has been investigated in literature [17], but which did not provide a way to find out the gain matrix K, and Corollary 1 will provide a criterion to find it out.

Theorem 2. *If there exist positive constants $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \eta$ and matrices $X > 0$, W of appropriate dimensions satisfying the following* LMI*:*

$$\begin{bmatrix} AX + XA^{\mathrm{T}} + BW^{\mathrm{T}} + WB^{\mathrm{T}} + 2\eta X + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) I & \delta_2 X & LX \\ \delta_2 X & -\varepsilon_1 I & 0 \\ LX & 0 & -\varepsilon_3 I \end{bmatrix} < 0, \tag{15}$$

then the error dynamical system with the feedback gain $K = W^{\mathrm{T}} X^{-1}$ and nonlinear feedback $\alpha = \frac{-(\varepsilon_2^{-1}\delta_3^2 x^{\mathrm{T}} x + \varepsilon_4^{-1} d^2)}{2\|e\|^2} P^{-1} e$ is globally exponentially stable, implying that the systems are globally exponentially synchronized.

Proof. By using Schur Complements [18], the (8) can be easily transformed to be

$$\begin{bmatrix} (A + BK)^{\mathrm{T}} P + P(A + BK) + 2\eta P + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) P^2 & \delta_2 I & LI \\ \delta_2 I & -\varepsilon_1 I & 0 \\ LI & 0 & -\varepsilon_3 I \end{bmatrix} < 0. \tag{16}$$

If we pre-multiplied $T = \mathrm{diag}\{P^{-1}, I\}$ and post-multiplied T to the left of (16), and let $X = P^{-1}$ and $K = W^{\mathrm{T}} X^{-1}$, then the result will be produced. This completes the proof. □

Reviewing the literature [17], we find that the author did not establish a criterion to find the feedback matrix gain K. Let us have a simple introduction to the problem that we will solve. The detailed information can be referred to in [17].

Based on active technique, Hao Zhang [17] transformed a class of chaos synchronization to the problem of stabilization of the following linear dynamical system with parameters perturbation but bounded,

$$\dot{e} = (A + K + \Delta A(t)) e \tag{17}$$

where e is error state vector, A is matrix with appropriate dimensions, $\Delta A(t)$ is time-varying parameters perturbation such that $\|\Delta A(t)\| \leq \delta$ and K is the feedback gain that will be found out via the following corollary.

Corollary 1. *If there exist positive constant ε_1 and matrices $X > 0$, W of appropriate dimensions satisfying the following* LMI

$$\begin{bmatrix} XA^{\mathrm{T}} + AX + W + W^{\mathrm{T}} + \varepsilon_1 I & \delta X \\ \delta X & -\varepsilon_1 I \end{bmatrix} < 0 \tag{18}$$

then the error dynamical system (17) with feedback gain matrix $K = WX^{-1}$ is asymptotically stable at $e = 0$, implying that the coupled systems in Ref. [17] *are globally exponentially synchronized*

Proof. According to the proof procedure of Theorems 1 and 2, similarly, we can easily obtain (18). □

4. Simulations

In this section, we will give two examples to illustrate the effectiveness of our two criteria: one is the chaotic Murali–Lakshmanan–Chua circuit simulated to illustrate the effectiveness of Theorem 2. The other is the chaotic Chen systems simulated to illustrate the effectiveness of the Corollary 1.

4.1. Murali–Lakshmanan–Chua circuit

The chaotic Murali–Lakshmanan–Chua circuit is described by [19]

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\beta & -\sigma\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}-f(x_1)\\ F\sin(\omega t)\end{bmatrix} \tag{19}$$

where $\sigma > 0$, $\beta > 0$, $F > 0$, $\omega > 0$ and $f(\cdot)$ is a piecewise linear function.

$$f(x_1) = bx_1 + \frac{1}{2}(a-b)(|x_1+1| - |x_1-1|) \tag{20}$$

with $a < b < 0$.

