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Robust synchronization of unified chaotic systems via sliding mode control

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Abstract

This paper investigates the chaos synchronization problem for a class of uncertain master–slave unified chaotic systems. Based on the sliding mode control technique, a robust control scheme is established which guarantees the occurrence of a sliding motion of error states even when the parameter uncertainty and external perturbation are present. Furthermore, a novel proportional–integral (PI) switching surface is introduced for determining the synchronization performance of systems in the sliding mode motion. Simulation results are proposed to demonstrate the effectiveness of the method.

1. Introduction

A chaotic system is a highly complex dynamic nonlinear system and its response exhibits a number of specific characteristics, including an excessive sensitivity to the initial conditions, broad Fourier transform spectra, and fractal properties of the motion in phase space. Since the pioneering work of Pecora and Carroll in 1990 [1], chaos synchronization has received increasing attention over the last few years. Chaos synchronization can be applied in the vast areas of physics and engineering systems such as in chemical reactions, power converters, biological systems, information processing, especially in secure communication [2–5].

In 1963, Lorenz found the first classical chaotic attractor [6]. In 1999, Chen and Ueta found another chaotic attractor, which is similar, but not topologically equivalent to the Lorenz attractor [7]. In 2002, Lü and Chen also found the critical attractor between the Lorenz and Chen attractors [8]. To bridge the gap between the Lorenz attractor and Chen attractor, Lü et al. presented a unified chaotic system [9]. It includes the Lorenz and Chen systems as two extremes, respectively, and Lü system as a transition system [2,9,14]. Although there are some results reported on the unified chaotic systems [10–14], it is very necessary to future explore the synchronization of the uncertain unified chaotic systems. For designing a robust control of uncertain systems, sliding mode control is frequently adopted due to its inherent advantages of easy realization, fast response, good transient performance and insensitive to variation in plant parameters or external disturbances [15]. In this paper, we examine the problem of synchronization based on sliding mode control for uncertain unified chaotic systems. To achieve this goal, a novel proportional–integral (PI) switching surface

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is proposed to simplify the task of assigning the performance of the closed-loop error system in sliding motion. Having established the PI switching surface, a sliding mode controller (SMC) is designed. This controller is robust to the parameter uncertainty and external perturbation and guarantees the occurrence of sliding motion and the synchronization of the master–slave unified chaotic systems.

The organization of this paper is as follows: Section 2 describes the synchronization problem of uncertain master–slave unified chaotic systems; the PI switching surface and the sliding mode controller are designed in Section 3; a numerical example to demonstrate the effectiveness of the proposed method is included in Section 4. For simplicity, in the following section, W^T denotes the transpose of W, and $\|W\|$ represents the Euclidean norm when W is a vector or the induced norm when W is a matrix. $\lambda_i(W)$ denotes an eigenvalue of W and $\lambda_{\max}(W)$ represents the eigenvalue of W with the maximum real part. $\mathrm{sign}(s(t)) = [\mathrm{sign}(s_1)\ \cdots\ \mathrm{sign}(s_m)]^T \in R^{m\times 1}$, $\mathrm{sign}(s_i) = 1$, if $s_i > 0$; $\mathrm{sign}(s_i) = 0$, if $s_i = 0$; $\mathrm{sign}(s_i) = -1$, if $s_i < 0$.

2. Synchronization for uncertain master–slave unified chaotic systems

Consider the unified chaotic system which is described by

$$\begin{cases} \dot{x} = (25\alpha + 10)(y - x), \\ \dot{y} = (28 - 35\alpha)x - xz + (29\alpha - 1)y, \\ \dot{z} = xy - \left(\dfrac{8+\alpha}{3}\right)z, \end{cases} \tag{1}$$

where x, y, z are state variable and $\alpha \in [0,1]$.

Obviously, system becomes the original Lorenz system for $\alpha = 0$ while system becomes the original Chen system for $\alpha = 1$. When $\alpha = 0.8$, system becomes the critical system. In particular, system (1) bridges the gap between Lorenz system and Chen system. Moreover, system (1) is always chaotic in the whole interval $\alpha \in [0,1]$.

