On the synchronization of neural networks containing time-varying delays and sector nonlinearity

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Abstract

We present a systematic design procedure for synchronization of neural networks subject to time-varying delays and sector nonlinearity in the control input. Based on the drive-response concept and the Lyapunov stability theorem, a memoryless decentralized control law is proposed which guarantees exponential synchronization even when input nonlinearity is present. The supplementary requirement that the time-derivative of time-varying delays must be smaller than one is released for the proposed control scheme. A four-dimensional Hopfield neural network with time-varying delays is presented as the illustrative example to demonstrate the effectiveness of the proposed synchronization scheme.

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1. Introduction

Since Pecora and Carroll [1] proposed drive-response concept and showed that it is possible to synchronize chaotic systems through a simple coupling, the control and synchronization problems of the chaotic systems have been thoroughly studied over the past two decades due to its potential applications in creating secure communication systems [2–7]. On the other hand, Hopfield neural networks (HNNs) and cellular neural networks (CNNs) have attracted considerable attention in recent decades [8–11] and have been widely applied in a number of engineering and scientific fields, including those of neuro-biology, population biology, computing technology, image processing, and so on. Researchers have identified and investigated various chaotic phenomena induced in HNNs and CNNs by different delays [12–15].

Neural networks are nonlinear high-dimensional systems composed of many interconnected neurons. Therefore, implementing a centralized control scheme for such systems is challenging. Furthermore, the control inputs of practical systems are frequently subject to nonlinearity as a result of physical limitations. It has been shown that input nonlinearity can cause a serious degradation of the system performance, a reduced rate of response, and in a worst-case scenario, system failure if the controller is not well designed [16]. Therefore, it is clear that the effects of input nonlinearity must be taken into account when designing and implementing a synchronization control scheme. However, a review of the published literature suggests that the problem of controlling and synchronizing neural networks subject to input nonlinearity has received relatively little attention.
Consequently, this study presents a systematic design procedure for synchronization of a general class of delayed chaotic neural networks with sector nonlinearity in the control inputs. Based on the drive-response concept and the Lyapunov stability theorem, a memoryless decentralized control law is proposed which guarantees global synchronization. This controller is robust to the nonlinear input and guarantees the synchronization of the drive-response neural networks with time-varying delays. A four-dimensional Hopfield neural network with time-varying delays is presented to demonstrate the effectiveness of the proposed synchronization scheme.

The notations $\Re$ and $\Re^n$ are used to denote real number field and the real vector space of dimension $n$, respectively. $|w|$ denotes the absolute value of $w$. Moreover, $M^T$ is used to denote the transpose of a square matrix $M$, while for $x \in \Re^n$, $\|x\| = (x^T x)^{1/2}$ and $\|x\|_1 = \sum_i |x_i|$ denotes the Euclidean norm and the 1-norm of the vector, respectively.

2. Synchronization problem formulation

A general class of neural networks, including HNNs and CNNs as special cases, can be described by the following delayed differential equations

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau_j(t))) + J_i, \quad i = 1, \ldots, n,$$

where $n \geq 2$ denotes the number of neurons in the network, $x_i$ is the state variable associated with the $i$th neuron, and $c_i x_i(t)$ is an appropriately behaved function remaining the solution of drive neural networks (1) bounded. $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$ indicate the interconnection strengths among the neurons without and with a time-varying delay

$$\tau_j(t) \geq \tau_j^0(t) \geq 0, \quad i = 1, \ldots, n,$$

and $\phi_i$ is an external constant input. The time-varying delay satisfies $\tau_j^0(t) = \max(\tau_j(t))$ for $1 \leq i \leq n$ and $t \geq 0$. The initial conditions associated with system (1) are given by $x_i(0) = x_i(t) = 0$. The initial conditions associated with system (2) are given by $z_i(0) = z_i(t) = 0$. The initial conditions associated with system (3) are given by $y_i(0) = y_i(t) = 0$.

Now define the error variable in the following way:

$$z_i(t) = -c_i z_i(t) + \sum_{j=1}^{n} a_{ij} f_j(z_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(z_j(t - \tau_j(t))) + \rho_i(t) + J_i + \phi_i(u_i(t)), \quad i = 1, \ldots, n,$$

where $z_i$ is the state variable associated with the response system (2). The response systems (2) are assumed to be subject to external disturbance $\rho_i(t) \in \Re$.