Consider master uncertain Murali–Lakshmanan–Chua circuit system as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -(\beta+\Delta\beta_1(t)) & -(\sigma+\Delta\sigma_1(t))\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}-f(x_1)\\ F\sin(\omega t)\end{bmatrix}+d_1(t) \tag{21}$$

where $d_1(t) = [0.28\sin(0.5t), -0.2\cos(5t)]^{\mathrm{T}}$, $\Delta\beta_1(t) = 0.1\sin(5t)$, $\Delta\sigma_1(t) = -0.11\cos(2t)$, and slave system as

$$\begin{bmatrix}\dot{\widehat{x}}_1\\ \dot{\widehat{x}}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -(\beta+\Delta\beta_2(t)) & -(\sigma+\Delta\sigma_2(t))\end{bmatrix}\begin{bmatrix}\widehat{x}_1\\ \widehat{x}_2\end{bmatrix}+\begin{bmatrix}-f(\widehat{x}_1)\\ F\sin(\omega t)\end{bmatrix}+d_2(t)+BK\begin{bmatrix}\widehat{x}_1-x_1\\ \widehat{x}_2-x_2\end{bmatrix}+\alpha \tag{22}$$

where $d_2(t) = [-0.1\cos(2t), 0.26\sin(3t)]^{\mathrm{T}}$, $\Delta\beta_2(t) = 0.13\cos(3t)$, $\Delta\sigma_2(t) = 0.098\sin(4t)$.

From Eqs. (21) and (22), we get

$$A=\begin{bmatrix}0 & 1\\ -\beta & -\sigma\end{bmatrix},\qquad f(x_1,t)=\begin{bmatrix}-f(x_1)\\ F\sin(\omega t)\end{bmatrix},\qquad \Delta A_1(t)=\begin{bmatrix}0 & 0\\ -\Delta\beta_1(t) & -\Delta\sigma_1(t)\end{bmatrix}$$

$$\text{and}\quad \Delta A_2(t)=\begin{bmatrix}0 & 0\\ -\Delta\beta_2(t) & -\Delta\sigma_2(t)\end{bmatrix}.$$

Since the above, we have

$$\|d_1(t)-d_2(t)\| \le \sqrt{(0.28+0.1)^2+(0.2+0.26)^2} < 0.6$$

$$\|\Delta A_2(t)\| \le \sqrt{0.13^2+0.098^2} < 0.17$$

$$\|\Delta A_1(t)-\Delta A_2(t)\| \le \sqrt{(0.13+0.1)^2+(0.11+0.098)^2} < 0.32. \tag{23}$$

According to Eqs. (14)–(16) in Ref. [16], we can rewrite Eq. (4) as

$$\dot{e} = (\bar{A}+BK)e + \bar{g}(\widehat{x}) - \bar{g}(x) + d_2(t) - d_1(t) + \Delta A_2(t)\,\widehat{x} - \Delta A_1(t)x + \alpha \tag{24}$$

where

$$\bar{A} = A + \begin{bmatrix}-\dfrac{a+b}{2} & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}-\dfrac{a+b}{2} & 1\\ -\beta & -\sigma\end{bmatrix}\quad\text{and}\quad \bar{g}(\widehat{x})-\bar{g}(x)=\begin{bmatrix}-\bar{k}_{\widehat{x}_1,x_1} & 0\\ 0 & 0\end{bmatrix}(\widehat{x}-x)$$

with $\frac{a-b}{2} \le \bar{k}_{\widehat{x}_1,x_1} \le -\frac{a-b}{2}$.

