Controlling system dimension: A class of real systems that obey the Kaplan–Yorke conjecture

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Edited by Harry L. Swinney, University of Texas, Austin, TX, and approved October 16, 2003 (received for review August 13, 2003)

The Kaplan–Yorke conjecture suggests a simple relationship between the fractal dimension of a system and its Lyapunov spectrum. This relationship has important consequences in the broad field of nonlinear dynamics where dimension and Lyapunov exponents are frequently used descriptors of system dynamics. We develop an experimental system with controllable dimension by making use of the Kaplan–Yorke conjecture. A rectangular steel plate is driven with the output of a chaotic oscillator. We controlled the Lyapunov exponents of the driving and then computed the fractal dimension of the plate's response. The Kaplan–Yorke relationship predicted the system's dimension extremely well. This finding strongly suggests that other driven linear systems will behave similarly. The ability to control the dimension of a structure's vibrational response is important in the field of vibration-based structural health monitoring for the robust extraction of damage-sensitive features.

Analysis of data collected from nonlinear systems is often accomplished by viewing the system response as a dynamical attractor in phase space (i.e., the space defined by the system's variables). The resulting portrait of the dynamics may be thought of as a geometric object in this space to which all trajectories, within a nearby set of initial conditions, will evolve. Based on this description, a variety of metrics can then be used to extract information about the underlying process, among them fractal dimension and Lyapunov exponents (LEs). Measures of fractal dimension quantify the variation in attractor geometry over many different length scales. The assumption underlying such measures is that the distribution of points on an attractor is the same at both small and large length scales, i.e., the object is "self-similar." LEs reflect the stability of a dynamical system to perturbation in various directions. Positive LEs indicate an instability or a "stretching" of the phase space, whereas negative exponents are associated with stable or contracting phase space directions. These two metrics are arguably the most important and widely used descriptors of nonlinear system dynamics. Researchers have used these quantities to describe the dynamics of animal populations (1–3), human locomotion (4), cardiac function (5, 6) geological systems (7), climate (8, 9), and mechanical systems (10).

This work offers strong experimental evidence of a conjectured relationship between LEs and fractal dimension. The Kaplan–Yorke conjecture (11) states that a system's complete Lyapunov spectrum may be used to give direct estimates of a system's fractal dimension (specifically, information dimension), implying that knowledge of one quantity can lead to estimates of the other. Although the conjecture has been shown to be false in certain instances, it remains a useful and widely used tool in the analysis of nonlinear system dynamics. For numerically generated data, the relationship is routinely invoked (12–14) while experimental agreement has been observed for laser data (15), a mechanical oscillator (16), and in fluid mechanics (Taylor–Couette flow) (17). In each of the above cited works, the Kaplan–Yorke conjecture was used as a check between estimates of LEs and dimension.

Perhaps more importantly, the relationship implies that control of the LEs can lead to control of a system's fractal dimension. Several researchers have shown numerically that varying a system's Lyapunov spectrum can produce varied dimension estimates in accordance with Kaplan–Yorke (18–20). In each of these studies, the conjecture was tested and found to hold over a narrow range of dimension. Recent works have demonstrated the utility of controlling dimension in both estimating system damping (21) and detecting damage in structures (22, 23). In these works the system's LEs were fixed such that the system response was forced to occupy a low-dimensional attractor. However, to date the ability to adjust a system's fractal dimension by controlling LEs has not been observed in an experimental system. In this work, we control the LEs of a driven steel plate and produce dimension estimates that show excellent agreement with those obtained via the Kaplan–Yorke conjecture over a wide range of fractal dimension.

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Dimension, LEs, and Kaplan–Yorke

Attractor-Based Measures. Attractor-based analysis begins by reconstructing the system's dynamical attractor. Assuming that some dynamical variable $x(n)$ $n = 1 \cdots N$ has been measured from the system of interest at discrete, fixed-time intervals, a delay coordinate embedding (24) may be used to qualitatively reconstruct the system's underlying attractor as

$$\vec{x}(n) = (x(n), x(n+T), \ldots, x(n+(m-1)T)). \quad [1]$$

Here T is a measure of time delay, chosen to maximize the information content of $x(n)$, and m is the embedding dimension that must be large enough to "unfold" the attractor. Choice of delay is typically accomplished by looking for the first minimum of the average mutual information function (25), whereas embedding dimension is often selected by using the empirical, geometric criteria of Kennel *et al.* (26) The following discussion assumes a reconstructed attractor $\vec{x}(n)$.

