Chaos synchronization of two stochastic Duffing oscillators by feedback control

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Abstract

This paper addresses chaos synchronization of two identical stochastic Duffing oscillators with bounded random parameters subject to harmonic excitations. In the analysis the stochastic Duffing oscillator is first transformed into an equivalent deterministic nonlinear system by Gegenbauer polynomial approximation, so that the chaos synchronization problem of stochastic Duffing oscillators can be reduced into that of the equivalent deterministic systems. Then a feedback control strategy is adopted to synchronize chaotic responses of two identical equivalent deterministic systems under different initial conditions. The feedback parameters are determined through analysis of the top Lyapunov exponent of the variational equation of the controlled responding system. Numerical analysis shows that the feedback control strategy is an effective way to synchronize two identical stochastic Duffing systems.

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1. Introduction

Since Pecora and Carroll proposed their method of synchronizing chaos [1], theoretical as well as experimental research on chaos synchronization has been carried out in a variety of nonlinear dynamic systems. It was thought impossible previously to synchronize two chaotic systems, but now synchronization of chaotic motion becomes tractable. Chaos synchronization has been widely investigated in many fields, such as physical [2], biological, chemical and medical science [3,4], secure communications [5], etc. Zhang et al. [6] studied chaos synchronization of two parametrically excited pendulums by feedback control. Yassen [7] applied the adaptive control to synchronize two modified Chua’s circuit systems. Xiao et al. [8] investigated turbulence control and synchronization in a modified complex Ginzburg–Landau system. There are a large number of papers dealing with chaos synchronization problems, such as some synchronization criteria for coupled chaotic systems by Sun et al. [9]; chaos synchronization between two chaotic

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2. Transformation of stochastic Duffing system into its deterministic equivalent

Consider a dissipative Duffing oscillator with a bounded random parameter and subject to a harmonic excitation. The differential equation of the response can be expressed as

\[ \ddot{x} + a\dot{x} + b(x + cx^3) = p(t), \quad a > 0, \]

where \( a, c \) are deterministic constants, \( p(t) = F\sin(\omega t) \), and \( b \) is a random parameter which can be expressed in standard form:

\[ b = \bar{b} + \sigma \xi, \]

where \( \bar{b} \) is the mean value of \( b \); \( \sigma \) is a constant coefficient; and \( \xi \) is a bounded random variable defined on \([-1, 1]\) with a given \( \lambda \)-PDF. Since the system itself is stochastic, so is the response even under deterministic excitations only. Hence, the response of Eq. (1) should be a function of time \( t \) and the random variable \( \xi \). According to the orthogonal polynomial approximation, the response can be expressed in the following form:

\[ x(t, \xi) = \sum_{l=0}^{N} x_l(t) G_l^j(\xi), \]

where \( G_l^j(\xi) \) represents the \( l \)th Gegenbauer Polynomial. We note that only if \( N \to \infty \), the sum of Eq. (3) can precisely stand for the solution of the stochastic system. Otherwise, the sum of finite number of terms in Eq. (3) just stands for an approximate solution with a minimal mean square residual error.

Substituting Eqs. (2) and (3) into Eq. (1) and applying the orthogonal relationship of Gegenbauer polynomials, we can follow the equivalent nonlinear deterministic system [13]:

\[
\begin{align*}
\ddot{x}_0(t)/dr^2 + a\dot{x}_0(t)/dr + \bar{b}x_0(t) + |\sigma x_1(t)x_1' + \bar{b}cg_0 + \sigma cg_1x_1' = p(t), \\
\ddot{x}_1(t)/dr^2 + a\dot{x}_1(t)/dr + \bar{b}x_1(t) + \sigma (x_2(t)x_2' + x_0(t)\beta_0) + bcg_1 + \sigma cg_2(x_2'x_2' + g_2^j\beta_0) = 0, \\
\vdots \\
\ddot{x}_N(t)/dr^2 + a\dot{x}_N(t)/dr + \bar{b}x_N(t) + \sigma (x_{N-1}(t)\beta_{N-1} + bcg_N + \sigma cg_{N-1}x_{N-1}' + g_{N-1}\beta_{N-1}) = 0,
\end{align*}
\]

where \( x_j', \beta_j' \ (j = 0, 1, \ldots, N + 1) \) are functions of \( \lambda \), and \( g_j' \ (j = 0, 1, \ldots, N + 1) \) are functions of \( x_0, x_1, \ldots, x_N \). The detailed information about \( x_j', \beta_j', g_j' \) is given in Ref. [13].