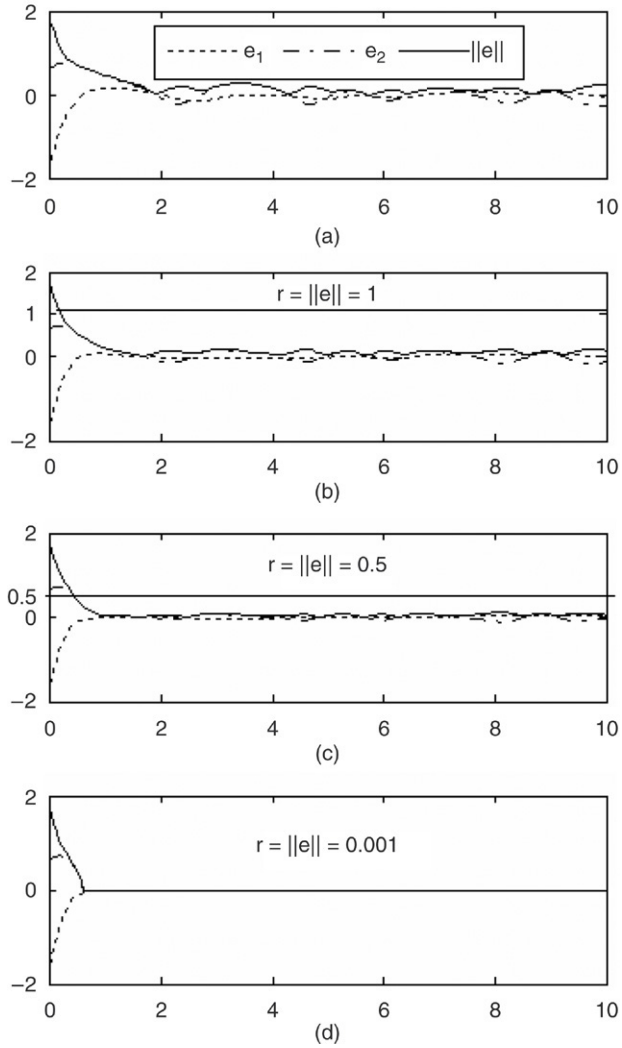

Fig. 1. Synchronization of chaotic Murali–Lakshmanan–Chua circuit with parameter perturbation and external disturbances with different controller. (a) Controller in [16], (b) our controller with $r = 1$, (c) our controller with $r = 0.5$, (d) our controller with $r = 0.001$.

From Eq. (24), we can get

$$\|\bar{g}(\widehat{x}) - \bar{g}(x)\| = \left\| \begin{bmatrix} -\bar{k}_{\widehat{x_1},x_1} & 0 \\ 0 & 0 \end{bmatrix} \right\| \|e\| \leq \left| \frac{a-b}{2} \right| \|e\| = L\|e\| \tag{25}$$

where $L = \left|\frac{a-b}{2}\right|$.

In the following, chaos synchronization based on the chaotic Murali–Lakshmanan–Chua circuit is illustrated. The parameters of the circuit used are $\sigma = 1.015$, $\beta = 1.0$, $F = 0.15$, $\omega = 0.75$, $a = -1.02$ and $b = -0.55$. The initial value of state variables of two systems is assigned as $[x_1, x_2, \widehat{x_1}, \widehat{x_2}]^{\mathrm{T}} = [1, 0.6, -0.8, 1.2]^{\mathrm{T}}$.

We have $d = 0.6$, $\delta_2 = 0.17$ and $\delta_3 = 0.32$ from (23). Choosing $B = [1, 0]^{\mathrm{T}}$ and $\eta = 0.5$, obviously, $(\bar{A}, B)$ is controllable, and one easily obtains

$$K = [-4.0880, 1.0075], \qquad P = \begin{bmatrix} 1.0018 & -0.4237 \\ -0.4237 & 0.7084 \end{bmatrix},$$

$\varepsilon_2 = 0.6309$ and $\varepsilon_4 = 0.6309$ from (15) by using Matlab LMI Toolbox. In the light of Remark 2, we select $r = 1, 0.5, 0.001$, respectively, chaos synchronization error will be bounded within $\|e\| \leq 1$, $\|e\| \leq 0.5$ and $\|e\| \leq 0.001$, respectively, as shown in Fig. 1(b)–(d). Fig. 1(a) shows the error curves by using controller in [16].