It is well know that system (1) plays a very important role in the investigation of the generalized Lorenz system family. Therefore, it is very necessary to future investigate the chaos synchronization for the unified chaotic systems.

In this paper, our problem undertaken here is to consider the synchronization problem of system (1) based on the sliding mode control. For the unified chaotic system (1), the master and slave systems are defined below, respectively,

$$\begin{cases} \dot{x}_m = (25\alpha + 10)(y_m - x_m), \\ \dot{y}_m = (28 - 35\alpha)x_m - x_m z_m + (29\alpha - 1)y_m, \\ \dot{z}_m = x_m y_m - \left(\dfrac{8+\alpha}{3}\right)z_m \end{cases} \tag{2}$$

and

$$\begin{cases} \dot{x}_s = (25\alpha + 10)(y_s - x_s), \\ \dot{y}_s = (28 - 35\alpha)x_s - x_s z_s + (29\alpha - 1)y_s + u_1 + \Delta f_1(x_s, y_s, z_s, p), \\ \dot{z}_s = x_s y_s - (\dfrac{8+\alpha}{3})z_s + u_2 + \Delta f_2(x_s, y_s, z_s, p), \end{cases} \tag{3}$$

where the lower scripts 'm' and 's' stand for the master (or drive) system and the slave (or response) one, respectively. u_1 and u_2 are the sliding mode controllers such that the two chaotic systems can be synchronized. $p \in R$ is the external perturbation. Δf_1 and Δf_2 are the uncertainties including parameter uncertainty and external perturbation applied to the slave system. In general it is assumed that Δf_1 and Δf_2 are bounded by

$$\|\Delta f\| = \left\| \begin{bmatrix} \Delta f_1 \\ \Delta f_2 \end{bmatrix} \right\| \leqslant \beta_1 \left\| \begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix} \right\| + \beta_2 = \beta_1 \|X_s\| + \beta_2, \tag{4}$$

where $\beta_1, \beta_2 > 0$ are given.

Define the error signal as

$$\begin{cases} e_1(t) = x_s(t) - x_m(t), \\ e_2(t) = y_s(t) - y_m(t), \\ e_3(t) = z_s(t) - z_m(t), \end{cases} \tag{5}$$

which gives that

$$\begin{cases} x_m z_m - x_s z_s = -z_m e_1 - x_s e_3, \\ -x_m y_m + x_s y_s = y_m e_1 + x_s e_2. \end{cases} \tag{6}$$

From Eqs. (5) and (6), we have the following error dynamics:

$$\begin{cases} \dot{e}_1 = -(25\alpha + 10)e_1 + (25\alpha + 10)e_2, \\ \dot{e}_2 = (28 - 35\alpha)e_1 + (29\alpha - 1)e_2 - z_m e_1 - x_s e_3 + u_1 + \Delta f_1, \\ \dot{e}_3 = -\frac{(8+\alpha)}{3} e_3 + y_m e_1 + x_s e_2 + u_2 + \Delta f_2, \end{cases} \tag{7}$$

or in the form of matrix as

$$\dot{e} = Ae + Bf + Bu + B\Delta f, \tag{8}$$

where

$$A = \begin{bmatrix} -(25\alpha + 10) & (25\alpha + 10) & 0 \\ (28 - 35\alpha) & (29\alpha - 1) & 0 \\ 0 & 0 & -\frac{(8+\alpha)}{3} \end{bmatrix}; \quad e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix};$$

$$f = \begin{bmatrix} -z_m e_1 - x_s e_3 \\ y_m e_1 + x_s e_2 \end{bmatrix}; \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}; \quad \Delta f = \begin{bmatrix} \Delta f_1 \\ \Delta f_2 \end{bmatrix}.$$

The control goal considered in this paper is that the two unified chaotic systems can be synchronized such that the resulting error vector satisfies

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|x_s(t) - x_m(t)\| \to 0. \tag{9}$$

3. Switching surface and controller design

Using a sliding mode control method to synchronize the coupled unified chaotic systems with perturbations involves two basic steps: (1) selecting an appropriate switching surface such that the sliding motion on the sliding manifold is stable and ensures $\lim_{t\to\infty}\|e(t)\| = 0$; and (2) establishing a robust control law which guarantees the existence of the sliding manifold $s(t) = 0$ even in the event of parameter uncertainty and external perturbation.