Without loss of generality, the external disturbance is bounded, i.e. $|\rho_i(t)| \leq \delta_i$. $u_i(t)$ in (2) is the unidirectionally coupled term, which is regarded as the control input and appropriately designed such that the synchronization is achieved. The function $\phi_i(u_i(t))$ is a continuous nonlinear sector function, generally caused by physical limitations, in which $\phi_i(0) = 0$ and $u_i(t) \rightarrow \phi_i(u_i(t))$ lies within sector $[\beta_{i,1}, \beta_{i,2}]$, i.e.

$$\beta_{i,1} u_i^2(t) \geq u_i(t) \cdot \phi_i(u_i(t)) \geq \beta_{i,1} u_i^2(t), \quad i = 1, \ldots, n,$$

where $\beta_{i,1}$ and $\beta_{i,2}$ are non-zero positive constants. The nonlinear function $\phi_i(u_i(t))$ is shown in Fig. 1.

The synchronization problem considered in this Letter is that for different initial conditions of drive system (1) and response system (2), the two coupled systems can be synchronized by designing the appropriate control $u_i(t)$ introduced to the response system (2) such that $\lim_{t \rightarrow \infty} |z_i(t)| = 0$, $i = 1, 2, \ldots, n$.

Before proceeding to main results, the following assumption is taken for the function $f_j$ in the considered neural networks:

**Assumption 1.** $f_j : \Re \rightarrow \Re, i \in \{1, 2, \ldots, n\}$ is bounded and satisfies the Lipschitz condition with a Lipschitz constant of $L_i > 0$, i.e. $|f_j(u) - f_j(v)| \leq L_i |u - v|$ for all $u, v \in \Re$.

Now let us define the synchronization error vector $E(t)$ as $E(t) = [e_1(t), e_2(t), \ldots, e_i(t), \ldots, e_n(t)]^T$, where $e_i(t) = x_i(t) - z_i(t)$. Therefore, the error dynamics between the neural networks given in (1) and (2) can be described by

$$\dot{e}_i(t) = -c_i e_i(t) - \sum_{j=1}^{n} a_{ij} \left(f_j(e_j(t) + z_j(t)) - f_j(z_j(t))\right) - \sum_{j=1}^{n} b_{ij} \left(f_j(e_j(t - \tau_j(t)) + z_j(t - \tau_j(t))) - f_j(z_j(t - \tau_j(t)))\right) - \rho_i(t) - \phi_i(u_i(t)), \quad i = 1, \ldots, n,$$
Eq. (4) can be rewritten in the following compact form:

\[ \dot{e}_i(t) = -\left( c_ie_i(t) - \sum_{j=1}^{n} a_{ij}\theta_j(e_j(t)) - \sum_{j=1}^{n} b_{ij}\theta_j(e_j(t - \tau_j(t))) \right) - \rho_i(t) - \phi_i(u_i(t)), \quad i = 1, \ldots, n, \tag{5} \]

where

\[ \theta_j(e_j(t)) = f_j(e_j(t) + z_j(t)) - f_j(z_j(t)) \in \mathbb{R}. \]

The problem in realizing synchronization between two coupled neural networks now transforms to another problem on how to design the control input \( u_i(t) \), robust to external disturbance and input sector nonlinearity such that the resulting synchronization error vector, \( E(t) \), asymptotically converges to zero, i.e. \( \lim_{t \to \infty} \| E(t) \| = 0 \).

3. Main results

The main result of this Letter is stated in the following theorem.

**Main theorem.** For the drive-response structure of the neural networks given in (1) and (2), if the control input \( u_i(t) \) in (2) is suitably designed as

\[ u_i(t) = \gamma_i \eta_i \frac{e_i(t)}{|e_i(t)|}, \quad \gamma_i > \frac{1}{\beta_i,1}, \tag{6} \]

where

\[ \eta_i = \left( |c_i| + \sum_{j=1}^{n} L_i |a_{ji}| |i| + \delta_i + \sum_{j=1}^{n} L_i |b_{ji}| |e_i(t)|_{\text{max}} \right) \quad \text{and} \quad e_i(t)|_{\text{max}} = \max_{t \in [-\tau^*, t]} |e_i(t)| \]

is the maximum absolute value of each element of \( E(t) \), then the neural networks given in (1) and (2) are synchronized, i.e. the trajectory of the error dynamics system given in (5) converges to \( E(t) = [e_1(t), e_2(t), \ldots, e_n(t)]^T = 0 \).