Eq. (4) is just the equivalent nonlinear deterministic system reduced by the Gegenbauer polynomial approximation for the stochastic Duffing system with a random parameter \( \xi \) of a \( \lambda \)-PDF. The equivalent deterministic system determines the time functions \( x_j(t) \) alone, playing a significant role in the following analysis. Remembering the approximate expression of Eq. (3), one can easily see that once the \( x_j(t) \) \((j = 0, 1, \ldots, N)\) are found, the approximate stochastic response of the original stochastic dynamic system is at hand. Namely, after we obtain the solution \( x_j(t) \) of Eq. (4) by any effective numerical methods, say the Runge–Kutta method [17], the approximate stochastic response of the original stochastic Duffing system can be expressed as.
\[ x(t, \xi) \approx \sum_{j=0}^{N} x_j(t) G_j^f(\xi). \]  

(5)

Corresponding to every sampled value of the random parameter, say \( \xi = \tilde{\xi} \), where \( \tilde{\xi} \in [-1, 1] \), there is a corresponding sample response:

\[ x(t, \tilde{\xi}) \approx \sum_{j=0}^{N} x_j(t) G_j^f(\tilde{\xi}). \]

(6)

3. Synchronization of two stochastic Duffing systems

By introducing the state variables:

\[ Y(t) = [y_1(t), y_2(t), \ldots, y_{2N+1}(t), y_{2N+1}(t)]^\top = [x_0(t), \ldots, x_N(t), \tilde{x}_0(t), \ldots, \tilde{x}_N(t)]^\top, \]

(7)

Eq. (4) can be rewritten into the first-order differential system:

\[ \dot{Y} = AY + G(Y) + F(t), \quad Y \in \mathbb{R}^{2(N+1)}, \]

(8)

where \( A \) is a constant matrix, \( G(Y) \) is the nonlinear part of the system, and \( F(t) \) is the driving signal, which can be written respectively as follows:

\[
A = \begin{bmatrix} 0 & 1 \\ H & -aI \end{bmatrix},
\]

\[
G(Y) = [g_1(Y), g_2(Y), \ldots, g_{2N+2}(Y)]^\top,
\]

\[
F(t) = [f_1(t), f_2(t), \ldots, f_{2N+2}(t)]^\top,
\]

where \( I \) is a \((N+1) \times (N+1)\) unit matrix, and \( H, g_i(Y), f_i(t) \) can be expressed as

\[
H = \begin{bmatrix} -b & -\sigma x_1^i \\ -\sigma \beta_0^i & -b & \cdot & \cdot \\ -\cdot & -\cdot & -\sigma x_N^i \\ -\sigma \beta_{N-1}^i & -b \end{bmatrix},
\]

\[
g_i(Y) = \begin{cases} 0, & i = 1, \ldots, N+1, \\ -b c_i^g - \alpha c_i^f x_i^i, & i = N+2, \\ -b c_i^g - \alpha \left( g_i^j \left( x_i^j \right) + \sum_{j-(N+1)}^{j-(N+3)} \beta_i^j \right), & i \neq 1, \ldots, N+2, \end{cases}
\]

\[
f_i(t) = \begin{cases} p(t), & i = N+2, \\ 0, & i \neq N+2. \end{cases}
\]

Now introducing a responding system with the same form as Eq. (8), together with a controlling effort \( U(t) \) added on the right side of the equation, we have

\[ \dot{\overline{Y}} = A\overline{Y} + G(\overline{Y}) + F(t) + U(t), \quad \overline{Y} \in \mathbb{R}^{2(N+1)}, \]

(9)

where \( Y \) represents the state vector of the responding system.

In this paper we try to consider the synchronization through unitary feedback control, so we let

\[ U(t) = K(Y - \overline{Y}). \]

(10)

Thus, we have

\[ \dot{\overline{Y}} = A\overline{Y} + G(\overline{Y}) + F(t) + K(Y - \overline{Y}). \]

(11)

Let \( \overline{Y} = Y + \delta \overline{Y} \), and put Eq. (11) in the vicinity of \( \overline{Y} = Y \) into a Taylor series, which is then truncated beyond the linear term. Thus, we have the variational equation of the controlled responding system:
\[ \dot{\delta Y} = (A - K + \Omega)\delta Y, \]  

(12)

where

\[ \Omega = \left. \frac{\partial G(Y)}{\partial Y} \right|_{Y = Y}. \]

Now we have the following theorem: The systems (8) and (9) will synchronize only if the top Lyapunov exponent of Eq. (12),

\[ \lambda_{\text{top}} = \lim_{t \to \infty} \frac{1}{t} \ln \left| \frac{\delta Y(t)}{\delta Y(0)} \right| \]

is negative, i.e. only if \( \lambda_{\text{top}} < 0 \), and for any initial condition \( Y(0) \neq \overline{Y}(0) \) we have

\[ \lim_{t \to \infty} |Y(t) - \overline{Y}(t)| = 0. \]