Many different definitions of dimension exist in dynamical systems theory. Most of them in some way reflect the number of variables active in the dynamics. Here, we refer to dimension in the geometric sense, reflecting how an attractor's geometry varies over many orders of the attractor's length scales. Such measures collectively quantify fractal dimension, the most popular among them being the correlation dimension. Given a hypersphere of radius ε centered about some fiducial point f on the attractor, $\vec{x}(f)$, the correlation dimension reflects the way in which the average number of points within the hypersphere scales with ε. Estimating D_c, therefore, involves computing the correlation sum

This paper was submitted directly (Track II) to the PNAS office.

Abbreviation: LE, Lyapunov exponent.

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$$C(\varepsilon) = \lim_{N\to\infty} \frac{2}{N(N-1)} \sum_{i=1}^{N-1} \sum_{j>i}^{N} \Theta(\varepsilon - |\vec{x}(i) - \vec{x}(j)|) \qquad [2]$$

where

$$\Theta(\varepsilon - |\vec{x}(i) - \vec{x}(j)|) = \begin{cases} 1 : \varepsilon - |\vec{x}(i) - \vec{x}(j)| \geq 0 \\ 0 : \varepsilon - |\vec{x}(i) - \vec{x}(j)| < 0 \end{cases}.$$

By assuming a power law relationship between $C(\varepsilon)$ and ε the correlation dimension is given by

$$D_c = \lim_{\varepsilon\to 0} \log C(\varepsilon)/\log \varepsilon. \qquad [3]$$

Estimates of D_c therefore require the evaluation of **2** and extracting the slope of the log/log plots of $C(\varepsilon)$ v. ε over a well defined (more than one decade on the log scale) scaling region. For practical details regarding the estimation, the reader is referred to Theiler (27, 28).

LEs are also globally averaged quantities, reflecting the system stability in each of the principle directions. More specifically, given some initial separation $\delta_i(0)$ in direction i at time $n = 0$, the complete Lyapunov spectrum is written

$$\lambda_i = \lim_{N\to\infty} \frac{1}{N} \ln \frac{\delta_i(N)}{\delta_i(0)}. \qquad [4]$$

Each exponent specifies the average exponential rate at which the perturbations will grow or decay. The sum of these exponents therefore represents the long-term evolution of a volume element in phase space. For a chaotic system (defined by a positive LE) such an element will grow in certain directions and shrink in others. In fact, because an attractor represents steady-state dynamics, it may be thought of as existing in a balanced state between the contraction and expansion rates of a volume element in phase space. It is this general idea that gives rise to the Kaplan–Yorke conjecture (11).

The Kaplan–Yorke conjecture states that

$$D_L = k + \frac{\sum_{i}^{k} \lambda_i}{-\lambda_{k+1}}, \qquad [5]$$

where k is the maximum number of exponents, arranged in decreasing order, which may be added before the sum becomes negative, and D_L is referred to as the Lyapunov dimension, or Kaplan–Yorke dimension. D_L is conjectured to be equal to the information dimension, D_I (another measure of fractal dimension) and is typically very close, if not equal to, the correlation dimension. In this study we assume $D_I \approx D_c$. Although counterexamples to the conjecture have been found (for example the Feigenbaum attractor, ref. 29), in most cases the relationship holds. Furthermore, it has been proven by Ledrappier (30) that D_L is an upper bound on the information dimension ($D_I \leq D_L$). Control over a system's LEs can therefore be used to bound a system's dimension, regardless of whether or not the conjecture holds. The goal here is to vary the λ_i for a driven, experimental structure and then compare estimates of D_L to the computed values for D_c. In doing so we are directly testing the ability to control the dimension of the structure's response.

Filtering Chaotic Signals. From Eq. **5** it is evident that if the LEs can be controlled, so, too, can the Lyapunov dimension. In many instances, the complete Lyapunov spectrum is difficult to estimate, owing primarily to the well known difficulties associated with computing negative LEs (31). However, it is possible to construct an experiment in which the some of the LEs are known to a high degree of certainty. Consider a linear, stable $\mathcal{N}$-degree-of-freedom system, governed by the constant coefficient matrix **A** coupled to the output of a chaotic oscillator, described by the function **F**,

Fig. 1. Experimental setup.

$$\frac{d\vec{z}}{dt} = \mathbf{F}(\vec{z})$$
$$\frac{d\vec{x}}{dt} = \mathbf{A}\vec{x} + \mathbf{B}\vec{z}. \qquad [6]$$