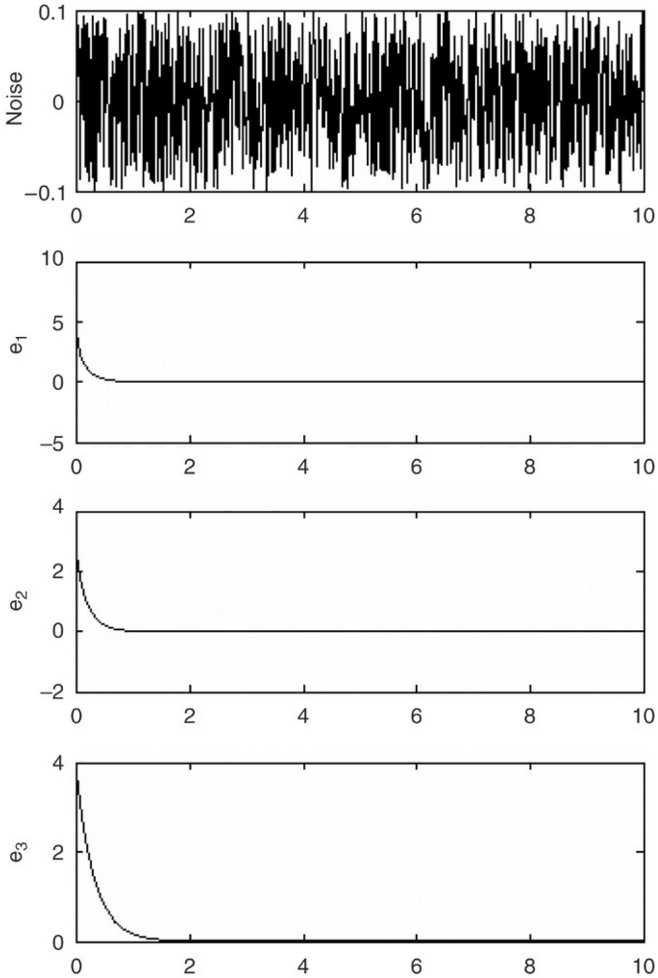

Fig. 2. Synchronization of two Chen's systems with parameter perturbation. (a) Noise, (b) error e_1, (c) error e_2 (d) error e_3.

From the Fig. 1, we can see that the controller in Ref. [16] cannot synchronize two uncertain chaotic systems. From Fig. 1(b)–(d) it can be seen that our controller can make the synchronization error within any desired bounded area. Especially, Fig. 1(d) shows that if we select r small enough, then the synchronization can be achieved. In the practical sense, our method can synchronize two uncertain systems with given accuracy. Thus, the proposed method is effective.

4.2. Chen chaotic system

Consider the Chen chaotic system as

$$\begin{cases} \dot{x}_1 = a(x_1 - x_2) \\ \dot{x}_2 = (c-a)x_1 - x_1x_3 + cx_2 \\ \dot{x}_3 = x_1x_2 - bx_3 \end{cases} \tag{26}$$

where a, b and c are three real positive parameters. When $a = 35$, $b = 3$ and $c = 28$, the Chen's system behaves chaotically.

According to the numerical simulation in Ref. [17], the system parameter b is perturbed by the stochastic noise $\varepsilon\delta(t)$ with the amplitude value of $\varepsilon = 0.1$, as is shown in Fig. 2(a).