Now, the proportional–integral switching surface is defined as

$$s = Ce - \int_0^t (CA + CBK)e(\tau)\,d\tau, \tag{10}$$

where $s \in R^{2\times1}$, $C \in R^{2\times3}$ and $K \in R^{2\times3}$. C is chosen to satisfy $CB \neq 0$ (i.e., CB is nonsingular) and K is chosen such that $\lambda_{max}(A + BK) < 0$ (i.e., $(A + BK)$ is stable).

It is well known that when the system operates in the sliding mode, it satisfies the following equations [16,17]:

$$s = Ce - \int_0^t (CA + CBK)e(\tau)\,d\tau = 0 \tag{11a}$$

and

$$\dot{s} = C\dot{e} - (CA + CBK)e = 0. \tag{11b}$$

Therefore, we can obtain the equivalent control $u_{eq}(t)$ in the sliding manifold by differentiating (10) with respect to time and substituting form (8):

$$\dot{s} = C\dot{e} - (CA + CBK)e = CAe + CBf + CBu_{eq} + CB\Delta f - CAe - CBKe = CB(u_{eq} + f + \Delta f - Ke) = 0. \tag{12}$$

Since CB is nonsingular, the equivalent control $u_{eq}(t)$ in the sliding mode is given by

$$u_{eq} = Ke - f - \Delta f. \tag{13}$$

Substituting $u_{eq}(t)$ into (8), the following sliding mode equation is obtained as

$$\dot{e} = Ae + Bf + BKe - Bf - B\Delta f + B\Delta f = (A + BK)e. \tag{14}$$

By (14), it is interesting to note that when the error dynamics between master–slave unified chaotic systems is in the sliding mode, the system is insensitive to parameter uncertainty and external perturbation. In other words, the controlled system is robust. Furthermore, we can easily assign the performance of error dynamics (14) in the sliding mode just by selecting an appropriate matrix K using any pole assignment method.

Before stating the scheme of the controller, the reaching condition of the sliding mode is given below.

Lemma 1. *The motion of the sliding mode is asymptotically stable, if the following reaching condition is held*:

$$s^{\mathrm{T}}(t)\dot{s}(t) < 0. \tag{15}$$

Proof. Let $V(t) = 0.5s^{\mathrm{T}}(t)s(t)$ be the Lyapunov function. Differentiating $V(t)$ with respect to time yields

$$\dot{V}(t) = s^{\mathrm{T}}(t)\dot{s}(t). \tag{16}$$

Therefore, according to the Lyapunov stability theorem, we known that if $s^{\mathrm{T}}(t)\dot{s}(t) < 0$, then equilibrium at the origin is asymptotically stable; i.e., the vector $s(t)$ will decay to zero. □

To achieve the reaching condition indicated in Lemma 1, a control law is proposed as

$$u = Ke - \gamma(CB)^{-1}[\|CB\|(\|f\| + \beta_1\|X_{\mathrm{s}}\| + \beta_2)]\,\mathrm{sign}(s), \tag{17}$$

where γ is an arbitrarily constant larger than 1.

In the following, the proposed scheme (17) will be proved to be able to derive the error dynamics (8) onto the sliding mode $s(t) = 0$.

Theorem 1. *The reaching condition of expression* (15) *of the sliding mode is satisfied, if the control* $u(t)$ *is given by* (17).