**Proof.** To confirm that the origin of (5) is asymptotically stable, a Lyapunov function \( V \) is defined as

\[ V(t) = \| E(t) \|_1 = \sum_{i=1}^{n} |e_i(t)|. \tag{7} \]

We can verify that \( V(t) \) is a nonnegative function over \([-\tau^*, +\infty)\) and is radially unbounded, i.e., \( V(t) \to +\infty \) as \( e_i(t) \to +\infty \). According to the definition of \( \theta_j(e_j(t)) \) and Assumption 1, it is shown that

\[ |\theta_j(e_j(t))| \leq L_j |e_j(t)| \quad \text{and} \quad |\theta_j(e_j(t - \tau_j(t)))| \leq L_j |e_j(t - \tau_j(t))|. \tag{8} \]
Taking the derivative of $V$ with respect to time $t$ and substituting from (5), one has

\[
\dot{V}(t) = \sum_{i=1}^{n} \frac{e_i(t)\dot{e}_i(t)}{|e_i(t)|} = \sum_{i=1}^{n} \frac{e_i(t)}{|e_i(t)|} \left[ -c_i e_i(t) - \sum_{j=1}^{n} a_{ij} \theta_j(e_j(t)) + \sum_{j=1}^{n} b_{ij} \theta_j(e_j(t - \tau_j(t))) - \rho_i(t) - \phi_i(u_i(t)) \right].
\]  

(9)

Since

\[
\sum_{i=1}^{n} \frac{e_i(t)}{|e_i(t)|} \left( -c_i e_i(t) - \sum_{j=1}^{n} a_{ij} \theta_j(e_j(t)) + \sum_{j=1}^{n} b_{ij} \theta_j(e_j(t - \tau_j(t))) - \rho_i(t) - \phi_i(u_i(t)) \right) \leq \sum_{i=1}^{n} \left( |c_i e_i(t)| + |\theta_j(e_j(t))| + |\theta_j(e_j(t - \tau_j(t)))| + |\phi_i(u_i(t))| \right) \leq \sum_{i=1}^{n} \left( |c_i e_i(t)| + \delta_i \right),
\]

(10)

\[
\sum_{i=1}^{n} \frac{e_i(t)}{|e_i(t)|} \left( -c_i e_i(t) - \sum_{j=1}^{n} a_{ij} \theta_j(e_j(t - \tau_j(t))) \right) \leq \sum_{i=1}^{n} \frac{|e_i(t)|}{|e_i(t)|} \sum_{j=1}^{n} \left( a_{ij} \theta_j(e_j(t)) \right) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} L_j |a_{ij}| |e_j(t)| = \sum_{i=1}^{n} \sum_{j=1}^{n} L_i |b_{ji}| |e_i(t)|,
\]

(11)

and

\[
\sum_{i=1}^{n} \frac{e_i(t)}{|e_i(t)|} \left( -c_i e_i(t) - \sum_{j=1}^{n} b_{ij} \theta_j(e_j(t - \tau_j(t))) \right) \leq \sum_{i=1}^{n} \frac{|e_i(t)|}{|e_i(t)|} \sum_{j=1}^{n} \left( b_{ij} \theta_j(e_j(t - \tau_j(t))) \right) \leq \sum_{i=1}^{n} \sum_{j=1}^{n} L_i |b_{ji}| |e_i(t)|,
\]

(12)

(9) can be further derived as

\[
\dot{V}(t) \leq \sum_{i=1}^{n} \left( \left| c_i \right| + \sum_{j=1}^{n} L_i |a_{ji}| \right) |e_i(t)| + \delta_i + \sum_{j=1}^{n} L_i |b_{ji}| |e_i(t)|_{\text{max}} - \sum_{i=1}^{n} \frac{e_i(t)\phi_i(u_i(t))}{|e_i(t)|} = \sum_{i=1}^{n} \left( \eta_i - \frac{e_i(t)\phi_i(u_i(t))}{|e_i(t)|} \right).
\]

(13)

If the control input $u_i(t)$ is suitably designed as shown in (6), i.e.

\[
u_i(t) = \gamma_i \eta_i \frac{e_i(t)}{|e_i(t)|}, \quad \gamma_i > \frac{1}{\beta_i,1}.
\]

(14)

Furthermore, from (3), we have

\[
u_i(t) \cdot \phi_i(u_i(t)) = \gamma_i \eta_i \frac{e_i(t)}{|e_i(t)|} \phi_i(u_i(t)) \geq \beta_i,1 \left( \gamma_i \eta_i \frac{e_i(t)}{|e_i(t)|} \right)^2 = \beta_i,1 (\gamma_i \eta_i)^2.
\]