In the light of controller design procedure in Ref. [17], we define the control function as

$$u(t) = V(t) - F(x, \widehat{x}) \tag{27}$$

where K is a 3×3 gain matrix, which will be found out by the Corollary 1 in this paper,

$$\begin{pmatrix} V_1(t) \\ V_2(t) \\ V_3(t) \end{pmatrix} = K \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \quad \text{and} \quad F(x, \widehat{x}) = \begin{pmatrix} 0 \\ -\widehat{x_1}\,\widehat{x_3} + x_1 x_3 \\ \widehat{x_1}\,\widehat{x_2} - x_1 x_3 \end{pmatrix} \quad \text{with } e_i = \widehat{x_i} - x_i, (i = 1, 2, 3).$$

Thus, the error system can be described by (17), where

$$A = \begin{pmatrix} -a & a & 0 \\ c-a & c & 0 \\ 0 & 0 & -b \end{pmatrix} \quad \text{and} \quad \Delta A(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\varepsilon\delta(t) \end{pmatrix}.$$

Hence, $\|\Delta A(t)\| \le 0.1$.

By using the previous Corollary 1 and Matlab LMI Toolbox, we can get

$$K = \begin{bmatrix} -0.0663 & 0.6256 & 0 \\ 3.1180 & -28.3106 & 0 \\ 0 & 0 & -0.1340 \end{bmatrix}.$$

According to Corollary 1, chaos synchronization is achieved as shown in Fig. 2(b)–(c). As expected, one can observe that the trajectories of the slave system asymptotically approach the ones of the master system in Fig. 2(b)–(d). Fig. 2(a) shows the noise we added. Thus, the proposed method is effective.

5. Conclusion

A new LMI criterion (Theorem 2) has been derived for robust chaos synchronization for a class of uncertain chaotic systems with different parameters perturbation and different external disturbances on both master system and slave system via Lyapunov stability theory and linear matrix inequality technique. The criterion has been applied to the chaotic Murali–Lakshmanan–Chua circuit for illustration. The proposed criterion can ensure that the slave system synchronizes to the master system at an exponential rate. Meanwhile, we also develop a LMI criterion Corollary 1 to solve the pending problem in [17]. Moreover, the criterion has been illustrated by the Chen chaotic system. Finally, it is easy to find that we only discuss about a class of system with parameters perturbation which is the coefficient of a state variable. How to cope with the case where the parameters perturbation is the coefficient of a nonlinear item in state equation is another topic that will be discussed in another paper.

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Appendix

In this appendix, we will prove that with the new controller α^*, the coupled systems (1) and (3) are globally synchronized with arbitrary given error.

Lemma A.1. *Given a continuous nonlinear system as*

$$\dot{x} = f(x, t). \tag{28}$$

If there exists a positive function $V(x, t)$ satisfying

$$\lambda_1 \|x\|^2 \le V(x, t) \le \lambda_2 \|x\|^2 \quad \forall (x, t) \in U \times R \tag{29}$$

$$\dot{V}(x, t) \le -\lambda_3 V(x, t) \quad \forall (x, t) \in U \times R \tag{30}$$

where $x \in U$, λ_1, λ_2, λ_3 are positive scalar constants, then the solution of system (28) satisfies the following condition.

$$\|x(t)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}} \|x(t_0)\| \exp(-\lambda_3 (t - t_0)/2). \tag{31}$$

Proof. By comparing theorem, we have

$$V(x,t) \le V(x,t_0)\exp(-\lambda_3(t-t_0)). \tag{32}$$

From (29), we get

$$\|x(t)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}}\,\|x(t_0)\|\exp(-\lambda_3(t-t_0)/2).$$

This completes the proof. □

Lemma A.2. *Suppose the controller α is replaced by α^*. Then, for the error system (4), there must exist $0 < \varepsilon < 1$ and area $U(e) = \{r\sqrt{1-\varepsilon} < \|e\|\}$ such that*

$$\dot{V} \le -2\eta' V \quad e \in U(e) \tag{33}$$

where $V = e^{\mathrm{T}}Pe$ and $0 < \eta' < \eta$.