Proof. Substituting (8) and (17) into the derivative $s^{\mathrm{T}}(t)\dot{s}(t)$, we get the following result:

$$\begin{aligned} s^{\mathrm{T}}\dot{s} &= s^{\mathrm{T}}[CBf + CBu + CB\Delta f - CBKe] \\ &= s^{\mathrm{T}}[CBf + CBKe - \gamma[\|CB\|(\|f\| + \beta_1\|X_{\mathrm{s}}\| + \beta_2)]\cdot \mathrm{sign}(s) + CB\Delta f - CBKe] \\ &\leqslant -\gamma[\|CB\|(\|f\| + \beta_1\|X_{\mathrm{s}}\| + \beta_2)]\cdot s^{\mathrm{T}}\,\mathrm{sign}(s) + \|CB\|(\|f\| + \|\Delta f\|)\|s\|. \end{aligned} \tag{18}$$

Furthermore, $s^{\mathrm{T}}\,\mathrm{sign}(s) = |s_1| + |s_2| \geqslant \|s\| = \sqrt{s_1^2 + s_2^2}$, we have

$$s^{\mathrm{T}}\dot{s} \leqslant (1-\gamma)[\|CB\|(\|f\| + \beta_1\|X_{\mathrm{s}}\| + \beta_2)]\|s\|. \tag{19}$$

Since $\gamma > 1$ has been selected in (17), one can conclude that reaching condition ($s^{\mathrm{T}}(t)\dot{s}(t) < 0$) is always satisfied. Thus the proof is achieved completely. □

4. A numerical example

In this section, to verify and demonstrate the effectiveness of the proposed method, we discuss the simulation results for Lorenz and Chen system. In simulation experiments, values of uncertain parameters are chosen as follows:

$$\Delta f_1(x_{\mathrm{s}}, y_{\mathrm{s}}, z_{\mathrm{s}}, p) = \Delta f_2(x_{\mathrm{s}}, y_{\mathrm{s}}, z_{\mathrm{s}}, p) = 0.5\cos(t)\|X_{\mathrm{s}}\| + 0.3\sin(2t). \tag{20}$$

Based on (4), $\beta_1 = 0.5$ and $\beta_2 = 0.3$ can be obtained. We choose $\gamma = 1.5 > 1$, $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ such that $CB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is nonsingular.

Case I: Lorenz system. When $\alpha = 0$, Eqs. (2) and (3) is Lorenz's system. First, we choose $K = \begin{bmatrix} -32.2 & 4 & 0 \\ 0 & 0 & 0.6667 \end{bmatrix}$ such that $\lambda_{\max}(A_{\alpha=0} + BK) = -2 < 0$. For this numerical simulation, we assume that the initial conditions $(x_{\mathrm{m}}(0), y_{\mathrm{m}}(0), z_{\mathrm{m}}(0)) = (1.5, 2, 1)$ and $(x_{\mathrm{s}}(0), y_{\mathrm{s}}(0), z_{\mathrm{s}}(0)) = (-1, -5, -10)$ are employed. The simulation results are shown in Figs. 1–3. Figs. 1 and 2 show, respectively, the corresponding $s(t)$ and the state responses of systems (2) and (3). Fig. 3 shows the synchronization error. It can be seen that the synchronization errors converge to zero rapidly.