(15)

From (15), it is shown that:

\[
\frac{e_i(t)}{|e_i(t)|} \phi_i(u_i(t)) \geq \beta_i,1 \gamma_i \eta_i.
\]

(16)

Substituting (16) into (13) yields

\[
\dot{V}(t) \leq -\sum_{i=1}^{n} (\gamma_i \beta_i,1 - 1) \eta_i.
\]

(17)

Since $\gamma_i > \frac{1}{\beta_i,1}$ has been specified in (6), it can be concluded that $\dot{V}(t) < 0$. Furthermore, according to Lyapunov theory, the inequality $\dot{V}(t) < 0$ indicates that $V(t)$ converges to zero, i.e. $E(t) = [e_1(t), e_2(t), \ldots, e_n(t)]^T = 0$. Hence, the proof is completed. □

**Remark 1.** The controller in (6) applies a discontinuous control law and hence chattering is evident. In order to eliminate this chattering, the controller can be modified to

\[
u_i(t) = \gamma_i \eta_i \frac{e_i(t)}{|e_i(t)|} + \omega, \quad \gamma_i > \frac{1}{\beta_i,1},
\]

(18)

where $\omega$ is a sufficiently small positive constant. From previous studies [17,18], the solution of the system (5) with (18) can be made arbitrarily close to the solution of system (5) with (6) by specifying a sufficiently small value of $\omega$. 

4. An illustrative example

This section of the Letter presents an illustrative example to demonstrate the effectiveness of the proposed synchronization scheme. The simulation is performed using MATLAB software and the fourth-order Runge–Kutta method with a fixed step size of 0.001.

Example. A four-dimensional Hopfield Neural Network with time-varying delays is introduced as the drive system. This HNN is described by

\[
\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{4} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{4} b_{ij} f_j(x_j(t - \tau_j(t))), \quad i = 1, 2, 3, 4,
\]  

(19)

where

\[ c_i = 1, \quad A = (a_{ij})_{4 \times 4} = \begin{bmatrix} 0.85 & -2 & -0.5 & 0.5 \\ 1.8 & 1.15 & 0.6 & 0.3 \\ 1.1 & 1.21 & 2.5 & 0.05 \\ 0.1 & -0.4 & -1.5 & 1.45 \end{bmatrix} \quad \text{and} \quad f_i(x_i) = \tanh(x_i). \]

The chaotic behavior of this system with \( B = (b_{ij})_{4 \times 4} = (0)_{4 \times 4} \) and \( \tau_j(t) = 0 \) has already been investigated in a previous study [19]. The present example chooses the case of

\[ B = (b_{ij})_{4 \times 4} = \begin{bmatrix} -0.4 & -1.1 & 0.4 & 0.2 \\ 0.5 & -0.1 & 0.3 & 0.3 \\ 0.1 & -0.8 & 0.2 & 0.1 \\ 0.1 & -0.4 & -0.6 & -0.5 \end{bmatrix} \]

and \( \tau_j(t) = 1.2(1 - \cos(t)) \) seconds, \( j = 1, 2, 3, 4 \), which satisfy \( 0 \leq \tau_j(t) \leq 2.4 = \tau_j^* \) seconds, \( -1.2 \leq \dot{\tau}_j(t) \leq 1.2 = \sigma_j^* \). The HNN (19) has a chaotic attractor which can be seen in Fig. 2.

The response system is designed as

\[
\dot{z}_i(t) = -c_i z_i(t) + \sum_{j=1}^{4} a_{ij} f_j(z_j(t)) + \sum_{j=1}^{4} b_{ij} f_j(z_j(t - \tau_j(t))) + \rho_i(t) + \phi_i(u_i(t)), \quad i = 1, 2, 3, 4.
\]  

(20)

Due to physical limitations, the control input is subjected to sector nonlinearities defined as

\[
\phi_1(u_1(t)) = \begin{bmatrix} 0.8 + 0.1 \sin(u_1(t)) \end{bmatrix} u_1(t), \quad \phi_2(u_2(t)) = \begin{bmatrix} 0.6 + 0.3 \cos(u_2(t)) \end{bmatrix} u_2(t),
\]

\[
\phi_3(u_3(t)) = \begin{bmatrix} 0.8 + 0.1 \sin(u_3(t)) \end{bmatrix} u_3(t), \quad \phi_4(u_4(t)) = \begin{bmatrix} 0.6 + 0.3 \cos(u_4(t)) \end{bmatrix} u_4(t)
\]  