Proof. From the proof procedure of Theorem 1, when $\|e\| < r$, it is easy to get

$$\begin{aligned}\dot{V} &\le -2\eta V - 2e^{\mathrm{T}}P\alpha + 2e^{\mathrm{T}}P\alpha^* \\ &= -2\eta V + (1-\|e\|^2/r^2)(\delta_3^2\,\|x\|^2/\varepsilon_2 + d^2/\varepsilon_4).\end{aligned} \tag{34}$$

Since the master system is a chaotic system, x is bounded. Therefore, there must exist M such that $\delta_3^2\|x\|^2/\varepsilon_2 + d^2/\varepsilon_4 \le M$. Defining $\varepsilon = \frac{\lambda_1 r^2\delta}{M+\lambda_1 r^2\delta}$ and $U_1(e) = \{r\sqrt{1-\varepsilon} < \|e\| < r\}$, we have

$$\dot{V} \le -2\eta V + \varepsilon M \quad \text{for every } e \in U_1(e). \tag{35}$$

Let $\delta = 2(\eta-\eta')$. From (35), we have

$$\dot{V} \le -2\eta' V - \delta V + \varepsilon M. \tag{36}$$

Obviously, there must exist constants λ_1, which is the minimum eigenvalue of the matrix P and λ_2, which is the maximum eigenvalue of the matrix P, such that $\lambda_1\|e\|^2 \le V \le \lambda_2\|e\|^2$.

Thus, from (36), we get

$$\dot{V} \le -2\eta' V - \delta\lambda_1 r^2(1-\varepsilon) + \varepsilon M. \tag{37}$$

Note that $\varepsilon = \frac{\lambda_1 r^2\delta}{M+\lambda_1 r^2\delta}$, we obtain $\dot{V} \le -2\eta' V$.

On the other hand, $\dot{V} \le -2\eta V < -2\eta' V$ for every $e \in \{\|e\| \ge r\}$. Thus we have

$$\dot{V} \le -2\eta' V \qquad e \in U(e).$$

This completes the proof. □

Theorem 3. *Suppose the controller α is replaced by α^*. Then the error system (4) is bounded stable with the bound as following*

$$\|e(t)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}}r \quad t \in [t_0+T, +\infty) \tag{38}$$

where T is a positive number, λ_1 and λ_2 are the minimum eigenvalue and maximum eigenvalue of the matrix P, respectively.

Proof. By Lemma A.2, there must exist $U_1 = \{\|e\| > r\sqrt{1-\varepsilon_1}\} \supset U_2 = \{\|e\| > r\sqrt{1-\varepsilon_2}\}$ such that $\dot{V} \le -2\eta' V$ for every $e \in U_1$. Now, in the following, we will prove that when every initial point $e(t_0) \in U_3 = \{r\sqrt{1-\varepsilon_2} < \|e\| < r\}$ then $\|e(t)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}}r$ for $t \in (t_0, +\infty)$.

Let the initial point $e(t_0) \in U_3$. By Lemma A.1, there must exist time T_0 such that $e(t_0 + T_0) \in \{\|e\| = r\sqrt{1-\varepsilon_2}\}$ and $e(t) \in \{\|e\| > r\sqrt{1-\varepsilon_2}\}$ where $t \in (t_0, t_0+T_0)$. By Lemma A.1 again, we get $\|e(t)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}}r$ for $t \in (t_0, t_0+T_0)$. If $e(t)$ enters the area $U_4 = \{\|e\| \le r\sqrt{1-\varepsilon_2}\}$ and never escapes this area, obviously, $\|e(t)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}}r$; if $e(t)$ enters the area U_4 and escapes this area, obviously it must enter the area U_3. Thus it is obvious that $\|e(t)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}}r$ for $e(t_0) \in U_3$.

As a matter of fact, by Lemma A.1, when the initial point $e(t_0) \in \{r \le \|e\|\}$, there must exist time T such that $e(t_0+T) \in U_3$. Thus $\|e(t)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}}r$ for $t \in [t_0+T, +\infty)$. When $e(t_0) \in U_4$ or $e(t_0) \in U_3$, obviously, $\|e(t)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}}r$ for $t \in [t_0, +\infty)$. This completes the proof. □

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