The LEs for the linear system are the logarithm of the real parts of the eigenvalues of **A**, which are all negative. In the absence of coupling ($\mathbf{B} = 0$) each system will possess its own set of LEs denoted $\lambda_j^F\ i = 1 \cdots \mathcal{M}$ for the chaotic system and $\lambda_k^A\ j = 1 \cdots \mathcal{N}$ for the linear system. It has been shown (32) that if the two systems are coupled through **B** such that one of the state variables $\vec{z}$ forces the system $\vec{x}$, the complete Lyapunov spectrum will be the union

$$\lambda_i^S = \lambda_j^F \cup \lambda_k^A \qquad [7]$$

arranged in decreasing order in accordance with Kaplan–Yorke. From Eq. **5** it is evident that changing the LEs associated with the forcing signal $\mathbf{B}\vec{z}(n)$ can alter the dimension of the response, $\vec{x}(n)$. One way to control the LEs is to alter the time scales (bandwidth) of the forcing system; this does not affect the LEs of the structure. This is easily accomplished by multiplying the state equations by a constant. Using the system described by **6** and knowing the state matrix **A**, the dimension of the output signal can be controlled, thus providing a test bed for manipulating the Kaplan–Yorke dimension.

Experiment

The structure studied herein is a thin rectangular steel plate measuring $664 \times 408 \times 3$ mm. (Fig. 1) and clamped along the two shorter edges. For small amplitude excitation the plate will obey the linear model described by Eq. **6**, where the state matrix **A** contains the mass, stiffness, and damping properties associated with the plate. Forcing for the structure is provided by a MB Dynamics (Cleveland) shaker, attached to the plate at the location shown in Fig. 1. Response data were recorded at three different locations on the structure by using fiber Bragg grating strain sensors (33) and at the forcing location by means of a force transducer.

To determine the LEs for the plate the structure was excited with broadband white noise. Impulse response time histories for each force/response pair were then obtained via the inverse Fourier transform of the corresponding transfer function estimates. The Eigensystem realization algorithm (34) was then used to extract estimates of the state matrix **A** from which the logarithms of the real parts of the eigenvalues were obtained as the LEs. The procedure was repeated for 60 different records, each of length 8,192 points. Based on these 60 values, the

Table 1. Estimates of the first five LEs for the steel plate

λ_1^A	λ_2^A	λ_3^A	λ_4^A	λ_5^A
−8.11	−8.42	−9.04	−9.47	−11.82

confidence intervals were set to span the interquartile range, i.e. the middle 50% of the values. Assigning confidence based on the interquartile range is one approach commonly used when the underlying distribution is unknown. The mean values for each of the first five plate LEs were taken as estimates for the λ_k^A and are shown in Table 1. Based on these estimates, a driving signal could be designed to generate a structural response of varying dimension. The excitation signal was chosen as the first state variable, z_1, of the chaotic Lorenz oscillator,

$$\eta^{-1}\frac{dz_1}{dt} = 10(z_2 - z_1)$$
$$\eta^{-1}\frac{dz_2}{dt} = 28z_1 - z_2 - z_1 z_3 \quad [8]$$
$$\eta^{-1}\frac{dz_3}{dt} = -(8/3)z_3 + z_1 z_2,$$

where the tuning parameter η is used to control the bandwidth of the oscillator to produce the desired spectrum of the driving LEs. Because the driving equations of motion are known, the exponents may be computed directly by using the method described in Wolf *et al*. (35). Fourteen different forcing scenarios (bandwidths) were used for the excitation. The coefficients used and the resulting values for the λ_j^F are listed in Table 2. Note that the Lorenz system is a continuous time "flow" and therefore will always possess one exponent that is zero (perturbations in time will neither grow or decay), i.e., $\lambda_2^F = 0.0$ regardless of η. The various forcing scenarios were designed to cover a large spread of dimension values ($2 \le D_L \le 7$) while giving insight into the way dimension transitions from one integer value to the next (five cases were run for $3 \le D_L \le 4$).