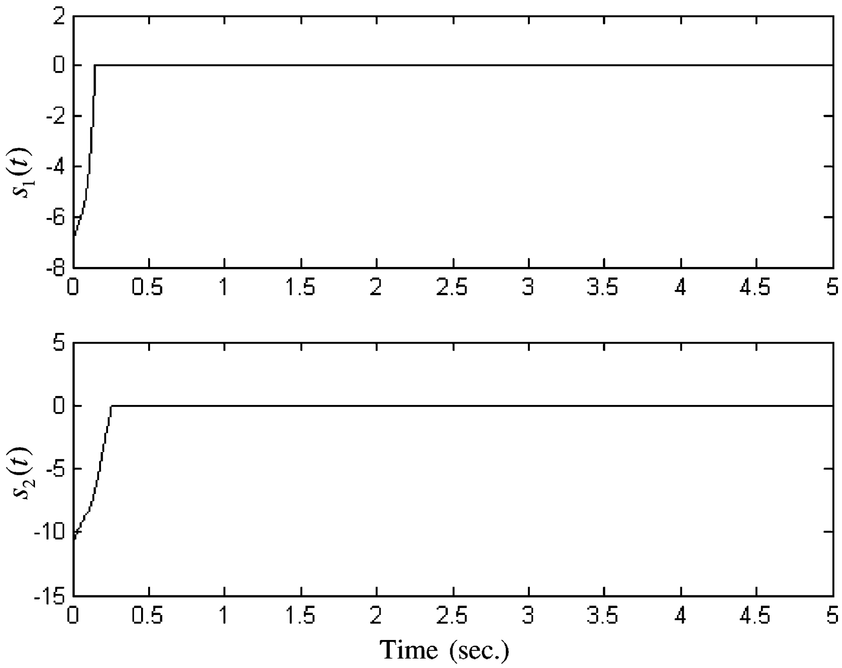

Fig. 1. Time response of $s(t)$ for Lorenz's systems ($\alpha = 0$).

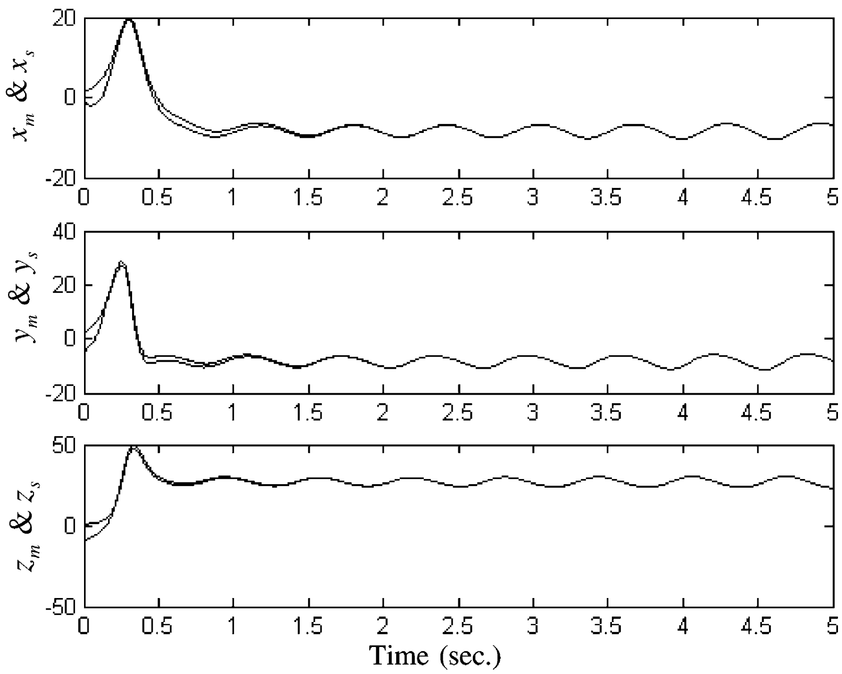

Fig. 2. State trajectories of driver system with $\alpha = 0$.

Case II: Chen system. When $\alpha = 1$, Eqs. (2) and (3) is Chen's system. We choose $K = \begin{bmatrix} -21.34 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ such that $\lambda_{\max}(A_{\alpha=1} + BK) = -2 < 0$. Under the same simulation condition of Case I, the simulation results are given in Figs. 4–6.