(13)

4. Numerical simulation

The following numerical simulation will demonstrate the effectiveness of the proposed control scheme. The fourth-order Runge–Kutta method is used to integrate the differential equations with time step 0.01. The parameters of the system are chosen as

\[ N = 4, \quad a = 0.3, \quad \bar{b} = -1.0, \quad \sigma = 0.1, \quad c = -1.0, \quad \omega = 1.2, \quad P = 0.36, \quad \lambda = 1.0. \]

For simplicity, the feedback matrix \( K \) is selected as a diagonal matrix, i.e.

\[ K = \text{diag}[k_1, k_2, \ldots, k_{10}], \]

where

\[ k_1 = k_2 = k_3 = k_4 = k_5 = k', \quad k_6 = k_7 = k_8 = k_9 = k_10 = k''. \]

The phase trajectory of the ensemble mean response with respect to \( \xi \), i.e. \( E[X(t, \xi)] \) is shown in Fig. 1, which is typically chaotic.

Figs. 2 and 3 show the variation of top Lyapunov exponent versus the feedback parameters \( k' \) and \( k'' \). It is clearly shown that the sign of top Lyapunov exponent depends on matching pairs of \( k' \) and \( k'' \).

Without lose of generality the different initial conditions are taken as

\[ Y = [-1.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0]^T, \]

\[ \overline{Y} = [1.0, 0.0, 0.0, 0.0, 1.0, 0.0, 0.0, 0.0, 0.0, 0.0]^T \]

for the driving system and the responding system respectively in the above chaos synchronization problem. The simulation results for the mean responses of system (8) and system (9) without the feedback control are shown in Fig. 4, while Fig. 5 shows the mean responses of the driving system (8) and the responding system (9) under the feedback con-
Fig. 2. The variation of top Lyapunov exponent versus $k'$.

Fig. 3. The variation of top Lyapunov exponent versus $k''$.

Fig. 4. The mean responses without feedback control. Curve A: for driving system; curve B: for responding system.

Fig. 5. The mean responses with feedback control ($k' = 0.34, k'' = 0.2$). Curve A: for driving system; curve B: for responding system.
control. It is clearly seen that the chaotic responses of two identical systems under different initial conditions get synchronized after a short transient period. Fig. 6 shows the evolutions of state synchronization errors, which all converge to zero as time $t$ goes to infinity.

We have studied the chaos synchronization of the ensemble mean responses of the two stochastic Duffing systems. The following work will focus on the chaos synchronization of two sample responses, each of which belongs to an ensemble of infinite numbers of deterministic sample responses. As we have pointed out in Section 1, stochastic chaos synchronization is to synchronize two ensembles of chaotic motion in two stochastic systems. Therefore, it can be deduced if the chaos synchronization of two stochastic Duffing systems is realized, then every pair of sample motions of the driving system and the responding system will synchronize in view of Eq. (6).

It is clearly shown in Figs. 7 and 8 that the sample responses, $x(t,0)$, of two identical stochastic Duffing systems are also synchronized through the feedback control, which verifies what we have deduced.

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**Fig. 6.** The time response for the state synchronization errors $e_i = \bar{y}_i - y_i$ ($i = 1, \ldots, 5$).

**Fig. 7.** The sample responses ($\xi = 0$) without feedback control, Curve A: for driving system; curve B: for responding system.

**Fig. 8.** The sample responses ($\xi = 0$) with feedback control ($k' = 0.34, k'' = 0.2$), Curve A: for driving system; curve B: for responding system.
5. Conclusion

Stochastic chaos synchronization stands for an infinitely large amount of realizations, each of which is a deterministic chaos synchronization of two identical deterministic Duffing oscillators. Since the random parameters are modeled by bounded random variables with $\lambda$-PDF, fit for practical engineering meaning, so the stochastic system can be transformed into an equivalent deterministic system via the Gegenbauer polynomial approximation. Then the problems of stochastic chaos synchronization can be transferred into those of deterministic chaos synchronization of two deterministic equivalent systems. Hence any available effective method for solving problems of deterministic chaos synchronization can be applied to the latter one. Thus, the equivalent deterministic system bridges the gap between the problem of stochastic chaos synchronization and that of the deterministic one.

The feedback control approach is an effective way to synchronize two equivalent deterministic systems resulting from two identical stochastic Duffing oscillators respectively. To guarantee synchronization, the entrainment zone, where $\Lambda_{\text{top}} < 0$, in feedback control parameter space is discussed. As long as the parameters are selected in the entrainment zone, the synchronized chaotic motion of two stochastic Duffing systems can be realized, which has been demonstrated by a numerical example.

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