As an example of how the excitation is designed, assume we want to construct a forcing signal such that the plate's response has a dimension of $D_L = 3.5$. According to the Kaplan–Yorke formula (see Eq. **5**) this requires k (the maximum number of exponents that can be added before the sum becomes negative) to be 3. We therefore must choose λ_1^F such that $\lambda_1^F + 0.0 - 8.11 > 0.0$. However, we must also maintain that $\lambda_1^F + 0.0 - 8.11 - 8.42 < 0.0$ so that $k \ngeq 4$. If these two inequalities hold, and $\lambda_3^F < -8.42$ (so that the negative exponent associated with the driving doesn't factor into the computation), then $\lambda_{k+1} = -8.42$ in the Kaplan–Yorke formula. The next step is to then consider the fractional part of Eq. **5**. Forcing the fractional part of the dimension to be 0.5 simply requires that $0.5 = (\lambda_1^F + 0.0 - 8.11)/8.42$ or $\lambda_1^F = 12.32$. We may therefore adjust η to produce a driving signal with precisely this value for the dominant LE.

One interesting implication of the Kaplan–Yorke formula is that all exponents greater in magnitude than λ_{k+1} do not play a role in the computation, regardless of the number of degrees of freedom associated with the system. It should therefore be noted that the dominant exponent λ_1^F is the key to influencing dimension in this case. The negative exponent associated with the driving is of such a large magnitude compared with the plate LEs that it plays no role in computing D_L (see Tables 1 and 2). Because $\lambda_2^F = 0.0$ for all cases, and λ_3^F does not affect D_L we may describe the results in terms of λ_1^F only.

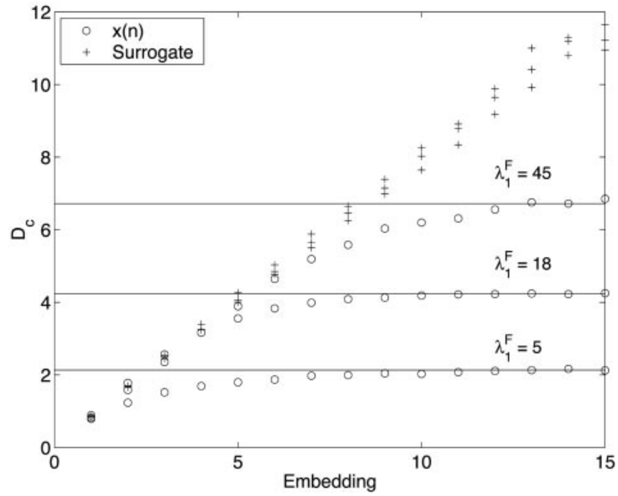

Fig. 2. Progression of D_c with embedding dimension for data taken at select values of λ_1^F and the corresponding surrogate data.

Results

Select results from the computation of D_c are displayed in Fig. 2. Computations of the correlation dimension were performed by using the algorithm of Grassberger and Proccacia (36). Each response time history consisted of $N = 50{,}000$ points. It was determined through average mutual information that the appropriate delay was $T = 12$; the sampling rate was continually adjusted to match the excitation speed so the delay was the same in each of the 14 cases. The correlation dimension algorithm was applied, evaluating the integral in Eq. **2** over a subset of $M = 5{,}000$ points as the embedding was varied from $m = 1 \cdots 15$. A scaling region of 1.5 decades was used in each of the computations.

In each of the three cases presented, the progression of D_c with embedding shows a clear plateau with the horizontal lines indicating the estimated value of D_c. Also shown is the progression for the associated phase-randomized surrogate data sets (see ref. 15 for details). Each time series was transformed into the frequency domain via the fast Fourier transform (FFT). A random phase on the interval $[0, 2\pi)$ is then introduced, and the data are transformed back into the time domain by using the inverse FFT. The resulting signal matches the mean, variance, and autocorrelation (hence the power spectrum) of the original data, yet is stochastic. In theory, stochastic signals should not exhibit a plateau, but rather fill the size of the space in which they are embedded. Estimating dimension for both a signal and its surrogate is therefore a convenient check against indications of "false determinism," which can sometimes arise because of

Table 2. Coefficients η and the resulting LEs for Lorenz excitation

λ/η	5.54	11.08	13.29	15.51	16.61	17.72	19.94	22.15	27.69	33.23	38.76	44.30	49.84	55.38
λ_1^F	5.00	10.00	12.00	14.00	15.00	16.00	18.00	20.00	25.00	30.00	35.00	40.00	45.00	50.00
λ_2^F	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$-\lambda_3^F$	80.5	161.0	193.2	225.4	241.5	257.6	289.8	322.0	402.0	483.0	563.5	644.0	724.5	805.0