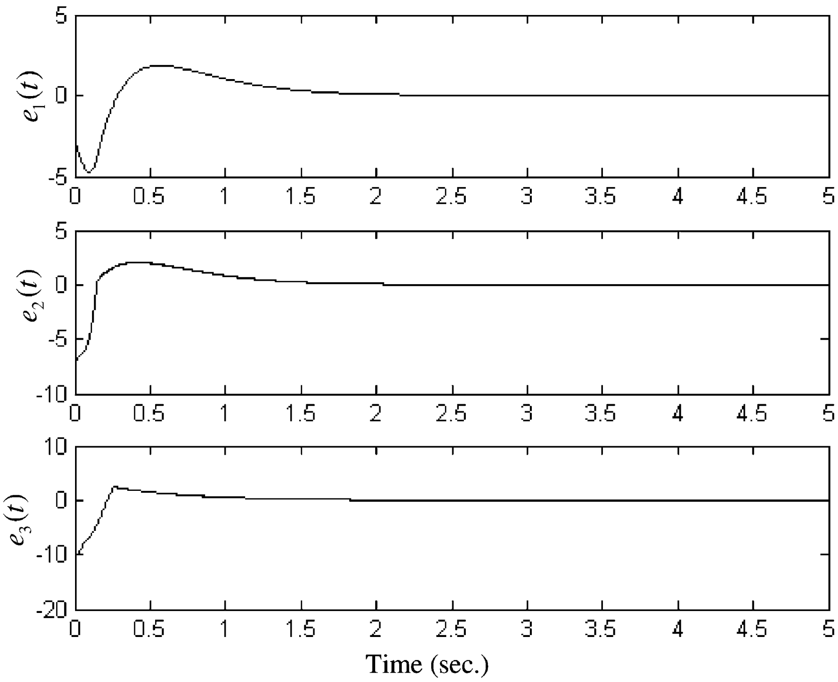

Fig. 3. Synchronization errors between master and slave chaotic systems ($\alpha = 0$).

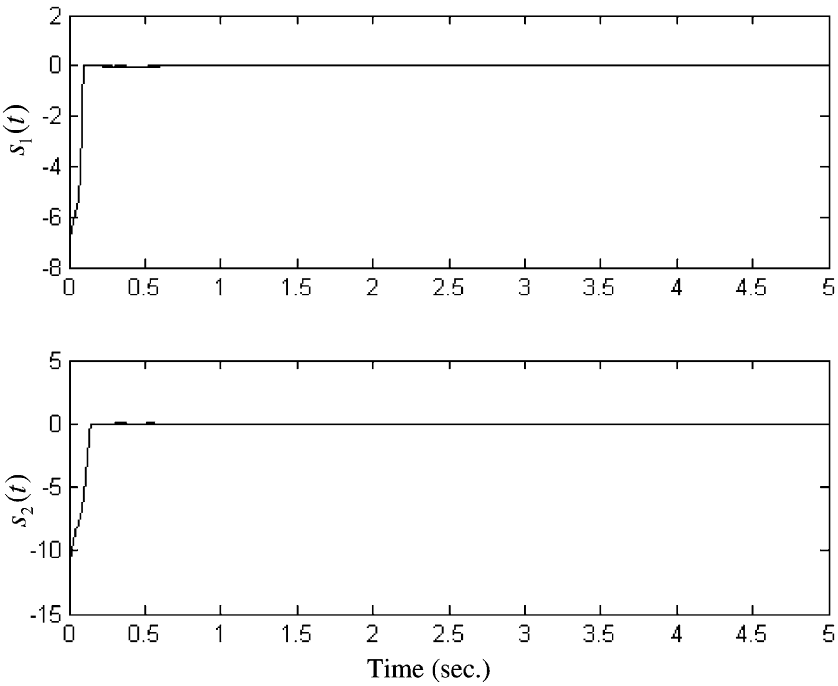

Fig. 4. Time response of $s(t)$ for Chen's system ($\alpha = 1$).