(21)

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Fig. 2. Chaotic trajectory of the HNN (19).
and the external disturbance $\rho_i(t)$ ($i = 1, 2, 3, 4$) are chosen as $\rho_i(t) = 0.2G(t)$. The function $G(t)$ is a uniform distribution on the interval $(0, 1)$. Therefore, we have $\delta_i = 0.2$ ($i = 1, 2, 3, 4$). Based on (3), we can derive $\beta_{1,1} = 0.7$, $\beta_{1,2} = 0.9$, $\beta_{2,1} = 0.3$, $\beta_{2,2} = 0.9$ and $\beta_{3,1} = 0.7$, $\beta_{3,2} = 0.9$, $\beta_{4,1} = 0.3$, $\beta_{4,2} = 0.9$. Obviously, the system satisfies Assumption 1 with $L_1 = L_2 = L_3 = L_4 = 1$.

From (18), the continuous control inputs are determined as:

$$u_1(t) = \gamma_1 \left( \left| c_1 + \sum_{j=1}^{4} L_1 |a_{j1}| e_1(t) \right| + \delta_1 + \sum_{j=1}^{4} L_1 |b_{j1}| e_1(t)_{\max} \right) \frac{e_1(t)}{|e_1(t)| + 0.01}, \quad \gamma_1 = 1.8 > \frac{1}{0.7}. \quad (22)$$

$$u_2(t) = \gamma_2 \left( \left| c_2 + \sum_{j=1}^{4} L_2 |a_{j2}| e_2(t) \right| + \delta_2 + \sum_{j=1}^{4} L_2 |b_{j2}| e_2(t)_{\max} \right) \frac{e_2(t)}{|e_2(t)| + 0.01}, \quad \gamma_2 = 3.5 > \frac{1}{0.5}. \quad (23)$$

$$u_3(t) = \gamma_3 \left( \left| c_3 + \sum_{j=1}^{4} L_3 |a_{j3}| e_3(t) \right| + \delta_3 + \sum_{j=1}^{4} L_3 |b_{j3}| e_3(t)_{\max} \right) \frac{e_3(t)}{|e_3(t)| + 0.01}, \quad \gamma_3 = 1.8 > \frac{1}{0.7}. \quad (24)$$

and

$$u_4(t) = \gamma_4 \left( \left| c_4 + \sum_{j=1}^{4} L_4 |a_{j4}| e_4(t) \right| + \delta_4 + \sum_{j=1}^{4} L_4 |b_{j4}| e_4(t)_{\max} \right) \frac{e_4(t)}{|e_4(t)| + 0.01}, \quad \gamma_4 = 3.5 > \frac{1}{0.5}. \quad (25)$$

In numerical simulations, second is used for the time unit. The simulation results with initial conditions of $x(s) = [0.5 0.1 2]^T$ and $z(s) = [0 1 0 -1]^T$ for $-1.2(1 - \cos(t)) \leq s \leq 0$ second are shown in Figs. 3–5. Fig. 3 shows the state responses of drive and response systems corresponding to the proposed continuous control inputs in (22)–(25). It shows that the response system and drive system can reach synchronization in about 0.3 second after control operation at $t = 2$ second. Meanwhile, Fig. 4 depicts the synchronization error. It can be seen that the synchronization errors are regulated to zero asymptotically. Fig. 5 illustrates the continuous control inputs $u_1(t)$, $u_2(t)$, $u_3(t)$ and $u_4(t)$. From the simulation result, it shows that the proposed controller works well and robust to both external disturbance and the input nonlinearity. In particular, it is worthy of note that the controller is localized since it does not adopt any information of coupling states. This property makes the proposed controller easy to implement.

5. Conclusion

In this Letter, a systematic procedure is presented for control design to guarantee the global synchronization of a class of neural networks including Hopfield neural networks and cellular neural networks with/without time delays. It is worthy of note that the controller is memoryless and uses only the local state rather than coupling states interactive in the neural networks. Also the supplementary requirement that the time-derivative of time-varying delays must be smaller than one is released for the proposed
control scheme. Therefore, the proposed scheme is easily implemented. The results have shown that the proposed controller is robust to external disturbance and the input nonlinearity caused by physical limitations.

References