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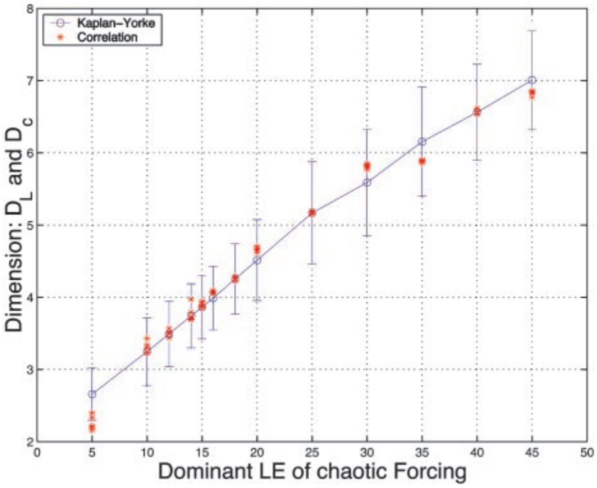

Fig. 3. Progression of D_c and D_L with dominant forcing LE.

limited amounts of data (28). Fig. 3 displays the progression of correlation dimension with the dominant forcing exponent along with the calculated Kaplan–Yorke estimate. The estimates of D_L are computed directly from Eq. **5** by using the known values of the driving exponents in combination with the estimated values of λ_k^A. For example, examining forcing scenario 5 we have $\lambda_1^F = 15.0$, $\lambda_2^F = 0.0$, $\lambda_1^A = -8.11$, $\lambda_2^A = -8.42$, giving $D_L = 3 + (15.0 + 0.0 - 8.11)/(8.42) = 3.82$. The uncertainty associated with this estimate results from the uncertainty in the estimates of λ_k^A. To account for experimental variability, the process was repeated for five different time series at each speed, resulting in five values of D_c at each speed. Each of these estimates are shown in Fig. 3. Estimates of the Kaplan–Yorke dimension clearly follow the same trend as do estimates of the correlation dimension. In fact, using the mean values of the estimates of D_L yield values that are extremely close to those obtained by direct estimates of D_c. Agreement between predicted and observed data provides strong evidence that this system obeys the Kaplan–Yorke conjecture across a broad spectrum of phase space dimension. Differences in the two trends are most likely a combination of error in estimating **A** and in the numerical difficulties associated with computing D_c. Because LEs are intimately tied to the way a system dissipates energy their estimation suffers the same well known difficulties associated with obtaining damping estimates (37). In addition, although it is difficult to assess confidence in D_c, the quality of estimates will (for a fixed amount of data) necessarily decrease for higher dimensions.

Discussion

This study illustrates experimental evidence supporting the Kaplan–Yorke conjecture over a wide range of dimension. Estimates of a system's LEs may be used to infer the dimensionality of the process. Conversely, knowledge of a system's fractal dimension may be used to gain insight into the relative magnitudes of a chaotic system's positive and negative LEs. In confirming this relationship we have also provided a mechanism for controlling (in the worst case bounding) the dimension of driven linear systems. The result holds broad implications for problems of nonlinear system identification where one has control or knowledge of the system's LEs, including autonomous systems (although it may be more difficult to experimentally manipulate the LEs of an autonomous system). The ability to control the dimensionality of a structure's response has already proven valuable in the use of nonlinear time-series analysis techniques in vibration-based structural health monitoring. The vibration-based paradigm involves exciting the structure with some prescribed input and then analyzing the structural response for damage-induced changes. Attractor-based analysis has proven extremely effective in detecting the presence and magnitude of structural degradation but is predicated on a response that is (*i*) low dimensional, yet (*ii*) sufficiently influenced by the structure's dynamics so that changes can be clearly identified. Interrogating a structure with a sinusoid, for example, produces a low-dimensional ($D_c = 1$) attractor but is unable to sufficiently resolve dynamic change. At the other extreme, the current practice of exciting structures with broadband Gaussian noise produces response attractors that are too high dimensional for performing attractor-based analysis. By using the technique described in this work, the dimension of the response can be maintained at an appropriate level, allowing for robust feature extraction. This approach also allows the practitioner the freedom to adjust the time scales of the excitation. Certain kinds of damage are visible only at higher frequencies (shorter time scales). One can therefore use this approach to interrogate the desired time scales without altering the dimension of the response provided, of course, that the relative magnitudes of the LEs are not changed.

We thank the National Research Council for funding a postdoctoral fellowship for J.M.N. and the U.S. Naval Research Laboratory for funding this work under a 6.1 Advanced Research Initiative.

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