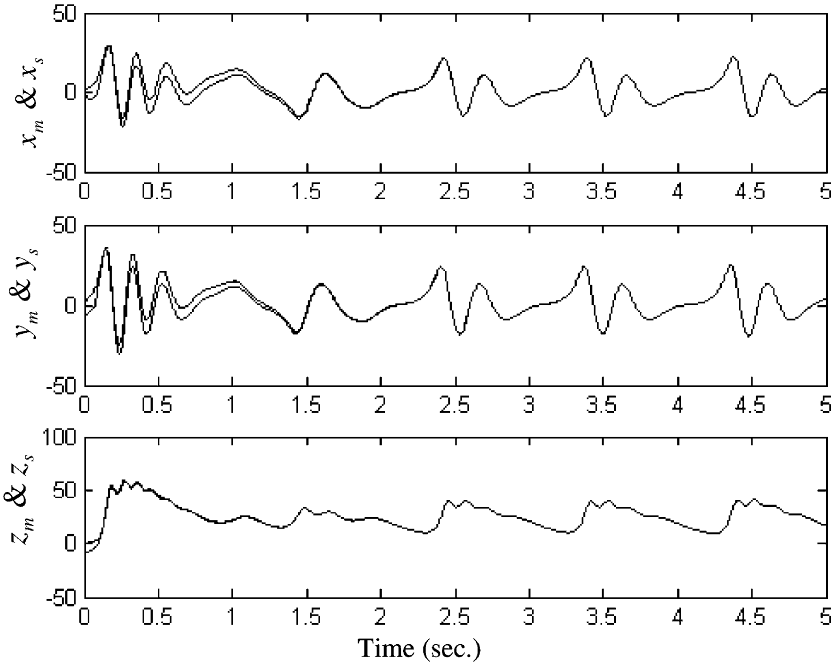

Fig. 5. State trajectories of driver system with $\alpha = 1$.

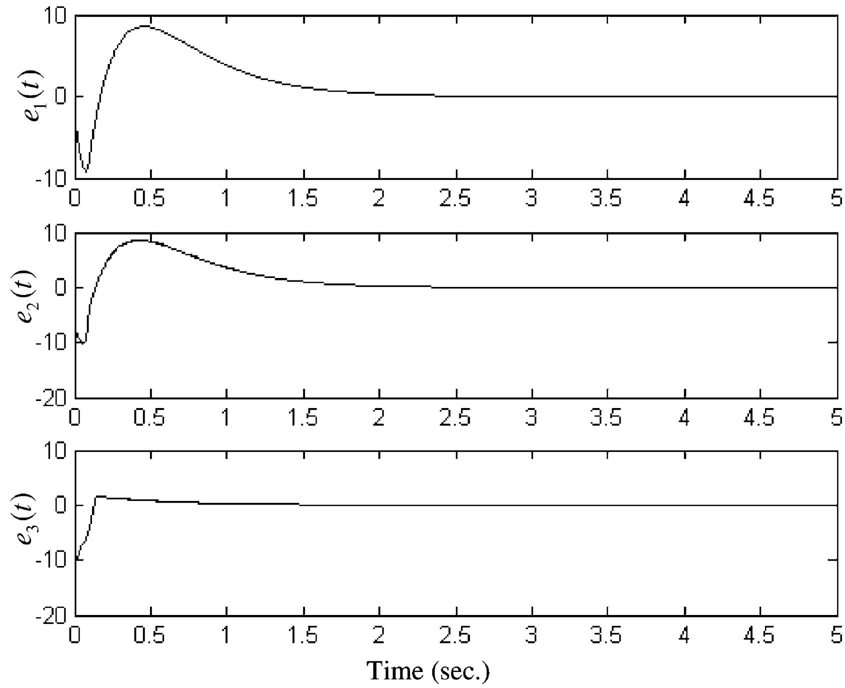

Fig. 6. Synchronization errors between master and slave chaotic systems ($\alpha = 1$).

5. Conclusions

In this paper, we investigate the synchronization problem of the uncertain unified chaotic systems. By the new PI switching surface, it is found the stability of the error dynamics in the sliding mode is easily ensured. A novel sliding mode controller has also been proposed to guarantee the occurrence of the sliding motion even when parameter uncertainty and external perturbation are present. Finally, a numerical simulation is provided to show the effectiveness of our method.

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